Review of *Combinatorial Matrix Theory* by Richard A. Brualdi and Herbert J. Ryser*

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Herb Ryser concluded his Carus Monograph [2] by stating, "We believe that the greatest truths of combinatorial analysis are yet to be revealed." In a review of the same monograph G.-C. Rota [1] conjectured that "the next decades will witness an explosion of combinatorial activity." Now, nearly 30 years after these statements were made, we marvel at the numerous advances in combinatorial mathematics, and yet we find that these thoughts are just as applicable today.

Combinatorial Matrix Theory celebrates the success of combinatorial mathematics by highlighting the beneficial relationship between matrix theory and combinatorics. The authors view combinatorial matrix theory as "the use of combinatorics in matrix theory (and vice versa) and the study of intrinsic properties of matrices viewed as arrays of numbers rather than algebraic objects." This view is fostered by their early chapters on Incidence Matrices, Matrices and Graphs, Matrices and Digraphs, Matrices and Bipartite Graphs, and Some Special Graphs; and is further advanced by chapters on Network Flows, the Permanent, Latin Squares, and Combinatorial Matrix Algebra.

The introductory chapter exhibits the power of combinatorial matrix theory. With a minimum of definitions, the authors show how basic tools of matrix theory lead to fundamental results related to (0, 1) matrices, designs, projective planes, systems of distinct representatives, and doubly stochastic matrices.

The authors have wisely elected to include the most fundamental results

*Encyclopedia of Mathematics and Its Applications, Vol. 39 (G.-C. Rota, Ed.), Cambridge University Press, 40 West 20th Street, New York, NY 10011.

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655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

from each topic. Yet each chapter guides the reader from basic definitions to the frontiers of research. For example, the second chapter, entitled Matrices and Graphs, covers much of standard algebraic graph theory, the rudiments of matching theory, and the theory of Laplacian matrices of graphs. An elegant proof of Tutte's theorem on matchings in general graphs, which is gleaned from the techniques of at least four authors, is presented in Section 2.6.

In their third chapter, Matrices and Digraphs, the authors discuss basic properties of irreducible and nearly reducible matrices, and then use digraphs as a tool to illuminate numerous results in nonnegative matrix theory. The discussions of the index of imprimitivity of a matrix and of exponents of primitive matrices and an excellent section on extensions of the Geršgorin theorem make this chapter extremely useful to graph theorists, matrix theorists, and numerical linear algebraists.

The chapter on matrices and bipartite graphs begins with a concise exposition of the basic structure of fully indecomposable and nearly decomposable matrices. The authors relate this treatment to that of irreducible and nearly reducible matrices presented in the previous chapter. Many seemingly unrelated results are united under the authors' concept of a decomposition theorem. For example, both König's theorem, which gives necessary and sufficient conditions for an m by n $(0,1)$ matrix M to have m independent ones, and Vizing's theorem, which provides an upper bound for the number of colors needed to color the edges of a given graph so that no two incident edges have the same color, can be viewed as decomposition theorems. König's theorem gives necessary and sufficient conditions in order that M can be decomposed as the sum $M = M_1 + M_2$ of $(0,1)$ matrices where M_1 has a unique 1 in each row and at most one 1 in each column; whereas Vizing's theorem concerns decompositions of the edges of a graph into disjoint matchings. Numerous decompositions theorems are stated throughout the book, and Sections 4.4 and 6.4 are exclusively devoted to these types of theorems.

Properties of regular graphs, strongly regular graphs, and generalizations to digraphs are the topics of Chapter 5.

A technique used in proving many existence theorems for matrices (and hence graphs and digraphs) with certain combinatorial properties is based in the theory of network flows. Chapter 6 develops the basic theory of network flows and then utilizes this powerful tool to establish existence theorems for digraphs with prescribed in- and outdegree sequences and for graphs with a prescribed degree sequence. In addition several matrix decomposition theorems are discussed.

The permanent function is the focus of Chapter 7. A self-contained introduction to the classical *problème des recontres* and *problème des ménages* develops the basic properties of the permanent. The section on inequalities exhibits the numerous techniques that are used to study the permanent function. General inequalities as well as inequalities for special classes of matrices [for example, (0, 1) matrices with a constant line sum] are given. The known methods of computation and the complexity of computing the permanent, as well as the evaluation of permanents via determinants, are also discussed.

A latin square of order n can be viewed as n permutation matrices of order n which sum to the all 1's matrix and can also be viewed as a decomposition of the complete bipartite graph $K_{n,n}$ into n perfect matchings. Many constructions of latin squares and mutually orthogonal latin squares are best described in terms of matrices. Thus latin squares are a natural topic of combinatorial matrix theory. Chapter 8 discusses the theory of partial transversals, partial latin squares, and mutually orthogonal and self-orthogonal latin squares, as well as some enumerative results.

The final chapter illustrates the wonderful interplay between algebra and combinatorial properties of matrices. Notions such as term rank, full indecomposability, and irreducibility are shown to have purely algebraic characterizations. In addition classical results such as the Cayley-Hamilton theorem and Jordan canonical form, as well as more recent results on polynomial identities for matrices, are proven by purely combinatorial methods.

An ample number of exercises of varying difficulty are provided for those wishing to use the book as a graduate text. The reference lists alone (one at the end of each section and a master list with over 400 items) make this book a valuable resource. Yet, the book's true worth is found in the authors' careful unification and exposition of the results. Proofs of the classical results are cast in a more modern setting. In numerous instances a novel proof which incorporates the strengths of several known proofs is presented. The book brings together many results that were previously unrelated in the research literature. *Combinatorial Matrix Theory* will have a tremendous impact by encouraging and influencing research for many years to come.

The death of Herb Ryser in 1985 was a loss to the entire mathematical community. While the authors were planning this book, Herb stated that above all this book "should reveal the great power and beauty of matrix theory in combinatorial settings." Any reader will agree that this goal has been brilliantly accomplished. A second book, entitled *Combinatorial Matrix Classes*, is planned, and we expectantly await its completion.

REFERENCES

1 G.-C. Rota, Review in "Recent Publications and Presentations" *Amer. Math. Monthly* 72:211 (1965).
2 H. J. Ryser, *Combinatorial Mathematics*, Carus Math. Monographs, Math. Assoc. Amer., 1963.

Received 9 January 1992; final manuscript accepted 10 January 1992