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A trace formula for canonical differential expressions

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Abstract

We prove a trace formula for pairs of self-adjoint operators associated to canonical differential expressions. An important role is played by the associated Weyl function.

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1. Introduction

In this paper we prove a trace formula for *canonical differential expressions* of the form

$$-iJ\frac{df}{dt}(t,z) = zf(t,z) + V(t)f(t,z), \quad t\geqslant 0,\ z\in\mathbb{C}, \tag{1.1}$$

where

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 & k(t) \\ k(t)^* & 0 \end{pmatrix}$$

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and where the function k, called the *potential*, is $\mathbb{C}^{n\times n}$-valued and with entries in $\mathbf{L}_1(0,\infty)$. The solution $f(t,z)$ is $\mathbb{C}^{2n\times p}$-valued (typically, p will be equal to 1, n or $2n$). Canonical differential expressions have a long history and have been studied in particular by Kreĭn and his coworkers. See e.g. [1,25,29–32]. They play an important role in inverse scattering, generalized Fourier analysis and related topics.

We associate to the differential expression (1.1) a pair (H_+, H_-) of self-adjoint operators as follows: they both have as domain the subspace of functions $f\in\mathbf{L}_2^{2n}(0,\infty)$ which are absolutely continuous with respect to Lebesgue measure and for which $(I_n \quad -I_n)f(0)=0$ and are defined by

$$H_+f(t) = -iJ\frac{df}{dt}(t,z) - \begin{pmatrix} 0 & k(t) \\ k(t)^* & 0 \end{pmatrix} f(t,z) \tag{1.2}$$

and

$$H_-f(t) = -iJ\frac{df}{dt}(t,z) + \begin{pmatrix} 0 & k(t) \\ k(t)^* & 0 \end{pmatrix} f(t,z), \tag{1.3}$$

respectively.

We prove that for every non-real z,

$$\operatorname{rank}\{(H_+ - zI)^{-1} - (H_- - zI)^{-1}\} = n$$

and give a formula for the trace of the operator

$$(H_+ - zI)^{-1} - (H_- - zI)^{-1}$$

in terms of the *Weyl function* of the canonical differential expression, see Theorem 1.2. The Weyl function is one of a number of functions of z associated to (1.1) and which we called in [8] the *characteristic spectral functions* of the canonical differential expression. Two other functions which will play an important role in the sequel are the *scattering function* and the *spectral function*.

We gather the main properties of the spectral function in the next theorem. We first recall that the Wiener algebra $\mathscr{W}^{n\times n}$ consists of the functions of the form

$$f(z) = D + \int_{-\infty}^{\infty} e^{izt}u(t)\,dt, \tag{1.4}$$

where $D\in\mathbb{C}^{n\times n}$ and where $u\in\mathbf{L}_1^{n\times n}(\mathbb{R})$. The subalgebra $\mathscr{W}_+^{n\times n}$ (resp. $\mathscr{W}_-^{n\times n}$) consists of the functions of the form (1.4) for which the support of u is in $\mathbb{R}_+$ (resp. in $\mathbb{R}_-$).

Theorem 1.1. *The canonical differential expression has a unique* $\mathbb{C}^{2n\times 2n}$*-valued solution* $\Theta(t,z)$ *such that* $\Theta(0,z)=I_{2n}$. *The limit*

$$\begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} = \lim_{t\to+\infty} e^{-itzJ}\Theta(t,z) \tag{1.5}$$

exists for every real z *and the function*

$$W(z)=(\alpha(z)-\beta(z))^{-1}(\alpha(z)-\beta(z))^{-1*}, \quad z\in\mathbb{R}$$

is a spectral function for the canonical differential expression in the following sense: the map which to $f\in\mathbf{L}_2^{2n}(\mathbb{R}_+)$ *associates the function*

$$(U_+f)(z)=\frac{1}{\sqrt{2\pi}}(I_n \quad I_n)\int_0^\infty \Theta(t,z^*)^*f(t)\,dt \tag{1.6}$$

defines a unitary map onto $\mathbf{L}_2^n(W)$:

$$\int_0^\infty f(t)^*f(t)\,dt=\int_{\mathbb{R}}(U_+f)(t)^*W(t)(U_+f)(t)\,dt.$$

Moreover,

$$(U_+H_+f)(z)=z(U_+f)(z)$$

for $f\in\operatorname{dom} H_+$. *The spectral function* W *is in the Wiener algebra and satisfies* $W(\infty)=I_n$.

See [24,17, pp. 1.6, p. 6.5] for a proof and see [22] for a direct proof when the function W is rational. When $k(t)\equiv 0$, we have $\Theta(t,z)=e^{iztJ}$ and map (1.6) reduces to the classical Fourier transform.

The Weyl function of the canonical differential expression is the unique function $N(z)$ analytic in the open upper half-plane, with $N(\infty)=iI_n$ and such that

$$W(z)=\operatorname{Im} N(z), \quad z\in\mathbb{R}.$$

Thus, if $W(z)=I_n+\int_{-\infty}^{\infty} e^{izt}u_W(t)\,dt$ with $u_W\in\mathbf{L}_2^{n\times n}(\mathbb{R})$ we have

$$N(z)=i\left\{I_n+2\int_0^\infty e^{itz}u_W(t)\,dt\right\}.$$

We note that N is bounded in the closed upper half-plane.

Theorem 1.2. *Let* H_+ *and* H_- *be the differential operators defined by* (1.2) *and* (1.3). *Then for any* z *off the real line the operator*

$$(H_+-zI)^{-1}-(H_--zI)^{-1}$$

has rank n. Let N be the Weyl function associated to the canonical differential expression (1.1). *Then*,

$$\mathrm{Tr}((H_+ - zI)^{-1} - (H_- - zI)^{-1}) = \mathrm{Tr}\, N(z)^{-1}N'(z) \tag{1.7}$$

and

$$\widetilde{\det}_{z_0}(H_- - zI)(H_+ - zI)^{-1} = \det N(z)N(z_0)^{-1}, \quad z \in \mathbb{C}_+, \tag{1.8}$$

where $z_0 \in \mathbb{C} \backslash \mathbb{R}$ *and* $\widetilde{\det}_{z_0}$ *is the generalized perturbation determinant associated to the pair* (H_+, H_-).

The definition and properties of generalized perturbation determinants are recalled in the appendix. The proof of the theorem is given in Section 5.2. In the proof we will make use of results of [24]. An important role in the arguments of [24] is played by transformations of the form

$$f_-(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\mathrm{Im}\; N(t)f(t)\,dt}{t - z} + iN(z)f(z).$$

These transformations were introduced and used in [2–4] to solve the inverse scattering problem associated to a function analytic and with a positive real part in the open upper half-plane (i.e. a Carathéodory function). Since [24] is not widely available we will present proofs of the results of [24] which we use. We also use extensively the paper [17].

We also give detailed formulas in the case where the Weyl function N is rational and analytic at infinity. This is equivalent to the fact that the spectral function W or the scattering function is rational. Potentials $k(t)$ corresponding to this case were characterized in [5]. The rational case was further studied in [6–8,19–22].

If

$$N(z) = i(I_n + c(zI - a)^{-1}b)$$

is a minimal realization of N, formula (1.7) implies that

$$\mathrm{Tr}((H_+ - wI)^{-1} - (H_- - wI)^{-1}) = \mathrm{Tr}(a - wI)^{-1} - \mathrm{Tr}(a^{\times} - wI)^{-1},$$

and from (1.8) we obtain

$$\widetilde{\det}_{z_0}(H_+ - wI)(H_- - wI)^{-1} = \frac{\det(a - wI)^{-1}(a^{\times} - wI)}{\det(a - z_0I)^{-1}(a^{\times} - z_0I)},$$

where $a^{\times} = a - bc$.

The outline of the paper is as follows. In Section 2 of the paper we recall the necessary background on reproducing kernel Hilbert spaces. In Section 3 we review some preliminaries on differential expressions of the form (1.1). Section 4 contains the proof of a result of A. Iacob which is used in the sequel. The proof of

Theorem 1.2 is given in Section 5. Section 6 deals with the rational case. Finally, we review in an appendix the definition and main properties of perturbation determinants.

A word on notation. For a function $f \in \mathbf{L}_2(\mathbb{R})$ the Fourier transform and its inverse are denoted by

$$\hat{f}(z) = \int_{\mathbb{R}} e^{itz} f(t)\, dt \quad \text{and} \quad \check{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} f(z)\, dz,$$

respectively.

The Hardy space of the open upper half-plane is denoted by $\mathbf{H}_2$ and the Hardy space of the open lower half-plane will be denoted by $\overline{\mathbf{H}_2}$.

2. Preliminaries on reproducing kernel Hilbert spaces

Reproducing kernel Hilbert spaces of analytic functions will play an important role in this paper and we here review the basic definitions and some properties which will be used in the sequel.

Definition 2.1. A $\mathbb{C}^{n \times n}$-valued function $K(z, w)$ defined for z and w in some set Ω is said to be positive in Ω if it is hermitian: $K(z, w) = K(w, z)^*$, and if for every choice of integer r and points $w_1, \ldots, w_r \in \Omega$ the $r \times r$ block matrix with ℓ, j entry $K(w_\ell, w_j)$ is positive.

Associated to a positive function is a uniquely defined Hilbert space (which we will denote by $\mathscr{H}(K)$) of functions from Ω into $\mathbb{C}^n$ with the following two properties:

1. For every choice of $w \in \Omega$ and $c \in \mathbb{C}^n$ the function $z \mapsto K(z, w)c$ belongs to $\mathscr{H}(K)$.
2. For every $f \in \mathscr{H}(K)$ and w, c as above,

$$\langle f(z), K(z, w)c \rangle_{\mathscr{H}(K)} = c^* f(w). \tag{2.1}$$

See e.g. [10,33]. The space $\mathscr{H}(K)$ is called the reproducing kernel Hilbert space with reproducing kernel $K(z, w)$. As an example take $\Omega = \mathbb{C}_+$ (the open upper half-plane); the function $K(z, w) = \frac{1}{-2\pi i (z - w^*)}$ is positive in $\mathbb{C}_+$ and the associated reproducing kernel Hilbert space is the Hardy space of the open upper half-plane. Formula (2.1) is then Cauchy's formula for Hardy functions; see [18, p. 34]. A slight extension of this example is:

Example 2.2. Let

$$N(z) = \begin{cases} i, & \forall z \in \mathbb{C}_+, \\ -i, & \forall z \in \mathbb{C}_-, \end{cases}$$

where $\mathbb{C}_-$ denotes the open lower half-plane. The function $\frac{N(z)-N(w)^*}{4\pi(z-w^*)}$ is positive in $\mathbb{C}\backslash\mathbb{R}$ and the associated reproducing kernel Hilbert space is the set of all functions of the form

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)\,dt}{t-z},$$

where $f \in \mathbf{L}_2(\mathbb{R})$ and with norm

$$\|F\|_{\mathscr{L}(N)} = \|f\|_{\mathbf{L}_2}.$$

This example is a particular case of Theorem 2.5. Writing $f(t) = f_-(t) + f_+(t)$ where f_+ (resp. f_-) is the restriction to the real line of a function of the Hardy space of the open upper half-plane (resp. of the Hardy space of the open lower half-plane) and using Cauchy's formula for Hardy functions we have

$$F(z) = \begin{cases} f_+(z), & z \in \mathbb{C}_+, \\ -f_-(z), & z \in \mathbb{C}_-. \end{cases}$$

The following simple fact will be used a number of times in the paper and we write it as a proposition.

Proposition 2.3. *Let $K(z,w)$ be a $\mathbb{C}^{n\times n}$-valued function positive in some set Ω. The linear span of the functions $z \mapsto K(z,w)c$ with $w \in \Omega$ and $c \in \mathbb{C}^n$ is dense in $\mathscr{H}(K)$.*

Indeed, let $F \in \mathscr{H}(K)$ be orthogonal to all the functions $K(z,w)c$. By the reproducing kernel property

$$\langle F(z), K(z,w)c \rangle_{\mathscr{H}(K)} = c^* F(w) = 0$$

and hence $F(z) \equiv 0$.

Another important property we will need is the following result:

Proposition 2.4. *Let $K(z,w)$ be a $\mathbb{C}^{n\times n}$-valued function positive in a set Ω and assume that the associated reproducing kernel Hilbert space $\mathscr{H}(K)$ is separable. Let $F_1(z), F_2(z), \ldots$ be an orthonormal basis of $\mathscr{H}(K)$. Then,*

$$K(z,w) = \sum_{\ell=1}^{\infty} F_\ell(z) F_\ell(w)^*,$$

where the convergence is in norm and pointwise.

See for instance [10,33] for a proof. The hypothesis of separability in the above proposition is satisfied in particular when the kernel $K(z,w)$ is analytic in z and w^*, as is easily seen from Proposition 2.3

In order to prove the trace formula (1.7) we first need to recall some results on the theory of reproducing kernel Hilbert spaces with reproducing kernel $\frac{N(z)-N(w)^*}{4\pi(z-w^*)}$.

These spaces have been introduced by L. de Branges and we refer to [3, Section 6, pp. 629–630] for more information and references; for the original works see [13,14]. Note that in these works functions with positive real part rather than positive imaginary part are considered.

Let N be a $\mathbb{C}^{n\times n}$-valued function analytic in the open upper half-plane $\mathbb{C}_+$ and with a positive imaginary part there. Such a function is called a Nevanlinna function, and we will write $N\in\mathscr{C}_n$. The Herglotz representation theorem (see e.g. [15]) asserts that $N(z)$ can be expressed as

$$N(z) = a + zb + \frac{1}{\pi}\int_{\mathbb{R}}\left\{\frac{1}{t-z} - \frac{t}{t^2+1}\right\} d\sigma(t), \tag{2.2}$$

where $a\in\mathbb{C}^{n\times n}$ is an hermitian matrix, $b\in\mathbb{C}^{n\times n}$ is a positive matrix and σ is a $\mathbb{C}^{n\times n}$-valued increasing function such that $\int_{\mathbb{R}}\frac{d\sigma(t)}{t^2+1}<\infty$.

Theorem 2.5. *Let N be a Nevanlinna function with Riesz–Herglotz representation* (2.2), *and extend N via formula* (2.2) *to the lower open half-plane. Then,*

$$N(z) = N(z^*)^*, \quad z\in\mathbb{C}\backslash\mathbb{R},$$

and the function

$$K_N(z,w) = \frac{N(z)-N(w)^*}{4\pi(z-w^*)}$$

is positive (*in the sense of reproducing kernels*) *in $\mathbb{C}\backslash\mathbb{R}$. Let $\mathscr{L}(N)$ be the associated reproducing kernel Hilbert space of functions analytic in $\mathbb{C}\backslash\mathbb{R}$ with reproducing $K_N(z,w)$. Then, $\mathscr{L}(N)$ is the set of functions of the form*

$$F(z) = \frac{1}{4\pi}bc + \frac{1}{2\pi}\int_{\mathbb{R}}\frac{d\sigma(t)f(t)}{t-z}, \tag{2.3}$$

where $c\in\mathbb{C}^n$ and $f\in\mathbf{L}_2^n(d\sigma)$, with norm

$$||F||^2_{\mathscr{L}(N)} = \frac{c^*bc}{4\pi} + ||f||^2_{\mathbf{L}_2^n(d\sigma)}. \tag{2.4}$$

It is invariant under the resolvent-like operators

$$R_\alpha F(z) = \frac{F(z)-F(\alpha)}{z-\alpha}.$$

Proof. We have

$$\frac{N(z)-N(w)^*}{4\pi(z-w^*)} = \frac{b}{4\pi} + \frac{1}{4\pi^2}\int_{\mathbb{R}}\frac{d\sigma(t)}{(t-z)(t-w^*)}. \tag{2.5}$$

Hence

$$\left(\frac{N(w_\ell)-N(w_j)^*}{4\pi(w_\ell-w_j^*)}\right)_{\ell,j=1,\dots,r}$$
$$=\begin{pmatrix}\frac{b}{4\pi} & \frac{b}{4\pi} & \cdots\\ \frac{b}{4\pi} & \frac{b}{4\pi} & \cdots\\ \cdot & \cdot & \cdot\\ \frac{b}{4\pi} & \frac{b}{4\pi} & \cdots\end{pmatrix}+\frac{1}{4\pi^2}\int_{\mathbb{R}}\begin{pmatrix}\frac{1}{t-w_1}\\ \vdots\\ \frac{1}{t-w_r}\end{pmatrix}d\sigma(t)\begin{pmatrix}\frac{1}{t-w_1}\\ \vdots\\ \frac{1}{t-w_r}\end{pmatrix}^*\geqslant 0.$$

From (2.5) we have

$$\sum_{j=1}^r K_N(z,w_j)c_j=\frac{b(\sum_{j=1}^r c_j)}{4\pi}+\frac{1}{4\pi^2}\int_{\mathbb{R}}\frac{d\sigma(t)}{t-z}\left(\sum_{j=1}^r\frac{c_j}{t-w_j^*}\right)$$

for every choice of $r\in\mathbb{N}$, $w_1,\dots,w_r\in\mathbb{C}\backslash\mathbb{R}$ and $c_1,\dots,c_r\in\mathbb{C}^n$, which is of the form (2.3) with

$$c=\sum_{j=1}^r c_j\quad\text{and}\quad f(t)=\frac{1}{2\pi}\sum_{j=1}^r\frac{c_j}{t-w_j^*}.$$

Moreover,

$$\left\|\sum_{j=1}^r K_N(z,w_j)c_j\right\|^2_{\mathscr{L}(N)}=\frac{(\sum_1^r c_j)^*b(\sum_1^r c_j)}{4\pi}+\left\|\frac{1}{2\pi}\left(\sum_{j=1}^r\frac{c_j}{z-w_j^*}\right)\right\|^2_{\mathbf{L}_2^n(d\sigma)}.$$

It follows by continuity that $\mathscr{L}(N)$ consists of the functions of the form (2.3) with norm (2.4). □

Lemma 2.6. *Let N is a $\mathbb{C}^{n\times n}$-valued Nevanlinna function such that* $\det N(z)\not\equiv 0$. *Then $-N^{-1}(z)$ is also a Nevanlinna function and the map $F\mapsto -N^{-1}F$ is a unitary map from $\mathscr{L}(N)$ into $\mathscr{L}(-N^{-1})$.*

Proof. Let $F(z)=\sum_{j=1}^r\frac{N(z)-N(w_j)^*}{4\pi(z-w_j^*)}c_j$ be an element of $\mathscr{L}(N)$ with $w_1,\dots,w_r\in\mathbb{C}\backslash\mathbb{R}$ and $c_1,\dots,c_r\in\mathbb{C}^n$. Then

$$-N(z)^{-1}F(z)=\sum_{j=1}^r\frac{N(z)^{-1}-N(w_j)^{-1*}}{4\pi(z-w_j^*)}\cdot N(w_j)^*c_j.$$

Thus $-N^{-1}F \in \mathscr{L}(N^{-1})$ and we have

$$\begin{aligned}||-N^{-1}F||^2_{\mathscr{L}(N^{-1})} &= \sum_{\ell,j=1}^{r} c_\ell N(w_\ell)\frac{-N(w_\ell)^{-1}+N(w_j)^{-1*}}{4\pi(w_\ell-w_j^*)}N(w_j^*)^*c_j^* \\ &= \sum_{\ell,j=1}^{r} c_\ell\frac{N(w_\ell)-N(w_j)^*}{4\pi(w_\ell-w_j^*)}c_j^* \\ &= ||F||^2_{\mathscr{L}(N)},\end{aligned}$$

where we used the reproducing kernel property in $\mathscr{L}(N)$ and $\mathscr{L}(N^{1-})$ to compute the various norms.

By Proposition 2.3 the linear span of the functions $z \mapsto K_N(z,w)c$ (with c and w as above) is dense in $\mathscr{L}(N)$ and the lemma is proved. □

The operators R_α satisfy the resolvent identity

$$(\alpha-\beta)R_\alpha R_\beta = R_\alpha - R_\beta,$$

and are continuous in $\mathscr{L}(N)$. Thus, when $\ker R_\alpha = \{0\}$ there is a closed densely defined self-adjoint transformation H such that

$$R_\alpha = (H-\alpha I)^{-1}$$

see [34].

The factor 4π in (2.5) should not disturb the reader. When $N(z)=i$ the left-hand side of (2.5) is equal to $\frac{1}{-2i\pi(z-w^*)}$ and so the restrictions of the functions of $\mathscr{L}(N)$ to the upper half-plane coincide with the functions of the classical Hardy space.

Stieltjes' inversion formula allows us to recover the function σ in (2.2): assuming σ normalized by

$$\sigma(t) = \frac{\sigma(t_+)+\sigma(t_-)}{2}$$

and $b=a=0$, we have

$$\sigma(t_2)-\sigma(t_1) = \frac{1}{\pi}\lim_{\substack{y\to+\infty\\ y>0}} \int_{t_1}^{t_2} \text{Im } N(x+iy)\,dx$$

(see e.g. [12]).

We also recall the formula for obtaining the matrix b in the Riesz–Herglotz representation (2.2):

$$b = \lim_{\substack{y\to+\infty\\ y>0}} \frac{\text{Im } N(iy)}{y}.$$

When N is continuous on the real line the function σ is thus absolutely continuous with respect to the Lebesgue measure and $\sigma'(t) = \text{Im } N(t)$.

We also note that the space $\mathscr{L}(N)$ is finite dimensional if and only if $d\sigma$ is a jump measure with a finite number of jumps.

We conclude this section with some simple examples.

Example 2.7. Let $N(z) = \frac{1+z}{1-z}$. Then

$$\frac{N(z) - N(w)^*}{z - w^*} = \frac{2}{(1-z)(1-w^*)}$$

and the space $\mathscr{L}(N)$ is the one-dimensional space spanned by the function $z \mapsto \frac{1}{1-z}$. Similarly,

$$\frac{-N^{-1}(z) + N^{-1}(w)^*}{z - w^*} = \frac{2}{(1+z)(1+w^*)}$$

and the space $\mathscr{L}(N^{-1})$ is spanned by the function $z \mapsto \frac{1}{z+1}$ and the map of multiplication by N^{-1} is one-to-one for $\mathscr{L}(N)$ onto $\mathscr{L}(-N^{-1})$. It is also readily seen to be an isometry.

Example 2.8. Let $N(z) = -\frac{1}{z}$. Then the space $\mathscr{L}(N)$ is the one-dimensional space spanned by the function $\frac{1}{z}$ and thus contains no non-zero constants. On the other hand, $-N(z)^{-1} = z$ and the space $\mathscr{L}(-N^{-1})$ is equal to $\mathbb{C}$.

Indeed, we have

$$\frac{N(z) - N(w)^*}{z - w^*} = \frac{1}{zw^*} \quad \text{and} \quad \frac{-N^{-1}(z) + N^{-1}(w)^*}{z - w^*} = 1.$$

Example 2.9. Let

$$N(z) = a + \frac{m_1}{t_1 - z} + \frac{m_2}{t_2 - z}$$

where $a, t_1, t_2 \in \mathbb{R}$ and m_1 and m_2 are strictly positive numbers. Then the space $\mathscr{L}(-N^{-1})$ contains no non-zero constant functions if and only if $a \neq 0$.

Indeed, we have

$$N(z) = \frac{a(z-t_1)(z-t_2) + m_1(z-t_2) + m_1(z-t_1)}{i(z-t_1)(z-t_2)}.$$

Thus $\lim_{z \to +\infty} \frac{N(z)^{-1}}{z} = 0$ if and only if $a = 0$.

3. Preliminaries on canonical differential expressions

3.1. The fundamental solution

Eq. (1.1) has a $\mathbb{C}^{2n\times 2n}$-valued solution $\Theta(t,z)$ uniquely defined by the condition $\Theta(0,z)=I_{2n}$ and called the *fundamental solution* or *matrizant*. The function $\Theta(t,z)$ admits an integral representation of the form

$$\Theta(t,z)=e^{itzJ}+\int_{-t}^{t}k(t,s)e^{iszJ}\,ds, \tag{3.1}$$

where the kernel $k(t,s)$ is continuous. See [17, (2.16), p. 150].

The first result of this section can be found in [17, p. 150]. We outline the proof for completeness.

Proposition 3.1. *Let $\Theta(t,z)$ be the matrizant of a canonical differential expression. Then, the function*

$$K_T(z,w)=\frac{-J+\Theta(T,z^*)^*J\Theta(T,w^*)}{-2\pi i(z-w^*)} \tag{3.2}$$

is positive in the complex plane. Let $\mathscr{H}(T)$ denote the associated reproducing kernel Hilbert space; the map

$$f\mapsto f^{\blacksquare}(z)=\frac{1}{\sqrt{2\pi}}\int_0^T\Theta(t,z^*)^*f(t)\,dt \tag{3.3}$$

is a unitary operator from $\mathbf{L}_2^{2n}(0,T)$ onto $\mathscr{H}(T)$.

Proof. From the differential equation (1.1) we have

$$\begin{aligned}
&\int_0^T z\Theta(t,z^*)^*\Theta(t,w^*)\,dt\\
&\quad=\int_0^T\left(-iJ\frac{\partial\Theta(t,z^*)}{\partial t}-V(t)\Theta(t,z^*)\right)^*\Theta(t,w^*)\,dt\\
&\quad=i\int_0^T\frac{\partial\Theta(t,z^*)^*}{\partial t}J\Theta(t,w^*)\,dt-\int_0^T\Theta(t,z^*)^*V(t)\Theta(t,w^*)\,dt
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_0^T\Theta(t,z^*)^*w^*\Theta(t,w^*)\,dt=&-i\int_0^T\Theta(t,z^*)^*J\frac{\partial\Theta(t,w^*)}{\partial t}\,dt\\
&-\int_0^T\Theta(t,z^*)^*V(t)\Theta(t,w^*)\,dt.
\end{aligned}$$

Hence,

$$(z-w^*)\int_0^T \Theta(t,z^*)^*\Theta(t,w^*)\,dt = i\int_0^T \frac{\partial}{\partial t}(\Theta(t,z^*)^*J\Theta(t,w^*))\,dt$$

and hence

$$\frac{-J+\Theta(T,z^*)^*J\Theta(T,w^*)}{-2\pi i(z-w^*)} = \frac{1}{2\pi}\int_0^T \Theta(t,z^*)^*\Theta(t,w^*)\,dt.$$

Thus the map

$$f \mapsto f^{\blacksquare}(z) = \frac{1}{\sqrt{2\pi}}\int_0^T \Theta(t,z^*)^*f(t)\,dt$$

is an isometry from the closure in $\mathbf{L}_2^{2n}(0,T)$ of the functions of the form $t\mapsto\Theta(t,w^*)\xi$ (where w runs through $\mathbb{C}$ and ξ runs through $\mathbb{C}^{2n}$) into $\mathscr{H}(T)$. To conclude one has to show that the indicated closure is in fact all of $\mathbf{L}_2^{2n}(0,T)$. Let $f(t)$ be orthogonal to the indicated span. Then, in view of representation (3.1),

$$\int_0^T\left(e^{-itz^*J}+\int_{-t}^t e^{-isz^*J}k(t,s)^*\,ds\right)f(t)\,dt = 0,\quad z\in\mathbb{C},$$

so that

$$\int_0^T e^{-itz^*J}\left(f(t)+\int_t^T k(t,s)^*f(s)\,ds\right)dt = 0,\quad z\in\mathbb{C}, \tag{3.4}$$

and so

$$f(t)+\int_t^T k(s,t)^*f(s)\,ds = 0,\quad 0\leqslant t\leqslant T,$$

where $k(t,s)$ is the kernel in representation (3.1). This latter is a Volterra equation whose only solution is $f(t)\equiv 0$. $\square$

The following corollary will be used in the sequel in the proof of the trace formula.

Corollary 3.2. *Let $T_1\leqslant T_2$. Then the space $\mathscr{H}(T_1)$ is isometrically included in the space $\mathscr{H}(T_2)$.*

Indeed, let $F(z)\in\mathscr{H}(T_1)$. It can be written as

$$F(z) = \int_0^{T_1}\Theta(t,z^*)^*f(t)\,dt = \int_0^{T_2}\Theta(t,z^*)^*f^{\wedge}(t)\,dt,$$

where $f^\wedge(t) = f(t)$ for $t \in [0, T_1]$ and $f^\wedge(t) = 0$ for $t \in (T_1, T_2]$. Thus $F(z) \in \mathscr{H}(T_2)$ and

$$||F||^2_{\mathscr{H}(T_1)} = ||F||^2_{\mathscr{H}(T_2)} = \int_0^T f(t)^* f(t)\, dt.$$

We recall the definition of the *Weyl coefficient function*.

Definition 3.3. The Weyl coefficient function $N(z)$ is defined in the open upper half-plane; it is the unique $\mathbb{C}^{n \times n}$-valued function such that

$$\int_0^\infty \begin{pmatrix} iN(z)^* & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \Theta(t,z)^* \Theta(t,z) \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \begin{pmatrix} -iN(z) \\ I_n \end{pmatrix} dt < \infty \tag{3.5}$$

for Im $z > 0$.

For the existence of square integrable solutions to the canonical differential expression, see [17, Section 8, p. 204].

Proposition 3.4. *Let N be the Weyl function of the canonical differential expression* (1.1). *Then, $-N(z)^{-1}$ is the Weyl function associated to the canonical differential expression with potential $-k(t)$.*

Indeed, the matrizant associated to the canonical differential expression with potential $-k(t)$ is $J\Theta(t,z)J$. Thus, we have to prove that

$$\int_0^\infty \begin{pmatrix} iN(z)^{-1*} & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} J\Theta(t,z)^* JJ\Theta(t,z)J \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \begin{pmatrix} -iN(z)^{-1} \\ I_n \end{pmatrix} dt < \infty.$$

Multiplying this expression on the left by $iN(z)^*$ and on the right by $-iN(z)$ we see that we have to verify that

$$\int_0^\infty \begin{pmatrix} I_n & iN(z)^* \end{pmatrix} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} J\Theta(t,z)^* JJ\Theta(t,z)J \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \begin{pmatrix} I_n \\ -iN(z) \end{pmatrix} dt < \infty.$$

This in turn is equivalent to (3.5) since

$$\begin{pmatrix} I_n & iN(z)^* \end{pmatrix} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} J = \begin{pmatrix} iN(z)^* & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}.$$

3.2. Fourier analysis associated to a canonical differential expression

Recall that the operator H_+ has been defined by (1.2) and the map U_+ defined by (1.6) is unitary from $\mathbf{L}_2^{2n}(0,\infty)$ onto $\mathbf{L}_2^n(\mathrm{Im}\ N)$ and is such that

$$U_+(H_+f)(z) = z(U_+f)(z)$$

for all $f \in \mathbf{L}_2^{2n}(0,\infty)$ such that $f' \in \mathbf{L}_2^{2n}(0,\infty)$ and $(I_n \quad -I_n)f(0) = 0$. The inverse map is given by

$$f(t) = \int_{\mathbb{R}} \Theta(t,z)\begin{pmatrix} I_n \\ I_n \end{pmatrix} \mathrm{Im}\ N(z)g(z)\,dz. \tag{3.6}$$

We set U_- to be the map

$$f \to \frac{1}{\sqrt{2\pi}} \int_0^\infty (I_n \quad I_n)\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}\Theta(t,z)^*\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} f(t)\,dt. \tag{3.7}$$

U_- is a unitary map from $\mathbf{L}_2^{2n}(0,\infty)$ onto $\mathbf{L}_2^n(\mathrm{Im}\ (-N^{-1}))$ such that

$$(U_-(H_-f))(z) = z(U_-f)(z)$$

for all $f \in \mathbf{L}_2^{2n}(0,\infty)$ such that $f' \in \mathbf{L}_2^{2n}(0,\infty)$ and $(I_n \quad -I_n)f(0) = 0$.

The operators $(H_+ - wI)^{-1}$ and $(H_- - wI)^{-1}$ can be explicitly computed using the generalized Fourier analysis.

Proposition 3.5. *Let $f \in \mathbf{L}_2^{2n}$. Then,*

$$((H_+ - wI)^{-1}f)(t)$$

$$= \int_{\mathbb{R}} \frac{\Theta(t,z)\begin{pmatrix} I_n \\ I_n \end{pmatrix} \mathrm{Im}\ N(z)(\int_0^\infty (I_n\ I_n)\Theta(u,z)^* f(u)\,du)}{z-w}\,dz,$$

$$((H_- - wI)^{-1}f)(t)$$

$$= \int_{\mathbb{R}} \frac{J\Theta(t,z)J\begin{pmatrix} I_n \\ I_n \end{pmatrix} \mathrm{Im}\ N(z)^{-1}\left(\int_0^\infty (I_n\ I_n)J\Theta(u,z)^* Jf(u)\,du\right)}{z-w}\,dz.$$

It seems quite difficult to prove directly from these formulas (for instance using the definition of the trace in terms of an orthonormal basis) that the operator

$$(H_+ - wI)^{-1} - (H_- - wI)^{-1}$$

has rank n.

Proposition 3.6. *Let $f \in \mathbf{L}_2^{2n}(0,\infty)$. Then,*

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \Theta(t,z)^* f(t)\,dt = \frac{1}{2}\begin{pmatrix} U_+f + U_-(Jf) \\ U_+f - U_-(Jf) \end{pmatrix}(z). \tag{3.8}$$

Proof. By definition of U_- we have

$$\begin{aligned}(U_-(Jf))(z) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty (I_n \;\; I_n)J\Theta(t,z)^* JJf(t)\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty (I_n \;\; -I_n)\Theta(t,z)^* f(t)\,dt\end{aligned}$$

and so we have

$$\begin{pmatrix} U_+f \\ U_-(Jf) \end{pmatrix}(z) = \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \times \left(\frac{1}{\sqrt{2\pi}}\int_0^\infty \Theta(t,z)^* f(t)\,dt\right)$$

and hence the result. □

3.3. The Hilbert transform

The unitary operator $M_{-N^{-1}}$ of multiplication by $-N^{-1}$ from $\mathscr{L}(N)$ onto $\mathscr{L}(-N^{-1})$ induces a corresponding unitary transformation in the related weighted spaces $\mathbf{L}_2(\text{Im } N)$ and $\mathbf{L}_2(\text{Im } (-N^{-1}))$. This transformation is the counterpart of the transformation relating orthogonal polynomials of the first and second kind in the discrete case (see e.g. [35]) and is called the *Hilbert transform* in [12]. We also refer to [11, Theorem IV, p. 548].

The following result appears in [24]; see [24, (5.52), p. 5.20].

Theorem 3.7. *Let f_- be defined by*

$$f_-(z) = \int_{\mathbb{R}} \frac{(\text{Im } N(t))f(t)\,dt}{\pi i(t-z)} + iN(z)f(z). \tag{3.9}$$

Then,

$$\int_{\mathbb{R}} \frac{(\mathrm{Im}\ N(t))f(t)\,dt}{2\pi i(t-z)} = N(z)\int_{\mathbb{R}} \frac{(\mathrm{Im}\ (-N(t)^{-1}))f_-(t)\,dt}{2\pi i(t-z)}. \tag{3.10}$$

Moreover, f_- is given by the formula

$$f_-(z) = ipN^*pf - iqNqf, \tag{3.11}$$

where $p = I - q$ denotes the orthogonal projection from the Lebesgue space $\mathbf{L}_2^n(\mathbb{R})$ onto the Hardy space of the open upper half-plane.

Proof. We first consider f of the form $f(t) = \frac{c}{-2\pi i(t-w^*)}$ with $w \in \mathbb{C}_+$ and $c \in \mathbb{C}^n$. Then

$$\int_{\mathbb{R}} \frac{(\mathrm{Im}\ N(t))f(t)\,dt}{\pi i(t-z)} = \int_{\mathbb{R}} \frac{(\mathrm{Im}\ N(t))c\,dt}{2\pi^2(t-z)(t-w^*)} = \frac{N(z)-N(w)^*}{2\pi(z-w^*)}c$$

and the Hilbert transform of $\frac{c}{-2\pi i(t-w^*)}$ (as given by (3.9)) is equal to

$$\left(\frac{N(z)-N(w)^*}{2\pi(z-w^*)}\right)c - \frac{N(z)c}{2\pi(z-w^*)} = -\frac{N(w)^*c}{2\pi(z-w^*)}.$$

Hence for $f(t) = \frac{c}{-2\pi i(t-w^*)}$ we have $f_-(t) = -\frac{N(w)^*c}{2\pi(t-w^*)}$.

The left-hand side of (3.10) is equal to $\frac{(N(z)-N(w)^*)c}{4\pi(z-w^*)}$ and the right-hand side of (3.10) is equal to

$$\begin{aligned} N(z)\int_{\mathbb{R}} \frac{\mathrm{Im}\ (-N(t))^{-1}N(w)^*c\,dt}{4\pi^2(t-z)(t-w^*)} &= N(z)\left(\frac{-N(z)^{-1}+N(w)^{-1*}}{4\pi(z-w^*)}\right)N(w)^*c \\ &= \frac{(N(z)-N(w)^*)c}{4\pi(z-w^*)}, \end{aligned}$$

where we have used the equality

$$\int_{\mathbb{R}} \frac{\mathrm{Im}\ (-N^{-1}(t))\,dt}{4\pi^2(t-z)(t-w^*)} = \frac{-N(z)^{-1}+N(w)^{-1*}}{4\pi(z-w^*)}.$$

The case of functions of the form $\frac{c}{-2\pi i(z-w)}$ with w in the open loer half-plane is proved in the same way and the result is proved by continuity and density.

The proof of formula (3.11) is done in the same way. We note that the operator $ipN^*p - iqNq$ is bounded since, as already remarked, N is bounded in the closed upper half-plane. We first consider a function f of the form $f(z) = \frac{c}{-2\pi i(z-w^*)}$ where $c \in \mathbb{C}^n$ and $w \in \mathbb{C}_+$. Then (3.11) is equal to $\frac{N(w)^*c}{-2\pi(z-w^*)}$, which coincides with (3.9).

Similarly, for $f(z) = \frac{c}{-2\pi i(z-w)}$, formula (3.11) gives $f_-(z) = \frac{N(w)c}{2\pi(z-w)}$. The general case follows by a density argument. $\square$

When f has its components in the Hardy space of the upper half-plane and $z \in \mathbb{C}_+$ we note that

$$f_-(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{N(t)^* f(t)}{t-z} dt. \tag{3.12}$$

4. A theorem of Iacob

In this section we prove a result, Theorem 4.2, taken from Iacob's thesis (see [24, Theorem 5.5′, p. 5.18]). The result is based on a result of [3], but no proofs for the specific result we need appeared besides in [24], and we will repeat the proof. The result in [24] is in the setting of J-inner function while here the functions we consider are inverses of J-inner functions. This explains the difference of signs in some formulas between the present work and [24].

We begin with a preliminary proposition. Recall that $\Theta(T,z)$ denotes the matrizant of the canonical differential expression (1.1). We set

$$\Theta(T,z) = \begin{pmatrix} \Theta_{11}(T,z) & \Theta_{12}(T,z) \\ \Theta_{21}(T,z) & \Theta_{22}(T,z) \end{pmatrix},$$

where the $\Theta_{ij}(T,z)$ are $\mathbb{C}^{n\times n}$-valued and

$$E_+(T,z) = -\Theta_{21}(T,z^*)^* + \Theta_{22}(T,z^*)^*, \tag{4.1}$$

$$E_-(T,z) = \Theta_{11}(T,z^*)^* - \Theta_{12}(T,z^*)^*, \tag{4.2}$$

$$F_+(T,z) = \Theta_{21}(T,z^*)^* + \Theta_{22}(T,z^*)^*, \tag{4.3}$$

$$F_-(T,z) = \Theta_{11}(T,z^*)^* + \Theta_{12}(T,z^*)^*. \tag{4.4}$$

Proposition 4.1. *Let $T \geqslant 0$. The following hold*:

1. *The function $E_-(T,z)$ is invertible in the open upper half-plane and $((z+i)E_-(T,z))^{-1} \in \mathbf{H}_2^{n\times n}$.*
2. *The function $E_+(T,z)$ invertible in the open lower half-plane and $((z-i)E_+(T,z))^{-1} \in \overline{\mathbf{H}_2}^{n\times n}$.*
3. *The function*

$$N_T(z) = \begin{cases} iF_-(T,z)E_-(T,z)^{-1} & \text{if } z \in \mathbb{C}_+, \\ iF_+(T,z)E_+(T,z)^{-1} & \text{if } z \in \mathbb{C}_- \end{cases}$$

is a Nevanlinna function.

4. *For* $z \in \mathbb{R}$ *it holds that*

$$\mathrm{Im}\ N_T(z) = E_-(T,z)^{-1*}E_-(T,z)^{-1} = E_+(T,z)^{-1*}E_+(T,z)^{-1}, \tag{4.5}$$

5. *and the function* $z \mapsto \mathrm{Im}\ N_T(z)$ *belongs to the Wiener algebra and takes value* I_n *at infinity*.

A proof can be found in [17]. For completeness we outline some of the arguments. It follows from (3.2) that the function $z \mapsto \Theta(T,z^*)^*$ is J-expansive in the open upper half-plane and J-unitary on the real line:

$$\Theta(T,z^*)^* J \Theta(T,z^*) \begin{cases} \geqslant J & \text{if } z \in \mathbb{C}_+, \\ = J & \text{if } z \in \mathbb{R}. \end{cases}$$

It follows that

$$\Theta(T,z^*) J \Theta(T,z^*)^* \geqslant J$$

for $z \in \mathbb{C}_+$; see e.g. [16, pp. 13–16]. Multiplying this inequality by $(I_n\ 0)$ on the left and by $(I_n\ 0)^*$ on the right we obtain

$$\Theta_{11}(T,z^*)\Theta_{11}(T,z^*)^* - \Theta_{12}(T,z^*)\Theta_{12}(T,z^*)^* \geqslant I_n$$

and so

$$F_-(T,z)^*E_-(T,z) + E_-(T,z)^*F_-(T,z) \geqslant 2\,I_n. \tag{4.6}$$

Assume that $E_-(T,z)c = 0$ for some vector $c \in \mathbb{C}^n$. Then, multiplying both sides of (4.6) by c^* on the left and c on the right we get $0 \geqslant cc^*$ and so $c = 0$ and $E_-(T,z)$ is invertible in the open upper half plane. Thus (4.6) also leads to

$$\frac{N_T(z) - N_T(z)^*}{2i} \geqslant E_-(T,z)^{-1*}E_-(T,z)^{-1}. \tag{4.7}$$

We now show that the function $((z+i)E_-(T,z))^{-1} \in \mathbf{H}_2^{n\times n}$. Let $c \in \mathbb{C}^n$ and $\varepsilon > 0$. Using (4.7) and the Poisson formula for harmonic function we obtain

$$\begin{aligned} \int_{\mathbb{R}} c \frac{E_-(T,t+i\varepsilon)^{-1*}E_-(T,t+i\varepsilon)^{-1}}{t^2+(1+\varepsilon)^2} c^*\,dt &\leqslant \int_{\mathbb{R}} c \frac{\mathrm{Im}\ N_T(t+i\varepsilon)}{t^2+(1+\varepsilon)^2} c \\ &= \frac{\pi}{1+\varepsilon} c(\mathrm{Im}\ N_T(i(1+\varepsilon))c^*\,dt \end{aligned}$$

and so

$$\sup_{\varepsilon>0}\int_{\mathbb{R}} c\frac{E_-(T,t+i\varepsilon)^{-1*}E_-(T,t+i\varepsilon)^{-1}}{t^2+(1+\varepsilon)^2}c^*\,dt\leqslant\pi c(\mathrm{Im}\ N_T(i))c^*$$

and so $((z+i)E_-(T,z))^{-1}\in\mathbf{H}_2^{n\times n}$.

Similarly, since $z\mapsto\Theta(T,z^*)^*$ is J-contractive in the lower half-plane we have

$$J\geqslant\Theta(T,z^*)J\Theta(T,z^*)^*,\quad z\in\mathbb{C}_-.$$

Multiplying this inequality on the left by $(0\ I_n)$ and by $(0\ I_n)^*$ on the right we obtain

$$\Theta_{22}(T,z^*)\Theta_{22}(T,z^*)^*-\Theta_{21}(T,z^*)\Theta_{21}(T,z^*)^*\geqslant I_n$$

and so

$$F_+(T,z)E_+(T,z)^*+E_+(T,z)F_+(T,z)^*\geqslant 2\,I_n$$

and so for z in the open lower half-plane we have

$$\frac{N_T(z)^*-N_T(z)}{2i}\geqslant E_+(T,z)^{-1*}E_+(T,z)^{-1}. \tag{4.8}$$

Thus

$$\frac{N_T(z)-N_T(z)^*}{(z-z^*)}\geqslant 0$$

in the open lower half-plane. Eq. (4.5) is obtained by considering (4.7) and (4.8) for z on the real line. The function $z\mapsto\Theta(T,z)$ is then unitary and the inequalities are replaced by equalities in (4.7) and (4.8). $\square$

Multiplying (3.2) by $(I_n\ -I_n)$ on the left and by $(I_n\ -I_n)^*$ on the right we see that the function

$$\Lambda_T(z,w)=\frac{E_-(T,z)E_-(T,w)^*-E_+(T,z)E_+(T,w)^*}{-2\pi i(z-w^*)} \tag{4.9}$$

is positive in the complex plane. We will denote by $\mathscr{B}_T$ the associated reproducing kernel Hilbert space.

Theorem 4.2. *Let $\mathscr{H}(T)$ denote the reproducing kernel Hilbert space with reproducing kernel $K_T(z,w)$ defined by* (3.2). *Then*

$$\mathscr{H}(T)=\left\{\frac{1}{2}\begin{pmatrix}g+\widetilde{g}\\ g-\widetilde{g}\end{pmatrix}\middle|\,g\in\mathscr{B}_T\right\}, \tag{4.10}$$

where

$$\widetilde{g}(z) = -\left(\int_{\mathbb{R}} \frac{\Delta_T(t)g(t)\,dt}{\pi i(t-z)} + iN_T(z)g(z)\right). \tag{4.11}$$

In this expression

$$\Delta_T(z) = (E_-(T,z)E_-(T,z)^*)^{-1} = (E_+(T,z)E_+(T,z)^*)^{-1} \tag{4.12}$$

and

$$N_T(z) = i\left(I_n - 2\int_0^\infty e^{izt}h(T,t)\,dt\right) \tag{4.13}$$

with $\Delta_T(z) = I_n - \int_{\mathbb{R}} e^{itz}h(T,t)\,dt$ *and* $t \mapsto h(T,t) \in \mathbf{L}_1^{n\times n}(\mathbb{R})$.

Proof. The proof is built in a number of steps. We first give some notations. Π denotes the projection

$$\Pi = \frac{1}{2}\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}.$$

We also define the spaces (where "c.l.s." stands for the closed linear span in the space $\mathscr{H}(T)$):

$$\mathscr{H}_\Pi(T) = \text{c.l.s.}\{K_T(z,w)(I_{2n}-\Pi)\xi;\ w\in\mathbb{C},\ \xi\in\mathbb{C}^{2n}\},$$

$$\mathscr{N}_\Pi = \{F\in\mathscr{H}(T); (I_{2n}-\Pi)F(z)\equiv 0\}.$$

We also recall that the elements of $\mathscr{H}(T)$ were characterized in Proposition 3.1 (see Eq. (3.3)).

Step 1: *The space* $\mathscr{N}_\Pi = \{0\}$.

Indeed, let $F(z) = \int_0^T \Theta(t,z^*)^* f(t)\,dt$ be such that

$$(I_{2n}-\Pi)F(z)\equiv 0.$$

Then, as in the proof of Proposition 3.1 (see Eq. (3.4)) we have

$$(I_{2n}-\Pi)\int_0^T e^{-itz^*J}\left(f(t) + \int_t^T k(t,s)^* f(s)\,ds\right)dt = 0.$$

But $(I_{2n}-\Pi)J = J\Pi$ and $\Pi J = J(I_{2n}-\Pi)$ so that

$$(I_{2n}-\Pi)e^{-itz^*J} = (\cosh itz^*J)(I_{2n}-\Pi) - (\sinh itz^*J)\Pi.$$

We are thus lead to

$$\int_0^T (\cosh itz^* J)(I_{2n}-\Pi)\left(f(t)+\int_t^T k(t,s)^* f(s)\,ds\right)dt=0$$

and

$$\int_0^T (\sinh itz^* J)\Pi\left(f(t)+\int_t^T k(t,s)^* f(s)\,ds\right)dt=0$$

for all $z\in\mathbb{C}$. It follows that

$$f(t)+\int_t^T k(t,s)^* f(s)ds\equiv 0$$

and so $f(t)\equiv 0$.

It follows from the previous step that $\mathscr{H}(T)=\mathscr{H}_\Pi(T)$.

Step 2: *The map*

$$\Lambda_T(z,w)c\to Y(\Lambda_T(z,w)c)=\frac{1}{2}\begin{pmatrix}\Lambda_T(z,w)c+\Gamma_T(z,w)c\\ \Lambda_T(z,w)c-\Gamma_T(z,w)c\end{pmatrix},$$

where

$$\Gamma_T(z,w)c=\frac{-2I_n+F_-(T,z)E_-(T,w)^*+F_+(T,z)E_+(T,w)^*}{-2\pi i(z-w^*)}c$$

extends to an isomorphism from $\mathscr{B}_T$ *onto* $\mathscr{H}(T)$.

Indeed, let $\xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ with ξ_1 and ξ_2 in $\mathbb{C}^n$. Then,

$$\begin{aligned}&\Pi K_T(z,w)(I_{2n}-\Pi)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix}I_n & -I_n\\ -I_n & I_n\end{pmatrix}\frac{-J+\Theta(T,z^*)^*J\Theta(T,w^*)}{-2\pi i(z-w^*)}\frac{1}{2}\begin{pmatrix}I_n & I_n\\ I_n & I_n\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix}\Gamma_T(z,w)\\ -\Gamma_T(z,w)\end{pmatrix}\left(\frac{\xi_1+\xi_2}{2}\right)\end{aligned}$$

and

$$(I_{2n}-\Pi)K_T(z,w)(I_{2n}-\Pi)\xi=\frac{1}{2}\begin{pmatrix}\Lambda_T(z,w)\\ \Lambda_T(z,w)\end{pmatrix}\left(\frac{\xi_1+\xi_2}{2}\right).$$

Hence

$$\begin{aligned}K_T(z,w)(I_{2n}-\Pi)\xi &= \Pi K_T(z,w)(I_{2n}-\Pi)\xi + (I_{2n}-\Pi)K_T(z,w)(I_{2n}-\Pi)\xi\\ &= \frac{1}{2}\begin{pmatrix}\Lambda_T(z,w)+\Gamma_T(z,w)\\ \Lambda_T(z,w)-\Gamma_T(z,w)\end{pmatrix}\left(\frac{\xi_1+\xi_2}{2}\right)\end{aligned}$$

and the map Y sends the linear span of the $\Lambda_T(z,w)c$ into $\mathscr{H}(T)$. Moreover, for every choice of complex numbers w_j and of vectors $\xi^{(j)} = \begin{pmatrix}\xi_1^{(j)}\\ \xi_2^{(j)}\end{pmatrix} \in \mathbb{C}^{2n}$ (with $j = 1, \dots, r$) we have (using the reproducing kernel property in $\mathscr{H}(T)$ and in $\mathscr{B}_T$)

$$\begin{aligned}&\left\|\sum_{\ell=1}^r K_T(z,w_\ell)(I_{2n}-\Pi)\xi^{(\ell)}\right\|^2_{\mathscr{H}(T)}\\ &= \sum_{\ell,j=1}^r \xi^{(\ell)*}(I_{2n}-\Pi)K_T(w_\ell,w_j)(I_{2n}-\Pi)\xi^{(j)}\\ &= \begin{pmatrix}\dfrac{\xi_1^{(\ell)^*}+\xi_2^{(\ell)^*}}{2} & \dfrac{\xi_2^{(\ell)^*}+\xi_1^{(\ell)^*}}{2}\end{pmatrix}\\ &\quad \frac{1}{2}\begin{pmatrix}\Lambda_T(w_\ell,w_j)+\Gamma_T(w_\ell,w_j)\\ -\Lambda_T(w_\ell,w_j)+\Gamma_T(w_\ell,w_j)\end{pmatrix}\left(\frac{\xi_1^{(j)}+\xi_2^{(j)}}{2}\right)\\ &= \sum_{\ell,j=1}^r \frac{(\xi_1^{(\ell)}+\xi_2^{(\ell)})^*}{2}\Lambda_T(w_\ell,w_j)\frac{(\xi_1^{(j)}+\xi_2^{(j)})}{2}\\ &= \left\|\sum_{\ell=1}^r \Lambda_T(z,w_\ell)\left(\frac{\xi_1^{(\ell)}+\xi_2^{(\ell)}}{2}\right)\right\|^2_{\mathscr{B}_T}.\end{aligned}$$

Thus with $f(z) = \sum_\ell \Lambda_T(z,w_\ell)(\frac{\xi_1^{(\ell)}+\xi_2^{(\ell)}}{2})$ we have $\|Y(f)\|_{\mathscr{H}(T)} = \|f\|_{\mathscr{B}_T}$ and hence the result by density and continuity.

Step 3: *It holds that*

$$-\Gamma_T(z,w) = \int_{\mathbb{R}} \frac{\Delta_T(t)\Lambda_T(t,w)\,dt}{\pi i(t-z)} + iN_T(z)\Lambda_T(z,w). \tag{4.14}$$

We prove (4.14) for $z, w \in \mathbb{C}_+$. The other cases are treated in a similar manner. We have $N_T(z) = iF_-(T,z)E_-(T,z)^{-1}$ and so

$$\begin{aligned}&iN_T(z)\Lambda_T(z,w)\\ &= -\frac{F_-(T,z)E_-(T,w)^*}{-2\pi i(z-w^*)} + \frac{F_-(T,z)E_-(T,z)^{-1}E_+(T,z)E_+(T,w)^*}{-2\pi i(z-w^*)}.\end{aligned}$$

Hence, proving (4.14) is equivalent to proving that

$$\frac{2I_n - F_+(T,z)E_+(T,w)^*}{-2\pi i(z-w^*)} = \frac{F_-(T,z)E_-(T,z)^{-1}E_+(T,z)E_+(T,w)^*}{-2\pi i(z-w^*)} + \int_{\mathbb{R}} \frac{\Delta_T(t)\Lambda_T(t,w)\,dt}{\pi i(t-z)}.$$

We now compute

$$\int_{\mathbb{R}} \frac{\Delta_T(t)\Lambda_T(t,w)\,dt}{\pi i(t-z)} = 2\int_{\mathbb{R}} \frac{E_-(T,t)^{-1*}E_-(T,t)^{-1}E_-(T,t)E_-(T,w)^*}{(2\pi i(t-z))(-2\pi i(t-w^*))}\,dt - 2\int_{\mathbb{R}} \frac{E_+(T,t)^{-1*}E_+(T,t)^{-1}E_+(T,t)E_+(T,w)^*}{(2\pi i(t-z))(-2\pi i(t-w^*))}\,dt.$$

Using Cauchy's formula for Hardy function we see that

$$\begin{aligned}\left(2\int_{\mathbb{R}} \frac{E_-(T,t)^{-1*}E_-(T,t)^{-1}E_-(T,t)E_-(T,w)^*}{(2\pi i(t-z))(-2\pi i(t-w^*))}\,dt\right)^* &= 2\int_{\mathbb{R}} \frac{E_-(T,w)E_-(T,t)^{-1}}{(-2\pi i(t-z^*))(2\pi i(t-w))}\,dt \\ &= \frac{2I_n}{-2\pi i(w-z^*)}\end{aligned}$$

so that

$$2\int_{\mathbb{R}} \frac{E_-(T,t)^{-1*}E_-(T,t)^{-1}E_-(T,t)E_-(w)^*}{(2\pi i(t-z))(-2\pi i(t-w^*))}\,dt = \frac{2I_n}{-2\pi i(z-w^*)}.$$

Similarly

$$\begin{aligned}-2\int_{\mathbb{R}} \frac{E_+(T,t)^{-1*}E_+(T,t)^{-1}E_+(T,t)E_+(T,w)^*}{(2\pi i(t-z))(-2\pi i(t-w^*))}\,dt &= -2\int_{\mathbb{R}} \frac{E_+(T,t)^{-1*}E_+(T,w)^*}{(2\pi i(t-z))(-2\pi i(t-w^*))}\,dt \\ &= \frac{-2E_+(T,z^*)^{-1*}E_+(T,w)^*}{-2\pi i(z-w^*)}\end{aligned}$$

since the function $z \mapsto \frac{E_+(T,z^*)^{-1*}}{z-w^*}$ has its components in $\mathbf{H}_2$. Hence to prove (4.14) boils down to verifying that

$$-F_+(T,z) = F_-(T,z)E_-(T,z)^{-1}E_+(T,z) - 2E_+(T,z^*)^{-1*}.$$

We multiply on the right both sides by $E_+(T, z^*)^*$ and recall that $E_+(T,z)E_+(T,z^*)^* = E_-(T,z)E_-(T,z^*)^*$. We are lead to check that

$$-F_+(T,z)E_+(T,z^*)^* = F_-(T,z)E_-(T,z^*)^* - 2I_n.$$

This in turn is easily checked by multiplying both sides of the equality $\Theta(T,z^*)^* J\Theta(T,z) = J$ on the left by $(I_n \ I_n)$ and on the right by $\begin{pmatrix} I_n \\ -I_n \end{pmatrix}$. $\square$

We note that Step 3 in the proof is the key to Iacob's result and to the trace formula.

5. The proof of the trace formula (1.7)

We will start with some preliminaries.

5.1. The spaces $\mathscr{B}_T$

The next step toward the proof of the trace formula for the pair of operators (H_+, H_-) is:

Proposition 5.1. *In formula* (4.11) *one can replace Δ_T by* Im $N(t)$ *and $N_T(z)$ by $N(z)$.*

Since the $\mathscr{B}_T$ form a nested sequence, a formal proof consists in letting $T \to \infty$ in (4.11). The rigourous proof makes use of various properties of the spaces $\mathscr{B}_T$. We proceed in a number of steps. Recall that the space $\mathscr{B}_T$ was defined in Theorem 4.2.

Step 1: *The space $\mathscr{B}_T$ is the closed linear span in $\mathbf{L}_2^n(W)$ of the functions*

$$z \mapsto \frac{e^{izt} - 1}{z} c,$$

where $|t| \leqslant T$ and c runs in $\mathbb{C}^n$.

For a proof see [17,24, Theorem 5.4, p. 185; Theorem 6.7, p. 6.30]. Under our assumptions on the weight functions, we have that

$$\varepsilon_1 I_n \leqslant W(t) \leqslant \varepsilon_2 I_n, \quad t \in \mathbb{R}$$

and so $\mathscr{B}_T$, as a vector space, is the same in the $\mathbf{L}_2^n(\mathbb{R})$ norm and in the norm of $\mathbf{L}_2^n(W)$. Hence, as a vector space, $\mathscr{B}_T$ is the set of functions of the form

$$\int_{-T}^{T} e^{izu} g(u)\, du \quad \text{with} \quad \int_{-T}^{T} g(u)^* g(u)\, du < \infty.$$

Step 2: *Let $T_1 \leqslant T_2$. The space $\mathscr{B}_{T_1}$ is isometrically included in $\mathscr{B}_{T_2}$.*

This is a direct consequence of Corollary 3.2 and of Step 2 of the proof of Theorem 4.2. A proof may also be found in [17, Corollary 5.2, p. 180].

Step 3: *The space $\mathscr{B}_T$ is isometrically included in $\mathbf{L}_2^n(\Delta_T)$.*

Indeed, the reproducing kernel $\Lambda_T(z,w)$ of $\mathscr{B}_T$ (given by (4.9)) can be written as

$$\Lambda_T(z,w) = E_-(T,z)\frac{I_n - V_T(z)V_T(w)^*}{-2\pi i(z-w^*)}E_-(T,w)^*, \tag{5.1}$$

where the function $V_T(z) = E_-(T,z)^{-1}E_+(T,z)$ is inner. It follows that

$$\mathscr{B}_T = \{F(z) = E_-(T,z)G(z),\ \ G \in \mathbf{H}_2^n \ominus V_T\mathbf{H}_2^n\}$$

with norm

$$||F||_{\mathscr{B}_T} = ||G||_{\mathbf{H}_2^n}$$

and the result follows.

Step 4: *The spaces* $\mathscr{B}_T$ *are isometrically included in* $\mathbf{L}_2^n(\mathrm{Im}\ N)$.

Indeed, from the two previous steps we have that $\mathscr{B}_T$ is isometrically included in $\mathbf{L}_2(\Delta_{T'})$ for all $T' \geqslant T$. To conclude we recall that

$$\mathrm{Im}\ N(t) = \lim_{T' \to \infty} \Delta_{T'}(t). \tag{5.2}$$

Indeed, let $\Theta(t,z)$ denote the matrizant of the canonical differential expression (1.1) and let $E_-(t,z)$ and $E_+(t,z)$ be defined by (4.1) and (4.2). Then it holds that

$$\lim_{t \to \infty} (E_-(t,z)E_-(t,z)^*)^{-1} = \mathrm{Im}\ N(z).$$

Indeed, using (1.5)

$$\begin{aligned}
&\lim_{t \to \infty} E_-(t,z)E_-(t,z)^* \\
&\quad = \lim_{t \to \infty} (\Theta_{11}(t,z) - \Theta_{12}(t,z))^*(\Theta_{11}(t,z) - \Theta_{12}(t,z)) \\
&\quad = \lim_{t \to \infty} ((\Theta_{11}(t,z) - \Theta_{12}(t,z))^* e^{itz})((\Theta_{11}(t,z) - \Theta_{12}(t,z))e^{-itz}) \\
&\quad = (\alpha(z) - \beta(z))^*(\alpha(z) - \beta(z))
\end{aligned}$$

and hence we have (5.2).

Step 5: *Let* Δ_T *be as above*; *the inverse Fourier transform of the function* $\Delta_T - \mathrm{Im}\ N$ *vanishes in the interval* $(-2T, 2T)$.

Indeed, we have for all $t, s \in [-T, T]$

$$\int_{\mathbb{R}} (\mathrm{Im}\ N(u) - \Delta_T(u))\frac{e^{iut}-1}{u}\frac{e^{-ius}-1}{u}\,du = 0. \tag{5.3}$$

The function $\mathrm{Im}\, N(u) - \Delta_T(u)$ is in the Wiener algebra and vanishes at infinity. Thus we have

$$\mathrm{Im}\ N(u) - \Delta_T(u) = \int_{\mathbb{R}} e^{iux} v_T(x)\, dx,$$

where the entries of v_T are in $\mathbf{L}_1(\mathbb{R})$. Differentiating (5.3) with respect to t and s we have

$$\int_{\mathbb{R}} e^{i(t-s)u} \left(\int_{\mathbb{R}} e^{iux} v_T(x)\, dx \right) du = 0$$

for $t, s \in [-T, T]$. Hence, using the properties of the inverse Fourier transform we have that $v_T(x) = 0$ for $x \in [-2T, 2T]$. It follows that

$$N(u) - N_T(u) = 2 \int_T^{\infty} e^{iux} v_T(x) dx. \tag{5.4}$$

Step 6: *We now conclude the proof and first check that one can replace $\Delta_T(t)$ and $N_T(z)$ by* Im $N(t)$ *and* $N(z)$, *respectively*, *in* (4.14).

By Step 1, $\Lambda_T(z, w)$ is of the form $\int_{-T}^{T} g(u) e^{izu} du$ where the entries of g are in $\mathbf{L}_2(-T, T)$. We show that in (3.11) we can replace N by N_T. Indeed, using (5.4)

$$\begin{aligned} & p(N - N_T)^* p \Lambda_T - q(N - N_T) q \Lambda_T \\ & = 2p\left(\left(\int_T^{\infty} e^{-iux} v_T(x)\, dx \right) \int_0^T g(u) e^{izu} du \right) \\ & \quad -\ 2q\left(\left(\int_T^{\infty} e^{iux} v_T(x)^* dx \right) \int_{-T}^{0} g(u) e^{izu} du \right) \\ & = 0. \end{aligned}$$

Here we used the easily verified facts that

$$\left(\int_T^{\infty} e^{-iux} v_T(x)\, dx \right) \int_0^T g(u) e^{izu}\, du \in \mathscr{W}_+$$

and

$$\left(\left(\int_T^{\infty} e^{iux} v_T(x)^* dx \right) \int_{-T}^{0} g(u) e^{izu} du \right) \in \mathscr{W}_-.$$

Thus for $w \in \mathbb{C}$ and $c \in \mathbb{C}^n$ we have

$$\Gamma_T(z,w)c = -\left(\int_{\mathbb{R}} \frac{\text{Im } N(t)\Lambda_T(t,w)c\,dt}{\pi i(t-z)} + iN(z)\Lambda_T(z,w)c\right)$$

$$= -i(pN^*p - qNq)\Lambda_T(z,w)c$$

$$= -(\Lambda_T(z,w)c)_- \quad \text{(in view of (3.11))}.$$

Since the span of the functions of the form $\Lambda(z,w)c$ ($w \in \mathbb{C},\ c \in \mathbb{C}^n$) is dense in $\mathscr{B}_T$ and since the operator $i(pN^*p - qNq)$ is bounded, (4.11) holds for all $g \in \mathscr{B}_T$ by continuity.

5.2. *Proof of the trace formula*

We begin with a proposition. We recall that the operators U_+ and U_- are defined by (1.6) and (3.7), respectively.

Proposition 5.2. *Let*

$$(V_+x)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{(\text{Im } N(t))x(t)\,dt}{t-z}, \quad \mathbf{L}_2^n(\text{Im } N) \to \mathscr{L}(N)$$

and

$$(V_-x)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{(\text{Im } (-N(t))^{-1})x(t)\,dt}{t-z}, \quad \mathbf{L}_2^n(\text{Im } (-N^{-1})) \to \mathscr{L}((-N^{-1})).$$

Then

$$-M_N V_- U_- J = V_+ U_+. \tag{5.5}$$

Proof. Eq. (5.5) can be rewritten as

$$-N(z)\int_{\mathbb{R}} \frac{\text{Im } (-N(t)^{-1})(U_-Jf)(t)\,dt}{2\pi i(t-z)} = \int_{\mathbb{R}} \frac{(\text{Im } N(t))(U_+f)(t)\,dt}{2\pi i(t-z)}.$$

Thus we have to prove that $-U_-Jf$ is the Hilbert transform of U_+f:

$$U_-Jf(z) = -(U_+f)_-(z)$$

$$= -\left(\int_{\mathbb{R}} \frac{(\text{Im } N)(t)(U_+f)(t)\,dt}{(t-z)} + iN(z)(U_+f)(z)\right).$$

From Proposition 5.1 we replace in (4.11) Δ_T by Im N and N_T by N. We thus have $\tilde{g} = -g_-$ for $g \in \mathscr{B}_T$. Comparing (3.8) and (4.10) we then obtain

$$(U_+ f)_- = -U_- Jf \tag{5.6}$$

for a function of support in $(0, T)$. Thus, equality (5.5) holds for every f with compact support and by continuity it holds for all $f \in \mathbf{L}_2^{2n}(\mathbb{R}_+)$. $\square$

We now turn to the proof of the trace formula. We first remark that we can choose an orthonormal basis of $\mathbf{L}_2^{2n}(\mathbb{R}_+)$ which consists of functions of the form

$$f_p(t) = \begin{pmatrix} x_p(t) \\ 0 \end{pmatrix} \tag{5.7}$$

and

$$f_p(t) = \begin{pmatrix} 0 \\ x_p(t) \end{pmatrix}, \tag{5.8}$$

i.e. such that $Jf_p = \pm f_p$ and so $(U_+ f_p)_- = \pm U_- f_p$ and $NV_- U_- f_p = \pm V_+ U_+ f_p$.

The family $F_p = V_+ U_+ f_p$ is an orthonormal basis of $\mathscr{L}(N)$ since $V_+ U_+$ is a unitary transformation from $\mathbf{L}_2^{2n}(\mathbb{R}_+)$ onto $\mathscr{L}(N)$. Recall that

$$\langle (H_+ - wI)^{-1} f, f \rangle_{\mathbf{L}_2^{2n}(\mathbb{R}_+)} = \langle R_w F, F \rangle_{\mathscr{L}(N)}, \quad F = V_+ U_+ f \tag{5.9}$$

for every $f \in \mathbf{L}_2^{2n}(\mathbb{R}_+)$ and that similarly,

$$\langle (H_- - wI)^{-1} f, f \rangle_{\mathbf{L}_2^{2n}(0,\infty)} = \langle R_w F', F' \rangle_{\mathscr{L}(-N^{-1})}, \quad F' = V_- U_- f.$$

On the other hand,

$$\langle R_w F', F' \rangle_{\mathscr{L}(-N^{-1})} = \langle N R_w N^{-1} N F', N F' \rangle_{\mathscr{L}(N)}.$$

Assume that f is such that $NV_- U_- f = \pm V_+ U_+ f$. Then, the formula

$$(R_w(uv))(z) = (R_w u)(z) v(w) + u(z)(R_w v)(z)$$

for u and v of appropriate dimensions and analytic in a neighborhood of the point w leads to:

$$\begin{aligned}
&\langle (H_- - wI)^{-1} f, f \rangle_{\mathbf{L}_2^{2n}(\mathbb{R}_+)} \\
&\quad = \langle N (R_w N^{-1})(z)(V_+ U_+ f)(w), V_+ U_+ f \rangle_{\mathscr{L}(N)} \\
&\quad\quad + \langle N N^{-1} (R_w V_+ U_+ f)(z), V_+ U_+ f \rangle_{\mathscr{L}(N)}
\end{aligned}$$

$$= -\langle (R_w N)(z)N(w)^{-1}(V_+U_+f)(w), V_+U_+f\rangle_{\mathscr{L}(N)}$$

$$+ \langle (H_+ - wI)^{-1}f, f\rangle_{\mathbf{L}_2^{2n}(\mathbb{R}_+)},$$

where we have used (5.9). Thus if $f = f_p$ is an element of a basis of $\mathbf{L}_2^{2n}(0,\infty)$ of the form (5.7) or (5.8)

$$\langle ((H_+ - w)^{-1} - (H_- - wI)^{-1})f_p, f_p\rangle_{\mathbf{L}_2^{2n}(0,\infty)}$$

$$= \left\langle \frac{N(z)-N(w)}{z-w}N(w)^{-1}F_p(w), F_p\right\rangle_{\mathscr{L}(N)}$$

$$= 4\pi\,\mathrm{Tr}\{N(w)^{-1}F_p(w)F_p(w^*)^*\}.$$

To prove the last equality, remark that $\frac{N(z)-N(w)}{z-w} = 4\pi K_N(z,w^*)$ so that we have

$$N'(w) = 4\pi\ K_N(w,w^*)$$

and

$$\left\langle \frac{N(z)-N(w)}{z-w}N(w)^{-1}F_p(w), F_p\right\rangle_{\mathscr{L}(N)}$$

$$= \langle F_p, 4\pi K_N(z,w^*)N(w)^{-1}F_p(w)\rangle^*_{\mathscr{L}(N)}$$

$$= 4\pi\Big(F_p(w)^*N(w)^{-1*}F_p(w^*)^*\Big)^*$$

$$= 4\pi\mathrm{Tr}\,N(w)^{-1}F_p(w)F_p(w^*)^*.$$

By Proposition 2.4,

$$K_N(z,w) = \sum_0^\infty F_p(z)F_p(w)^*, \tag{5.10}$$

where $F_1, F_2, \ldots$ form an orthonornal basis of $\mathscr{L}(N)$. Setting $z = w \in \mathbb{C}\backslash\mathbb{R}$ in (5.10) we obtain

$$N'(w) = 4\pi\left(\sum_0^\infty F_p(w)F_p(w^*)^*\right).$$

So,

$$
\begin{aligned}
&\mathrm{Tr}\{(H_+ - zI)^{-1} - (H_- - zI)^{-1}\}\\
&\quad = \sum_0^\infty \langle ((H_+ - w)^{-1} - (H_- - wI)^{-1})f_p, f_p \rangle_{\mathbf{L}_2^{2n}(0,\infty)}\\
&\quad = \sum_0^\infty \left\langle \frac{N(z) - N(w)}{z - w} N(w)^{-1} F_p(w), F_p \right\rangle_{\mathscr{L}(N)}\\
&\quad = \sum_0^\infty \langle F_p, 4\pi K_N(z, w^*) N(w)^{-1} F_p(w) \rangle^*_{\mathscr{L}(N)}\\
&\quad = 4\pi\, \mathrm{Tr}\, N(w)^{-1} \left(\sum_0^\infty F_p(w) F_p(w^*)^* \right)\\
&\quad = \mathrm{Tr}\, N(w)^{-1} N'(w)
\end{aligned}
$$

and this finishes the proof of the trace formula (1.7). We now turn to the proof of (1.8). First note that

$$
\frac{d}{dz} \ln \det N(z) = \mathrm{Tr}\ N(z)^{-1} N'(z). \tag{5.11}
$$

See [27, Lemma, p. 129]. Next we start from (A.3):

$$
\frac{d}{dz} \ln \widetilde{\det}_{z_0} (H_- - zI)(H_+ - zI)^{-1} = \mathrm{Tr}((H_+ - zI)^{-1} - (H_- - zI)^{-1}).
$$

Using (5.11) we see that (1.7) can be rewritten as

$$
\frac{d}{dz} \ln \widetilde{\det}_{z_0} (H_- - zI)(H_+ - zI)^{-1} = \frac{d}{dz} \ln \det N(z),
$$

from which we obtain

$$
\ln \widetilde{\det}_{z_0} (H_- - zI)(H_+ - zI)^{-1} = \ln \det N(z) + k
$$

for some constant k. Hence

$$
\widetilde{\det}_{z_0} (H_- - zI)(H_+ - zI)^{-1} = e^k \det N(z).
$$

The choice $z = z_0$ leads to $e^k = \det N(z_0)^{-1}$ and hence we obtain (1.8). $\square$

Definitions and the main properties of perturbation determinants are recalled in the appendix. We also refer to the paper [26], which can be found in [28].

6. The rational case

6.1. Introduction

In this section we study the rational case, that is the case where one (and hence all) of the characteristic spectral functions of expression (1.1) is rational. A rational function analytic at infinity and on the real line belongs to the Wiener algebra:

Proposition 6.1. *Let W be a $\mathbb{C}^{n\times n}$-valued rational function analytic on the real line and at infinity, and let $W(z) = D + C(zI - A)^{-1}B$ be a minimal realization of W. Then, W belongs to the Wiener algebra $\mathscr{W}^{n\times n}$ and $W(z) = D - \int_{\mathbb{R}} k(u)e^{izu}\,du$ where*

$$k(u) = \begin{cases} iCe^{-iuA}(I-P)B & \text{if } u>0, \\ -iCe^{-iuA}PB & \text{if } u<0, \end{cases}$$

where P is the Riesz projection corresponding to the eigenvalues of A in $\mathbb{C}_+$.

In [5] we characterized the class of potentials associated to a rational spectral function:

Theorem 6.2. *Let $W(z)$ be a $\mathbb{C}^{n\times n}$-valued rational function analytic and strictly positive on the real line and such that $W(\infty) = I_n$. Then, W is the spectral function of a canonical differential expression of the form* (1.1). *Let $W(z) = I_n + C(I_N - zA)^{-1}B$ be a minimal realization of W. Then the potential $k(t)$ is given by the formula*

$$k(t) = 2C(Pe^{-2itA^{\times}}|_{\operatorname{Im}\,P})^{-1}PB,$$

where $A^{\times} = A - BC$ and where P is the Riesz projection corresponding to the eigenvalues of A in $\mathbb{C}_+$.

These potentials are called *strictly exponential potentials*. The next theorem, proved in [6] and [9] expresses the spectral function in terms of a minimal realization of the spectral factor $S_-(z)$.

Proposition 6.3. *Let $W(z) = S_-(z)^{-1}S_-(z)^{-1*}$ be the spectral function associated to the canonical differential expression* (1.1) *and let $S_-(z) = I_n + c(zI - a)^{-1}b$ be a minimal realization of the spectral factor S_-. Then the potential associated to W is given by*

$$k(t) = 2ce^{2ita}(I_m + \Omega(Y - e^{-2ita^*}Ye^{2ita}))^{-1}(b + i\Omega c^*),$$

where Ω and Y are the unique solutions of the Lyapunov equations

$$i(\Omega a^{\times *} - a^{\times}\Omega) = bb^*,$$

$$i(Ya - a^* Y) = -c^* c.$$

One can obtain a minimal realization of N from various minimal realizations of other spectral data associated to the weight function. The following result is taken from [8,9].

Proposition 6.4. *Let $W(z) = S_-(z)^{-1} S_-(z)^{-1*}$ be the spectral function associated to the canonical differential expression* (1.1) *and let et $S_-(z) = I_n + c(zI - a)^{-1} b$ be a minimal realization of the spectral factor S_-. Then the function*

$$N(z) = i(I_n + 2(-b^* + ic\Omega)(zI_p - a^{\times *})^{-1} c^*)$$

is analytic in the open upper half-plane and such that

$$W(z) = \mathrm{Im}\ N(z).$$

6.2. The trace formula in the rational case

Let r be a rational function with no poles on the real line and vanishing at infinity. It can be written as a finite sum of the form

$$r(z) = \sum \frac{c_j}{(z - w_j)^{n_j}},$$

where the $c_j \in \mathbb{C}$ and the w_j are not real. If H is a self-adjoint operator we define

$$r(H) = \sum c_j (H - w_j I)^{-n_j}.$$

Theorem 6.5. *Let N be a $\mathbb{C}^{n\times n}$-valued rational function analytic in the closed upper half-plane and at infinity and such that $N(\infty) = I_n$ and with a positive real part in the upper half-plane. Then, N is the Nevanlinna function of a canonical differential expression. Let H_+ and H_- be the self-adjoint operators defined by* (1.2) *and* (1.3) *and let*

$$N(z) = i(I_n + c(zI - a)^{-1} b)$$

be a minimal realization of the Nevanlinna function N. Then, for any rational function r with no poles in the closed upper half-plane and vanishing at infinity, the operator

$$r(H_+) - r(H_-)$$

has rank n and

$$\mathrm{Tr}\{r(H_+) - r(H_-)\} = \mathrm{Tr}\, r(a) - \mathrm{Tr}\, r(a^\times),$$

where $a^\times = a - bc$. *Finally,*

$$\widetilde{\det}_{z_0}(H_+ - zI)(H_- - zI)^{-1} = \frac{\det(a - zI)^{-1}(a^\times - zI)}{\det(a - z_0 I)^{-1}(a^\times - z_0 I)}. \tag{6.1}$$

Proof. Let $N(z) = i(I_n + c(zI - a)^{-1}b)$ be a realization of N. Then,

$$N'(z) = -ic(zI - a)^{-2}b$$

and thus, with $a^\times = a - bc$ we have

$$\begin{aligned} N^{-1}(z)N'(z) &= (I - c(zI - a^\times)^{-1}b)(-c(zI - a)^{-2}b) \\ &= -c(zI - a)^{-2}b \\ &\quad + c(z - a^\times)^{-1}bc(zI - a)^{-2}b \\ &= -c(zI - a)^{-2}b \\ &\quad + c(zI - a^\times)^{-1}(a - a^\times)(zI - a)^{-2}b \\ &= -c(zI - a^\times)^{-1}(zI - a)^{-1}b. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Tr}\, N(z)^{-1}N'(z) &= \mathrm{Tr}(-c(zI - a^\times)^{-1}(zI - a)^{-1}b) \\ &= \mathrm{Tr}(zI - a)^{-1}(-bc)(zI - a^\times)^{-1} \\ &= \mathrm{Tr}(zI - a)^{-1}(a^\times - a)(zI - a^\times)^{-1} \\ &= \mathrm{Tr}(zI - a)^{-1}(a^\times - zI + zI - a)(zI - a^\times)^{-1}, \end{aligned}$$

and thus

$$\mathrm{Tr}\, N(z)^{-1}N'(z) = \mathrm{Tr}((zI - a^\times)^{-1} - (zI - a)^{-1}) \tag{6.2}$$

and (1.7) leads to

$$\mathrm{Tr}((H_+ - zI)^{-1} - (H_- - zI)^{-1}) = \mathrm{Tr}\ ((a - zI)^{-1} - (a^\times - zI)^{-1}) \tag{6.3}$$

and hence the result for r with simple poles. The general result is obtained by an approximation argument.

We now prove (6.1). Formula (A.1) with $B = a$ and $A = a^\times$ gives

$$\frac{d}{dz}\ln\det(a - zI)(a^\times - zI)^{-1} = \mathrm{Tr}\,\{(a^\times - zI)^{-1} - (a - zI)^{-1}\}.$$

On the other hand, recall (A.3):

$$\frac{d}{dz}\ln\widetilde{\det}_{z_0}(H_- - zI)(H_+ - zI)^{-1} = \mathrm{Tr}\,((H_+ - zI)^{-1} - (H_- - zI)^{-1}).$$

In view of (6.3) we have

$$\frac{d}{dz}\{\ln\det(a - zI)(a^\times - zI)^{-1} + \ln\widetilde{\det}_{z_0}(H_- - zI)(H_+ - zI)^{-1}\} = 0$$

and hence the result. $\square$

Appendix. Perturbation determinants

In this appendix we review some facts on perturbation determinants. First the definition. If A and B are (possibly unbounded operators) such that $B - A$ is trace class, the determinant $\det(B - zI)(A - zI)^{-1}$ makes sense and is called the perturbation determinant of the pair (A, B). See [23, Chapter 4, Section 3]. Furthermore, it holds that

$$\frac{d}{dz}\ln\det(B - zI)(A - zI)^{-1} = \mathrm{Tr}\{(A - zI)^{-1} - (B - zI)^{-1}\}, \tag{A.1}$$

see [27, p. 132].

One can extend the notion of perturbation determinant to the case where $B - A$ is not of trace class but where for some $z_0 \in \mathbb{C}$ both $(A - z_0 I)^{-1}$ and $(B - z_0 I)^{-1}$ exist and are such that $(B - z_0 I)^{-1} - (A - z_0)^{-1}$ is of trace class. One sets

$$\begin{aligned}&\widetilde{\det}_{z_0}\ (B - zI)(A - zI)^{-1}\\ &\quad\stackrel{\text{def}}{=} \det\left((B - z_0 I)^{-1} - \frac{I}{z - z_0}\right)\left((A - z_0 I)^{-1} - \frac{I}{z - z_0}\right)^{-1}.\end{aligned} \tag{A.2}$$

Formula (A.1) still holds when the perturbation determinant is replaced by this generalized perturbation determinant. See [27, p. 132]. For completeness we prove this fact in the next proposition.

Proposition A.1. *It holds that*

$$\frac{d}{dz}\ln \widetilde{\det}_{z_0}(B-zI)(A-zI)^{-1} = \mathrm{Tr}((A-zI)^{-1}-(B-zI)^{-1}). \tag{A.3}$$

In particular the value of $\widetilde{\det}_{z_0}$ *does not depend, up to a multiplicative constant, on the choice of* z_0 *and reduces to the perturbation determinant (still up to a multiplicative constant) when* $B-A$ *is of trace class.*

Proof. First note that formula (A.1) holds for

$$B_0 = (B-z_0I)^{-1},$$

$$A_0 = (A-z_0I)^{-1}.$$

Hence,

$$\frac{d}{dz}\ln \det((B_0-z_0I)^{-1}-zI)((A_0-z_0I)^{-1}-zI)^{-1}$$

$$= \mathrm{Tr}\{((A_0-z_0I)^{-1}-zI)^{-1}-((B_0-z_0I)^{-1}-zI)^{-1}\}.$$

Thus,

$$\frac{d}{dz}\ln \widetilde{\det}_{z_0}(B_0-zI)(A_0-zI)^{-1}$$

$$= \frac{d}{dz}\left((B_0-z_0I)^{-1}-\frac{I}{z-z_0}\right)\left((A_0-z_0I)^{-1}-\frac{I}{z-z_0}\right)^{-1}$$

$$= -\frac{1}{(z-z_0)^2}\mathrm{Tr}\left\{\left((A_0-z_0I)^{-1}-\frac{I}{z-z_0}\right)^{-1}-\left((B_0-z_0I)^{-1}-\frac{I}{z-z_0}\right)^{-1}\right\},$$

where we used the rule of differentiation for composition of functions to go from the second to the third line.

If z is also in the resolvent set of A, it is easily shown using the resolvent identity

$$(A-zI)^{-1}-(A-z_0I)^{-1} = (z-z_0)(A-zI)^{-1}(A-z_0I)^{-1}$$

that

$$\left((A-z_0I)^{-1}-\frac{I}{z-z_0}\right)^{-1} = (z-z_0)(-I+(z-z_0)(zI-A)^{-1}),$$

and similarly for B. Hence,

$$-\frac{1}{(z-z_0)^2}\mathrm{Tr}\left\{\left((A-z_0I)^{-1}-\frac{I}{z-z_0}\right)^{-1}-\left((B-z_0I)^{-1}-\frac{I}{z-z_0}\right)^{-1}\right\}$$

$$=-\frac{1}{(z-z_0)^2}\mathrm{Tr}\{(z-z_0)(-I+(z-z_0)(zI-A)^{-1})$$

$$-\ (z-z_0)\left(-I+(z-z_0)(zI-B)^{-1}\right)\}$$

$$=\mathrm{Tr}((A-zI)^{-1}-(B-zI)^{-1})$$

which ends the proof of (A.3). □

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