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Independent subbases and non-redundant codings of separable metrizable spaces

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ABSTRACT

The notion of an independent subbase was introduced by H. Tsuiki to apply non-redundant $\{0,1,\perp\}^\omega$-code representations to topological spaces. We prove that every dense in itself, separable, metrizable space X has an independent subbase and, if $\dim X \leqslant n$ in addition, then X has an independent subbase of dimension n. We also study other properties of subbases related to non-redundant $\{0,1,\perp\}^\omega$-codings.

1. Introduction

Let $\mathbb{T}$ be the set $\{0,1,\perp\}$, where $\perp$ is called the bottom character which means undefinedness. The notion of an independent subbase was introduced by Tsuiki [9] to apply non-redundant $\mathbb{T}^\omega$-code representations to topological spaces, in particular, separable metrizable spaces. Let ω be the first infinite ordinal. As usual, an element of ω is identified with the set of smaller elements, for example, $2 = \{0,1\}$, and ${}^\Gamma 2$ denotes the set of all maps from a finite subset Γ of ω to 2.

Definition 1. An *independent subbase* S of a space X is a subbase $\{S_{n,i}\colon\ n<\omega,\ i<2\}$ of X, such that

$$(\forall n<\omega)\quad (S_{n,0}\cap S_{n,1}=\emptyset), \tag{1.1}$$

and

$$(\forall n<\omega)\ (\forall\sigma\in {}^n 2)\ \left(\bigcap_{k<n} S_{k,\sigma(k)}\neq\emptyset\right). \tag{1.2}$$

We show its equivalence to the definition in [9] in the next section. In particular, we show in Lemma 7 that $S_{n,i}$ $(n<\omega,\ i<2)$ are regular open and $S_{n,0}$ and $S_{n,1}$ are exteriors of each other for an independent subbase S. It is obvious

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from (1.2) that for a finite subset Γ of ω and a function $\sigma \in^{\Gamma} 2$, $\bigcap_{k\in\Gamma} S_{k,\sigma(k)}$ is not empty. Therefore, we can say that an independent subbase is a subbase which generates through finite intersections and unions a free boolean subalgebra of the boolean algebra of regular open sets of X.

From an independent subbase S, we can define a mapping φ_S from X to $\mathbb{T}^\omega$ as follows

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \bot & (\text{otherwise}). \end{cases}$$

Actually, it is a topological embedding of X in $\mathbb{T}^\omega$ and it assignes a unique $\mathbb{T}^\omega$-code to each element of X. It has the property that each index has an "independent" meaning in that for any subset Γ of ω and $n \notin \Gamma$, any assignment σ of digits (i.e., 0 or 1) to Γ do not determine the digit at the index n because $\bigcap_{k\in\Gamma\cup\{n\}} S_{k,\sigma_i(k)} \neq \emptyset$ for both of the extensions σ_i $(i = 0, 1)$ of σ to $\Gamma \cup \{n\}$ which assigns i to n. We explain in detail the relation between an independent subbase and a non-redundant $\mathbb{T}^\omega$-code representation in Section 2.

In this paper, we prove Theorems 1, 3 and their corollaries below, which answer the questions asked in [9, Section 6]. Moreover, we study other notions of subbases closely related to an independent subbase and prove some results which fill a gap in a statement in [9, p. 679].

In [9, Proposition 6.1], it is shown that every space having an independent subbase is dense in itself. Then, examples of independent subbases are given on the Cantor set, the unit interval $\mathbb{I}$, the products $\mathbb{I}^n$, the Hilbert cube $\mathbb{I}^\omega$, the circle S^1, and several surfaces such as S^2, the torus T^2 and the n-torus nT^2, and it is asked if every dense in itself, separable, metrizable space has an independent subbase. Theorem 1 answers this question positively.

Theorem 1. *Every dense in itself, separable, metrizable space has an independent subbase.*

From this theorem and Proposition 6.1 of [9], we have the following.

Corollary 2. *A separable metrizable space X is dense in itself if and only if X has an independent subbase.*

For a $\mathbb{T}^\omega$-code representation of a topological space X, the maximum number of copies of $\bot$ which may appear in a code sequence has a computational meaning as is explained in [6] and also in Section 2 of this paper. For the case φ_S of a $\mathbb{T}^\omega$-code representation derived from an independent subbase S, such a number is equal to the dimension of an independent subbase defined as follows.

Definition 2. For a non-negative integer m, an independent subbase $S = \{S_{n,i}\colon\ n < \omega,\ \ i < 2\}$ is *of dimension m* if $\operatorname{ord}\{X \setminus (S_{n,0} \cup S_{n,1})\colon\ n < \omega\} \leqslant m - 1$, where $\operatorname{ord}\mathcal{A}$ means the largest integer m such that the collection $\mathcal{A}$ contains $m + 1$ sets with a non-empty intersection.

Note that if the dimension of S is m, then φ_S is an embedding of X in $\mathbb{T}^\omega_m$, which is the subspace of $\mathbb{T}^\omega$ only with those sequences with at most m copies of $\bot$. $\mathbb{T}^\omega_m$ is a T_0-space with the small inductive dimension m [8]. Therefore, if X is a separable metrizable space, $\dim X \leqslant m$ for $\dim X$ the covering dimension of X, because the covering dimension and the small inductive dimension coincide for separable metrizable spaces. It was an open question of [9] whether there is an independent subbase of dimension $\dim X$ for a dense in itself, separable, metrizable space X. Theorem 3 answers this question positively.

Theorem 3. *Every dense in itself, separable, metrizable space X with* $\dim X \leqslant m$ *has an independent subbase of dimension m.*

From this theorem, with the above discussion and Proposition 6.1 of [9], we can derive the next corollary.

Corollary 4. *A separable metrizable space X is dense in itself and* $\dim X \leqslant m$ *if and only if X has an independent subbase of dimension m.*

In the next section, we explain the relation between an independent subbase and a non-redundant $\mathbb{T}^\omega$-code representation of a topological space. We give the proof of Theorem 1 in Sections 3 and 4, and the proof of Theorem 3 in Section 5. After that, in Section 6, we investigate other properties of subbases related to non-redundant $\mathbb{T}^\omega$-codings.

Preliminaries and notations. Throughout this paper, X denotes a second countable space. If X is regular, then X is separable metrizable by Urysohn's metrization theorem. Recall that a subset U of X is *regular open* if U is the interior of its closure. The terms '(regular) open set' and 'closed set' always mean '(regular) open set in X' and 'closed set in X', respectively, and $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ denote the interior, closure, boundary of a set A in X, respectively. The notation and terminology on topology will be used as in [2].

Each element of $\mathbb{T}^\omega$ is called a *bottomed sequence* and each copy of 0 and 1 which appears in a bottomed sequence σ is called a *digit* of σ. A *finite element* of $\mathbb{T}^\omega$ is a bottomed sequence with a finite number of digits, and the set of all finite elements of $\mathbb{T}^\omega$ is denoted by $K(\mathbb{T}^\omega)$. We write $\mathrm{dom}(\sigma) = \{k\colon\ \sigma(k) \neq \bot\}$ for $\sigma \in \mathbb{T}^\omega$ and $t^\omega = (t, t, \ldots) \in \mathbb{T}^\omega$ for $t \in \mathbb{T}$.

We define the partial order relation $\bot \preccurlyeq 0$ and $\bot \preccurlyeq 1$ on $\mathbb{T}$, and denote its product order on $\mathbb{T}^\omega$ by the same symbol $\preccurlyeq$, i.e., for every $\sigma, \tau \in \mathbb{T}^\omega$, $\sigma \preccurlyeq \tau$ if $\sigma(n) \preccurlyeq \tau(n)$ for each $n < \omega$. Then 2^ω is the set of maximal elements of $\mathbb{T}^\omega$. We consider the T_0-topology $\{\emptyset, \{0\}, \{1\}, \{0,1\}, \mathbb{T}\}$ on $\mathbb{T}$, and its product topology of $\mathbb{T}^\omega$. We say that two elements σ and σ' of $\mathbb{T}^\omega$ are *compatible* if $\sigma \preccurlyeq \tau$ and $\sigma' \preccurlyeq \tau$ for some $\tau \in \mathbb{T}^\omega$, and write $\sigma \uparrow \sigma'$ if σ and σ' are compatible. For $\sigma \in \mathbb{T}^\omega$, we define $\uparrow\sigma = \{\tau\colon\ \tau \succcurlyeq \sigma\}$, $\downarrow\sigma = \{\tau\colon\ \tau \preccurlyeq \sigma\}$, and $\Downarrow\!\!\Uparrow\sigma = \bigcup\{\downarrow\sigma'\colon\ \sigma' \in \uparrow\sigma\}$. Therefore, we have $\Downarrow\!\!\Uparrow\sigma = \{\tau\colon\ \tau \uparrow \sigma\}$. The family $\{\uparrow\sigma\colon\ \sigma \in K(\mathbb{T}^\omega)\}$ is a base of the topology on $\mathbb{T}^\omega$.

A surjective partial function from 2^ω to X, i.e., a surjection defined on a subset of 2^ω, is called a *representation* of X [10]. A representation can be considered as a coding which may assign more than one codes to each element. That is, a representation ρ of X assigns elements of $\rho^{-1}(x)$ as codes of x.

The letters i, j, k, l, m, n will be used to denote finite ordinal numbers (= non-negative integers), and σ and τ will be used to denote bottomed sequences.

2. Independent subbases and non-redundant $\{0, 1, \bot\}^\omega$-codings

The idea of an independent subbase comes from a $\mathbb{T}^\omega$-coding of a topological space which is non-redundant in that (1) it assigns a unique code to each element as opposed to a representation which may assign more than one codes, and that (2) there is no redundancy among the meanings of the digits of each code.

In the introductory explanation of this section, we consider the case X is a separable metrizable space though an independent subbase will be defined generally for second countable spaces which are not necessarily regular. First, we consider a unique 2^ω-coding of a separable metrizable space X which is expressed as an injective function φ' from X to 2^ω. Such a coding induces computation on X with a Type2 machine, which is an extension of the Turing machine with input and output tapes of infinite length [10]. Here, for the extension in the next paragraph, we consider a variant of a Type2 machine which fills the cells in any order. That is, in order to output x, a machine starts with the output tape filled with $\bot$ and fills $\bot$-cells with 0 or 1 in any order, and the tape becomes $\varphi'(x)$ after an infinite time of execution. Correspondingly, as an input, we consider that a sequence of pairs (n_k, i_k) $(k = 0, 1, \ldots)$ for $n_k < \omega$ and $i_k < 2$ is given, where (n, i) means that the n-th cell of the input tape is filled with i. We have the restriction that each cell of the tape is filled only once, that is, $n_k \neq n_{k'}$ for $k \neq k'$. Therefore, the states of the input/output tapes change monotonically as $\bot^\omega \preccurlyeq \sigma_1 \preccurlyeq \sigma_2 \preccurlyeq \sigma_3 \preccurlyeq \cdots$ for $\sigma_i \in K(\mathbb{T}^\omega)$, where σ_i is the state of the input/output tape when the i-th input/output operation is made. Let $S'(\sigma) = \bigcap_{k \in \mathrm{dom}(\sigma)} \{x\colon\ \varphi'(x)(k) = \sigma(k)\}$. According to the change of the states of a tape, we have the information that x is in the sets $X = S'(\bot^\omega) \supseteq S'(\sigma_1) \supseteq S'(\sigma_2) \supseteq \cdots$ and $\{x\} = S'(\varphi'(x)) = \bigcap_{n<\omega} S'(\sigma_n)$. In order that this process is regarded as producing better and better topological approximation of x, it is natural to impose the condition that $\{S'(\sigma_n)\colon\ n < \omega\}$ forms a neighbourhood base of x. Since it holds for every $x \in X$, $\{S'(\sigma)\colon\ \sigma \in K(\mathbb{T}^\omega)\}$ should be a base of the topology of X. That is, we follow the idea that open sets are finitely observable properties [5]. Since $S'(\sigma) = \varphi'^{-1}(\{\tau \in 2^\omega\colon\ \tau \succcurlyeq \sigma\})$ for $\sigma \in K(\mathbb{T}^\omega)$ and the family of sets $\{\tau \in 2^\omega\colon\ \tau \succcurlyeq \sigma\}$ for $\sigma \in K(\mathbb{T}^\omega)$ forms a base of 2^ω, it means that φ' is an embedding of X in 2^ω.

However, there is no embedding of X in 2^ω for a non-zero-dimensional space X because 2^ω is zero-dimensional, and we cannot define computation on a non-zero-dimensional space X in this way. Usually, this problem is solved by considering a representation, which may assign more than one codes to each element [10]. Here, we consider a different approach, that is, by changing the code space from 2^ω to $\mathbb{T}^\omega$ and considering a coding function $\varphi : X \to \mathbb{T}^\omega$. It is immediate to show that separable metrizable spaces can be embedded in $\mathbb{T}^\omega$, and more generally that a topological space can be embedded in $\mathbb{T}^\omega$ if and only if it is a second countable T_0-space. On the other hand, by allowing some of the cells of the output tape to be unfilled even after an infinite-time of computation, the same kind of machine as above can input and output $\mathbb{T}^\omega$ elements. Therefore, with such a coding function, we can define computation over X. In particular, for $n \leqslant \omega$, a separable metrizable space X is n-dimensional if and only if there is an embedding of X in $\mathbb{T}^\omega_n$, which is the set of bottomed sequences with at most n copies of $\bot$ [6,8], and on the other hand, one can define in a concise way an IM2-machine, which input and output bottomed sequences in $\mathbb{T}^\omega_n$ for $n < \omega$ [6,7]. Therefore, coding functions with the code space $\mathbb{T}^\omega_n$ for $n < \omega$ are particularly important.

Now, suppose that an embedding $\varphi : X \to \mathbb{T}^\omega$ of a topological space X is given. We define $P_{n,i} = \{\sigma \in \mathbb{T}^\omega\colon\ \sigma(n) = i\}$ and $S_{n,i} = \{x\colon\ \varphi(x)(n) = i\} = \varphi^{-1}(P_{n,i})$. Since $\{P_{n,i}\colon\ n < \omega,\ i < 2\}$ forms a subbase of $\mathbb{T}^\omega$ such that $P_{n,0} \cap P_{n,1} = \emptyset$, $\{S_{n,i}\colon\ n < \omega,\ i < 2\}$ forms a subbase of X such that $S_{n,0} \cap S_{n,1} = \emptyset$ for every $n < \omega$. Conversely, if $S = \{S_{n,i}\colon\ n < \omega,\ i < 2\}$ is a subbase indexed with $\omega \times 2$ such that $S_{n,0} \cap S_{n,1} = \emptyset$ for $n < \omega$, then we have an embedding $\varphi_S : X \to \mathbb{T}^\omega$ defined as

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \bot & (\text{otherwise}). \end{cases}$$

Therefore, we identify an embedding of X in $\mathbb{T}^\omega$ with such a subbase indexed with $\omega \times 2$. When $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ is such a subbase and $\sigma \in \mathbb{T}^\omega$, we write

$$S(\sigma) = \bigcap_{n \in \mathrm{dom}(\sigma)} S_{n,\sigma(n)}.$$

Then the family $\{S(\sigma)\colon \sigma \in K(\mathbb{T}^\omega)\}$ forms a base of X corresponding to the subbase S.

In [9], an independent subbase is defined in three steps as a special case of such a subbase with index.

Definition 3. ([9]) A *dyadic subbase* of a space X is a subbase $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ indexed with $\omega \times 2$ such that every element is a regular open set and $S_{n,1} = X \setminus \mathrm{cl}\, S_{n,0}$ for $n < \omega$.

When $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ is a dyadic subbase of X, pairings of $S_{n,0}$ and $S_{n,1}$ are fixed and differences of indexings with $\omega \times 2$ are caused only by reindexings on ω and 2. Since the corresponding difference on the coding functions are caused only by reindexings on ω and inversions on the values, which are not essential when we consider properties of codings, we ignore the indexing and call the subbase itself a dyadic subbase.

If S is a dyadic subbase, $\{x\colon \varphi_S(x)(n) = \bot\}$ is the boundary of both $S_{n,0}$ and $S_{n,1}$ and is a nowhere dense subset for every n. This fact has the following computational meaning. As we have noted, for each $n < \omega$, one cannot obtain the information $\varphi_S(x)(n) = \bot$ by observing the n-th cell of the tape, because even if it has the value $\bot$ at the time of observation, it may be filled with 0 or 1 afterwards. If $\{x\colon \varphi_S(x)(n) = \bot\}$ is nowhere-dense, this information is not obtained even from the observation of the whole tape, because the information obtained from the tape state σ is that the value is in $S(\sigma)$, and $\{x\colon \varphi_S(x)(n) = \bot\}$ does not contain any subset of the form $S(\sigma)$ for $\sigma \in K(\mathbb{T}^\omega)$.

Next, we introduce a proper dyadic subbase. When $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ is a dyadic subbase of X and $\sigma \in \mathbb{T}^\omega$, we write

$$\bar{S}(\sigma) = \bigcap_{n \in \mathrm{dom}(\sigma)} \mathrm{cl}\, S_{n,\sigma(n)}.$$

Through the embedding φ_S, we can consider X as a subspace of $\mathbb{T}^\omega$ and we have the following characterization of $S : \mathbb{T}^\omega \to \mathcal{P}(X)$ and $\bar{S} : \mathbb{T}^\omega \to \mathcal{P}(X)$ via the order structure of $\mathbb{T}^\omega$.

Lemma 5 *(Proposition 3.5 of [9]). Suppose that S is a dyadic subbase of a space X and $\sigma \in \mathbb{T}^\omega$.*

(1) $S(\sigma) = \varphi_S^{-1}(\uparrow\sigma)$.
(2) $\bar{S}(\sigma) = \varphi_S^{-1}(\downarrow\uparrow\sigma)$.

Proof. (1) $x \in S(\sigma)$ if and only if $\varphi_S(x)(n) = \sigma(n)$ for every $n \in \mathrm{dom}(\sigma)$, if and only if $\varphi_S(x) \succcurlyeq \sigma$.

(2) $x \in \bar{S}(\sigma)$ if and only if $\varphi_S(x)(n)$ is $\sigma(n)$ or $\bot$ for every $n \in \mathrm{dom}(\sigma)$, if and only if $\varphi_S(x) \uparrow \sigma$. $\square$

Definition 4. ([9]) We say that a dyadic subbase is *proper* if $\mathrm{cl}\, S(\sigma) = \bar{S}(\sigma)$ for every $\sigma \in K(\mathbb{T}^\omega)$.

We investigate more about proper dyadic subbases in Section 6.

Finally, we introduce 'independent subbase' of [9]. Before that we explain two examples of proper dyadic subbases of the closed unit interval $\mathbb{I} = [0, 1]$ given in [9]. One is the *Dedekind subbase* $D = \{D_{n,i}\colon n < \omega,\ i < 2\}$ with $D_{n,0} = [0, q_n)$ and $D_{n,1} = (q_n, 1]$ for a numbering q_n $(n < \omega)$ of rational numbers in $(0, 1)$. The other one is the Gray subbase which corresponds to the Gray embedding [3,7]. Let the tent function $t : \mathbb{I} \to \mathbb{I}$ be

$$t(x) = \begin{cases} 2x & (0 \leqslant x \leqslant 1/2), \\ 2(1-x) & (1/2 < x \leqslant 1). \end{cases}$$

For the sets $X_0 = [0, 1/2)$ and $X_1 = (1/2, 1]$, we define the *Gray subbase* $G = \{G_{n,i}\colon n < \omega,\ i < 2\}$ as $G_{n,i} = t^{-n}(X_i)$. Fig. 1 shows the Gray subbase, with the gray lines representing $G_{n,0}$ and the black lines representing $G_{n,1}$.

One can see that each code sequence of the Dedekind subbase contains redundant information. For example, suppose that $q_n = 3/4$ and $q_m = 1/2$. Then, $D_{n,0} \supset D_{m,0}$ and therefore $\varphi_D(x)(n)$ always has the value 0 when $\varphi_D(x)(m) = 0$, and we do not need the n-th value in identifying x if the m-th value is 0. On the other hand, Gray subbase is efficient in that there is no such redundancy in each code sequence. In [9], in order to express such a non-redundancy, three notions are introduced on proper dyadic subbases.

Definition 5 *(Independent subbase of [9]).* A proper dyadic subbase S is an *independent subbase* if $S(\sigma) \neq \emptyset$ for every $\sigma \in K(\mathbb{T}^\omega)$.

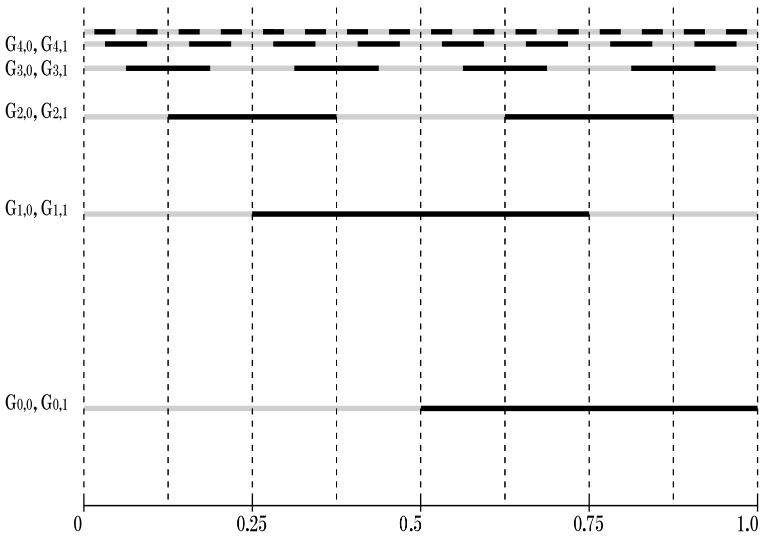

Fig. 1. The Gray subbase of the unit interval [0, 1].

This definition of an independent subbase is equivalent to the one in the previous section by the lemmas below. One can see that the Gray subbase is independent whereas the Dedekind subbase is not. Since $S(\sigma) = \varphi_S^{-1}(\uparrow\sigma)$ by Lemma 5(1) and $\{S(\sigma)\colon \sigma \in K(\mathbb{T}^\omega)\}$ is a base of $\mathbb{T}^\omega$, we have the following.

Proposition 6. *A proper dyadic subbase S is an independent subbase if and only if φ_S is a dense embedding.*

We defer the definitions of the other two properties, full-representing and canonically representing, to Section 6.

Lemma 7. *If $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ is a subbase of a space X satisfying* (1.1) *and* (1.2), *then S is a dyadic subbase.*

Proof. When $\sigma(n) = \bot$, let $\sigma_{n=i}$ denote the bottomed sequence obtained by assigning i to the n-th component of σ. Suppose that $x \notin S_{n,0} \cup S_{n,1}$ and $x \in S(\sigma)$ for $\sigma \in K(\mathbb{T}^\omega)$. We have $\sigma(n) = \bot$. From (1.2), $S(\sigma_{n=0})$ and $S(\sigma_{n=1})$ are both non-empty and they are subsets of $S(\sigma)$. Thus, in every neighbourhood of x, there are points of $S_{n,0}$ and $S_{n,1}$. That is, x is on the boundary of both $S_{n,0}$ and $S_{n,1}$. Therefore, $S_{n,0}$ and $S_{n,1}$ are regular open and $S_{n,1} = X \setminus \operatorname{cl} S_{n,0}$. □

Lemma 8. *If $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ is a subbase of a space X satisfying* (1.1) *and* (1.2), *then S is proper, and therefore S is an independent subbase in the sense of Definition* 5.

Proof. Suppose that $\operatorname{cl} S(\sigma) \subsetneq \bar{S}(\sigma)$ for $\sigma \in K(\mathbb{T}^\omega)$, and thus $x \in \bar{S}(\sigma)$ and $x \notin \operatorname{cl} S(\sigma)$ for some $x \in X$. Since $x \notin \operatorname{cl} S(\sigma)$, $x \in S(\tau)$ and $S(\sigma) \cap S(\tau) = \emptyset$ for some $\tau \in K(\mathbb{T}^\omega)$. If $\sigma \uparrow \tau$, $S(\sigma) \cap S(\tau) = S(\sigma \sqcup \tau)$, which is non-empty by (1.2). Here, $\sigma \sqcup \tau$ is the least upper bound of σ and τ. Therefore, we have $\sigma \not\uparrow \tau$. On the other hand, since $x \in \bar{S}(\sigma)$, we have $\sigma \uparrow \varphi_S(x)$ by Lemma 5(2). Since we also have $x \in S(\tau)$, we have $\varphi_S(x) \succcurlyeq \tau$. Therefore, $\sigma \uparrow \tau$ and we have contradiction.

Lemma 9. *Let $S = \{S_{n,i}\colon n < \omega,\ i < 2\}$ be a dyadic subbase of a space X. S is proper if and only if S satisfies*

$$(\forall n < \omega)\ \big(\forall \sigma \in {}^n2\big)\quad \Bigl(\operatorname{cl} \bigcap_{k<n} S_{k,\sigma(k)} = \bigcap_{k<n} \operatorname{cl} S_{k,\sigma(k)}\Bigr). \tag{2.1}$$

Proof. We only need to show the if part. Let $\sigma \in K(\mathbb{T}^\omega)$ and $\Gamma = \operatorname{dom}(\sigma)$. Fix $n < \omega$ with $\Gamma \subseteq n$. Put $T = \{\tau \in K(\mathbb{T}^\omega)\colon \operatorname{dom}(\tau) = n \text{ and } \tau \succcurlyeq \sigma\}$. Then, since $S(\sigma) \supseteq S(\tau) = \bigcap_{k<n} S_{k,\tau(k)}$ for each $\tau \in T$, it follows from (2.1) that

$$\begin{aligned}
\operatorname{cl} S(\sigma) \supseteq \bigcup_{\tau\in T} \operatorname{cl} S(\tau) &= \bigcup_{\tau\in T} \Bigl(\bigcap_{k<n} \operatorname{cl} S_{k,\tau(k)}\Bigr) \\
&= \bigcup_{\tau\in T} \Bigl(\bigcap_{k\in\Gamma} \operatorname{cl} S_{k,\tau(k)} \cap \bigcap_{k\in n\setminus\Gamma} \operatorname{cl} S_{k,\tau(k)}\Bigr) \\
&= \bigcap_{k\in\Gamma} \operatorname{cl} S_{k,\sigma(k)} \cap \bigcup_{\tau\in T} \Bigl(\bigcap_{k\in n\setminus\Gamma} \operatorname{cl} S_{k,\tau(k)}\Bigr).
\end{aligned}$$

This implies that $\operatorname{cl} S(\sigma) \supseteq \bigcap_{k\in\Gamma} \operatorname{cl} S_{k,\sigma(k)} = \bar{S}(\sigma)$, because the last union in the above formula is equal to X by the fact that $\mathcal{S}$ is dyadic. □

3. Proof of Theorem 1

For a subset A of a space X, A^e stands for the exterior of A in X. For open sets V and S in a space X, we write $V \gg S$ if $\operatorname{cl}(V \cap S) = \operatorname{cl} V \cap \operatorname{cl} S$ and $\operatorname{cl}(V^e \cap S) = \operatorname{cl}(V^e) \cap \operatorname{cl} S$. Note that the first equality does not necessarily imply the second, and $V \gg S$ does not necessarily imply $S \gg V$ even if both V and S are regular open (see Remark 1 at the end of this section).

Proof of Theorem 1. Let X be a dense in itself, separable, metrizable space. We show that X has a subbase $\{S_{n,i}\colon\ n<\omega,\ i<2\}$ satisfying (1.1) and (1.2). We may assume that X is non-empty. Then there exists a collection $\{U_{n,i}\colon\ n<\omega,\ i<2\}$ of non-empty regular open sets in X such that $\operatorname{cl} U_{n,0} \subseteq U_{n,1}$ for each $n<\omega$ and, for each $x\in X$ and each neighborhood G of x, there exists $n<\omega$ such that $x\in U_{n,0}$ and $U_{n,1}\subseteq G$. We inductively define, for each $n<\omega$, regular open sets $S_{n,0}$ and $S_{n,1}$ satisfying that

$$S_{n,1} = (S_{n,0})^e \quad (= X \setminus \operatorname{cl} S_{n,0}), \tag{3.1}$$

$$\left(\forall \sigma \in {}^{n+1}2\right) \quad \left(\operatorname{cl} \bigcap_{k\leqslant n} S_{k,\sigma(k)} = \bigcap_{k\leqslant n} \operatorname{cl} S_{k,\sigma(k)}\right), \tag{3.2}$$

$$\left(\forall \sigma \in {}^{n+1}2\right) \quad \left(\bigcap_{k\leqslant n} S_{k,\sigma(k)} \neq \emptyset\right), \tag{3.3}$$

and

$$(\forall x \in U_{n,0})\ (\exists \Gamma \subseteq n+1)\ \left(\exists \sigma \in {}^{\Gamma}2\right) \quad \left(x \in \bigcap_{k\in\Gamma} S_{k,\sigma(k)} \subseteq U_{n,1}\right). \tag{3.4}$$

If we construct $S_{n,0}$ and $S_{n,1}$ for all $n<\omega$, then $\{S_{n,i}\colon\ n<\omega,\ i<2\}$ is an independent subbase of X by (3.1), (3.3) and (3.4). Although (3.1) is stronger than (1.1) and (3.2) is not required in the definition of an independent subbase, we need them to proceed the induction and it is natural to require them in view of Lemmas 7, 8 and 9. For $n=0$, we may assume that $U_{0,0}\neq X$. Put $S_{0,0}=U_{0,0}$ and $S_{0,1}=(S_{0,0})^e$. Fix $n\geqslant 1$ and assume that $S_{k,0}$ and $S_{k,1}$ have been defined for all $k<n$. For each $\sigma\in{}^n2$, put $S(\sigma)=\bigcap_{k<n} S_{k,\sigma(k)}$. Then, by the inductive hypothesis, we have

$$\left(\forall \sigma \in {}^{n}2\right) \quad \left(\operatorname{cl} S(\sigma) = \bigcap_{k<n} \operatorname{cl} S_{k,\sigma(k)}\right), \tag{3.5}$$

$$\left(\forall \sigma \in {}^{n}2\right) \quad \left(S(\sigma) \neq \emptyset\right), \tag{3.6}$$

and

$$\left(\forall \sigma, \sigma' \in {}^{n}2\right) \quad \left(\text{if } \sigma \neq \sigma',\ \text{then } S(\sigma) \cap S(\sigma') = \emptyset\right). \tag{3.7}$$

Now, we need the following lemma.

Lemma 10. *Let $\{S_i\colon\ i\in I\}$ be a finite collection of disjoint open sets in X. Let F be a closed set and G an open set such that $F\subseteq G$. Then there exists a regular open set W such that $F\subseteq W\subseteq \operatorname{cl} W\subseteq G$ and $W\gg S_i$ for each $i\in I$.*

Leaving the proof of Lemma 10 to the next section, let us continue the proof of Theorem 1. By Lemma 10 there exists a regular open set V_n such that $\operatorname{cl} U_{n,0}\subseteq V_n\subseteq \operatorname{cl} V_n\subseteq U_{n,1}$ and

$$V_n \gg S(\sigma) \quad \text{for each } \sigma \in {}^n2. \tag{3.8}$$

Put $A=\{\sigma\in{}^n2\colon\ S(\sigma)\subseteq V_n\}$ and $B=\{\sigma\in{}^n2\colon\ S(\sigma)\cap \operatorname{cl} V_n=\emptyset\}$. Then, since V_n is regular open, we have

$$\left(\forall \sigma \in {}^n2 \setminus (A\cup B)\right) \quad \left(S(\sigma)\cap V_n \neq \emptyset \text{ and } S(\sigma)\cap V_n^e \neq \emptyset\right). \tag{3.9}$$

For every $\sigma\in A\cup B$, by (3.6), we can take a non-empty regular open set $G(\sigma)$ such that

$$\operatorname{cl} G(\sigma) \subseteq S(\sigma) \quad \text{and} \quad S(\sigma)\setminus \operatorname{cl} G(\sigma) \neq \emptyset. \tag{3.10}$$

Then, by the definitions of the sets A, B and (3.7), we have

$$(\forall \sigma \in A\cup B) \quad \left(\operatorname{bd} V_n \cap \operatorname{cl} G(\sigma) = \emptyset\right), \tag{3.11}$$

and

$$\big(\forall\sigma,\sigma'\in A\cup B\big)\quad\big(\text{if }\sigma\neq\sigma',\text{ then }\operatorname{cl}G(\sigma)\cap\operatorname{cl}G\big(\sigma'\big)=\emptyset\big). \tag{3.12}$$

Define

$$S_{n,0}=\Big(V_n\setminus\bigcup_{\sigma\in A}\operatorname{cl}G(\sigma)\Big)\cup\bigcup_{\sigma\in B}G(\sigma)$$

and

$$S_{n,1}=\Big(V_n^e\setminus\bigcup_{\sigma\in B}\operatorname{cl}G(\sigma)\Big)\cup\bigcup_{\sigma\in A}G(\sigma).$$

Then it follows from (3.6), (3.9) and (3.10) that $S_{n,i}\cap S(\sigma)\neq\emptyset$ for each $\sigma\in{}^n2$ and $i=0,1$. Thus, $S_{n,0}$ and $S_{n,1}$ satisfy (3.3). To show that $S_{n,0}$ and $S_{n,1}$ are regular open and satisfy (3.1), (3.2) and (3.4) we prove the following claim.

Claim 11. *We have*

$$\operatorname{cl}S_{n,0}=\Big(\operatorname{cl}V_n\setminus\bigcup_{\sigma\in A}G(\sigma)\Big)\cup\bigcup_{\sigma\in B}\operatorname{cl}G(\sigma) \tag{3.13}$$

and

$$\operatorname{cl}S_{n,1}=\Big(\operatorname{cl}V_n^e\setminus\bigcup_{\sigma\in B}G(\sigma)\Big)\cup\bigcup_{\sigma\in A}\operatorname{cl}G(\sigma). \tag{3.14}$$

Proof. We prove only the first equality (3.13), since the second can be proved similarly. Let R be the right hand set of (3.13). Then $\operatorname{cl}S_{n,0}\subseteq R$, since R is closed and includes $S_{n,0}$. Thus, it is enough to show that $R\setminus S_{n,0}\subseteq\operatorname{cl}S_{n,0}$. For every $x\in R\setminus S_{n,0}$, either $x\in\operatorname{cl}V_n\setminus V_n$ or $x\in\operatorname{cl}G(\sigma)\setminus G(\sigma)$ for some $\sigma\in A\cup B$. If $x\in\operatorname{cl}V_n\setminus V_n$, then $x\notin\bigcup_{\sigma\in A}\operatorname{cl}G(\sigma)$ by (3.11), and hence,

$$x\in\operatorname{cl}\Big(V_n\setminus\bigcup_{\sigma\in A}\operatorname{cl}G(\sigma)\Big)\subseteq\operatorname{cl}S_{n,0}.$$

If $x\in\operatorname{cl}G(\sigma)\setminus G(\sigma)$ for some $\sigma\in A$, then $x\in\operatorname{cl}(S(\sigma)\setminus\operatorname{cl}G(\sigma))$, because $G(\sigma)$ is regular open and $\operatorname{cl}G(\sigma)\subseteq S(\sigma)$. Hence, $x\in\operatorname{cl}S_{n,0}$ since $S(\sigma)\setminus\operatorname{cl}G(\sigma)\subseteq S_{n,0}$. If $x\in\operatorname{cl}G(\sigma)\setminus G(\sigma)$ for some $\sigma\in B$, then $x\in\operatorname{cl}S_{n,0}$ since $G(\sigma)\subseteq S_{n,0}$. Hence, we have (3.13). □

By Claim 11, $S_{n,1}=(S_{n,0})^e$ and both $S_{n,0}$ and $S_{n,1}$ are regular open. Hence, we have (3.1). Next, we show that $S_{n,i}$, $i<2$, satisfy (3.2). By (3.5), it is enough to prove that

$$\operatorname{cl}\big(S_{n,i}\cap S(\sigma)\big)=\operatorname{cl}S_{n,i}\cap\operatorname{cl}S(\sigma)$$

for each $\sigma\in{}^n2$ and $i<2$. We prove only the case $i=0$, since the proof for $i=1$ goes quite similarly if one replace V_n by V_n^e. We distinguish three cases. If $\sigma\in{}^n2\setminus(A\cup B)$, then by the definition of $S_{n,0}$ and Claim 11,

$$S_{n,0}\cap S(\sigma)=V_n\cap S(\sigma)$$

and

$$\operatorname{cl}S_{n,0}\cap\operatorname{cl}S(\sigma)=\operatorname{cl}V_n\cap\operatorname{cl}S(\sigma).$$

Hence, it follows from (3.8) that

$$\begin{aligned}\operatorname{cl}\big(S_{n,0}\cap S(\sigma)\big)&=\operatorname{cl}\big(V_n\cap S(\sigma)\big)\\&=\operatorname{cl}V_n\cap\operatorname{cl}S(\sigma)=\operatorname{cl}S_{n,0}\cap\operatorname{cl}S(\sigma).\end{aligned}$$

If $\sigma\in A$, then by the definition of $S_{n,0}$ and Claim 11,

$$S_{n,0}\cap S(\sigma)=S(\sigma)\setminus\operatorname{cl}G(\sigma)$$

and

$$\operatorname{cl}S_{n,0}\cap\operatorname{cl}S(\sigma)=\operatorname{cl}S(\sigma)\setminus G(\sigma).$$

Since $G(\sigma)$ is regular open and $\operatorname{cl} G(\sigma) \subseteq S(\sigma)$,

$$\begin{aligned}\operatorname{cl}\bigl(S_{n,0} \cap S(\sigma)\bigr) &= \operatorname{cl}\bigl(S(\sigma) \setminus \operatorname{cl} G(\sigma)\bigr)\\ &= \operatorname{cl} S(\sigma) \setminus G(\sigma) = \operatorname{cl} S_{n,0} \cap \operatorname{cl} S(\sigma).\end{aligned}$$

If $\sigma \in B$, then by the definition of $S_{n,0}$,

$$S_{n,0} \cap S(\sigma) = G(\sigma).$$

Since $V_n \cap S(\sigma) = \emptyset$, it follows from (3.8) that $\operatorname{cl} V_n \cap \operatorname{cl} S(\sigma) = \emptyset$, which implies that $\operatorname{cl} S_{n,0} \cap \operatorname{cl} S(\sigma) = \operatorname{cl} G(\sigma)$ by Claim 11. Hence,

$$\operatorname{cl}\bigl(S_{n,0} \cap S(\sigma)\bigr) = \operatorname{cl} G(\sigma) = \operatorname{cl} S_{n,0} \cap \operatorname{cl} S(\sigma).$$

Consequently, we have (3.2). Finally, to prove that $S_{n,i}$, $i < 2$, satisfy (3.4), let $x \in U_{n,0}$ be fixed. If $x \in S(\sigma)$ for some $\sigma \in A$, then $x \in S(\sigma) \subseteq V_n \subseteq U_{n,1}$. On the other hand, if $x \notin S(\sigma)$ for each $\sigma \in A$, then

$$x \in V_n \setminus \bigcup_{\sigma \in A} \operatorname{cl} G(\sigma) \subseteq S_{n,0}. \tag{3.15}$$

For each $\sigma \in B$, we now define $k(\sigma) < n$ and $i(\sigma) < 2$ as follows. To do this, let $\sigma \in B$ be fixed for a while. Since

$$V_n \cap \Bigl(\bigcap_{k<n} \operatorname{cl} S_{k,\sigma(k)}\Bigr) = V_n \cap \operatorname{cl} S(\sigma) = \emptyset$$

by (3.5), $x \notin \bigcap_{k<n} \operatorname{cl} S_{k,\sigma(k)}$, and hence, $x \notin \operatorname{cl} S_{k(\sigma),\sigma(k(\sigma))}$ for some $k(\sigma) < n$. Define $i(\sigma) = 1 - \sigma(k(\sigma))$. Then,

$$x \in S_{k(\sigma),i(\sigma)} \quad \text{and} \quad S_{k(\sigma),i(\sigma)} \cap S(\sigma) = \emptyset. \tag{3.16}$$

Define such $k(\sigma)$ and $i(\sigma)$ for each $\sigma \in B$. Then we have

$$\bigl(\forall \sigma, \sigma' \in B\bigr) \quad \bigl(\text{if } k(\sigma) = k\bigl(\sigma'\bigr), \text{ then } i(\sigma) = i\bigl(\sigma'\bigr)\bigr),$$

because x is in exactly one of $S_{k(\sigma),0}$ and $S_{k(\sigma),1}$. Put $\Gamma = \{k(\sigma)\colon \sigma \in B\}$, and define $\tau \in {}^{\Gamma}2$ by $\tau(k(\sigma)) = i(\sigma)$ for $\sigma \in B$. Then, by (3.16),

$$x \in \bigcap_{k \in \Gamma} S_{k,\tau(k)} \quad \text{and} \quad \bigcap_{k \in \Gamma} S_{k,\tau(k)} \cap \bigcup_{\sigma \in B} S(\sigma) = \emptyset. \tag{3.17}$$

Finally, put $\Lambda = \Gamma \cup \{n\}$ and define $\upsilon \in {}^{\Lambda}2$ by $\upsilon|_{\Gamma} = \tau$ and $\upsilon(n) = 0$. Then it follows from (3.15) and (3.17) that

$$x \in \bigcap_{k \in \Lambda} S_{k,\upsilon(k)} = \bigcap_{k \in \Gamma} S_{k,\tau(k)} \cap S_{n,0} \subseteq V_n \subseteq U_{n,1}.$$

Hence, we have (3.4). □

Remark 1. Consider the subspace $X = (-\infty, -1] \cup [0, +\infty)$ of the real line with the usual topology, and define regular open sets U and S in X by

$$U = (-\infty, -1] \cup \bigcup_{n<\omega} \bigl(1/(4n+3), 1/(4n+2)\bigr)$$

and

$$S = \bigcup_{n<\omega} \bigl(1/(4n+4), 1/(4n+1)\bigr).$$

Then it is easily checked that $U \gg S$ and $\operatorname{cl}(U^e \cap S^e) = \operatorname{cl}(U^e) \cap \operatorname{cl}(S^e)$ but $\operatorname{cl}(U \cap S^e) \neq \operatorname{cl} U \cap \operatorname{cl}(S^e)$. Hence, $U \gg S$ does not imply $S \gg U$, in general.

4. Proof of Lemma 10

We prove Lemma 10 used in the proof of Theorem 1. Let X be the same space as in the preceding section, and fix a metric on X which induces the topology of X. For a point $x \in X$ and $\varepsilon > 0$, $B(x, \varepsilon)$ denotes the ε-neighborhood of x in X.

Lemma 12. *Let S be a non-empty open set in X. Let K be a non-empty closed set such that $K \subseteq \operatorname{bd} S$, and let G be an open set with $K \subseteq G$. Then there exist open sets P and Q such that*

(1) $P \cup Q \subseteq S \cap G$, $\operatorname{cl} P \cup \operatorname{cl} Q \subseteq G$,
(2) $\operatorname{cl} P \cap \operatorname{cl} Q \cap S = \emptyset$, *and*
(3) $\operatorname{cl} P \cap \operatorname{bd} S = \operatorname{cl} Q \cap \operatorname{bd} S = K$.

Proof. Take open sets G_i, $i < \omega$, such that

$$G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \supseteq \cdots \supseteq K$$

and $\bigcap_{i<\omega} \operatorname{cl} G_i = K$. We only prove the case that K is infinite, since the finite case can be proved similarly. Let D be a countable dense set of K and enumerate the points of D as $D = \{x_k\colon k < \omega\}$, where $x_k \neq x_l$ whenever $k \neq l$. By induction on $k < \omega$ and $i < \omega$, we can define a collection $\{H(k,i)\colon k < \omega,\ i < \omega\}$ of non-empty open sets such that

$$\operatorname{cl} H(k,i) \subseteq S \cap G_{k+i} \cap B\bigl(x_k, 2^{-i}\bigr), \tag{4.1}$$

for each $k < \omega$ and $i < \omega$, and

$$\operatorname{cl} H(k,i) \cap \operatorname{cl} H(l,j) = \emptyset, \quad \text{whenever } (k,i) \neq (l,j). \tag{4.2}$$

Indeed, for each $k < \omega$, we can define a sequence $\{H(k,i)\colon i < \omega\}$ of non-empty open sets satisfying (4.1) and such that $\operatorname{cl} H(k,i) \cap \operatorname{cl} H(k,j) = \emptyset$ whenever $i \neq j$ by induction on i. Then $\bigcup_{i<\omega} \operatorname{cl} H(k,i) \cup \{x_k\}$ is closed in X and contains no points of $D \setminus \{x_k\}$. Hence, by induction on k, we can define such sequence for each $k < \omega$ so as to satisfy (4.2). Since each $X \setminus \operatorname{cl} G_j$ intersects only finitely many $H(k,i)$'s, the collection $\{H(k,i)\colon k < \omega,\ i < \omega\}$ is discrete at each point of $X \setminus K$. For each $k < \omega$ and $i < \omega$, take non-empty open sets $P(k,i)$ and $Q(k,i)$ such that $P(k,i) \cup Q(k,i) \subseteq H(k,i)$ and $\operatorname{cl} P(k,i) \cap \operatorname{cl} Q(k,i) = \emptyset$. Put

$$P = \bigcup_{k<\omega} \bigcup_{i<\omega} P(k,i) \quad \text{and} \quad Q = \bigcup_{k<\omega} \bigcup_{i<\omega} Q(k,i).$$

Then P and Q satisfy (1) by their definitions. Since $D \subseteq \operatorname{cl} P$ and D is dense in K, $K \subseteq \operatorname{cl} P$, and $K \subseteq \operatorname{cl} Q$ similarly. Hence, (2) and (3) follow from the fact that $\{P(k,i), Q(k,i)\colon k < \omega,\ i < \omega\}$ is discrete at each point of $X \setminus K$. □

Lemma 13. *Let V and S be open sets in X such that $V \gg S$. If we put $W = \operatorname{int}(\operatorname{cl} V)$, then $W \gg S$.*

Proof. Since $V \gg S$, $\operatorname{cl}(V \cap S) = \operatorname{cl} V \cap \operatorname{cl} S$ and $\operatorname{cl}(V^e \cap S) = \operatorname{cl}(V^e) \cap \operatorname{cl} S$. Since $V \cap S$ is dense in $W \cap S$ and $\operatorname{cl} V = \operatorname{cl} W$,

$$\operatorname{cl}(W \cap S) = \operatorname{cl}(V \cap S) = \operatorname{cl} V \cap \operatorname{cl} S = \operatorname{cl} W \cap \operatorname{cl} S.$$

On the other hand, since $V^e = W^e$,

$$\operatorname{cl}\bigl(W^e \cap S\bigr) = \operatorname{cl}\bigl(V^e \cap S\bigr) = \operatorname{cl}\bigl(V^e\bigr) \cap \operatorname{cl} S = \operatorname{cl}\bigl(W^e\bigr) \cap \operatorname{cl} S.$$

Hence, $W \gg S$. □

Proof of Lemma 10. Take an open set U with $F \subseteq U \subseteq \operatorname{cl} U \subseteq G$. For each $i \in I$, we put $K_i = \operatorname{bd} U \cap \operatorname{bd} S_i$, and define open sets P_i and Q_i as follows. If $K_i = \emptyset$, put $P_i = Q_i = \emptyset$. If $K_i \neq \emptyset$, then by Lemma 12 there exist open sets P_i and Q_i such that

$$P_i \cup Q_i \subseteq S_i \cap (G \setminus F), \qquad \operatorname{cl} P_i \cup \operatorname{cl} Q_i \subseteq G \setminus F, \tag{4.3}$$

$$\operatorname{cl} P_i \cap \operatorname{cl} Q_i \cap S_i = \emptyset, \tag{4.4}$$

and

$$\operatorname{cl} P_i \cap \operatorname{bd} S_i = \operatorname{cl} Q_i \cap \operatorname{bd} S_i = K_i. \tag{4.5}$$

Put $P = \bigcup_{i \in I} P_i$, $Q = \bigcup_{i \in I} Q_i$ and $T = X \setminus \bigcup_{i \in I} S_i$. Then, by (4.5),

$$\operatorname{cl} P \cap T = \operatorname{cl} Q \cap T = \bigcup_{i \in I} K_i \subseteq \operatorname{cl} U \setminus U. \tag{4.6}$$

Define $V = (U \cup P) \setminus \mathrm{cl}\, Q$. Then V is an open set and $F \subseteq V \subseteq \mathrm{cl}\, V \subseteq G$ by (4.3). If we prove that $V \gg S_i$ for each $i \in I$, then $W = \mathrm{int}(\mathrm{cl}\, V)$ is a required regular open set by Lemma 13. To this end, we need the following claims:

Claim 14. *$V \cap T = U \cap T$ and $\mathrm{cl}\, V \cap T = \mathrm{cl}\, U \cap T$.*

Proof. First, observe that $P \cap T = \emptyset$ by (4.3), and $(\mathrm{cl}\, Q \cap T) \cap U = \emptyset$ by (4.6). Hence, we have

$$\begin{aligned} V \cap T &= \big((U \cup P) \setminus \mathrm{cl}\, Q\big) \cap T \\ &= \big((U \cap T) \cup (P \cap T)\big) \setminus (\mathrm{cl}\, Q \cap T) \\ &= (U \cap T) \setminus (\mathrm{cl}\, Q \cap T) = U \cap T. \end{aligned}$$

Since $\mathrm{cl}\, P \cap T \subseteq \mathrm{cl}\, U$ by (4.6), we have

$$\begin{aligned} \mathrm{cl}\, V \cap T &\subseteq \mathrm{cl}(U \cup P) \cap T \\ &= (\mathrm{cl}\, U \cap T) \cup (\mathrm{cl}\, P \cap T) = \mathrm{cl}\, U \cap T. \end{aligned}$$

Conversely, since $\mathrm{cl}\, Q \cap T = \mathrm{cl}\, P \cap T$ by (4.6), we have

$$\begin{aligned} \mathrm{cl}\, U \cap T &\subseteq \mathrm{cl}\big((U \setminus \mathrm{cl}\, Q) \cup \mathrm{cl}\, Q\big) \cap T \\ &\subseteq (\mathrm{cl}\, V \cap T) \cup (\mathrm{cl}\, Q \cap T) \\ &= (\mathrm{cl}\, V \cap T) \cup (\mathrm{cl}\, P \cap T) = \mathrm{cl}\, V \cap T. \end{aligned}$$

Hence, $\mathrm{cl}\, V \cap T = \mathrm{cl}\, U \cap T$. □

Claim 15. *$\mathrm{bd}\, V \cap T \subseteq \mathrm{bd}\, U$ and $\mathrm{bd}(V^e) \cap T \subseteq \mathrm{bd}\, U$.*

Proof. The first inclusion is an immediate consequence of Claim 14, and the second follows from the first since $\mathrm{bd}\, V^e \subseteq \mathrm{bd}\, V$. □

Fix $i \in I$. To show that $V \gg S_i$, observe that

$$\mathrm{cl}\, V \cap \mathrm{cl}\, S_i = (\mathrm{cl}\, V \cap S_i) \cup (V \cap \mathrm{cl}\, S_i) \cup (\mathrm{bd}\, V \cap \mathrm{bd}\, S_i).$$

Since V and S_i are open,

$$(\mathrm{cl}\, V \cap S_i) \cup (V \cap \mathrm{cl}\, S_i) \subseteq \mathrm{cl}(V \cap S_i).$$

By Claim 15 and (4.5), $\mathrm{bd}\, V \cap \mathrm{bd}\, S_i \subseteq \mathrm{bd}\, U \cap \mathrm{bd}\, S_i = K_i \subseteq \mathrm{cl}\, P_i$. Since $P_i \subseteq V \cap S_i$, $\mathrm{bd}\, V \cap \mathrm{bd}\, S_i \subseteq \mathrm{cl}(V \cap S_i)$. Hence, we have $\mathrm{cl}(V \cap S_i) = \mathrm{cl}\, V \cap \mathrm{cl}\, S_i$. Similarly, since V^e and S^i are open, $(\mathrm{cl}(V^e) \cap S_i) \cup (V^e \cap \mathrm{cl}\, S_i) \subseteq \mathrm{cl}(V^e \cap S_i)$. By Claim 15, $\mathrm{bd}(V^e) \cap \mathrm{bd}\, S_i \subseteq \mathrm{bd}\, U \cap \mathrm{bd}\, S_i = K_i$. Since $K_i \subseteq \mathrm{cl}\, Q_i$ and $Q_i \subseteq V^e \cap S_i$, $\mathrm{bd}(V^e) \cap \mathrm{bd}\, S_i \subseteq \mathrm{cl}(V^e \cap S_i)$. Hence, we have $\mathrm{cl}(V^e \cap S_i) = \mathrm{cl}(V^e) \cap \mathrm{cl}\, S_i$, which completes the proof. □

5. Proofs of Theorem 3 and Corollary 4

For a collection $\mathcal{A}$ of subsets of a metric space, $\mathrm{mesh}\, \mathcal{A}$ denotes the least upper bound of the diameters of all members of $\mathcal{A}$. Morita [4] proved that if a metric space X satisfies $\dim X \leqslant m$, then X has a base $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ such that each $\mathcal{B}_n$ is locally finite and $\mathrm{ord}\{\mathrm{bd}\, B\colon B \in \mathcal{B}\} \leqslant m - 1$. First, we show that if X is dense in itself, then the collections $\mathcal{B}_n$, $n < \omega$, in his theorem can be defined so as to satisfy

$$\mathcal{B}_n \cap \mathcal{B}_{n'} = \emptyset, \quad \text{whenever } n \neq n', \tag{5.1}$$

and, moreover, we can make $\mathcal{B}_n$ a locally finite cover, consisting of regular open sets, of X with $\mathrm{mesh}\, \mathcal{B}_n \leqslant 2^{-n}$ for each $n < \omega$. To show this, we need the following lemma, which is the essence of Morita's theorem.

Lemma 16 *(Lemma 4.2.1 in [1]). If a normal space X satisfies the inequality $\dim X \leqslant m \geqslant 0$, then for every σ-locally finite family $\{U_s\colon s \in S\}$ of open sets in X and every family $\{F_s\colon s \in S\}$ of closed sets in X such that $F_s \subseteq U_s$ for each $s \in S$, there exists a family $\{V_s\colon s \in S\}$ of open sets in X such that $F_s \subseteq V_s \subseteq \mathrm{cl}\, V_s \subseteq U_s$ for each $s \in S$ and $\mathrm{ord}\{\mathrm{bd}\, V_s\colon s \in S\} \leqslant m - 1$.*

For covers $\mathcal{U}$ and $\mathcal{V}$ of X, we write $\mathcal{U} \lhd \mathcal{V}$ if $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $V \not\subseteq U$ for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Lemma 17. *If X is a dense in itself, paracompact space, then for every locally finite cover $\mathcal{V}$ of X consisting of non-empty open sets, there exists a locally finite open cover $\mathcal{U}$ of X such that $\mathcal{U} \lhd \mathcal{V}$.*

Proof. Let $\mathcal{V}$ be a locally finite cover of X consisting of non-empty open sets. Then the set $\mathcal{V}_x = \{V \in \mathcal{V}\colon\ x \in \operatorname{cl} V\}$ is finite for each $x \in X$. Since X is dense in itself, we can pick a point $y(x, V) \in V$ with $y(x, V) \neq x$ for each $V \in \mathcal{V}_x$. For each $x \in X$, fix $V(x) \in \mathcal{V}$ with $x \in V(x)$, and put

$$U(x) = V(x) \setminus \Bigl(\bigcup \{\operatorname{cl} V\colon\ V \in \mathcal{V} \setminus \mathcal{V}_x\} \cup \bigl\{ y(x, V)\colon\ V \in \mathcal{V}_x \bigr\} \Bigr).$$

Then $U(x)$ is an open set and $V \nsubseteq U(x)$ for each $V \in \mathcal{V}$. Hence, every locally finite open refinement $\mathcal{U}$ of $\{U(x)\colon\ x \in X\}$ is a required cover of X. □

Now, let X be a dense in itself, metric space with $\dim X \leqslant m$. By Lemma 17, we can define inductively locally finite open covers $\mathcal{U}_n = \{U_{n,s}\colon\ s \in S_n\}$ and $\mathcal{V}_n = \{V_{n,s}\colon\ s \in S_n\}$ of X for each $n < \omega$ such that

$$\emptyset \neq V_{n,s} \subseteq \operatorname{cl} V_{n,s} \subseteq U_{n,s} \quad \text{for each } s \in S_n,$$

$$\operatorname{mesh} \mathcal{U}_n < 2^{-n} \quad \text{and} \quad \mathcal{U}_{n+1} \lhd \mathcal{V}_n.$$

By Lemma 16, there exists a family $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$, where $\mathcal{B}_n = \{B_{n,s}\colon\ s \in S_n\}$ for $n < \omega$, of open sets in X such that

$$\operatorname{cl} V_{n,s} \subseteq B_{n,s} \subseteq \operatorname{cl} B_{n,s} \subseteq U_{n,s} \quad \text{for each } s \in S_n \text{ and } n < \omega,$$

and

$$\operatorname{ord}\{\operatorname{bd} B\colon\ B \in \mathcal{B}\} \leqslant m - 1.$$

Then $\mathcal{B}$ is a base for X satisfying all requirements of Morita's theorem stated above. Observe that each $\mathcal{B}_n$ is a locally finite open cover of X with $\operatorname{mesh} \mathcal{B}_n < 2^{-n}$. Moreover, we can assume that all members of $\mathcal{B}$ are regular open by replacing $B \in \mathcal{B}$ by $\operatorname{int}(\operatorname{cl} B)$ if necessary. Finally, we show that the covers $\mathcal{B}_n$, $n < \omega$, satisfy (5.1). Suppose not; then there exist $n < n' < \omega$ such that $B_{n,s} = B_{n',s'}$ for some $s \in S_n$ and $s' \in S_{n'}$. Since $B_{n',s'} \subseteq U_{n',s'} \in \mathcal{U}_{n'}$ and $\mathcal{U}_{n'}$ is a refinement of $\mathcal{U}_{n+1}$, there exists $U \in \mathcal{U}_{n+1}$ such that $B_{n',s'} \subseteq U$. Thus, $V_{n,s} \subseteq B_{n,s} = B_{n',s'} \subseteq U$, which contradicts the fact that $\mathcal{U}_{n+1} \lhd \mathcal{V}_n$. Hence, we have (5.1). Now, we are ready to prove Theorem 3.

Proof of Theorem 3. Let X be a non-empty, dense in itself, separable, metrizable space with $\dim X \leqslant m$, and fix a metric d on X which induces the topology of X. We show that X has an independent subbase of dimension m. Using the above observation about Morita's theorem and a bijection between $\omega \times \omega$ and ω, we have a collection $\mathcal{B} = \bigcup_{n<\omega} \bigcup_{i<\omega} \mathcal{B}_{n,i}$, where each $\mathcal{B}_{n,i}$ is a locally finite cover of X consisting of regular open sets, such that

$$\inf\{\operatorname{mesh} \mathcal{B}_{n,i}\colon\ i < \omega\} = 0 \quad \text{for each } n < \omega, \tag{5.2}$$

$$\operatorname{ord}\{\operatorname{bd} B\colon\ B \in \mathcal{B}\} \leqslant m - 1, \tag{5.3}$$

and

$$\mathcal{B}_{n,i} \cap \mathcal{B}_{n',i'} = \emptyset, \quad \text{whenever } (n, i) \neq \bigl(n', i'\bigr). \tag{5.4}$$

For each $n < \omega$, put $\mathcal{B}_n = \bigcup_{i<\omega} \mathcal{B}_{n,i}$. Then $\mathcal{B}_n$ is a base for X by (5.2), and

$$\mathcal{B}_n \cap \mathcal{B}_{n'} = \emptyset, \quad \text{whenever } n \neq n' \tag{5.5}$$

by (5.4). Now, we trace the proof of Theorem 1 and define $S_{n,0}$ and $S_{n,1}$ more carefully. In the first step of the proof, we may assume that

$$d(\operatorname{cl} U_{n,0}, X \setminus U_{n,1}) > 0 \quad \text{for each } n < \omega. \tag{5.6}$$

Further, we may assume that $U_{0,0} \in \mathcal{B}_0$, which implies that $\operatorname{bd} S_{0,0} = \operatorname{bd} B$ for some $B \in \mathcal{B}_0$. For $n \geqslant 1$, we show that the regular open set V_n can be chosen so as to satisfy

$$\operatorname{bd} V_n \subseteq \bigcup \{\operatorname{bd} B\colon\ B \in \mathcal{B}_n\}. \tag{5.7}$$

To show this, let $n \geqslant 1$ be fixed, and look at the proof of Lemma 10. By (5.6), we can assume that $d(F, X \setminus G) > 0$ in the proof of Lemma 10. Thus, by (5.2), we can take the set U as the union of members of $\mathcal{B}_{n,i}$ for some $i < \omega$. Then we have

$$\operatorname{bd} U \subseteq \bigcup \{\operatorname{bd} B\colon\ B \in \mathcal{B}_{n,i}\}, \tag{5.8}$$

because $\mathcal{B}_{n,i}$ is locally finite. Moreover, in the proof of Lemma 12, we can take all the sets $P(k,i)$ and $Q(k,i)$ from $\mathcal{B}_n$. Remember that the collection $\{P(k,i), Q(k,i)\colon k<\omega,\ i<\omega\}$ was discrete at each point of $X\setminus K$ in the proof of Lemma 12. Hence, by (4.6) and (5.8), we then have

$$\operatorname{bd} P \cup \operatorname{bd} Q \subseteq \bigcup\{\operatorname{bd} B\colon B\in\mathcal{B}_n\} \tag{5.9}$$

in the proof of Lemma 10. Let us also remember that the set V_n above was defined by $V_n=\operatorname{int}(\operatorname{cl} V)$, where $V=(U\cup P)\setminus \operatorname{cl} Q$. Hence, it follows from (5.8) and (5.9) that V_n satisfies (5.7). Now, we back to the definition of $S_{n,0}$ in proof of Theorem 1. We can take all the sets $G(\sigma)$, $\sigma\in A\cup B$, from $\mathcal{B}_n$. Then we have

$$\operatorname{bd} S_{n,0} \subseteq \bigcup\{\operatorname{bd} B\colon B\in\mathcal{B}_n\}. \tag{5.10}$$

Hence, it follows from (5.3), (5.5) and (5.10) that $\operatorname{ord}\{\operatorname{bd} S_{n,0}\colon n<\omega\}\leqslant m-1$, which means that the resulting independent subbase $S=\{S_{n,i}\colon n<\omega,\ i<2\}$ is of dimension m. □

Finally, Corollary 4 follows from Theorem 3, Corollary 2 and the following lemma.

Lemma 18. *If a separable metrizable space X has a subbase $S=\{S_{n,i}\colon n<\omega,\ i<2\}$ such that $S_{n,0}\cap S_{n,1}=\emptyset$ for each $n<\omega$ and $\operatorname{ord}\{X\setminus(S_{n,0}\cup S_{n,1})\colon n<\omega\}\leqslant m-1$, then $\dim X\leqslant m$.*

This lemma seems to be known and can also be derived from the fact that $\mathbb{T}^\omega_m$ is m-dimensional [6,8], because φ_S is an embedding in $\mathbb{T}^\omega_m$ under the assumption of this lemma. We give a direct proof here for reader's convenience.

Proof of Lemma 18. Note that some $S_{n,i}$ may be an empty set. We prove this by induction on $m<\omega$. If $m=0$, then our assumption implies that each $X\setminus(S_{n,0}\cup S_{n,1})$ is empty, and hence, all members of S are open and closed. This implies that $\dim X\leqslant 0$. Now, we assume that the statement holds for all $l<m$, and let X be a separable metrizable space with a subbase $S=\{S_{n,i}\colon n<\omega,\ i<2\}$ such that $S_{n,0}\cap S_{n,1}=\emptyset$ for each $n<\omega$ and $\operatorname{ord}\{X\setminus(S_{n,0}\cup S_{n,1})\colon n<\omega\}\leqslant m-1$. Put $T_n=X\setminus(S_{n,0}\cup S_{n,1})$ for each $n<\omega$. Then, $\dim T_n\leqslant m-1$ for each $n<\omega$ by inductive hypothesis, because $\operatorname{ord}\{T_k\cap T_n\colon k<\omega \text{ and } k\neq n\}<m-2$ and $\{S_{k,i}\cap T_n\colon k<\omega,\ k\neq n \text{ and } i<2\}$ is a subbase of T_n. Observe that, for every finitely many members $S_{n(j),i(j)}$, $j<k$, of S if we put $B=\bigcap_{j<k}S_{n(j),i(j)}$, then $\operatorname{bd} B\subseteq\bigcup_{j<k}T_{n(j)}$, and hence, $\dim\operatorname{bd} B\leqslant m-1$. This means that X has a countable base $\mathcal{B}$ such that $\dim\operatorname{bd} B\leqslant m-1$ for each $B\in\mathcal{B}$. Hence, $\dim X\leqslant m$. □

6. Canonically representing subbase and weakly canonically representing subbase

In this section, we consider the case X is a Hausdorff space. In [9], in addition to independent subbase, two more properties which are expressing non-redundancy of a proper dyadic subbase are introduced. They are full-representing subbase and canonically representing subbase. In this section, we introduce yet another one called weakly canonically representing subbase, and investigate relations among them.

Proposition 19 *(Propositions 3.8 and 3.10 of [9]). Suppose that $S=\{S_{n,i}\colon n<\omega,\ i<2\}$ is a proper dyadic subbase of a Hausdorff space X.*

(1) *If $x\neq y\in X$, then x and y are separated by $S_{n,i}$ and $S_{n,1-i}$ for some n and i, and therefore $\varphi_S(x)\not\uparrow\varphi_S(y)$.*
(2) *If $x\in X$ and $\sigma\succcurlyeq\varphi_S(x)$, then $\bar{S}(\sigma)=\{x\}$.*
(3) *If $\tau\in 2^\omega$, then $\bar{S}(\tau)$ is either a one-point set $\{x\}$ for some $x\in X$ or the empty set.*

Proof. (1) Suppose that $x\neq y\in X$. Since X is Hausdorff, there exists $\sigma\in K(\mathbb{T}^\omega)$ such that $x\in S(\sigma)$ and $y\in X\setminus\operatorname{cl} S(\sigma)$. Since S is proper,

$$\operatorname{cl} S(\sigma)=\bar{S}(\sigma)=\bigcap_{n\in\operatorname{dom}(\sigma)}\operatorname{cl} S_{n,\sigma(n)}.$$

Therefore, for some n, $x\in S_{n,\sigma(n)}$ and $y\notin\operatorname{cl} S_{n,\sigma(n)}$, and thus $y\in S_{n,1-\sigma(n)}$.

(2) From (1) and Lemma 5(2), $\bar{S}(\varphi_S(x))=\{y\colon \varphi_S(y)\uparrow\varphi_S(x)\}=\{x\}$. Since $\bar{S}$ is antimonotonic, we only need to show $x\in\bar{S}(\sigma)$ for $\sigma\succcurlyeq\varphi_S(x)$. It is immediate because $\varphi_S(x)\uparrow\sigma$ and Lemma 5(2).

(3) From (2), $\bar{S}(\tau)$ is a one-point set if $\tau\succcurlyeq\varphi_S(x)$ for some $x\in X$. Since $\tau\in 2^\omega$, $\downarrow\tau=\downarrow\!\uparrow\tau$. Thus, by Lemma 5(2), $\bar{S}(\tau)=\varphi_S^{-1}(\downarrow\tau)$. Therefore, it is an empty set if $\tau\not\succcurlyeq\varphi_S(x)$ for every $x\in X$. □

From Proposition 19(3), when S is a proper dyadic subbase of a Hausdorff space X, there is a surjective partial function ρ_S from 2^ω to X, which assigns to $\tau\in 2^\omega$ the unique element of $\bar{S}(\tau)$ when it is non-empty. Therefore, a proper dyadic subbase S induces a representation ρ_S of X.

Definition 6. ([9]) A *full-representing subbase* of a Hausdorff space X is a proper dyadic subbase S such that ρ_S is a total function.

Definition 7. ([9]) A *canonically representing subbase* of a Hausdorff space X is a proper dyadic subbase S such that, for every $\sigma \in \mathbb{T}^\omega$ and $x \in X$, $\bar{S}(\sigma) = \{x\}$ if and only if $\sigma \succcurlyeq \varphi_S(x)$.

That is, a canonically representing subbase is a proper dyadic subbase for which the converse of Proposition 19(2) also holds. It is proved in [9] that full-representing implies canonically representing and canonically representing implies independent, and these three are equivalent when the space is compact. We show in the next proposition some properties equivalent to canonically representing. The domain of a representation ρ_S of X is denoted by $\mathrm{dom}(\rho_S)$. For $\sigma \in \mathbb{T}^\omega$ and $n \in \mathrm{dom}(\sigma)$, $\mathrm{inv}(\sigma, n)$ denotes the sequence obtained by inverting the value of the n-th element of σ, and $\mathrm{erase}(\sigma, n)$ denotes the sequence obtained by replacing the n-th element of σ with $\bot$. We write $s \prec t$ for $s, t \in \mathbb{T}$ if $s \preccurlyeq t$ and $s \neq t$, and $\sigma \prec \tau$ for $\sigma, \tau \in \mathbb{T}^\omega$ if $\sigma \preccurlyeq \tau$ and $\sigma \neq \tau$.

Proposition 20. *Suppose that S is a proper dyadic subbase of a Hausdorff space X. The following are equivalent.*

(1) *S is a canonically representing subbase.*
(2) *If $\sigma \uparrow \varphi_S(x)$ and $\sigma \not\succcurlyeq \varphi_S(x)$ for $x \in X$ and $\sigma \in \mathbb{T}^\omega$, then $\bar{S}(\sigma)$ contains at least one point other than x.*
(3) *If $\tau \in \mathrm{dom}(\rho_S)$ for $\tau \in 2^\omega$, then $\mathrm{inv}(\tau, n) \in \mathrm{dom}(\rho_S)$ for every $n < \omega$.*

Proof. (1) → (2): If $\sigma \uparrow \varphi_S(x)$, then $x \in \bar{S}(\sigma)$ by Lemma 5(2). If $\sigma \not\succcurlyeq \varphi_S(x)$, then $\bar{S}(\sigma) \neq \{x\}$ by (1). Hence, we have (2).

(2) → (3): Let $x = \rho_S(\tau)$, i.e., $\bar{S}(\tau) = \{x\}$. If $n \notin \mathrm{dom}(\varphi_S(x))$, then $\rho_S(\mathrm{inv}(\tau, n)) = x$, and hence, $\mathrm{inv}(\tau, n) \in \mathrm{dom}(\rho_S)$. Suppose that $n \in \mathrm{dom}(\varphi_S(x))$. Let $\sigma = \mathrm{erase}(\tau, n)$. Then we have $\sigma \prec \tau$ and therefore $\sigma \uparrow \varphi_S(x)$, because $\tau \succcurlyeq \varphi_S(x)$. Since $n \in \mathrm{dom}(\varphi_S(x))$ and $\sigma(n) = \bot$, we have $\sigma \not\succcurlyeq \varphi_S(x)$. Therefore, $\bar{S}(\sigma)$ contains at least one point other than x by (2). Since $\tau \in 2^\omega$, $\uparrow\sigma = \{\sigma, \tau, \mathrm{inv}(\tau, n)\}$. Hence, we have

$$\begin{aligned}\bar{S}(\sigma) = \varphi_S^{-1}(\downarrow\uparrow\sigma) &= \varphi_S^{-1}\big(\downarrow\tau \cup \downarrow\mathrm{inv}(\tau, n)\big) \\ &= \varphi_S^{-1}(\downarrow\tau) \cup \varphi_S^{-1}\big(\downarrow\mathrm{inv}(\tau, n)\big) \\ &= \bar{S}(\tau) \cup \bar{S}\big(\mathrm{inv}(\tau, n)\big) = \{x\} \cup \bar{S}\big(\mathrm{inv}(\tau, n)\big).\end{aligned}$$

Therefore, $\bar{S}(\mathrm{inv}(\tau, n))$ is not empty, and thus, $\rho_S(\mathrm{inv}(\tau, n))$ exists, which means that $\mathrm{inv}(\tau, n) \in \mathrm{dom}(\rho_S)$.

(3) → (1): We fix $\sigma \in \mathbb{T}^\omega$ and $x \in X$. By Proposition 19(2), it suffices to show that $\bar{S}(\sigma) = \{x\}$ implies $\sigma \succcurlyeq \varphi_S(x)$. Now, suppose on the contrary that $\bar{S}(\sigma) = \{x\}$ but $\sigma \not\succcurlyeq \varphi_S(x)$. Then $\sigma \uparrow \varphi_S(x)$ by Lemma 5(2), and $\sigma(n) = \bot \prec \varphi_S(x)(n)$ for some $n < \omega$. Thus, we can choose $\tau \in 2^\omega$ such that $\sigma \preccurlyeq \tau$ and $\varphi_S(x) \preccurlyeq \tau$. Put $\sigma' = \mathrm{erase}(\tau, n)$. Since $\sigma \preccurlyeq \sigma'$, we have

$$\bar{S}(\sigma) \supseteq \bar{S}(\sigma') = \varphi_S^{-1}(\downarrow\uparrow\sigma') = \varphi_S^{-1}\big(\downarrow\tau \cup \downarrow\mathrm{inv}(\tau, n)\big) = \big\{x, \rho_S\big(\mathrm{inv}(\tau, n)\big)\big\}.$$

Here, $\rho_S(\mathrm{inv}(\tau, n))$ exists by (3), and it differs from x since $\tau(n) = \varphi_S(x)(n) \succ \bot$. This contradicts the fact that $\bar{S}(\sigma) = \{x\}$. □

The intuitive meaning of Proposition 20(3) is that if $\tau \in 2^\omega$ is a ρ_S-code of $x \in X$, then every digit of τ is either meaningless (i.e., if we invert its value, then the result is also a code of x), or indispensable in identifying x (i.e., if we invert its value, then the result is a code of another point).

In this article, we define yet another property of a proper dyadic subbase.

Definition 8. A *weakly canonically representing subbase* of a Hausdorff space X is a proper dyadic subbase S such that, if $\sigma \prec \varphi_S(x)$ for $x \in X$ and $\sigma \in \mathbb{T}^\omega$, then $\bar{S}(\sigma)$ contains at least one point other than x.

Since $\bar{S}$ is antimonotonic, we only need to consider the case σ is maximal among those bottomed sequences strictly smaller than $\varphi_S(x)$. Therefore, we can also restate as follows.

Proposition 21. *A proper dyadic subbase S is weakly canonically representing if and only if, for all $x \in X$ and $n \in \mathrm{dom}(\varphi_S(x))$, we have $\bar{S}(\mathrm{erase}(\varphi_S(x), n)) \neq \{x\}$.*

It means that all the digits of $\varphi_S(x)$ are indispensable in identifying x, and if we erase one of them and replace it with $\bot$, then it becomes ambiguous and denotes more than one elements with respect to $\bar{S}$.

If S is a full-representing subbase, then ρ_S is a total function and therefore S is canonically representing by Proposition 20(3). If $\sigma \prec \varphi_S(x)$, we have $\sigma \uparrow \varphi_S(x)$ and $\sigma \not\succcurlyeq \varphi_S(x)$. Therefore, a canonically representing subbase is weakly canonically representing. The proof of Proposition 4.7 in [9], which says that a canonically representing subbase S of a non-empty space is independent, only uses the fact that S is weakly canonically representing, and therefore, a weakly canonically representing subbase is independent if the space is non-empty.

In [9], it is written without a proof that weakly canonically representing is equivalent to canonically representing. However, it is not correct and weakly canonically representing is strictly weaker. We show an example of a proper dyadic subbase which is weakly canonically representing but not canonically representing. Let $\mathbb{H} = \mathbb{I}^\omega$ be the Hilbert cube. We consider the dyadic subbase H obtained as the infinite product of the Gray-subbase G of $\mathbb{I}$, through some encoding of $\mathbb{T}^{\omega\times\omega}$ in $\mathbb{T}^\omega$. Then H is a full-representing subbase. Let $z_0 = (1/2, 1/2, \dots) \in \mathbb{H}$ and $\tau_0 \in 2^\omega$ be the infinite sequence obtained by filling all the $\perp$ in $\varphi_H(z_0)$ with 0. Consider the subset

$$Y = \{y \in 2^\omega\colon\ y \text{ and } \tau_0 \text{ differ at infinitely many coordinates}\}$$

of the Cantor space 2^ω. Let C be the dyadic subbase of Y obtained as the restriction of the obvious one on 2^ω. That is, we have $\varphi_C(\sigma) = \sigma$ for $\sigma \in Y$. C is canonically representing by Proposition 20(3) and thus weakly canonically representing. We define our space as the topological sum $X = Y \oplus \mathbb{H}$. We consider the dyadic subbase S defined as $S_{0,0} = Y$, $S_{0,1} = \mathbb{H}$, and $S_{n+1,i} = C_{n,i} \cup H_{n,i}$ $(n \geqslant 0,\ i = 0, 1)$. S is a proper dyadic subbase of X.

Proposition 22. *The dyadic subbase S defined above is weakly canonically representing.*

Proof. Since H and C are weakly canonically representing, we only need to consider in Proposition 21 the cases $\varphi_S(x) = 1\mu$ and $n = 0$, and $\varphi_S(x) = 0\mu$ and $n = 0$, and show that $\bar{S}(\mathrm{inv}(\varphi_S(x), n)) \neq \emptyset$.

First, consider the case $\varphi_S(x) = 1\mu$ and $n = 0$. We have $\mu = \varphi_H(y)$ for some $y \in \mathbb{H}$. Let σ be the element in 2^ω obtained by filling all the bottoms of μ with 1. If infinitely many coordinates of y are different from 1/2, σ is different from τ_0 at infinitely many coordinates. If finitely many coordinates of y are different from 1/2, μ contains infinitely many $\perp$ and therefore σ is different from τ_0 at infinitely many coordinates. Therefore, σ is in Y in both cases. Therefore, $0\sigma \in \varphi_S(X)$ and $0\sigma \uparrow 0\mu$. Thus, $\bar{S}(0\mu) \neq \emptyset$.

Next, consider the case $\varphi_S(x) = 0\mu$ and $n = 0$. Since $\mu \in 2^\omega$ and H is full-representing, there is an element $y \in \mathbb{H}$ such that $\varphi_H(y) \preccurlyeq \mu$. We have $1\varphi_H(y) \in \varphi_S(X)$ and $1\varphi_H(y) \uparrow 1\mu$. Therefore, $\bar{S}(1\mu) \neq \emptyset$. □

Proposition 23. *The dyadic subbase S is not canonically representing.*

Proof. For $z_0 \in \mathbb{H}$ and $\tau_0 \in 2^\omega$ in the definition of X, we have $\rho_S(1\tau_0) = z_0$. Since $\tau_0 \notin Y$, $\bar{S}(0\tau_0) = \emptyset$. Therefore, from Proposition 20(3), we have the result. □

It is an interesting problem to determine a space which has a subbase discussed in this section. By the definition, a separable metrizable space X is dense in itself and compact if and only if X has a full-representing subbase. However, the authors do not know such characterizations for a canonically representing subbase and for a weakly canonically representing subbase.

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