Available online at www.sciencedirect.com
ScienceDirect
Topology 45 (2006) 887–918
TOPOLOGY
www.elsevier.com/locate/top

A topological view of Gromov–Witten theory

D. Maulik*, R. Pandharipande

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Received 19 May 2005; accepted 18 May 2006

Abstract

We study relative Gromov–Witten theory via universal relations provided by the degeneration and localization formulas. We find relative Gromov–Witten theory is completely determined by absolute Gromov–Witten theory. The relationship between the relative and absolute theories is guided by a strong analogy to classical topology.

As an outcome, we present a mathematical determination of the Gromov–Witten invariants (in all genera) of the Calabi–Yau quintic 3-fold in terms of known theories.
© 2006 Elsevier Ltd. All rights reserved.

MSC: 14N35

Keywords: Gromov–Witten relative Calabi–Yau symplectic

0. Introduction

0.1. Overview

Let V be a nonsingular projective variety containing a nonsingular divisor W. All our varieties are defined over $\mathbb{C}$.

The *absolute* Gromov–Witten theory of V is defined by integrating descendent classes over the moduli space of stable maps to V. The *relative* Gromov–Witten theory of the pair (V, W) is defined by descendent integration over the space of stable relative maps to V with prescribed tangency conditions along W.

We present here a systematic study of relative Gromov–Witten theory via universal relations. We find the relative theory does *not* provide new invariants. The relative theory is completely determined by the

* Corresponding author.
E-mail addresses: dmaulik@math.princeton.edu (D. Maulik), rahulp@math.princeton.edu (R. Pandharipande).

0040-9383/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.top.2006.06.002

absolute theory. The relation between the relative and absolute theories is guided by a strong analogy to classical topology.

Our results open new directions in the subject. For example, we present a complete mathematical determination of the Gromov–Witten theory (in all genera) of the Calabi–Yau quintic hypersurface in $\mathbb{P}^4$.

0.2. Leray–Hirsch

Let X be a nonsingular projective variety equipped with a line bundle L. Let Y be the projective bundle $\mathbb{P}(L \oplus \mathcal{O}_X)$, and let π be the projection map,

$$\pi : Y \to X.$$

The summands L and $\mathcal{O}_X$ respectively determine divisors

$$D_0, D_\infty \subset Y$$

isomorphic to X via π.

We consider four descendent Gromov–Witten theories (in all genera) of the projective bundle Y: the absolute Gromov–Witten theory of Y and the relative Gromov–Witten theories of the three pairs

$$(Y, D_0), \qquad (Y, D_\infty), \qquad (Y, D_0 \cup D_\infty).$$

The first result of the paper is a reconstruction theorem for the Gromov–Witten theories of Y in terms of X.

Theorem 1. *All four theories of Y can be uniquely and effectively reconstructed from the Gromov–Witten theory of X and the class $c_1(L) \in H^2(X, \mathbb{Q})$.*

Theorem 1 is proven in Section 1 by exhibiting an explicit set of recursions. The two main techniques used are localization (with respect to the natural fiberwise $\mathbb{C}^*$-action on Y) and degeneration. Equivariant relative invariants of $\mathbb{P}^1$ appear as constants in the recursions. We view Theorem 1 as a Leray–Hirsch result in Gromov–Witten theory.

0.3. Relative in terms of absolute

Let V be a nonsingular projective variety containing a nonsingular divisor W. Let

$$N \to W$$

be the normal bundle of W in V.

Let $\mathcal{F}$ be the degeneration to the normal cone of W, the blow-up of $V \times \mathbb{C}$ along the subvariety $W \times 0$. Let

$$\epsilon : \mathcal{F} \to \mathbb{C}$$

be the projection to the second factor. We find

$$\epsilon^{-1}(0) = V \cup_W \mathbb{P}(N \oplus \mathcal{O}_W)$$

where the inclusion

$$W \subset \mathbb{P}(N \oplus \mathcal{O}_W)$$

is determined by summand N. The degeneration formula [2,12,14,15] expresses the absolute Gromov–Witten theory of V in terms of the relative theories of the pairs (V, W) and $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$.

Theorem 1 together with an inversion of the degeneration formula yields the following result proven in Section 2.

Theorem 2. *The relative Gromov–Witten theory of the pair (V, W) can be uniquely and effectively reconstructed from the absolute theory of V, the absolute theory of W, and the restriction map $H^*(V, \mathbb{Q}) \to H^*(W, \mathbb{Q})$.*

We view Theorem 2 as a Gromov–Witten analogue of the standard long exact sequence relating absolute and relative cohomology theories.

0.4. Mayer–Vietoris

Let V be a nonsingular projective variety, and let

$$\epsilon : \mathcal{V} \to \Delta$$

be a flat family over a disk $\Delta \subset \mathbb{C}$ at the origin satisfying:

(i) $\mathcal{V}$ is nonsingular,
(ii) ϵ is smooth over the punctured disk $\Delta^* = \Delta \setminus \{0\}$,
(iii) $\epsilon^{-1}(1) \cong V$,
(iv) $\epsilon^{-1}(0) = V_1 \cup_W V_2$ is a normal crossings divisor in $\mathcal{V}$.

The family ϵ defines a canonical map

$$H^*(V_1 \cup_W V_2, \mathbb{Q}) \to H^*(V, \mathbb{Q})$$

with image defined to be the *nonvanishing* cohomology of V.

The degeneration formula and Theorem 2 together yield a Mayer–Vietoris result.[1]

Theorem 3. *The Gromov–Witten theory of the nonvanishing cohomology of V can be uniquely and effectively reconstructed from the absolute theories of V_1, V_2, and W and the restriction maps*

$$H^*(V_1, \mathbb{Q}) \to H^*(W, \mathbb{Q}), \qquad H^*(V_2, \mathbb{Q}) \to H^*(W, \mathbb{Q}).$$

0.5. Hypersurfaces

0.5.1. Hypersurface pairs

A *hypersurface pair* (V, W) is a nonsingular hypersurface $V \subset \mathbb{P}^r$ together with a nonsingular divisor $W \subset V$ defined by a complete intersection in $\mathbb{P}^r$.

Let $\beta \in H_2(V, \mathbb{Z})$ be a curve class. Let $\vec{\mu}$ be an ordered partition,

$$\sum_j \mu_j = \int_\beta [W],$$

[1] If $H_2(V, \mathbb{Z})$ has torsion τ, the Gromov–Witten theory of V determined by Theorem 3 is defined with curve classes valued in $H_2(V, \mathbb{Z})/\tau$. Torsion differences can be lost in degeneration. However, torsion collapsing is *not* required for Theorems 1 and 2.

with positive parts. The moduli space $\overline{M}_{g,n}(V/W, \beta, \vec{\mu})$ parameterizes stable relative maps from genus g, n-pointed curves to V of class β with multiplicities along W determined by $\vec{\mu}$.

The relative conditions in the theory correspond to partitions *weighted* by the cohomology of W. Let $\delta_1, \ldots, \delta_{m_W}$ be a basis of $H^*(W, \mathbb{Q})$. A cohomology weighted partition ν consists of an *unordered* set of pairs,

$$\left\{(\nu_1, \delta_{s_1}), \ldots, (\nu_{\ell(\nu)}, \delta_{s_{\ell(\nu)}})\right\},$$

where $\sum_j \nu_j$ is an *unordered* partition of $\int_\beta [W]$. The automorphism group, $\mathrm{Aut}(\nu)$, consists of permutation symmetries of ν.

The *standard* order on the parts of ν is

$$(\nu_i, \delta_{s_i}) > (\nu_{i'}, \delta_{s_{i'}})$$

if $\nu_i > \nu_{i'}$ or if $\nu_i = \nu_{i'}$ and $s_i > s_{i'}$. Let $\vec{\nu}$ denote the partition $(\nu_1, \ldots, \nu_{\ell(\nu)})$ obtained from the standard order.

The descendent Gromov–Witten invariants of the hypersurface pair are defined by integration against the virtual class of the moduli of maps. Let $\gamma_1, \ldots, \gamma_{m_V}$ be a basis of $H^*(V, \mathbb{Q})$, and let

$$\left\langle \tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_n}(\gamma_{l_n}) \mid \nu \right\rangle_{g,\beta}^{V/W} = \frac{1}{|\mathrm{Aut}(\nu)|} \int_{[\overline{M}_{g,n}(V/W,\beta,\vec{\nu})]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_{l_i}) \cup \prod_{j=1}^{\ell(\nu)} \mathrm{ev}_j^*(\delta_{s_j}).$$

Here, the second evaluations,

$$\mathrm{ev}_j : \overline{M}_{g,n}(V/W, \beta, \vec{\nu}) \to W$$

are determined by the relative points.

Gromov–Witten invariants are defined (up to sign) for *unordered* weighted partitions ν. To fix the sign, the integrand on the right side requires an ordering. The ordering is corrected by the automorphism prefactor.

0.5.2. Simple classes

A class $\gamma \in H^*(V, \mathbb{Q})$ is *simple* if γ lies in the image of the restriction map

$$H^*(\mathbb{P}^r, \mathbb{Q}) \to H^*(V, \mathbb{Q}).$$

The simple Gromov–Witten theory of V consists of the integrals of descendents of simple classes. Similarly, the simple Gromov–Witten theory of the pair (V, W) consists of integrals of descendents of simple classes with *no restrictions on the cohomology classes of* W *in the relative constraints*.

A refinement of Theorem 2 proven in Section 2 is valid for the geometry of the hypersurface pair (V, W).

Corollary 1. *The simple Gromov–Witten theory of a hypersurface pair* (V, W) *can be uniquely and effectively reconstructed from the simple Gromov–Witten theory of* V*, the full Gromov–Witten theory of* W*, and the restriction map* $H^*(V, \mathbb{Q}) \to H^*(W, \mathbb{Q})$.

0.5.3. Curves, surfaces, and 3-folds

Nonsingular curves have a rich Gromov–Witten theory including descendents of odd classes. The Gromov–Witten theory of curves is fully determined in [20–22].

Nonsingular surfaces in $\mathbb{P}^3$ of degree up to 3 are rational with Gromov–Witten theories determined by localization [9,10]. The $K3$ surface in degree 4 is holomorphic symplectic — hence all Gromov–Witten invariants of the $K3$ vanish for nonconstant maps [1]. We will present a complete scheme for calculating the simple Gromov–Witten theory of surfaces of degree 5 and higher.

Nonsingular 3-folds in $\mathbb{P}^4$ are determined by localization methods only in degrees 1 and 2. We present a complete scheme for calculating the simple Gromov–Witten theory of hypersurfaces of degree 3, 4, and 5 in $\mathbb{P}^4$.

Gromov–Witten theory is not interesting for 3-fold hypersurfaces of degree greater than 5 in $\mathbb{P}^4$ since the moduli spaces of nonconstant maps have negative dimension.

0.5.4. Calculation scheme

Let $X \subset \mathbb{P}^r$ be a generic hypersurface of degree $d \geq 2$ with equation f. Let

$$h = h_1 h_2$$

be a product of generic polynomials of degree $d-1$ and 1. Let t be a coordinate on $\mathbb{C}$. The ideal

$$(tf - h)$$

determines a subvariety

$$\mathcal{X} \subset \mathbb{P}^r \times \mathbb{C}$$

flat over $\mathbb{C}$. The generic element of the family is a nonsingular hypersurface. The fiber over $0 \in \mathbb{C}$ is a union,

$$\mathcal{X}_0 = X_1 \cup_I X_2,$$

of a hypersurface X_1 with equation h_1 and a hyperplane X_2 with equation h_2 along a complete intersection $I \subset \mathbb{P}^r$ of type $(d-1, 1)$.

The total space of $\mathcal{X}$ is singular. The singular locus of $\mathcal{X}$ over $0 \in \mathbb{C}$ is a complete intersection $S \subset \mathbb{P}^r$ of type $(d, d-1, 1)$ contained in I. Locally analytically, the singularities of $\mathcal{X}$ over $0 \in \mathbb{C}$ are translates of the 3-fold double point.

Let $\widetilde{\mathcal{X}}$ denote the blow-up of $\mathcal{X}$ along the Weil divisor X_2. The total space $\widetilde{\mathcal{X}}$ is nonsingular over $0 \in \mathbb{C}$. The fiber over $0 \in \mathbb{C}$ is a union,

$$\widetilde{\mathcal{X}}_0 = X_1 \cup_I \widetilde{X}_2,$$

where X_1 and I are as before and $\widetilde{X}_2$ is the blow-up of X_2 along S.

By the degeneration formula, the simple Gromov–Witten theory of X is determined by the simple Gromov–Witten theories of the pairs (X_1, I) and $(\widetilde{X}_2, I)$. By Corollary 1, the simple Gromov–Witten theory of (X_1, I) is determined by the simple theory of X_1 and the full theory of I. The simple Gromov–Witten theory of $(\widetilde{X}_2, I)$ requires, in addition, the simple theory of $\widetilde{X}_2$.

By the application of Lemma 1 of Section 0.5.5 below to $\widetilde{X}_2$, we conclude the simple Gromov–Witten theory of X is determined by the simple theories of the hypersurfaces

$$X_1, X_2 \subset \mathbb{P}^r$$

of lower degree and the full theories of the varieties I and S of lower dimension.

0.5.5. Blow-up lemma

Let V be a nonsingular, projective variety. Let $Z \subset V$ be the nonsingular complete intersection of two nonsingular divisors

$$W_1, W_2 \subset V,$$

and let $\widetilde{V}$ be the blow-up of V along Z.

Lemma 1. *The Gromov–Witten theory of $\widetilde{V}$ is uniquely and effectively determined by the Gromov–Witten theories of V, W_1, and Z and the restriction maps*

$$H^*(V, \mathbb{Q}) \to H^*(W_1, \mathbb{Q}) \to H^*(Z, \mathbb{Q}).$$

Only the absolute Gromov–Witten theory of *one* of the divisors is needed in Lemma 1. However, an optimal result should avoid both divisors. Lemma 1 is proven in Section 3.

0.5.6. Surfaces

The calculation scheme determines the simple Gromov–Witten invariants of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$.

For example, the simple Gromov–Witten theory of the degree 5 surface S_5 is determined in terms of the Gromov–Witten theories of the following spaces:

$$\mathbb{P}^2,\ S_4,\ C_4,\ \mathbb{P}^0,$$

where S_4 is a $K3$ surface and C_4 is a nonsingular quartic plane curve.

For surfaces of general type, Gromov–Witten invariants in the adjunction genus with *primary* field insertions are determined by gauge theory: Taubes' tetrology connects these Gromov–Witten invariants to Seiberg–Witten theory [23–26]. In Section 3.3, we present a calculation of

$$\langle 1 \rangle^{S_5}_{6,K} = -1$$

by our method. The structure of Gromov–Witten theory in other genera or with descendent insertions is not known.[2]

0.5.7. 3-folds

The calculation scheme determines the simple Gromov–Witten invariants of hypersurfaces of degree 3, 4, and 5 in $\mathbb{P}^4$ in terms of known theories.

The Calabi–Yau quintic 3-fold $Q \subset \mathbb{P}^4$ appears to be the most difficult hypersurface captured by the scheme. The Gromov–Witten invariants of Q are determined in terms of the Gromov–Witten theories of the following spaces:

$$\mathbb{P}^3,\ \mathbb{P}^2,\ S_2,\ S_3,\ S_4,\ C_{1,2},\ C_{2,3},\ C_{3,4},\ C_{4,5}.$$

Here, $S_d \subset \mathbb{P}^3$ is a nonsingular degree d surface, and $C_{d_1,d_2} \subset \mathbb{P}^3$ is a nonsingular complete intersection curve of type (d_1, d_2).

The quintic scheme, the first mathematical determination of the Gromov–Witten theory of Q, is presented in Section 3.2.

[2] See [13,18] for recent progress.

0.5.8. Further directions

Our calculation scheme using degeneration and Mayer–Vietoris can be pursued in several other contexts — hypersurfaces are treated here as the first illustration of the method. For example, the simple Gromov–Witten theories of all surface and 3-fold complete intersections in projective space are determined similarly. See [18] for further directions.

0.6. Gathmann's proposal

Our last topic concerns Gathmann's proposal for the calculation of higher genus Gromov–Witten invariants of the quintic $Q \subset \mathbb{P}^4$.

Gathmann has studied the Gromov–Witten invariants of Q in genus 0 and 1 via a relation to the Gromov–Witten theory of $\mathbb{P}^4$ — the opposite direction of the scheme discussed in Section 0.5. Theorem 1 allows us to pursue Gathmann's proposal in all genera.

Let $\mathcal{G}$ be the blow-up of $\mathbb{P}^4 \times \mathbb{C}$ along the subvariety $Q \times 0$. Let

$$\epsilon : \mathcal{G} \to \mathbb{C}$$

be the projection to the second factor. We find

$$\epsilon^{-1}(0) = \mathbb{P}^4 \cup_Q \mathbb{P}(\mathcal{O}_Q(5) \oplus \mathcal{O}_Q)$$

where the inclusion

$$Q \subset \mathbb{P}(\mathcal{O}_Q(5) \oplus \mathcal{O}_Q)$$

is determined by the summand $\mathcal{O}_Q(5)$.

The degeneration formula [2,12,14,15] expresses the absolute Gromov–Witten theory of $\mathbb{P}^4$ in terms of the relative theories of the pairs $(\mathbb{P}^4, Q)$ and $(\mathbb{P}(\mathcal{O}_Q(5) \oplus \mathcal{O}_Q), D_0)$. The absolute Gromov–Witten theory of $\mathbb{P}^4$ may be computed by several methods [7,9,10]. Theorem 1 reduces the relative theory of $(\mathbb{P}(\mathcal{O}_Q(5) \oplus \mathcal{O}_Q), D_0)$ to the absolute theory of Q. The degeneration formula then provides a system of equations for the relative invariants of $(\mathbb{P}^4, Q)$ *and* the Gromov–Witten invariants $N_{g,d}$ of Q.

Conjecture 1. *The system of equations obtained from the degeneration formula and Theorem 1 determines both the relative theory of the pair $(\mathbb{P}^4, Q)$ and the Gromov–Witten invariants $N_{g,d}$ of Q.*

Gathmann pursued the above method in genus 0 and 1 [5,6,8] via a different approach to the reduction of the relative theory of the pair

$$(\mathbb{P}(\mathcal{O}_Q(5) \oplus \mathcal{O}_Q), D_0)$$

to the absolute theory of Q. Theorem 1, however, is valid for *all* genera. Gathmann's method and Theorem 1 together provide a computation scheme for all $N_{g,d}$ so long as the arising equations are nonsingular.

We have proven Conjecture 1 in genus 2.[3] Even if the nonsingularity is not known beforehand, the computation can be undertaken in genus $g \geq 3$. Gathmann's proposal, though difficult and not yet certain for all genera, appears more suitable for calculations than the complete quintic scheme discussed in Section 0.5.

[3] The proof is omitted here.

A very different approach to the Gromov–Witten theory of Q in genus 1 has been advanced in a series of papers by Zinger and collaborators [16,27–29].

1. Leray–Hirsch

1.1. Notation

1.1.1. Cohomology

Let X be a nonsingular projective variety equipped with a line bundle L, and let Y be the projective bundle

$$\pi : \mathbb{P}(L \oplus \mathcal{O}_X) \to X$$

with sections D_0, D_∞ corresponding to the summands L, $\mathcal{O}_X$ respectively.

Let $\delta_1, \ldots, \delta_{m_X}$ be a basis of $H^*(X, \mathbb{Q})$ containing the identity element. We will often denote the identity by δ_{Id}. The *degree* of δ_i is the real grading in $H^*(X, \mathbb{Q})$. We view δ_i as an element of $H^*(Y, \mathbb{Q})$ via pull-back by π.

Let $[D_0], [D_\infty] \in H^2(Y, \mathbb{Q})$ denote the cohomology classes associated to the divisors. Define classes in $H^*(Y, \mathbb{Q})$ by

$$\gamma_i = \delta_i,$$
$$\gamma_{m_X+i} = \delta_i \cdot [D_0],$$
$$\gamma_{2m_X+i} = \delta_i \cdot [D_\infty].$$

We will use the following notation:

$$\gamma_i^\delta = \delta_{i \bmod m_X},$$
$$\gamma_i^D = 1, \qquad [D_0], \quad \text{or} \quad [D_\infty].$$

The second assignment depends upon the integer part of $(i-1)/m_X$. The set $\{\gamma_1, \ldots, \gamma_{2m_X}\}$ determines a basis of $H^*(Y, \mathbb{Q})$.

1.1.2. Theorem 1 for the absolute theory of Y

There is a fiberwise $\mathbb{C}^*$-action on Y determined by scaling the second factor in the sum $L \oplus \mathcal{O}_X$. The absolute theory of Y can be directly computed via the virtual localization formula of [10].

The $\mathbb{C}^*$-action on Y induces a canonical $\mathbb{C}^*$-action on the moduli space of stable maps $\overline{M}_{g,n}(Y, \beta)$. The $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,n}(Y, \beta)$ are determined by bipartite graphs. The vertices correspond to spaces of stable maps to D_0 or D_∞ — both targets are canonically isomorphic to X. The virtual localization formula reduces the Gromov–Witten invariants of Y to Hodge integrals in the Gromov–Witten theory of X. The Hodge insertions may be removed by the relations of [3]. The proof of Theorem 1 for the absolute theory of Y is complete.

1.1.3. Brackets

We will use the following bracket notation for the Gromov–Witten invariants of the pair (Y, D_0):

$$\left\langle \mu \left| \prod_i \tau_{k_i}(\gamma_{l_i}) \right.\right\rangle_{g,\beta} = \frac{1}{|\mathrm{Aut}(\mu)|} \int_{[\overline{M}_{g,n}(Y/D_0,\beta,\vec{\mu})]^{\mathrm{vir}}} \prod_i \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_{l_i}) \cup \prod_j \mathrm{ev}_j^*(\delta_{r_j}),$$

where

$$\mu = \{(\mu_1, \delta_{r_1}), \ldots, (\mu_{\ell(\mu)}, \delta_{r_{\ell(\mu)}})\}$$

is a partition weighted by the cohomology of X and

$$\sum_j \mu_j = \int_\beta [D_0].$$

Relative invariants are defined only when $\int_\beta [D_0] \geq 0$.

For the pair (Y, D_∞), the relative conditions will be written on the right side of the bracket:

$$\left\langle \prod_i \tau_{k_i}(\gamma_{l_i}) \middle| \nu \right\rangle_{g,\beta}.$$

The invariants of the pairs (Y, D_0) and (Y, D_∞) are termed *type I.*

For the pair $(Y, D_0 \cup D_\infty)$, the relative conditions for D_0 will be written on the left side and the relative conditions for D_∞ will be written on the right side:

$$\left\langle \mu \middle| \prod_i \tau_{k_i}(\gamma_{l_i}) \middle| \nu \right\rangle_{g,\beta}.$$

The invariants of $(Y, D_0 \cup D_\infty)$ are termed *type II.*

The above brackets denote Gromov–Witten invariants with *connected* domain curves. Disconnected invariants arise naturally in the degeneration formula. We will treat disconnected invariants as products of connected invariants except in the study of rubber targets in Section 1.5. However, our proof of Theorem 1 is valid without assuming the product rule. The connected/disconnected issue will be discussed carefully in Section 1.8.

1.1.4. Partition terminology

The following constants associated to a weighted partition μ will arise often:

- $\deg(\mu) = \sum_i \deg(\delta_{r_i})$, the total degree of the cohomology weights,
- $Id(\mu)$ equals the number occurrences of the pair $(1, \delta_{Id})$ in μ,
- $\mathfrak{z}(\mu) = \prod_i \mu_i \cdot |\mathrm{Aut}(\mu)|$.

We assume the cohomology basis $\delta_1, \ldots, \delta_{m_X}$ is self dual with respect to the Poincaré pairing. Then, to each weighted partition μ, a dual partition $\mu^\vee$ is defined by taking the Poincaré duals of the cohomology weights.

1.1.5. Orderings

All Gromov–Witten invariants $\langle , \rangle_{g,\beta}$ vanish if

$$\beta \in H_2(Y, \mathbb{Z})$$

is not an effective curve class. We define a partial ordering on $H_2(Y, \mathbb{Z})$ as follows:

$$\beta' < \beta$$

if $\beta - \beta'$ is a nonzero effective curve class.

The set of pairs (m, δ) where $m \in \mathbb{Z}_{>0}$ and $\delta \in H^*(X, \mathbb{Q})$ is partially ordered by the following *size* relation

$$(m, \delta) > (m', \delta') \tag{1}$$

if $m > m'$ or if $m = m'$ and $\deg(\delta) > \deg(\delta')$.

Let μ be a partition weighted by the cohomology of X,

$$\mu = \{(\mu_1, \delta_{r_1}), \ldots, (\mu_{\ell(\mu)}, \delta_{r_{\ell(\mu)}})\}.$$

We may place the pairs of μ in decreasing order by size (1). A *lexicographic* ordering on weighted partitions is defined as follows:

$$\mu \overset{l}{>} \mu'$$

if, after placing μ and μ' in decreasing order by size, the first pair for which μ and μ' differ in size is larger for μ.

1.2. Fiber classes

Let $[F] \in H_2(Y, \mathbb{Z})$ denote the class of a fiber of π. The *fiber class* invariants of type I and II are those for which β is a (possibly zero) multiple of $[F]$. Our first goal is to calculate the fiber class invariants of both types in terms of the classical cohomology of X.

Consider a connected type I invariant of a fiber class,

$$\left\langle \mu \left| \prod_i \tau_{k_i}(\gamma_{l_i}) \right\rangle_{g,d[F]} \right. . \tag{2}$$

We determine the fiber class invariant (2) from the equivariant theory of $\mathbb{P}^1$.

The moduli space of stable relative maps

$$\overline{M}_Y = \overline{M}_{g,n}(Y/D_0, d[F], \vec{\mu}),$$

is fibered over X,

$$\pi : \overline{M}_Y \to X,$$

with fiber isomorphic to the moduli space of maps to $\mathbb{P}^1$ relative to 0,

$$\overline{M}_{\mathbb{P}^1} = \overline{M}_{g,n}(\mathbb{P}^1/0, d, \vec{\mu}).$$

In fact, $\overline{M}_Y$ is the fiber bundle constructed from the principal $\mathbb{C}^*$-bundle associated to L and a standard $\mathbb{C}^*$-action on $\overline{M}_{\mathbb{P}^1}$.

The π-relative obstruction theory of $\overline{M}_Y$ is obtained from the $\overline{M}_{\mathbb{P}^1}$-fiber bundle structure over X. The relationship between the π-relative virtual fundamental class $[\overline{M}_Y]^{\mathrm{vir}_\pi}$ and the virtual fundamental class $[\overline{M}_Y]^{\mathrm{vir}}$ is given by the equation

$$[\overline{M}_Y]^{\mathrm{vir}} = c_{top}(\mathbb{E} \boxtimes T_X) \cap [\overline{M}_Y]^{\mathrm{vir}_\pi} \tag{3}$$

where $\mathbb{E}$ is the Hodge bundle. We may rewrite (3) as

$$[\overline{M}_Y]^{\text{vir}} = \sum_q h_q\,(c_1(\mathbb{E}), c_2(\mathbb{E}), \ldots)\, t_q\,(c_1(T_X), c_2(T_X), \ldots) \cap [\overline{M}_Y]^{\text{vir}_\pi}$$

where h_q and t_q are polynomials.

The invariant (2) can then be computed by pairing cohomology classes in X with the results of equivariant integrations in the Gromov–Witten theory of $\overline{M}_{\mathbb{P}^1}$. We write

$$\left\langle \mu \left| \prod_i \tau_{k_i}(\gamma_{l_i}) \right. \right\rangle_{g,d[F]} = \frac{1}{|\text{Aut}(\mu)|} \sum_q \int_X \left(t_q \prod_i \gamma_{l_i}^{\delta} \prod_j \delta_{r_j} \cap \pi_* \left(h_q \prod_i \psi_i^{k_i} \text{ev}_i^*(\gamma_{l_i}^D) \cap [\overline{M}_Y]^{\text{vir}_\pi} \right) \right). \quad (4)$$

The interior push-forward

$$\pi_* \left(h_q \prod_i \psi_i^{k_i} \text{ev}_i^*(\gamma_{l_i}^D) \cap [\overline{M}_Y]^{\text{vir}_\pi} \right)$$

is obtained from the corresponding Hodge integral in the equivariant Gromov–Witten theory of $\mathbb{P}^1/0$ after replacing the hyperplane class on $\mathbb{CP}^\infty$ by $c_1(L)$.

The argument for the pairs (Y, D_∞) and $(Y, D_0 \cup D_\infty)$ is identical. The required Hodge integrals in the equivariant relative Gromov–Witten theory of $\mathbb{P}^1$ are fully determined by the Hodge removal equations of [3,17] and the results of [20,21].

1.3. Distinguished type II invariants

Distinguished invariants of type II are integrals for which there is a distinguished marked point p with a *pure* cohomology condition of the form $[D_0] \cdot \delta$ where

$$\deg(\delta) > 0.$$

We will write distinguished invariants as

$$\langle \mu | \tau_0([D_0] \cdot \delta) \cdot \omega | \nu \rangle_{g,\beta}$$

where $\tau_0([D_0] \cdot \delta)$ is the distinguished insertion and ω denotes the product of the non-distinguished insertions. Let $\|\omega\|$ denote the number of non-distinguished insertions.

We will compute distinguished type II invariants by an inductive algorithm. A partial ordering $\overset{\circ}{<}$ on the set of distinguished type II invariants is defined as follows:

$$\langle \mu' | \tau_0([D_0] \cdot \delta') \cdot \omega' | \nu' \rangle_{g',\beta'} \overset{\circ}{<} \langle \mu | \tau_0([D_0] \cdot \delta) \cdot \omega | \nu \rangle_{g,\beta}$$

if one of the conditions below holds

(1) $\beta' < \beta$,
(2) equality in (1) and $g' < g$,
(3) equality in (1)–(2) and $\|\omega'\| < \|\omega\|$,
(4) equality in (1)–(3) and $\deg(\mu') > \deg(\mu)$,

(5) equality in (1)–(4) and $\deg(\nu') > \deg(\nu)$,
(6) equality in (1)–(5) and $\deg(\delta') > \deg(\delta)$,
(7) equality in (1)–(6) and $\nu' \overset{l}{>} \nu$.

For any given distinguished invariant of type II, there are only finitely many distinguished invariants of type II lower in the partial ordering. Our algorithm consists of relations between type I and type II invariants which allow us to move down the partial ordering.

1.4. *Relation* 1

Relation 1 expresses a distinguished invariant of type II in terms of type I invariants and strictly lower distinguished invariants of type II with respect to $\overset{\circ}{<}$.

Fix g and $\beta > 0$. Type I and II invariants of genus g and class β will be viewed as *principal* terms of the equations below. All type I and II invariants of Y with

$$\beta' < \beta$$

or

$$\beta' = \beta \quad \text{and} \quad g' < g$$

are viewed as *non-principal* terms. The non-principal terms are inductively determined and, therefore, omitted in the equations.

Let R denote the distinguished type II invariant

$$\langle \mu | \tau_0([D_0] \cdot \delta) \cdot \omega | \nu \rangle_{g,\beta}$$

with $\deg(\delta) > 0$ and relative conditions

$$\mu = \{(\mu_i, \delta_{r_i})\}, \qquad \nu = \{(\nu_j, \delta_{s_j})\}.$$

Relation 1. *We have*

$$\begin{aligned} \langle \mu | \tau_0([D_0] \cdot \delta) \cdot \omega | \nu \rangle_{g,\beta}\, C = & \left\langle \mu \left| \tau_0([D_0] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta} \\ & - \sum_{\substack{\tilde{R}_{g,\beta} \text{ distinguished type II} \\ \tilde{R} \overset{\circ}{<} R}} \tilde{R}\, C_{\tilde{R},R} \\ & - \sum_{\substack{\|\omega'\| \le \|\omega\| \\ \deg(\mu') \ge \deg(\mu)+1}} C_{\mu',\omega'} \left\langle \mu' \left| \omega' \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta} \\ & - \cdots, \end{aligned}$$

where

$$C = \prod_j \frac{1}{(\nu_j - 1)!} \left(\int_\beta [D_\infty] \right)^{Id(\nu)} \neq 0$$

and the coefficients $C_{*,*}$ are fiber class integrals. The dots stand for non-principal terms of type I and II.

Proof. Consider the Gromov–Witten invariant of the pair (Y, D_0) obtained by replacing the relative conditions ν along D_∞ of R by the insertions

$$\theta = \prod_j \tau_{\nu_j-1}([D_\infty] \cdot \delta_{s_j}).$$

The resulting type I invariant,

$$\left\langle \mu \,\middle|\, \tau_0([D_0] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j-1}([D_\infty] \cdot \delta_{s_j}) \right\rangle_{g,\beta}, \tag{5}$$

is the first term on the right side of Relation 1. We will obtain Relation 1 by computing (5) via degeneration to the normal cone of D_∞.

The special fiber of the degeneration to the normal cone of D_∞ is a union of two copies of Y,

$$Y_1 \cup_D Y_2,$$

along a divisor D. The intersection D is identified with D_∞ on Y_1 and D_0 on Y_2.

The degeneration formula expresses (5) in terms of type II invariants on Y_1 and type I invariants on Y_2 relative to D_0:

$$\langle \mu \mid \tau_0([D_0] \cdot \delta) \cdot \omega\theta \rangle_{g,\beta} = \sum \langle \mu | \tau_0([D_0] \cdot \delta) \cdot \omega_1 | \eta \rangle^\bullet_{g_1,\beta_1} \mathfrak{z}(\eta) \langle \eta^\vee \mid \omega_2 \theta \rangle^\bullet_{g_2,\beta_2}.$$

The sum on the right is over all splittings of g and β, all distributions of the insertions of ω, all intermediate cohomology weighted partitions η, and all configurations of connected components yielding a connected total domain. The invariants on the right are possibly disconnected — indicated by the superscript $\bullet$. The subscript g_i denotes the arithmetic genus of the total map to Y_i.

Let $\{(\eta_k, \rho_k)\}$ be the parts of η. We may assume the insertions of ω with cohomology classes divisible by $[D_0]$ and $[D_\infty]$ are distributed to Y_1 and Y_2 respectively. Hence, the invariants of Y_1 are distinguished. For a given distribution,

$$\int_{\beta_1} [D_\infty] = \int_{\beta_2} [D_0].$$

Since we are omitting non-principal terms, we may assume either $\beta_1 = \beta$ or $\beta_2 = \beta$.

Case 1: $\beta_1 = \beta$.

The principal terms from Y_1 will be shown to be either R or a distinguished invariant of type II lower than R in our ordering.

Let $f_i : C_i \to Y_i$ be the elements of the relative moduli spaces for a fixed splitting. The condition $\beta_1 = \beta$ forces β_2 to be a multiple of the fiber class $[F]$. Let $\ell(\eta)$ denote the length of η. We find

$$g = g_1 + g_2 + \ell(\eta) - 1.$$

Since β_2 is a fiber class, every connected component of C_2 intersects D_0 and contains at least one relative marking. Hence,

$$g_2 \geq 1 - \ell(\eta).$$

We conclude $g \geq g_1$ with equality if and only if C_2 consists of rational components, each totally ramified over D_0. Since an invariant in the degeneration formula with $g > g_1$ is non-principal, we consider only the extremal configurations.

If any of the insertions of ω are distributed to Y_2, the type II invariant of Y_1 will be strictly lower than R with respect to $\overset{\circ}{<}$. These principal contributions appear in the second term on the right of Relation 1.

We must analyze the case in which C_2 consists of rational components totally ramified over D_0 and the only non-relative insertions on Y_2 are given by θ.

The distribution of the $\ell(\nu)$ insertions of θ among the $\ell(\eta)$ rational components of C_2 decomposes ν into $\ell(\eta)$ cohomology weighted partitions

$$\nu = \coprod_{k=1}^{\ell(\eta)} \pi^{(k)},$$

where we allow empty weighted partitions. Here,

$$\pi^{(k)} = \{(\nu^{(k)}_{n_1}, \delta_{i^{(k)}_1}), \ldots, (\nu^{(k)}_{n_s}, \delta_{i^{(k)}_s})\}.$$

Then, for each k,

$$\deg(\pi^{(k)}) = \sum \deg(\delta_{i^{(k)}_j}) \leq \deg(\rho_k). \tag{6}$$

We conclude

$$\deg(\nu) \leq \deg(\eta).$$

By the ordering $\overset{\circ}{<}$, a strict inequality in (6) implies a strictly lower invariant. We consider only the case where equality holds in (6) for each k.

The dimension constraint for Y_2 yields the equality

$$\eta_k - 1 = \sum_{j=1}^{\ell(\pi^{(k)})} (\pi^{(k)}_j - 1) \tag{7}$$

on each component of the domain C_2.

Consider the weighted partition $\pi^{(k)}$ containing the largest element (ν_1, δ_{i_1}) of ν in the size ordering. By formula (7), either

$$\eta_k > \nu_1$$

or $\eta_k = \nu_1$ and all the other pairs of $\pi^{(k)}$ are of the form $(1, \delta)$. In the second case, either

$$\deg(\rho_k) > \deg(\delta_{i_1})$$

or $\rho_k = \delta_{i_1}$ and all the other pairs of $\pi^{(k)}$ are of the form $(1, \delta_{Id})$. Therefore, either η is larger than ν in the lexicographic ordering and corresponds to a type II invariant strictly lower than R in the $\overset{\circ}{<}$ ordering, or the maximal pairs of η and ν agree.

We now repeat the above analysis for the second largest element of ν and continue until all the elements of η are exhausted. We find either a strictly smaller type II invariant in the $\overset{\circ}{<}$ ordering or

$$\eta = \nu.$$

In the latter case, we recover R.

The normalization of θ sets the coefficient of R in the degeneration formula to equal

$$\prod_j \frac{1}{(\nu_j - 1)!} \left(\int_\beta [D_\infty] \right)^{Id(\nu)}.$$

The coefficient is a product of $\mathfrak{z}(\nu)$ with genus 0 fiber class integrals. The fiber class integrals are evaluated using the Gromov–Witten/Hurwitz correspondence of [20] and the divisor equation.

Case 2: $\beta_2 = \beta$.

The principal terms from Y_2 will be shown to be type I invariants of the form of the third term on the right of Relation 1.

Let $f_i : C_i \to Y_i$ be the elements of the relative moduli spaces for a fixed splitting. The condition $\beta_2 = \beta$ forces β_1 to be a multiple of the fiber class $[F]$. As before, after neglecting lower terms, we may assume C_1 consists of $\ell(\eta)$ rational components, each totally ramified over D_∞.

The distribution of the $\ell(\mu)$ relative markings among the $\ell(\eta)$ rational components of C_1 decomposing μ into $\ell(\eta)$ cohomology weighted partitions

$$\mu = \coprod_{k=1}^{\ell(\eta)} \pi^{(k)},$$

where empty weighted partitions are *not* allowed.

If the kth component of C_1 does not contain the distinguished marked point, then

$$\deg(\pi^{(k)}) + \deg(\rho_k) \le \dim_{\mathbb{R}}(X)$$

since all these classes are pulled-back from the same projection map to X. If the kth component of C_1 does contain the distinguished marked point, then

$$\deg(\pi^{(k)}) + \deg(\rho_k) \le \dim_{\mathbb{R}}(X) - 1$$

since there is an additional class of nonzero degree from the distinguished marking. The result follows since the cohomology weights of $\eta^\vee$ are Poincaré dual to the classes ρ_k. □

1.5. *Rubber calculus*

1.5.1. *Rubber targets*

We will study the Gromov–Witten theory of the pair (Y, D_0) via virtual localization [10,11] with respect to the natural fiberwise $\mathbb{C}^*$-action on Y discussed in Section 1.1.2.

The $\mathbb{C}^*$-action on Y induces a canonical $\mathbb{C}^*$-action on the moduli space of stable relative maps to the pair (Y, D_0). The $\mathbb{C}^*$-fixed loci of the latter action involve stable relative maps to non-rigid targets. Let

$$\overline{M}^\bullet = \overline{M}^\bullet_{g,n}(Y/D_0 \cup D_\infty, \beta, \vec{\mu}, \vec{\nu})$$

denote the moduli space of stable maps to Y relative to both divisors, and let

$$\overline{M}^{\bullet\sim} = \overline{M}^{\bullet\sim}_{g,n}(Y/D_0 \cup D_\infty, \beta, \vec{\mu}, \vec{\nu})$$

denote the corresponding space of stable maps to a non-rigid target — termed a *rubber* target in [4,22]. Let

$$\epsilon : \overline{M}^\bullet \to \overline{M}^{\bullet\sim}$$

be the canonical forgetful map.

The superscripted $\bullet$ indicates the moduli spaces $\overline{M}^{\bullet}$ and $\overline{M}^{\bullet\sim}$ may parameterize maps with disconnected domains with specified genus and class distributions. The latter data is not made explicit in our notation. The subscripted g is the arithmetic genus of the total domain. Similarly, the brackets $\langle,\rangle^{\bullet}$ and $\langle,\rangle^{\bullet\sim}$ will denote invariants with possibly disconnected domains. There is no product rule relating connected and disconnected rubber invariants.

1.5.2. Cotangent classes

The moduli space $\overline{M}^{\bullet}$ and $\overline{M}^{\bullet\sim}$ carry tautological cotangent line bundles $\mathbb{L}_0$ and $\mathbb{L}_\infty$ determined by the relative divisors. The associated cotangent line classes

$$\Psi_0 = c_1(\mathbb{L}_0), \qquad \Psi_\infty = c_1(\mathbb{L}_\infty),$$

play an important role in relative Gromov–Witten theory.

Let pr_1 and pr_2 denote the projections onto the first and second factors of the product

$$D_0 \times \overline{M}^{\bullet}.$$

Let $\tau : T \to \overline{M}^{\bullet}$ denote the universal family of *targets* over the moduli space, and let

$$\iota : D_0 \times \overline{M}^{\bullet} \to T$$

denote the inclusion of the relative divisor. The cotangent line determined by D_0 is defined by

$$\mathbb{L}_0 = \mathrm{pr}_{2*}\left(\mathrm{Conorm}(\iota) \otimes \mathrm{pr}_1{}^*(\mathrm{Norm}(Y/D_0))\right), \tag{8}$$

where $\mathrm{Conorm}(\iota)$ is the conormal bundle of the embedding ι and

$$\mathrm{Norm}(Y/D_0) = L^*$$

is the normal bundle of D_0 in Y. The push-forward (8) is easily seen to define a *line bundle* $\mathbb{L}_0$.

The line bundle $\mathbb{L}_\infty$ on $\overline{M}^{\bullet}$ is similarly defined. The constructions in the rubber case are identical.

1.5.3. Rigidification

The following rigidification lemma plays a fundamental role in our localization analysis.

Lemma 2. *Let p be a non-relative marking with evaluation map*

$$\mathrm{ev}_p : \overline{M}^{\bullet} \to Y.$$

Then,

$$\begin{aligned}[\overline{M}^{\bullet\sim}]^{\mathrm{vir}} &= \epsilon_*\left(\mathrm{ev}_p^*([D_0]) \cap [\overline{M}^{\bullet}]^{\mathrm{vir}}\right)\\ &= \epsilon_*\left(\mathrm{ev}_p^*([D_\infty]) \cap [\overline{M}^{\bullet}]^{\mathrm{vir}}\right).\end{aligned}$$

Proof. The forgetful map ϵ is equivariant with respect to the canonical $\mathbb{C}^*$-action on $\overline{M}^{\bullet}$ induced from the fiberwise $\mathbb{C}^*$-action on Y and the trivial $\mathbb{C}^*$-action on $\overline{M}^{\bullet\sim}$. We prove the first equality by $\mathbb{C}^*$-localization.

A stable relative map corresponding to an element of a typical $\mathbb{C}^*$-fixed locus of $\overline{M}^{\bullet}$ is a union of three basic submaps:

(i) a nonrigid stable map to the degeneration of Y over D_0,
(ii) a nonrigid stable map to the degeneration of Y over D_∞,
(iii) a collection of $\mathbb{C}^*$-invariant, fiber class, rational Galois covers joining (i)–(ii).

The forgetful map simply contracts the intermediate rational curves (iii).

Assuming we have a proper degeneration on each side of Y, the virtual dimension of the $\mathbb{C}^*$-fixed locus is 2 less than the virtual dimension of $\overline{M}^\bullet$. Since the dimension of

$$\epsilon_*\left(\mathrm{ev}^*([D_0]) \cap [\overline{M}^\bullet]^{\mathrm{vir}}\right) \tag{9}$$

is only 1 less than the virtual dimension of $\overline{M}^\bullet$, the contributions of the above loci cancel in the computation of the push-forward (9).

The only fixed loci which may contribute are those with target degeneration on only one side of Y. Since the marking p is constrained by an insertion of $[D_0]$, we need only consider degenerations along D_0.

There is a unique $\mathbb{C}^*$-fixed locus which provides non-cancelling contributions to the push-forward (9). Moreover, the $\mathbb{C}^*$-fixed locus is isomorphic to $\overline{M}^{\bullet\sim}$. The contribution,

$$\frac{-\mathrm{ev}_p^*(c_1(L)) + t}{-\Psi_\infty + t} \cap [\overline{M}^{\bullet\sim}]^{\mathrm{vir}},$$

is obtained from the virtual localization formula. Here, Ψ_∞ is the cotangent line class on $\overline{M}^{\bullet\sim}$ at the relative divisor D_∞. By dimension considerations, the only non-cancelling part is $[\overline{M}^{\bullet\sim}]^{\mathrm{vir}}$ — proving the first equality. The proof of the second equality is identical. □

1.5.4. Dilaton and divisor

As before, let Ψ_∞ denote the cotangent line class on $\overline{M}^{\bullet\sim}$ at the relative divisor D_∞. The dilaton equation for rubber integrals is

$$\left\langle \mu \left| \tau_1(1) \prod_{i=1}^n \tau_{k_i}(\gamma_{l_i})\, \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} = (2g - 2 + n + \ell(\mu) + \ell(\nu)) \left\langle \mu \left| \tau_1(1) \prod_{i=1}^n \tau_{k_i}(\gamma_{l_i})\, \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim}.$$

The divisor equation for $H \in H^2(X, \mathbb{Q})$, however, takes a modified form:

$$\begin{aligned}
&\left\langle \mu \left| \tau_0(H) \prod_{i=1}^n \tau_{k_i}(\gamma_{l_i})\, \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} \\
&\quad = \left(\int_{\pi_*(\beta)} H \right) \cdot \left\langle \mu \left| \prod_{i=1}^n \tau_{k_i}(\gamma_{l_i})\, \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} + \sum_{j=1}^n \left\langle \mu \left| \ldots \tau_{k_j-1}(\gamma_{l_j} \cdot H) \ldots \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} \\
&\qquad + \sum_{j=1}^{\ell(\nu)} \left\langle \mu \left| \prod_{i=1}^n \tau_{k_i}(\gamma_{l_i})\, \Psi_\infty^{k-1} \right| \{\ldots (\nu_j, \delta_{s_j} \cdot H) \ldots\} \right\rangle_{g,\beta}^{\bullet\sim} \cdot \nu_j.
\end{aligned}$$

The dilaton and divisor equations are proven by the standard cotangent line comparison method.

1.5.5. Calculus I: Fiber class

The rubber calculus relates Gromov–Witten rubber invariants with Ψ_∞ insertions to Gromov–Witten invariants of the pair $(Y, D_0 \cup D_\infty)$.

Consider first a rubber integral with descendent insertions ω for which β is a multiple of the fiber class:

$$\left\langle \mu \left| \omega \ \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim}. \tag{10}$$

A contracted genus 0 component of the domain must carry at least 3 non-relative markings by stability. Similarly, a contracted genus 1 domain component must carry at least 1 non-relative marking. A non-contracted domain component must carry at least 2 relative markings — the intersection points with D_0 and D_∞. Finally, by target stability, not all domain components can be genus 0 and fully ramified over D_0 and D_∞. We conclude

$$2g - 2 + n + \ell(\mu) + \ell(\nu) > 0.$$

Therefore, the fiber class rubber integral (10) is determined by

$$\left\langle \mu \left| \tau_1(1) \cdot \omega \ \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} \tag{11}$$

and the dilaton equation.

Let p denote the marked point carrying the insertion $\tau_1(1)$ in the rubber integral (11). There is a canonical map to the Artin stack of genus 0, 3-pointed curves,

$$\alpha : \overline{M}^{\bullet\sim} \to \mathcal{M}_{0,3}.$$

Given $[f] \in \overline{M}^{\sim}$, $\alpha(f)$ is the genus 0 curve

$$C_f = \pi^{-1}(\pi(f(p)))$$

with the 3 markings determined by

$$D_0 \cap C_f, \qquad f(p), \qquad D_\infty \cap C_f.$$

The class

$$\Psi_\infty - \mathrm{ev}_p^*(c_1(L))$$

is the pull-back of the cotangent line of the third marking on the Artin stack.

The topological recursion relation with respect to the cotangent class of the third marking of $\mathcal{M}_{0,3}$ can be pulled-back via α:

$$\left\langle \mu \left| \tau_1(1) \cdot \omega \ \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} = \left\langle \mu \left| \tau_1(c_1(L)) \cdot \omega \ \Psi_\infty^{k-1} \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} + \sum \langle \mu \,|\tau_1(1) \cdot \omega_1 |\, \eta \rangle_{g_1,\beta_1}^{\bullet\sim} \mathfrak{z}(\eta) \left\langle \eta^\vee \left| \omega_2 \Psi_\infty^{k-1} \right| \nu \right\rangle_{g_2,\beta_2}^{\bullet\sim},$$

where the sum is over all splittings of g and β, all distributions of the insertions, and all intermediate cohomology weighted partitions.

The first term of the sum on the right can be expressed as a type II invariant by Lemma 2:

$$\langle \mu \,|\tau_1(1) \cdot \omega_1 |\, \eta \rangle_{g_1,\beta_1}^{\bullet\sim} = \langle \mu \,|\tau_1([D_0]) \cdot \omega_1 |\, \eta \rangle_{g_1,\beta_1}^{\bullet}.$$

For the application of Lemma 2 here, we require the compatibility of ϵ with the cotangent lines at the marked points.

We have reduced the original fiber class rubber invariant (10) to invariants of the same type with *fewer* Ψ_∞ insertions. Repeating the cycle yields rubber invariants without Ψ_∞ insertions. The latter are related to type II invariants by Lemma 2 after adding a dilaton insertion.

1.5.6. *Calculus II:* $\pi_*(\beta) \neq 0$

If β is not fiber class, then $\pi_*(\beta) \neq 0$. Consider a non-fiber class rubber integral:

$$\left\langle \mu \left| \omega\ \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim}. \tag{12}$$

If $H \in H^2(X, \mathbb{Q})$ is an ample class, then

$$\int_{\pi_*(\beta)} H > 0.$$

Therefore, by the divisor equation, the rubber integral

$$\left\langle \mu \left| \tau_0(H) \cdot \omega\ \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\bullet\sim} \tag{13}$$

determines (12) modulo rubber integrals with strictly fewer cotangent lines.

As in the fiber case, we may apply the topological recursion relations to (13). By repeating the cycle and applying Lemma 2 after all Ψ_∞ insertions are removed, we can express the original rubber invariant (12) in terms of type II invariants.

A refined consequence of the rubber calculus will be needed in the proof of Relation 2 in the following section.

Lemma 3. *A rubber invariant with connected domain and class satisfying $\pi_*(\beta) \neq 0$ is expressed by the calculus in terms of type II invariants as:*

$$\left\langle \mu \left| \omega\ \Psi_\infty^k \right| \nu \right\rangle_{g,\beta}^{\sim} = \sum_{\substack{\|\omega'\| \le \|\omega\| \\ \deg(\mu') \ge \deg(\mu) \\ \deg(\nu') \ge \deg(\nu) \\ m \ge 0}} C_{\mu',\omega',\nu'} \left\langle \mu' \left| \tau_0([D_0] \cdot H \cdot c_1(L)^m) \cdot \omega' \right| \nu' \right\rangle_{g,\beta} + \cdots. \tag{14}$$

The brackets $\langle , \rangle$ on the right denote connected invariants. The coefficients $C_{*,*,*}$ are determined by fiber class integrals, and the dots stand for non-principal terms of type II.

1.6. *Relation 2*

The next relation expresses the type I invariants occurring in Relation 1 in terms of type II invariants and the Gromov–Witten theory of X.

Consider the type I invariant

$$\left\langle \mu \left| \tau_0([D_0] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right.\right\rangle_{g,\beta}$$

with $\pi_*(\beta) \neq 0$, $\deg(\delta) > 0$, and relative conditions

$$\mu = \left\{(\mu_i, \delta_{r_i})\right\}, \qquad \nu = \left\{(\nu_j, \delta_{s_j})\right\}.$$

Relation 2. *We have,*

$$\left\langle \mu \left| \tau_0([D_0] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta}$$
$$= \sum_{\substack{\|\omega'\| \leq \|\omega\| \\ \deg(\mu') \geq \deg(\mu)+1 \\ \deg(\nu') \geq \deg(\nu) \\ m \geq 0}} C_{\mu',\nu',\omega'} \left\langle \mu' \left| \tau_0([D_0] \cdot H \cdot c_1(L)^m) \cdot \omega' \right| \nu' \right\rangle_{g,\beta}$$
$$+ \sum_{\substack{\|\omega'\| \leq \|\omega\| \\ \deg(\mu') \geq \deg(\mu) \\ \deg(\nu') \geq \deg(\nu)+1 \\ m \geq 0}} C_{\mu',\nu',\omega'} \left\langle \mu' \left| \tau_0([D_0] \cdot H \cdot c_1(L)^m) \cdot \omega' \right| \nu' \right\rangle_{g,\beta}$$
$$- \sum_{\substack{\|\omega'\| \leq \|\omega\| \\ \deg(\mu') \geq \deg(\mu) \\ \deg(\nu') \geq \deg(\nu) \\ m \geq 0}} C_{\mu',\nu',\omega'} \left\langle \mu' \left| \tau_0([D_0] \cdot c_1(L)^{m+1} \cdot \delta) \cdot \omega' \right| \nu' \right\rangle_{g,\beta} + \cdots$$

where $H \in H^2(X, \mathbb{Q})$ is an ample class and the coefficients $C_{*,*,*}$ are fiber class integrals. The dots stand for non-principal terms of type II and integrals in the Gromov–Witten theory of X

Proof. The first step is to use the basic divisor relation in $H^2(Y, \mathbb{Q})$,

$$[D_0] = [D_\infty] - c_1(L),$$

to rewrite the invariant on the left side of Relation 2:

$$\left\langle \mu \left| \tau_0([D_0] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta} = \left\langle \mu \left| \tau_0([D_\infty] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta}$$
$$- \left\langle \mu \left| \tau_0(c_1(L) \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta}.$$

We will calculate the two latter type I invariants by localization on (Y, D_0).

The fiberwise $\mathbb{C}^*$-action on Y induces a canonical action on the moduli space of stable relative maps to the pair (Y, D_0). The $\mathbb{C}^*$-fixed loci consist of moduli spaces of stable maps to rubber over D_0 connected by fiberwise rational Galois covers to moduli spaces of stable maps to D_∞. The connection data of the Galois covers is described by a sum over cohomology weighted partitions η specifying the rubber relative conditions on the connecting divisor.

We first study the localization calculation of the type I invariant

$$\left\langle \mu \left| \tau_0([D_\infty] \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta}. \tag{15}$$

The insertions of (15) all have canonical equivariant lifts. The marked points not included in ω are all distributed to the moduli space of stable maps to D_∞. By the localization formula, the contribution of the latter moduli space is simply a Hodge integral in the Gromov–Witten theory of X. The Hodge class may be removed by [3].

Since we neglect lower terms, we only need to consider $\mathbb{C}^*$-fixed loci for which the maps to rubber are of genus g and class β. Let C_0 be the subcurve of the domain mapping to rubber, and let C_∞ be the subcurve mapping to D_∞. Since all components of C_∞ are contracted, the argument used for rational fibers in the proof of Relation 1 shows

$$\deg(\eta) \geq \deg(\nu) + \deg(\delta).$$

The principal terms of the localization formula for (15) are therefore of the form

$$\left\langle \mu \left| \omega' \ \Psi_\infty^k \right| \eta \right\rangle_{g,\beta}^{\sim}$$

where $\|\omega'\| \leq \|\omega\|$ and $\deg(\eta) \geq \deg(\nu) + 1$.

By Lemma 3, the invariant (15) contributes only principal terms of the type of the second summand on the right side of Relation 2.

Next, we study the localization calculation of the type I invariant

$$\left\langle \mu \left| \tau_0(c_1(L) \cdot \delta) \cdot \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta}. \tag{16}$$

If the insertion $\tau_0(c_1(L) \cdot \delta)$ is distributed to D_∞, then, as above, we only obtain principal terms of the type of the second summand on the right of Relation 2.

If the insertion $\tau_0(c_1(L) \cdot \delta)$ is distributed to D_0, then, by the rubber calculus, we only obtain principal terms of the type of the first and third summands on the right of Relation 2. □

We will also require a version of Relation 2 *without* the distinguished insertion $\tau_0([D_0] \cdot \delta)$. The proof is identical.

Relation 2′. *We have*

$$\left\langle \mu \left| \omega \prod_j \tau_{\nu_j - 1}([D_\infty] \cdot \delta_{s_j}) \right. \right\rangle_{g,\beta} = \sum_{\substack{\|\omega'\| \leq \|\omega\| \\ \deg(\mu') \geq \deg(\mu) \\ \deg(\nu') \geq \deg(\nu) \\ m \geq 0}} C_{\mu',\nu',\omega'} \left\langle \mu' | \tau_0([D_0] \cdot H \cdot c_1(L)^m) \cdot \omega' | \nu' \right\rangle_{g,\beta} + \cdots,$$

where the dots stand for non-principal terms of type II and integrals in the Gromov–Witten theory of X.

1.7. Proof of Theorem 1

Our primary induction is on the pair (g, β) where

$$(g', \beta') < (g, \beta)$$

if $\beta' < \beta$ or if $\beta' = \beta$ and $g' < g$. If $\pi_*(\beta) = 0$, the invariants are fiber class and are determined for all g by Section 1.2.

By a secondary induction on the $\overset{\circ}{<}$ ordering, Relations 1, 2, and 2′ determine all distinguished invariants of type II. Consider an invariant

$$\langle \mu | \tau_0([D_0] \cdot \delta) \cdot \omega | \nu \rangle_{g,\beta} \tag{17}$$

for which $\pi_*(\beta) \neq 0$. We apply Relation 1 to (17). To the first term on the right, we apply Relation 2. To the third term on the right, we apply relation Relation 2′. The outcome modulo the primary induction is a determination of (17) in terms of distinguished type II invariants lower in the $\overset{\circ}{<}$ ordering. The proof of Theorem 1 for distinguished type II invariants is complete.

By localization and the rubber calculus, every type I and II invariant can be expressed in terms of distinguished type II invariants and Hodge integrals in the Gromov–Witten theory of X. □

1.8. Connected/disconnected invariants

For simplicity, we have assumed the disconnected invariants of the pairs (Y, D_0), (Y, D_∞), and $(Y, D_0 \cup D_\infty)$ factor as a product of connected invariants. Unfortunately, the product rule for disconnected invariants has not been included in the foundational treatments of the subject.

The entire proof of Theorem 1 is valid *without* assuming the product rule. The induction argument reduces the disconnected invariants of the three relative pairs to the Gromov–Witten theory of X.

If the product rule is not assumed, the only difference in the proof of Theorem 1 occurs in the treatment of the fiber class integrals in Section 1.2. Disconnected fiber class invariants must be considered. Let

$$P = \bigcup_{i=1}^{n} \mathbb{P}^1$$

be a disconnected set of projective lines. Denote the disconnected divisors

$$\{0, \ldots, 0\}, \qquad \{\infty, \ldots, \infty\} \subset P$$

by D_0 and D_∞. The disconnected fiber class invariants require the computation of the $(\mathbb{C}^*)^n$-equivariant Gromov–Witten theory of the pairs

$$(P, D_0), \qquad (P, D_\infty), \qquad (P, D_0 \cup D_\infty).$$

An elementary $(\mathbb{C}^*)^n$-localization argument reduces the study of P to the product of n copies of $\mathbb{P}^1$ — as predicted by the product rule. We leave the details to the reader.

Indeed, our study of relative Gromov–Witten theory can be used to *prove* the product rule. The argument will be presented elsewhere.

2. Relative in terms of absolute

2.1. Notation

Let V be a nonsingular projective variety containing a nonsingular divisor W,

$$\iota : W \to V.$$

Let ι^* denote the restriction map on cohomology,

$$\iota^* : H^*(V, \mathbb{Q}) \to H^*(W, \mathbb{Q}).$$

The cohomological push-forward

$$\iota_* : H^*(W, \mathbb{Q}) \to H^*(V, \mathbb{Q})$$

is determined by the restriction map ι^* and Poincaré duality.

Let N be the normal bundle of W in V. Since

$$c_1(N) = \iota^*(c_1(T_V)) - c_1(T_W) \in H^*(W, \mathbb{Q}),$$

the Chern class $c_1(N)$ is also determined by the restriction map ι^*.

We denote the Gromov–Witten invariants of the pair (V, W) by the right bracket

$$\left\langle \prod_i \tau_{k_i}(\gamma_{l_i}) \middle| \nu \right\rangle_{g,\beta}^{V/W} \tag{18}$$

where

$$\{\gamma_1, \ldots, \gamma_{m_V}\}, \ \{\delta_1, \ldots, \delta_{m_W}\}$$

are bases of $H^*(V, \mathbb{Q})$ and $H^*(W, \mathbb{Q})$,

$$\nu = \left\{(\nu_1, \delta_{s_1}), \ldots, (\nu_{\ell(\nu)}, \delta_{s_{\ell(\nu)}})\right\},$$

and $\sum_j \nu_j = \int_\beta [W]$.

The above bracket denotes Gromov–Witten invariants with connected domain curves. As before, we treat disconnected invariants as products of connected invariants. Our proof of Theorem 2 is valid without assuming the product rule — see the discussion of Section 1.8.

2.2. *Degeneration*

Let $\mathcal{F}$ be the degeneration to the normal cone of W. The degeneration formula [2,12,14,15] applied to $\mathcal{F}$ expresses the absolute Gromov–Witten theory of V in terms of the relative theories of (V, W) and $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$.

By Theorem 1, the relative theory of $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$ is determined by the absolute theory of W and the Chern class $c_1(N) \in H^2(W, \mathbb{Q})$. In order to prove Theorem 2, we view the degeneration formula as providing equations for the invariants (18) in terms of the Gromov–Witten theories of V and $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$.

2.3. *Ordering*

We partially order the invariants of (V, W) by a relation very similar to the ordering of distinguished type II invariants of Section 1.3:

$$\langle \omega' \mid \nu' \rangle_{g',\beta'} \overset{\circ}{<} \langle \omega \mid \nu \rangle_{g,\beta}$$

if one of the conditions below holds

(1) $\beta' < \beta$,
(2) equality in (1) and $g' < g$,
(3) equality in (1)–(2) and $\|\omega'\| < \|\omega\|$,
(4) equality in (1)–(3) and $\deg(\nu') > \deg(\nu)$,
(5) equality in (1)–(4) and $\nu' \overset{l}{>} \nu$.

Here, ω' and ω represent products of descendent insertions. For any given invariant of (V, W), there are only finitely many invariants of (V, W) lower in the partial ordering.

The degeneration equations for the invariants (V, W) will be proven to be lower triangular with respect to the $\overset{\circ}{<}$ ordering — and therefore nonsingular.

2.4. *Proof of Theorem 2*

To each relative invariant (18) of the pair (V, W), we associate the following absolute invariant of V

$$\left\langle \prod_i \tau_{k_i}(\gamma_{l_i}) \cdot \prod_j \tau_{\nu_j - 1}\left(\iota_*(\delta_{s_j})\right) \right\rangle_{g,\beta}^{V}. \tag{19}$$

In order to evaluate the absolute invariant (19) by the degeneration formula, we must lift the cohomology classes γ_{l_i} and $\iota_*(\delta_{s_j})$ to the total space of the family $\mathcal{F}$:

(i) the classes γ_{l_i} are lifted by pull-back via the first factor of the blow-down map

$$\mathcal{F} \to V \times \mathbb{C},$$

(ii) the classes $\iota_*(\delta_{s_j})$ are lifted by

$$\iota_{W\times\mathbb{C},*}(\delta_{s_j} \otimes Id) \in H^*(\mathcal{F}, \mathbb{Q})$$

where

$$\iota_{W\times\mathbb{C}} : W \times \mathbb{C} \to \mathcal{F}$$

is the inclusion in the blow-up via strict transform.

Lemma 4. *The principal terms of the degeneration equation are*

$$\left\langle \prod_i \tau_{k_i}(\gamma_{l_i}) \cdot \prod_j \tau_{\nu_j - 1}\left(\iota_*(\delta_{s_j})\right) \right\rangle_{g,\beta}^{V} = \left\langle \prod_i \tau_{k_i}(\gamma_{l_i}) \middle| \nu \right\rangle_{g,\beta}^{V/W} C + \sum \langle \omega' \mid \nu' \rangle_{g,\beta}^{V/W} C_{\omega',\nu'} + \cdots,$$

where

$$C = \prod_j \frac{1}{(\nu_j - 1)!} \left(\int_\beta [W] \right)^{Id(\nu)} \neq 0,$$

the coefficients $C_{*,*}$ are fiber class integrals, and the sum is over lower invariants in the $\overset{\circ}{<}$ ordering.

Following the notation of Section 1.4, the *principal terms* of the equation are the invariants of (V, W) of genus g and class β.

Proof. The strategy is identical to the proof of Relation 1. We need only consider degeneration splittings for which the (V, W) side carries genus g, class β, and *all* the insertions

$$\prod_i \tau_{k_i}(\gamma_{l_i}).$$

By the lifting choice, all the insertions

$$\prod_j \tau_{\nu_j - 1}\left(\iota_*(\delta_{s_j})\right)$$

are distributed to the $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$ side. The analysis of the rational fiber class integrals then exactly follows Case 1 of the proof of Relation 1. □

By Lemma 4, the degeneration equations form a lower triangular system determining the invariants of (V, W) in terms of the invariants of V and $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$. The proof of Theorem 2 is complete. □

2.5. *Proof of Corollary 1*

Let (V, W) be a hypersurface pair in $\mathbb{P}^r$. We first prove

$$\iota_*(\delta) \in H^*(V, \mathbb{Q})$$

is simple for any $\delta \in H^*(W, \mathbb{Q})$. If δ is simple, the result is clear. If δ is not simple, then

$$\deg(\delta) = r - 2$$

by Lefchetz. We conclude

$$\deg(\iota_*(\delta)) = r,$$

and thus $\iota_*(\delta)$ is simple by Lefchetz.

Consider the degeneration equation of Lemma 4 associated to a simple invariant of (V, W). The left side involves a simple invariant of V since $\iota_*(\delta)$ is always simple. The right side involves simple invariants of (V, W) and general invariants of $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$.

The degeneration equations form a lower triangular system determining the simple invariants of (V, W) in terms of the simple invariants of V and general invariants of $(\mathbb{P}(N \oplus \mathcal{O}_W), W)$. □

3. Mayer–Vietoris and the quintic scheme

3.1. *Proof of Lemma 1*

Our calculation scheme for hypersurfaces depends upon Theorem 3 and Lemma 1. Theorem 3 is an immediate consequence of the degeneration formula and Theorem 2.

Let V be a nonsingular, projective variety. Let $Z \subset V$ be the nonsingular complete intersection of two nonsingular divisors

$$W_1, W_2 \subset V,$$

and let $\widetilde{V}$ be the blow-up of V along Z.

Lemma 1. *The Gromov–Witten theory of $\widetilde{V}$ is uniquely and effectively determined by the Gromov–Witten theories of V, W_1, and Z and the restriction maps*

$$H^*(V,\mathbb{Q}) \to H^*(W_1,\mathbb{Q}) \to H^*(Z,\mathbb{Q}).$$

Proof. Let N_i be the normal bundle to W_i in V. Let $\mathcal{F}$ be the degeneration to the normal cone of W_1 in V. By strict transform, there is an inclusion

$$\iota_{W_1\times\mathbb{C}}: W_1 \times \mathbb{C} \to \mathcal{F}.$$

Let $\widetilde{\mathcal{F}}$ be the blow-up of $\mathcal{F}$ along $Z \times \mathbb{C}$ embedded via

$$Z \times \mathbb{C} \subset W_1 \times \mathbb{C} \stackrel{\iota_{W_1\times\mathbb{C}}}{\longrightarrow} \mathcal{F}.$$

The family $\mathcal{F}$ is a degeneration of V to (V, W_1) and $(\mathbb{P}(N_1 \oplus \mathcal{O}_{W_1}), D_0)$. The family $\widetilde{\mathcal{F}}$ is a degeneration of $\widetilde{V}$ to (V, W_1) and $(\widetilde{\mathbb{P}}(N_1\oplus\mathcal{O}_{W_1}), D_0)$ where $\widetilde{\mathbb{P}}(N_1\oplus\mathcal{O}_{W_1})$ is the blow-up of $\mathbb{P}(N_1\oplus\mathcal{O}_{W_1})$ along $Z \subset D_\infty$.

By the degeneration formula, the Gromov–Witten theory of $\widetilde{V}$ is determined by the Gromov–Witten theories of (V, W_1) and $(\widetilde{\mathbb{P}}(N_1 \oplus \mathcal{O}_{W_1}), D_0)$ since the nonvanishing cohomology for the family $\widetilde{\mathcal{F}}$ is all of $H^*(\widetilde{V},\mathbb{Q})$. By Theorem 2, the two relative theories are determined by the Gromov–Witten theories of V, W_1, and $\widetilde{\mathbb{P}}(N_1 \oplus \mathcal{O}_{W_1})$ and the classical restriction maps.

The projective bundle $\mathbb{P}(N_1|_Z \oplus \mathcal{O}_Z)$ over Z is a divisor in $\mathbb{P}(N_1 \oplus \mathcal{O}_{W_1})$ containing the center $Z \subset D_\infty$ of the blow-up. The normal bundle of

$$\mathbb{P}(N_1|_Z \oplus \mathcal{O}_Z) \subset \mathbb{P}(N_1 \oplus \mathcal{O}_{W_1})$$

is the pull-back of $N_2|_Z$.

By repeating the first construction of the proof, we find the Gromov–Witten theory of $\widetilde{\mathbb{P}}(N_1 \oplus \mathcal{O}_{W_1})$ is determined by the Gromov–Witten theories of

$$\mathbb{P}(N_1 \oplus \mathcal{O}_{W_1}), \qquad \mathbb{P}(N_1|_Z \oplus \mathcal{O}_Z),$$

and the blow-up of

$$\mathbb{P}(N_1|_Z \oplus \mathcal{O}_Z) \times_Z \mathbb{P}(N_2|_Z \oplus \mathcal{O}_Z)$$

along Z embedded as $D_\infty \times_Z D_\infty$.

Finally, the last blown-up variety can be studied via the virtual localization formula [10]. The variety

$$\mathbb{P}(N_1|_Z \oplus \mathcal{O}_Z) \times_Z \mathbb{P}(N_2|_Z \oplus \mathcal{O}_Z) \tag{20}$$

carries a fiberwise $\mathbb{C}^* \times \mathbb{C}^*$-action over Z. The action lifts to the blow-up of (20) along the $\mathbb{C}^* \times \mathbb{C}^*$-fixed locus $D_\infty \times_Z D_\infty$. The $\mathbb{C}^* \times \mathbb{C}^*$-action on the blown-up space has 5 fixed loci — each isomorphic to Z. The localization formula reduces the Gromov–Witten invariants of the blown-up space to Hodge integrals in the Gromov–Witten theory of Z. □

A direct generalization of the proof of Lemma 1 yields the following related result. Let $Z \subset V$ be the nonsingular complete intersection of n nonsingular divisors

$$W_1, \ldots, W_n \subset V,$$

and let $\widetilde{V}$ be the blow-up of V along Z. The Gromov–Witten theory of $\widetilde{V}$ is determined by the Gromov–Witten theories of

$$V, W_1, W_1 \cap W_2, W_1 \cap W_2 \cap W_3, \ldots, \cap_{i=1}^n W_i = Z$$

and the classical restriction maps.

Conjecture 2. *The Gromov–Witten theory of $\widetilde{V}$ is uniquely and effectively determined by the Gromov–Witten theories of V and Z and the restriction map $H^*(V, \mathbb{Q}) \to H^*(Z, \mathbb{Q})$.*

3.2. The Calabi–Yau quintic

3.2.1. Notation

The following notation for curves, surfaces, and 3-folds will be convenient for the study of the quintic:

(i) let $C_{d_1,d_2} \subset \mathbb{P}^3$ be a nonsingular complete intersection of type (d_1, d_2),
(ii) let $S_d \subset \mathbb{P}^3$ be a nonsingular surface of degree d,
(iii) let $T_d \subset \mathbb{P}^4$ be a nonsingular 3-fold of degree d.

Finally, let $\mathbb{P}^3[d_1, d_2]$ be the blow-up of $\mathbb{P}^3$ along C_{d_1,d_2}.

3.2.2. The quintic scheme

The calculation scheme for the quintic is diagrammed below by arrows showing the dependencies of Gromov–Witten theories.

(a) The superscript $*$ denotes simple Gromov–Witten theories,
(b) the arrow $\xrightarrow{k}$ denotes the application of Theorem k in simple or full form,
(c) the arrow $\xrightarrow{l}$ denotes the application of Lemma 1.

The quintic scheme:

$$T_5^* \xrightarrow{3} (T_4, S_4)^*, (\mathbb{P}^3[4,5], S_4),$$
$$(T_4, S_4)^* \xrightarrow{2} T_4^*, S_4,$$
$$T_4^* \xrightarrow{3} (T_3, S_3)^*, (\mathbb{P}^3[3,4], S_3)$$
$$(T_3, S_3)^* \xrightarrow{2} T_3^*, S_3,$$
$$T_3^* \xrightarrow{3} (T_2, S_2)^*, (\mathbb{P}^3[2,3], S_2)$$
$$(T_2, S_2)^* \xrightarrow{2} T_2^*, S_2,$$
$$T_2^* \xrightarrow{3} (T_1, S_1)^*, (\mathbb{P}^3[1,2], S_1)$$
$$(T_1, S_1)^* \xrightarrow{2} T_1^*, S_1,$$
$$(\mathbb{P}^3[4,5], S_4) \xrightarrow{2} \mathbb{P}^3[4,5], S_4,$$
$$\mathbb{P}^3[4,5] \xrightarrow{l} \mathbb{P}^3, S_4, C_{4,5},$$
$$(\mathbb{P}^3[3,4], S_3) \xrightarrow{2} \mathbb{P}^3[3,4], S_3,$$

$$\mathbb{P}^3[3,4] \xrightarrow{l} \mathbb{P}^3, S_3, C_{3,4},$$
$$(\mathbb{P}^3[2,3], S_2) \xrightarrow{2} \mathbb{P}^3[2,3], S_2,$$
$$\mathbb{P}^3[2,3] \xrightarrow{l} \mathbb{P}^3, S_2, C_{2,3},$$
$$(\mathbb{P}^3[1,2], S_1) \xrightarrow{2} \mathbb{P}^3[1,2], S_1,$$
$$\mathbb{P}^3[1,2] \xrightarrow{l} \mathbb{P}^3, S_1, C_{1,2}.$$

The end points of the quintic scheme are the following absolute Gromov–Witten theories:

$$\mathbb{P}^3, \mathbb{P}^2, S_2, S_3, S_4, C_{1,2}, C_{2,3}, C_{3,4}, C_{4,5},$$

all previously determined.

3.3. *The quintic surface S_5*

We end the paper with a basic calculation for the quintic surface

$$S_5 \subset \mathbb{P}^3.$$

Let K denote the canonical class of S_5. The adjunction genus of curves in class K is 6. The expected dimension of genus 6 curves of class K is 0. By Taubes' result equating Gromov and Seiberg–Witten invariants,

$$\langle 1\rangle^{S_5}_{6,K} = SW(-P_{S_5}).$$

Since S_5 is a minimal surface of general type,

$$SW(-P_{S_5}) = (-1)^{(1+p_g(S_5)+q(S_5))} = -1.$$

Here, P_{S_5} is the Spin^c-structure induced by the complex structure of S_5 and p_g and q are the geometric genus and irregularity of the surface S_5 [19]. We will calculate $\langle 1\rangle^{S_5}_{6,K}$ directly via our method.

The first step is the degeneration of the quintic to a union of a K3 surface and a blown-up projective plane along a plane quartic curve:

$$S_5^* \to (S_4, C_4)^*, (\mathbb{P}^2[4,5], C_4).$$

Let B denote $\mathbb{P}^2[4,5]$, the blow-up of $\mathbb{P}^2$ in 20 points. The degeneration formula yields

$$\langle 1\rangle^{S_5}_{6,K} = \sum_{g_i,\beta_i,\mu} \langle 1 \mid \mu\rangle^{S_4/C_4}_{g_1,\beta_1} \langle \mu^\vee \mid 1\rangle^{B/C_4}_{g_2,\beta_2} \tag{21}$$

where the summation is over all possible genus splittings $g_1 + g_2 + \ell(\mu) - 1 = 6$, class splitting $\beta_1 + \beta_2 = K$, and cohomology-weighted partitions μ.

Let H, L denote the hyperplane classes on S_4 and B respectively and let E_i denote the class of the ith exceptional divisor of B.

Although all configurations on the right side of (21) can be treated algorithmically, most can be easily ruled out without any computation. For example, configurations with $g_1 = 5$, $\beta_1 = H$, require four fixed point condition in μ_1. However, since the dimension of the linear system $|H|$ is 3, the solution set will be empty.

For configurations with $\beta_1 = 0$, we can show only

$$\beta_2 = 5L - \sum_{i=1}^{20} E_i$$

is allowed as follows. Consider a configuration with

$$\beta_2 = 5L - 2E_1 - E_3 - \cdots - E_{20}.$$

A curve mapped to B gives a section of $\mathcal{O}_C(5)$ on C with divisor

$$2p_1 + p_3 + \cdots + p_{20}.$$

However, a monodromy argument shows that such a linear equivalence is impossible for a generic choice of

$$p_1 + \cdots + p_{20} \in |\mathcal{O}_C(5)|.$$

Finally, for many configurations, the relative invariant for (B, C_4) is easily seen to vanish. For instance, when $g_1 = 3$, $\beta_2 = L$ with

$$\mu = \{(2, [p]), (1, 1), (1, 1)\},$$

the relative invariant for the curve mapped to B is the number of tangents to $C_4 \subset \mathbf{P}^2$ which pass through two fixed generic points.

After ruling out these easy cases, there is the single $\beta_1 = 0$ case discussed above and five configurations with $\beta_1 = H$. For $g_1 = 3$, the possible partitions with $\beta_1 = H$ are

$$\mu_1 = \{(1, [p]), (1, [p]), (1, 1), (1, 1)\},$$
$$\mu_2 = \{(1, \alpha), (1, \alpha^\vee), (1, [p]), (1, 1)\},$$
$$\mu_3 = \{(1, \alpha_1), (1, \alpha_1^\vee), (1, \alpha_2), (1, \alpha_2^\vee)\}$$

where $[p] \in H^2(C, \mathbb{Z})$ is the Poincaré-dual class to a point and $\alpha_i, \alpha_i^\vee$ are basis elements of $H^1(C, \mathbb{Z})$. For $g_1 = 4$, the possible partitions are

$$\nu_1 = \{(2, [p]), (1, [p]), (1, 1)\},$$
$$\nu_2 = \{(2, 1), (1, [p]), (1, [p])\}.$$

In each of these cases, we have $g_2 = 0$ and $\beta_2 = L$ for the curve mapped to B.

For each relative invariant of S_4, we study the associated absolute invariant on S_4 via the degeneration to

$$S_4 \cup_{C_4} \mathbf{P}(K_{C_4} \oplus \mathcal{O}_{C_4}).$$

We will denote the latter projective bundle by P and the sections corresponding to the summands K_C and $\mathcal{O}_C$ by D_0 and D_∞. For $g_1 = 3$, we obtain the equations

$$\langle \tau_0([p])\tau_0([p]) \rangle^{S_4}_{3,H} = \langle 1 \mid \mu_1 \rangle^{S_4/C_4}_{3,H} + \langle (0) \mid \tau_0([p])\tau_0([p]) \rangle^{P/D_\infty}_{3,[D_0]}, \tag{22}$$

$$\begin{aligned} &\langle \tau_0(\iota_*\alpha)\tau_0(\iota_*\alpha^\vee)\tau_0([p]) \rangle^{S_4}_{3,H} \\ &\quad = \langle 1 \mid \mu_2 \rangle^{S_4/C_4}_{3,H} + \langle 1 \mid \mu_1 \rangle^{S_4/C_4}_{3,H} + \langle (0) \mid \tau_0(\iota_*\alpha)\tau_0(\iota_*\alpha^\vee)\tau_0([p]) \rangle^{P/D_\infty}_{3,[D_0]}, \end{aligned} \tag{23}$$

and

$$\langle \tau_0(\iota_*\alpha_1)\tau_0(\iota_*\alpha_1^\vee)\tau_0(\iota_*\alpha_2)\tau_0(\iota_*\alpha_2^\vee)\rangle^{S_4}_{3,H} = \langle 1 \mid \mu_3\rangle^{S_4/C_4}_{3,H} + 2\langle 1 \mid \mu_2\rangle^{S_4/C_4}_{3,H} + \langle 1 \mid \mu_1\rangle^{S_4/C_4}_{3,H} + \langle (0) \mid \tau_0(\iota_*\alpha_1)\tau_0(\iota_*\alpha_1^\vee)\tau_0(\iota_*\alpha_2)\tau_0(\iota_*\alpha_2^\vee)\rangle^{P/D_\infty}_{3,[D_0]}. \tag{24}$$

Since there exist Kahler deformations of the K3 surface S_4 with no embedded curves, all absolute invariants vanish. The relative invariants of (P, C_∞) are immediately calculated via relative localization to yield 1, 0, and 0 respectively. The genus 3 relative invariants are

$$\langle 1 \mid \mu_1\rangle^{S_4/C_4}_{3,H} = -1, \qquad \langle 1 \mid \mu_2\rangle^{S_4/C_4}_{3,H} = 1, \qquad \langle 1 \mid \mu_3\rangle^{S_4/C_4}_{3,H} = -1.$$

The same procedure for the two genus 4 invariants shows they both vanish.

It remains to compute the relative invariants of the pair (B, C_4). For the relative invariants $\langle \mu_i^\vee \mid 1\rangle^{B/C_4}_{0,L}$, we argue directly. The moduli space

$$\overline{M}_{0,4}(B/C_4, (1, 1, 1, 1))$$

is a blowup of the two dimensional space

$$Z = \left\{(p_1, p_2, p_3, p_4) \in C \times C \times C \times C \mid \mathcal{O}_C\left(\sum p_i\right) \cong K_C\right\}.$$

Since the relative conditions are pulled back from C^4, we only need to cap these classes with the fundamental class of Z in $H_2(C^4, \mathbb{Z})$. The latter class is given by a degeneracy locus calculation, $[Z]$ equals the second Chern class of the rank 4 bundle whose fiber at (p_1, p_2, p_3, p_4) is $\bigoplus K|_{\sum p_i}$. The invariants are

$$\langle \mu_1^\vee \mid 1\rangle^{B/C_4}_{0,L} = 1, \qquad \langle \mu_2^\vee \mid 1\rangle^{B/C_4}_{0,L} = 1, \qquad \langle \mu_3^\vee \mid 1\rangle^{B/C_4}_{0,L} = 1.$$

The remaining contribution is the relative invariant for (B, C_4) where the entire genus 6 curve is mapped to B with homology class $\beta = 5H - \sum_{i=1}^{20} E_i$. The corresponding absolute invariant is

$$\langle 1\rangle^B_{6,\beta} = 1. \tag{25}$$

The invariant (25) is computed by considering the blow-up of $\mathbb{P}^2$ at 20 *general* points. Degenerating along C_4 gives the following relation between the relative invariants for (B, C_4) and (P, D_0)

$$1 = \langle 1\rangle^B_{6,\beta} = \sum_{g_i,\beta_i,\mu} \langle 1 \mid \mu\rangle^{P/D_0}_{g_1,\beta_1}\langle \mu^\vee \mid 1\rangle^{B/C_4}_{g_2,\beta_2}, \tag{26}$$

where again we sum over all configurations.

The situation is identical to our degeneration of S_5 with P in place of S_4. In particular, the same arguments allow us to reduce to the same five configurations with $g_1 = 3$ or 4, in addition to the configuration where the entire curve is mapped to B. Moreover, the relative invariants are computed by the same set of Eqs. (22)–(24) with P instead of S_4. The only difference is the absolute invariants on P no longer vanish identically and must be computed directly by localization. Omitting the details, we find for genus 3, the relative invariants are

$$\langle 1 \mid \mu_1\rangle^{P/D_0}_{3,[D_0]} = 7, \qquad \langle 1 \mid \mu_2\rangle^{P/D_0}_{3,[D_0]} = -3, \qquad \langle 1 \mid \mu_3\rangle^{P/D_0}_{3,[D_0]} = 1.$$

For genus 4, the relative invariants again vanish.

Finally, if we subtract equation Eq. (26) from equation Eq. (21), we have

$$\begin{aligned}\langle 1\rangle^{S_5}_{6,K} &= 1+\sum_i \left(\langle 1 \mid \mu_i\rangle^{S_4/C_4}_{3,H} - \langle 1 \mid \mu_i\rangle^{P/D_0}_{3,[D_0]}\right)\langle \mu_i^\vee \mid 1\rangle^{B/C_4}_{0,L}\\ &= 1+(-1-7)\cdot 1+3(1-(-3))\cdot 1+3(-1-1)\cdot 1\\ &= -1\end{aligned}$$

where the factors of 3 arise from the different combinations of odd cohomology conditions $\alpha \in H^1(C,\mathbb{Z})$.

Acknowledgments

We thank J. Bryan for many helpful discussions. In particular, the quintic surface calculation in Section 3.3 was motivated by conversations with him. We have also benefitted from discussions with T. Graber, E. Katz, A. Okounkov, and A. Zinger.

D.M. was partially supported by an NSF graduate fellowship. R.P. was partially supported by the NSF and the Packard Foundation.

References

[1] J. Bryan, N.C. Leung, The enumerative geometry of $K3$ surfaces and modular forms, J. Amer. Math. Soc. 13 (2000) 371–410.
[2] Y. Eliashberg, A. Givental, H. Hofer, Introduction to symplectic field theory, GAFA (2000) 560–673.
[3] C. Faber, R. Pandharipande, Hodge integrals and Gromov–Witten theory, Invent. Math. 139 (2000) 173–199.
[4] C. Faber, R. Pandharipande, Relative maps and tautological classes, J. Eur. Math. Soc. 7 (2005) 13–49.
[5] A. Gathmann, Absolute and relative invariants of hypersurfaces, Duke Math. J. 115 (2002) 171–203.
[6] A. Gathmann, Relative Gromov–Witten invariants and the mirror formula, Math. Ann. 325 (2003) 393–412.
[7] A. Gathmann, Topological recursion relations and Gromov–Witten invariants of higher genus, 2003 (preprint).
[8] A. Gathmann, Gromov–Witten invariants of hypersurfaces, Habilitation Thesis, Univ. of Kaiserslautern, 2003.
[9] A. Givental, Semisimple Frobenius manifolds at higher genus, IMRN (2001) 1265–1286.
[10] T. Graber, R. Pandharipande, Localization of virtual classes, Invent. Math. 135 (1999) 487–518.
[11] T. Graber, R. Vakil, Relative virtual localization and vanishing of tautological classes on moduli spaces of curves, 2003 (preprint).
[12] E. Ionel, T. Parker, Relative Gromov–Witten invariants, Ann. of Math. 157 (2003) 45–96.
[13] J. Lee, T. Parker, A structure theorem for the Gromov–Witten invariants of Kähler surfaces, 2005 (preprint).
[14] A.-M. Li, Y. Ruan, Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds I, Invent. Math. 145 (1) (2001) 151–218.
[15] J. Li, A degeneration formula for Gromov–Witten invariants, J. Differential Geom. 60 (2002) 199–293.
[16] J. Li, A. Zinger, On the genus 1 Gromov–Witten invariants of complete intersection threefolds, 2004 (preprint).
[17] D. Maulik, Hodge integrals and relative Gromov–Witten theory (in preparation).
[18] D. Maulik, R. Pandharipande, New calculations in Gromov–Witten theory, math.AG/0601395.
[19] J. Morgan, The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-manifolds, Princeton University Press, 1996.
[20] A. Okounkov, R. Pandharipande, Gromov–Witten theory, Hurwitz numbers, and completed cycles, Ann. of Math. (in press).
[21] A. Okounkov, R. Pandharipande, The equivariant Gromov–Witten theory of $\mathbb{P}^1$, Ann. of Math. (in press).
[22] A. Okounkov, R. Pandharipande, Virasoro constraints for target curves, Invent. Math. (in press).
[23] C. Taubes, SW $\Rightarrow$ Gr: from the Seiberg–Witten equations to pseudo-holomorphic curves, J. Amer. Math. Soc. 9 (1996) 845–918.

[24] C. Taubes, Counting pseudo-holomorphic submanifolds in dimension 4, J. Differential Geom. 44 (1996) 818–893.
[25] C. Taubes, Gr $\Rightarrow$ SW: from pseudo-holomorphic curves to Seiberg–Witten solutions, J. Differential Geom. 51 (1999) 203–334.
[26] C. Taubes, GR $=$ SW: counting curves and connections, J. Differential Geom. 52 (1999) 453–609.
[27] R. Vakil, A. Zinger, in preparation.
[28] A. Zinger, On a compactification of the moduli space of holomorphic maps from smooth genus 1 Riemann surfaces, 2004 (preprint).
[29] A. Zinger, On the structure of certain natural cones over moduli spaces of genus 1 holomorphic maps, 2004 (preprint).