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Paley–Wiener spaces for real reductive Lie groups

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1. INTRODUCTION

The Paley–Wiener theorem for K-finite compactly supported smooth functions on a real reductive group Lie group G of the Harish-Chandra class is due to J. Arthur [1] in general, and to O.A. Campoli [9] for G of split rank one.

In our paper [7] we established a similar Paley–Wiener theorem for smooth functions on a reductive symmetric space. In this paper we will show that Arthur's theorem is a consequence of our result if one considers the group G as a symmetric space for $G \times G$ with respect to the left times right action. At the same time we will formulate a Paley–Wiener theorem for K-finite *generalized functions* (in the sense of distribution theory) on G, and prove that it is a special case of the Paley–Wiener theorem for symmetric spaces established in our paper [8].

All mentioned Paley–Wiener theorems are formulated in the following spirit. A Fourier transform is defined by means of Eisenstein integrals for the minimal principal series for the group or space under consideration. The Eisenstein integrals depend on a certain spectral parameter and satisfy the so called Arthur–Campoli relations. A Paley–Wiener space is defined as a certain space of meromorphic functions in the spectral parameter, characterized by the Arthur–Campoli relations and by growth estimates. The Paley–Wiener theorem asserts that the Fourier transform is a topological linear isomorphism from a space of K-finite com-

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pactly supported smooth or generalized functions onto a particular Paley–Wiener space.

The Paley–Wiener space of Arthur's paper is defined in terms of Eisenstein integrals as introduced by Harish-Chandra [13]; we shall refer to these integrals as being *unnormalized*. Our Paley–Wiener theorems in [7] and [8] are defined in terms of the so-called *normalized* Eisenstein integrals defined in [3]. For G considered as a symmetric space the normalized Eisenstein integral differs from the unnormalized one. Consequently, the associated Fourier transforms and Paley–Wiener spaces are different. The final objective of this paper is to clarify the relationships between the various Paley–Wiener spaces.

Recently, P. Delorme [12] has proved a different Paley–Wiener theorem, involving the operator-valued Fourier transforms associated with all generalized principal series representations. In his result the Arthur–Campoli relations are replaced by intertwining relations. Moreover, the result is valid without the restriction of K-finiteness.

We shall now give a brief outline of the present paper. In Section 2 we introduce the basic concepts, in particular the space $C_c^\infty(G:\tau)$ of τ-spherical compactly supported smooth functions, for which Arthur's theorem is most conveniently formulated. We review the definition of Harish-Chandra's (unnormalized) Eisenstein integral $E(P:\lambda)$ for P a minimal parabolic subgroup of G and with spectral parameter $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$; here $\mathfrak{a}_0$ is the Lie algebra of the split component A_0 of P. Finally, we give the definition of the associated Fourier transform ${}^u\mathcal{F}_P$.

In Section 3 we recall, in Theorem 3.3, the formulation of Arthur's Paley–Wiener theorem [1]. This theorem deals with the family of Fourier transforms $({}^u\mathcal{F}_Q)_{Q\in\mathcal{P}_0}$ with Q ranging over the finite set $\mathcal{P}_0$ of parabolic subgroups with the same split component A_0. The theorem asserts that this family of transforms establishes an isomorphism from $C_c^\infty(G:\tau)$ onto a Paley–Wiener space ${}^u\mathrm{PW}(G,\tau,\mathcal{P}_0)$. The Fourier transforms in the family are completely determined by any single one among them. Accordingly, in Theorem 3.6, Arthur's Paley–Wiener theorem is reformulated by asserting that a single Fourier transform ${}^u\mathcal{F}_P$ defines a topological linear isomorphism from $C_c^\infty(G:\tau)$ onto a Paley–Wiener space ${}^u\mathrm{PW}(G,\tau,P)$.

In Section 4 we formulate, in Theorem 4.2, a distributional Paley–Wiener theorem, which asserts that ${}^u\mathcal{F}_P$ defines a topological linear isomorphism from the space $C_c^{-\infty}(G:\tau)$ of τ-spherical compactly supported generalized functions on G onto a Paley–Wiener space ${}^u\mathrm{PW}^*(G,\tau,P)$.

At this point of the paper, the main results have been stated. The rest of the paper is devoted to proofs. In Section 5 we prepare for this by introducing the C-functions and listing those of their properties that are needed.

In Section 6 we give the proof that Theorem 3.6 is indeed a reformulation of Arthur's theorem. We describe the relations between the different Fourier transforms in terms of C-functions. The equivalence of Theorems 3.3 and 3.6 is then captured by the assertion, in Proposition 6.4, that the natural map between the Paley–Wiener spaces ${}^u\mathrm{PW}(G,\tau,\mathcal{P}_0)$ and ${}^u\mathrm{PW}(G,\tau,P)$ is a topological isomorphism.

In the next Section, 7, a normalized Fourier transform $\mathcal{F}_P$ is defined in terms of the normalized Eisenstein integral

$$E^\circ(P:\lambda:\cdot) := E(P:\lambda:\cdot) \circ C_{P|P}(1:\lambda)^{-1};$$

This normalization is natural from the point of view of asymptotic expansions; it has the effect that the new, normalized C-functions become unitary for imaginary λ. It follows from the relation between the Eisenstein integrals that the normalized Fourier transform is related to the above Fourier transform by a relation of the form ${}^u\mathcal{F}_P = \mathcal{U}_P \circ \mathcal{F}_P$, where $\mathcal{U}_P$ denotes multiplication by a C-function $\lambda \mapsto C_{P|P}(1:-\bar{\lambda})^*$. An associated Paley–Wiener space $\mathrm{PW}(G, \tau, P)$ is defined, as well as a distributional Paley–Wiener space, indicated by superscript $*$. The main result of the section, Theorem 7.8, asserts that the map $\mathcal{U}_P$ induces isomorphisms $\mathrm{PW}(G, \tau, P) \to {}^u\mathrm{PW}(G, \tau, P)$ and $\mathrm{PW}^*(G, \tau, P) \to {}^u\mathrm{PW}^*(G, \tau, P)$. As a result, the associated Paley–Wiener theorems for the normalized Fourier transform are equivalent to those for the unnormalized transform.

Let ${}_*\mathrm{X}$ denote G viewed as a symmetric space for ${}_*G := G \times G$. In the final section the unnormalized Eisenstein integral $E(P:\lambda)$ for G is related to the unnormalized Eisenstein integral for ${}_*\mathrm{X}$ as defined in [3]. It follows from this relation that the normalized Eisenstein integrals, for G and ${}_*\mathrm{X}$, coincide; therefore, so do the normalized Fourier transforms. Likewise, it is shown that the Paley–Wiener spaces for G coincide with the similar spaces for ${}_*\mathrm{X}$. This finally establishes the validity of all mentioned Paley–Wiener theorems as special cases of the theorems in [7] and [8].

2. BASIC CONCEPTS

Let G be a real reductive Lie group of the Harish-Chandra class and let K be a maximal compact subgroup. Let V_τ be a finite-dimensional Hilbert space, and let τ be a unitary double representation of K in V_τ. By this we mean that $\tau = (\tau_1, \tau_2)$ with τ_1 a left and τ_2 a right unitary representation of K in V_τ; moreover, the representations τ_1 and τ_2 commute. We will often drop the subscripts on τ_1 and τ_2, writing

$$\tau(k_1)v\tau(k_2) = \tau_1(k_1)v\tau_2(k_2)$$

for all $v \in V_\tau$ and $k_1, k_2 \in K$. A function $f : G \to V_\tau$ is called τ-spherical if it satisfies the rule

$$f(k_1 g k_2) = \tau(k_1) f(g) \tau(k_2) \tag{2.1}$$

for all $g \in G$ and $k_1, k_2 \in K$. The space of smooth τ-spherical functions is denoted by $C^\infty(G:\tau)$ and equipped with the usual Fréchet topology. The subspace $C_c^\infty(G:\tau)$ of compactly supported smooth τ-spherical functions is equipped with the usual complete locally convex (Hausdorff) topology.

We shall first briefly establish some notation for the group. As usual, we denote Lie groups by Roman capitals, and their Lie algebras by the Gothic lower case equivalents.

Let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution associated with K. The associated infinitesimal involution of $\mathfrak{g}$ is denoted by the same symbol and the associated eigenspaces with eigenvalues $+1$ and -1 by $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Accordingly, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}$ and let Σ be the restricted root system of $\mathfrak{a}_0$ in $\mathfrak{g}$. Let $A_0 := \exp \mathfrak{a}_0$ be the associated vectorial subgroup of G and let $\mathcal{P}_0 = \mathcal{P}(A_0)$ be the collection of parabolic subgroups of G with split component A_0. Each element $P \in \mathcal{P}_0$ is minimal and has a Langlands decomposition of the form $P = M_0 A_0 N_P$, with M_0 equal to the centralizer of A_0 in K. Let $\Sigma(P)$ be the collection of $\mathfrak{a}_0$-roots in $\mathfrak{n}_P = \mathrm{Lie}(N_P)$. Then $P \mapsto \Sigma(P)$ defines a one-to-one correspondence between $\mathcal{P}(A_0)$ and the collection of positive systems for Σ.

We equip $\mathfrak{a}_0$ with a W-invariant positive definite inner product $\langle \cdot\, , \cdot \rangle$; the dual space $\mathfrak{a}_0^*$ is equipped with the dual inner product. The latter inner product is extended to a complex bilinear form on $\mathfrak{a}_{0\mathbb{C}}^*$. The norm associated with the inner product on $\mathfrak{a}_0^*$ is extended to a Hermitian norm on $\mathfrak{a}_{0\mathbb{C}}^*$, denoted by $|\cdot|$.

We shall now review the definition of the τ-spherical Eisenstein integral related to a given parabolic subgroup $P \in \mathcal{P}(A_0)$. Let τ_{M_0} denote the restriction of τ to M_0. As M_0 is a subgroup of K, the space $L^2(M_0 : \tau_{M_0})$ of square integrable τ_{M_0}-spherical functions $M_0 \to V_\tau$ is finite dimensional and equals the space of smooth τ_{M_0}-spherical functions. We equip M_0 with normalized Haar measure, and define the finite-dimensional Hilbert space $\mathcal{A}_2 = \mathcal{A}_2(\tau)$ by

$$\mathcal{A}_2 := L^2(M_0 : \tau_{M_0}) = C^\infty(M_0 : \tau_{M_0}). \tag{2.2}$$

Let $\psi \in \mathcal{A}_2$. For $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ we define the function $\psi_\lambda = \psi_{P,\lambda} : G \to V_\tau$ by

$$\psi_\lambda(namk) = a^{\lambda+\rho_P}\psi(m)\tau_2(k),$$

for $k \in K$, $m \in M_0$, $a \in A_0$ and $n \in N_P$. Here $\rho_P \in \mathfrak{a}_0^*$ is defined by

$$\rho_P(H) = \frac{1}{2}\,\mathrm{tr}\big(\mathrm{ad}(H)|\mathfrak{n}_P\big).$$

By the analytic nature of the Iwasawa decomposition $G = N_P A_0 K$, the function ψ_λ is analytic. We define the Eisenstein integral $E(P : \psi : \lambda) : G \to V_\tau$ by

$$E(P : \psi : \lambda : x) := \int_K \tau_1(k)\psi_\lambda\big(k^{-1}x\big)\, dk, \tag{2.3}$$

for $x \in G$. Then, clearly, $E(P : \psi : \lambda)$ is a function in $C^\infty(G : \tau)$, depending linearly on ψ and holomorphically on λ.

Remark 2.1. Here we have adopted the same convention as J. Arthur [1, §2], which differs from Harish-Chandra's. Let E^{HC} denote the Eisenstein integral as defined by Harish-Chandra [13, §9]. Then $E^{\mathrm{HC}}(P : \psi : \lambda) = E(P : \psi : i\lambda)$.

For the reader's convenience, we note that Arthur [1] uses the notation $\mathcal{A}_{\text{cusp}}(M_0, \tau)$ or $\mathcal{A}_0$ for the space (2.2).

In terms of the Eisenstein integral we define a Fourier transform ${}^u\mathcal{F}_P$ from $C_c^\infty(G : \tau)$ to the space $\mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ of holomorphic $\mathcal{A}_2$-valued functions on $\mathfrak{a}_{0\mathbb{C}}^*$. The superscript u indicates that this Fourier transform is defined in terms of the above unnormalized Eisenstein integral, in contrast with a normalized Fourier transform $\mathcal{F}_P$ to be defined later.

Let dx be a choice of Haar measure on G. We define the Fourier transform ${}^u\mathcal{F}_P f$ of $f \in C_c^\infty(G : \tau)$ by the formula

$$(2.4) \qquad \langle {}^u\mathcal{F}_P f(\lambda), \psi \rangle_{\mathcal{A}_2} = \int_G \langle f(x), E(P : \psi : -\bar{\lambda} : x) \rangle_{V_\tau} \, dx,$$

for $\psi \in \mathcal{A}_2$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$. It follows from the Paley–Wiener theorem in [1] that ${}^u\mathcal{F}_P$ is injective on $C_c^\infty(G : \tau)$. This injectivity can also be established by application of the subrepresentation theorem [11, Theorem 8.21].

3. ARTHUR'S PALEY–WIENER THEOREM

The image of $C_c^\infty(G : \tau)$ under Fourier transform is described by the Paley–Wiener theorem due to J. Arthur [1], which we shall now formulate.

It is convenient to rewrite the definition of the Fourier transform ${}^u\mathcal{F}_P$ in terms of a (unnormalized) dual Eisenstein integral. Given $x \in G$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$, we agree to define $E(P : \lambda : x) \in \mathrm{Hom}(\mathcal{A}_2, V_\tau)$ by the formula

$$E(P : \lambda : x)\psi := E(P : \psi : \lambda : x)$$

for $\psi \in \mathcal{A}_2$. Moreover, we define a dual Eisenstein integral by

$$(3.1) \qquad {}^uE^*(P : \lambda : x) := E(P : -\bar{\lambda} : x)^* \in \mathrm{Hom}(V_\tau, \mathcal{A}_2),$$

where the superscript $*$ indicates that the Hilbert adjoint has been taken. The superscript u serves to distinguish the present dual Eisenstein integral from a normalized version that will be introduced at a later stage.

The dual Eisenstein integral ${}^uE^*$ may be viewed as a smooth $\mathrm{Hom}(V_\tau, \mathcal{A}_2)$-valued function on $\mathfrak{a}_{0\mathbb{C}}^* \times G$ which is holomorphic in the first variable. In terms of this Eisenstein integral, the Fourier transform (2.4) may be expressed as an integral transform. Indeed, it readily follows from the given definitions that

$$(3.2) \qquad {}^u\mathcal{F}_P f(\lambda) = \int_G {}^uE^*(P : \lambda : x) f(x) \, dx,$$

for $f \in C_c^\infty(G : \tau)$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$. We now proceed to giving the definition of a suitable Paley–Wiener space.

Let V be a finite-dimensional real linear space. We denote by $S(V)$ the symmetric algebra of $V_{\mathbb{C}}$. This algebra is identified with the algebra of constant coefficient holomorphic differential operators on $V_{\mathbb{C}}$ in the usual way.

We denote by $\mathcal{O}(V_{\mathbb{C}})$ the space of holomorphic functions on $V_{\mathbb{C}}$ and, for $a \in V_{\mathbb{C}}$, by $\mathcal{O}_a = \mathcal{O}_a(V_{\mathbb{C}})$ the space of germs of holomorphic functions at a. Moreover, we denote by $\mathcal{O}_a(V_{\mathbb{C}})^*_{\text{tayl}}$ the space of linear functionals $\mathcal{O}_a \to \mathbb{C}$ of the form

$$f \mapsto \text{ev}_a(uf) = uf(a),$$

with $u \in S(V)$. The elements of $\mathcal{O}_a(V_{\mathbb{C}})^*_{\text{tayl}}$ will be called Taylor functionals at a, as they give linear combinations of coefficients of a Taylor series at a. Clearly, the map $u \mapsto \text{ev}_a \circ u$ is a linear isomorphism from $S(V)$ onto $\mathcal{O}_a(V_{\mathbb{C}})^*_{\text{tayl}}$.

We define the space of Taylor functionals on $V_{\mathbb{C}}$ as the algebraic direct sum

$$\mathcal{O}(V_{\mathbb{C}})^*_{\text{tayl}} := \bigoplus_{a \in V_{\mathbb{C}}} \mathcal{O}_a(V_{\mathbb{C}})^*_{\text{tayl}}.$$

Given $U \in \mathcal{O}(V_{\mathbb{C}})^*_{\text{tayl}}$, the finite set of $a \in V_{\mathbb{C}}$ with $U_a \neq 0$ is called the support of U, notation $\text{supp}\, U$. Given $f \in \mathcal{O}(V_{\mathbb{C}})$, we put

$$Uf := \sum_{a \in \text{supp}\, U} U_a f_a.$$

The map $U \otimes f \mapsto Uf$ defines an embedding of $\mathcal{O}(V_{\mathbb{C}})^*_{\text{tayl}}$ into the linear dual $\mathcal{O}(V_{\mathbb{C}})^*$; this justifies the notation. Note that in fact the elements of $\mathcal{O}(V_{\mathbb{C}})^*_{\text{tayl}}$ are continuous with respect to the usual Fréchet topology on $\mathcal{O}(V_{\mathbb{C}})$.

Finally, we note that a finitely supported function $U : V_{\mathbb{C}} \to S(V)$ may be viewed as a Taylor functional by the formula $Uf := \sum_a \text{ev}_a[U(a)f]$. Accordingly, the space of Taylor functionals may be identified with the space of finitely supported functions $V_{\mathbb{C}} \to S(V)$.

Definition 3.1. An (unnormalized, holomorphic) Arthur–Campoli functional for $(G, \tau, \mathcal{P}_0)$ is a family $(\mathcal{L}_P)_{P \in \mathcal{P}_0} \subset \mathcal{O}(\mathfrak{a}^*_{0\mathbb{C}})^*_{\text{tayl}} \otimes \mathcal{A}^*_2$ such that

$$\sum_{P \in \mathcal{P}_0} \mathcal{L}_P\big[{}^u E^*(P : \cdot : x) v_P\big] = 0$$

for all $x \in G$ and all $(v_P)_{P \in \mathcal{P}_0} \subset V_\tau$. The linear space of such families is denoted by ${}^u\text{AC}_{\text{hol}}(G, \tau, \mathcal{P}_0)$.

For $R > 0$ we define $\text{H}_R(\mathfrak{a}^*_{0\mathbb{C}})$ to be the space of holomorphic functions $\varphi : \mathfrak{a}^*_{0\mathbb{C}} \to \mathbb{C}$ such that for every $n \in \mathbb{N}$,

$$\nu_{R,n}(\varphi) := \sup_{\lambda \in \mathfrak{a}^*_{0\mathbb{C}}} \big(1 + |\lambda|\big)^n e^{-R|\text{Re}\,\lambda|} \big|\varphi(\lambda)\big| < \infty.$$

Equipped with the seminorms $\nu_{R,n}$, for $n \in \mathbb{N}$, this space is a Fréchet space.

Definition 3.2. Let $R > 0$. The Paley–Wiener space ${}^u\mathrm{PW}_R(G, \tau, \mathcal{P}_0)$ is defined to be the space of families $(\varphi_P)_{P\in\mathcal{P}_0} \subset \mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ such that

$$\sum_{P\in\mathcal{P}_0} \mathcal{L}_P \varphi_P = 0$$

for all $(\mathcal{L}_P)_{P\in\mathcal{P}_0} \in {}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, \mathcal{P}_0)$.

By continuity of the Taylor functionals, the Paley–Wiener space is a closed subspace of the direct sum of a finite number of copies of $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$, one for each $P \in \mathcal{P}_0$; it is therefore a Fréchet space of its own right.

For $R > 0$ we put $G_R := K \exp \bar{B}_R K$, where $\bar{B}_R$ denotes the closed ball of center 0 and radius R in $\mathfrak{a}_0$. Moreover, we define the following closed subspace of $C_c^\infty(G : \tau)$, and equip it with the relative topology:

$$C_R^\infty(G : \tau) = \{ f \in C^\infty(G : \tau) \mid \operatorname{supp} f \subset G_R \}.$$

We have now gathered the concepts and notation needed to formulate the Paley–Wiener theorem due to J. Arthur [1, p. 83, Theorem 3.3.1].

Theorem 3.3 (Arthur [1]). *The map $f \mapsto ({}^u\mathcal{F}_P f)_{P\in\mathcal{P}_0}$ is a topological linear isomorphism from $C_R^\infty(G : \tau)$ onto ${}^u\mathrm{PW}_R(G, \tau, \mathcal{P}_0)$.*

Each of the individual Fourier transforms ${}^u\mathcal{F}_P$, for $P \in \mathcal{P}_0$, is already injective on $C_c^\infty(G : \tau)$. It is therefore natural to reformulate Arthur's theorem in terms of a single Fourier transform.

Definition 3.4. Let $P \in \mathcal{P}_0$. An (unnormalized, holomorphic) Arthur–Campoli functional for the triple (G, τ, P) is a functional $\mathcal{L} \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)_{\mathrm{tayl}}^* \otimes \mathcal{A}_2^*$ such that

$$\mathcal{L}\big[{}^uE^*(P : \cdot : x)v\big] = 0, \tag{3.3}$$

for all $x \in G$ and all $v \in V_\tau$. The space of such functionals is denoted by ${}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, P)$.

Definition 3.5. Let $P \in \mathcal{P}_0$ and $R > 0$. We define the Paley–Wiener space ${}^u\mathrm{PW}_R(G, \tau, P)$ to be the space of functions $\varphi \in \mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ such that $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in {}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, P)$.

Arthur's Paley–Wiener theorem may now be reformulated as follows.

Theorem 3.6. *Let $P \in \mathcal{P}_0$ and $R > 0$. The map ${}^u\mathcal{F}_P$ is a topological linear isomorphism from $C_R^\infty(G : \tau)$ onto ${}^u\mathrm{PW}_R(G, \tau, P)$.*

The equivalence of Theorems 3.3 and 3.6 will be established in Section 6.

In this section we will formulate a Paley–Wiener theorem characterizing the image under Fourier transform of the space $C_c^{-\infty}(G : \tau)$ of compactly supported τ-spherical generalized functions.

We shall first define the mentioned space. The space $C_c^{-\infty}(G)$ of compactly supported generalized functions on G is defined as the topological linear dual of the Fréchet space of smooth densities on G. It is equipped with the strong dual topology.

Via the map $f \mapsto f\,dx$, the space $C_c^{-\infty}(G)$ is isomorphic with the space of compactly supported generalized densities on G. Via integration the latter space may in turn be identified with the continuous linear dual of $C^\infty(G)$, i.e., with the space of compactly supported distributions on G.

The pairing with smooth densities induces a natural embedding $C_c^\infty(G) \hookrightarrow C_c^{-\infty}(G)$; accordingly, the left and right regular representations of G in $C_c^\infty(G)$ extend to continuous representations of G in $C_c^{-\infty}(G)$. We now define $C_c^{-\infty}(G : V_\tau)$ as the space of $K \times K$-invariants in $C_c^{-\infty}(G) \otimes V_\tau$. Again, there is a natural embedding $C_c^\infty(G : \tau) \hookrightarrow C_c^{-\infty}(G : \tau)$.

The definition of ${}^u\mathcal{F}_P$ by (3.2), for a given $P \in \mathcal{P}_0$, has a natural interpretation for compactly supported τ-spherical generalized functions. Accordingly, ${}^u\mathcal{F}_P$ extends to a continuous linear map $C_c^{-\infty}(G : \tau) \to \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$.

Let $R > 0$ and $n \in \mathbb{N}$. We define $\mathrm{H}_{R,n}^*(\mathfrak{a}_{0\mathbb{C}}^*)$ to be the space of entire holomorphic functions $\varphi : \mathfrak{a}_{0\mathbb{C}}^* \to \mathbb{C}$ with

$$\nu_{R,-n}(\varphi) = \sup_{\lambda \in \mathfrak{a}_{0\mathbb{C}}^*} \big(1 + \|\lambda\|\big)^{-n} e^{-R|\mathrm{Re}\,\lambda|} \big|\varphi(\lambda)\big| < \infty.$$

Equipped with the given norm, this space is a Banach space. If $m < n$, then

$$\mathrm{H}_{R,m}^*(\mathfrak{a}_{0\mathbb{C}}^*) \subset \mathrm{H}_{R,n}^*(\mathfrak{a}_{0\mathbb{C}}^*),$$

with continuous inclusion map. The union $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$ of these spaces, for $n \in \mathbb{N}$, is equipped with the inductive limit locally convex topology.

Definition 4.1. Let $P \in \mathcal{P}_0$ and $R > 0$. The distributional Paley–Wiener space ${}^u\mathrm{PW}_R^*(G, \tau, P)$ is defined as the space of functions $\varphi \in \mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ satisfying the Arthur–Campoli relations $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in {}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, P)$.

By continuity of Taylor functionals, the space ${}^u\mathrm{PW}_R^*(G, \tau, P)$ is a closed subspace of $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$; we endow it with the relative topology.

The following result is the analogue of Arthur's Paley–Wiener theorem for K-finite compactly supported generalized functions on G. Given $R > 0$ we denote by $C_R^{-\infty}(G : \tau)$ the space of τ-spherical generalized functions on G with support contained in $G_R = K \exp \bar{B}_R K$. It is equipped with the relative topology.

Theorem 4.2 (The Paley–Wiener theorem for generalized functions). *Let $P \in \mathcal{P}_0$ and $R > 0$. The Fourier transform ${}^u\mathcal{F}_P$ extends to a topological linear isomorphism from $C_R^{-\infty}(G:\tau)$ onto ${}^u\mathrm{PW}_R^*(G, \tau, P)$.*

As mentioned in the introduction, this theorem will follow from the results of the present paper combined with the distributional Paley–Wiener theorem for reductive symmetric spaces proved in [8].

Remark 4.3. With the same arguments that will lead to the equivalence of Theorems 3.6 and 3.3, it can be shown that Theorem 4.2 is equivalent to a Paley–Wiener theorem for generalized functions involving the family $({}^u\mathcal{F}_P)_{P\in\mathcal{P}_0}$ in the same spirit as Theorem 3.3.

5. C-FUNCTIONS, SINGULAR LOCI AND ESTIMATES

To establish the equivalence of Theorems 3.3 and 3.6 we need relations between the Fourier transforms, which can be given in terms of the so-called C-functions. The latter arise as coefficients in asymptotic expansions of Eisenstein integrals.

Let $Q \in \mathcal{P}_0$. We denote by $\mathfrak{a}_Q^+$ the positive chamber determined by the positive system $\Sigma(Q)$ and by A_Q^+ the image in A_0 under the exponential map. Then KA_Q^+K is an open dense subset of G.

In view of its τ-spherical behavior, the Eisenstein integral $E(P:\lambda)$, for $P \in \mathcal{P}_0$, is completely determined by its restriction to A_Q^+. It follows from Harish-Chandra's result [14, Theorem 18.1], that, on $M_0A_Q^+$, the given Eisenstein integral behaves asymptotically as follows:

$$(5.1) \qquad E(P:\lambda:ma)\psi \sim \sum_{s\in W} a^{s\lambda-\rho_Q}\big[C_{Q|P}(s:\lambda)\psi\big](m) \quad (a \to \infty \text{ in } A_Q^+),$$

for every $\psi \in \mathcal{A}_2$, every $m \in M_0$, and $\lambda \in i\mathfrak{a}_0^{*\mathrm{reg}}$. Here W denotes the Weyl group of the root system Σ and the coefficients $C_{Q|P}(s:\cdot)$ are $\mathrm{End}(\mathcal{A}_2)$-valued analytic functions of $\lambda \in i\mathfrak{a}_0^{*\mathrm{reg}}$. The functions $C_{Q|P}(s:\cdot)$, for $s \in W$, are uniquely determined by these properties. A priori they have a meromorphic extension to an open neighborhood of $i\mathfrak{a}_0^*$ in $\mathfrak{a}_{0\mathbb{C}}^*$.

Remark 5.1. Harish-Chandra denotes the c-functions by lower case letters. In view of Remark 2.1, the C-functions introduced above are related to Harish-Chandra's by the formula $C_{Q|P}(s:i\lambda) = c_{Q|P}(s:\lambda)$.

For $P \in \mathcal{P}_0$ and $R \in \mathbb{R}$ we put

$$(5.2) \qquad \mathfrak{a}_0^*(P,R) := \big\{\lambda \in \mathfrak{a}_{0\mathbb{C}}^* \mid \langle \mathrm{Re}\,\lambda, \alpha\rangle < R,\ \forall \alpha \in \Sigma(P)\big\}.$$

A Σ-hyperplane in $\mathfrak{a}_{0\mathbb{C}}^*$ is a hyperplane of the form $\langle\lambda,\alpha\rangle = c$, with $\alpha \in \Sigma$ and $c \in \mathbb{C}$. The hyperplane is said to be real if $c \in \mathbb{R}$.

We define $\Pi_\Sigma(\mathfrak{a}_{0\mathbb{C}}^*)$ to be the set of polynomial functions that can be written as a product of a nonzero complex number and linear factors of the form $\lambda \mapsto \langle \lambda, \alpha \rangle - c$, with $\alpha \in \Sigma$ and $c \in \mathbb{C}$. The subset of polynomial functions which are products as above with $c \in \mathbb{R}$ is denoted by $\Pi_{\Sigma,\mathbb{R}}(\mathfrak{a}_{0\mathbb{C}}^*)$.

Lemma 5.2. *Let $P \in \mathcal{P}_0$. The endomorphism $C(1:\lambda) = C_{P|P}(1:\lambda) \in \mathrm{End}(\mathcal{A}_2)$ is invertible for generic $\lambda \in i\mathfrak{a}_0^*$. Both maps*

$$\lambda \mapsto C(1:\lambda)^{\pm 1} \tag{5.3}$$

extend to $\mathrm{End}(\mathcal{A}_2)$-valued meromorphic functions on $\mathfrak{a}_{0\mathbb{C}}^*$ that can be expressed as products of functions of the form $\lambda \mapsto c_\alpha(\langle \lambda, \alpha \rangle)$, for $\alpha \in \Sigma(P)$, with c_α a meromorphic function on $\mathbb{C}$ with real singular locus. Accordingly, each of the functions (5.3) has a singular locus equal to a locally finite union of real Σ-hyperplanes in $\mathfrak{a}_{0\mathbb{C}}^*$.

Let $R \in \mathbb{R}$. Only a finite number of the mentioned singular hyperplanes intersect $-\mathfrak{a}_0^*(P, R)$. There exist polynomial functions $q_\pm \in \Pi_{\Sigma,\mathbb{R}}(\mathfrak{a}_0^*)$ such that $\lambda \mapsto q_\pm(\lambda) C(1:-\lambda)^{\pm 1}$ are regular on the closure of the set $\mathfrak{a}_0^*(P, R)$. If $q_\pm$ is any pair of polynomials with these properties, there exist constants $n \in \mathbb{N}$ and $C > 0$ such that

$$\left\| q^\pm(\lambda) C(1:-\lambda)^{\pm 1} \right\| \leqslant C\big(1 + |\lambda|\big)^n \quad \big(\lambda \in \mathfrak{a}_0^*(P, R)\big).$$

Proof. All assertions readily follow from the arguments in [1], proof of Lemma 5.2, except possibly for the final estimate, which at first follows for a particular choice of $q_\pm$. A straightforward application of the Cauchy integral formula then gives the result for arbitrary $q_\pm$ satisfying the hypotheses. □

Following Harish-Chandra [15, §17], we define the following normalized C-functions, for $P, Q \in \mathcal{P}_0$ and $s \in W$:

$$^\circ C_{Q|P}(s:\lambda) := C_{Q|Q}(1:s\lambda)^{-1} C_{Q|P}(s:\lambda). \tag{5.4}$$

The following result, due to Harish-Chandra [15], will be of crucial importance to us.

Lemma 5.3 (Harish-Chandra [15]). *For all $P, Q \in \mathcal{P}_0$ and $s \in W$, the $\mathrm{End}(\mathcal{A}_2)$-valued function $\lambda \mapsto {^\circ C_{Q|P}}(s:\lambda)$ has a rational extension to $\mathfrak{a}_{0\mathbb{C}}^*$.*

The endomorphism $^\circ C_{Q|P}(s:\lambda)$ is invertible for generic $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ and $\lambda \mapsto {^\circ C_{Q|P}}(s:\lambda)^{-1}$ is a rational $\mathrm{End}(\mathcal{A}_2)$-valued function.

Finally, each of the functions $\lambda \mapsto {^\circ C_{Q|P}}(s:\lambda)^{\pm 1}$ is a product of functions of the form $\lambda \mapsto c_\alpha(\langle \lambda, \alpha \rangle)$, for $\alpha \in \Sigma$, with c_α a $\mathrm{End}(\mathcal{A}_2)$-valued rational function on $\mathbb{C}$.

Proof. The assertions for $^\circ C_{Q|P}(s:\lambda)$ follow from [15, Lemma 19.2] combined with the corollary to Lemma 17.2 and with Lemma 17.4 of the same article. For imaginary λ, the endomorphism $^\circ C_{Q|P}(s:\lambda)$ is unitary, by [15, Lemma 17.3]. The remaining assertions now follow by application of Cramer's rule. □

According to [15, Lemma 17.2], the Eisenstein integrals are related by the following functional equations:

$$E(P:\lambda:\cdot) = E(Q:s\lambda:\cdot)^{\circ}C_{Q|P}(s:\lambda), \tag{6.1}$$

for $P, Q \in \mathcal{P}_0$ and $s \in W$, as an identity of meromorphic $C^\infty(G) \otimes \mathrm{Hom}(\mathcal{A}_2, V_\tau)$-valued functions in the variable $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$.

Lemma 6.1. *Let $P, Q \in \mathcal{P}_0$. Then ${}^{\circ}C_{Q|P}(1:\lambda) \circ {}^{\circ}C_{P|Q}(1:\lambda) = I$, as an identity of* $\mathrm{End}(\mathcal{A}_2)$*-valued functions in the variable $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$.*

Proof. From the functional equation for the Eisenstein integral it follows that, for $x \in G$,

$$E(P:\lambda:x) = E(P:\lambda:x)^{\circ}C_{P|Q}(1:\lambda)^{\circ}C_{Q|P}(1:\lambda).$$

Using (5.1), we infer that this identity is valid with $C_{P|P}(1:\lambda)$ in place of $E(P:\lambda:x)$ on both sides. As $C_{P|P}(1:\lambda)$ is invertible for generic λ, the required identity follows. □

In view of (3.1) it follows immediately from (6.1) that the unnormalized dual Eisenstein integrals satisfy the following functional equations, for $P, Q \in \mathcal{P}_0$ and $s \in W$,

$${}^{\circ}C_{Q|P}(s:-\bar{\lambda})^{*}\,{}^{u}E^{*}(Q:s\lambda:\cdot) = {}^{u}E^{*}(P:\lambda:\cdot), \tag{6.2}$$

as an identity of meromorphic $C^\infty(G) \otimes \mathrm{Hom}(V_\tau, \mathcal{A}_2)$-valued functions of $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$. In view of the definition of the Fourier transform in (3.2), this in turn implies that, for every $f \in C_c^\infty(G, \tau)$,

$${}^{\circ}C_{Q|P}(s:-\bar{\lambda})^{*}\,{}^{u}\mathcal{F}_Q f(s\lambda) = {}^{u}\mathcal{F}_P f(\lambda), \tag{6.3}$$

as an identity of meromorphic $\mathcal{A}_2$-valued functions of $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$.

Lemma 6.2. *Let $\varphi = (\varphi_P)_{P \in \mathcal{P}_0} \subset \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$, and assume that $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in {}^{u}\mathrm{AC}(G, \tau, \mathcal{P}_0)$. Then for all $P, Q \in \mathcal{P}_0$ and $s \in W$,*

$${}^{\circ}C_{Q|P}(s:-\bar{\lambda})^{*}\varphi_Q(s\lambda) = \varphi_P(\lambda), \tag{6.4}$$

for generic $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$.

Proof. Let $P, Q \in \mathcal{P}_0$ and $s \in W$ be fixed. In view of Lemma 5.3 there exists a polynomial function $q \in \Pi_\Sigma(\mathfrak{a}_{0\mathbb{C}}^*)$ such that $\lambda \mapsto q(\lambda)^{\circ}C_{Q|P}(s:-\bar{\lambda})^{*}$ is polynomial. Let φ fulfill the hypothesis and let $\mu \in \mathfrak{a}_{0\mathbb{C}}^* \setminus q^{-1}(0)$.

We define Taylor functionals $\mathcal{L}_R \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)_{\mathrm{tayl}}^* \otimes \mathcal{A}_2^*$ by $\mathcal{L}_P\psi := -q(\mu)\psi(\mu)$ and $\mathcal{L}_Q\psi = \mathrm{ev}_\mu[\lambda \mapsto q(\lambda)^{\circ}C_{Q|P}(s:-\bar{\lambda})^{*}\psi(s\lambda)]$, for $\psi \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$, and by $\mathcal{L}_R = 0$

for $R \in \mathcal{P}_0 \setminus \{P, Q\}$. It follows from (6.2) that $\mathcal{L} \in {}^u\mathrm{AC}(G, \tau, \mathcal{P}_0)$. Hence, $\mathcal{L}\varphi = 0$. We conclude that φ satisfies (6.4) for $\lambda \in \mathfrak{a}_{0\mathbb{C}}^* \setminus q^{-1}(0)$. $\square$

If V is a finite-dimensional real linear space, we denote by $\mathcal{M}(V_\mathbb{C})$ the space of meromorphic functions on $V_\mathbb{C}$. Given $P, Q \in \mathcal{P}_0$, we define the endomorphism $\gamma_{P|Q}$ of $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ by

$$[\gamma_{P|Q}\psi](\lambda) = {}^\circ C_{Q|P}(1 : -\bar{\lambda})^* \psi(\lambda).$$

In particular, $\gamma_{P|P} = I$.

Lemma 6.3. *Let $P, Q \in \mathcal{P}_0$ and $R > 0$. Then $\gamma_{P|Q}$ maps ${}^u\mathrm{PW}_R(G, \tau, Q)$ continuously into $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$.*

Proof. It follows from Lemma 5.3 that there exists a polynomial $q \in \Pi_\Sigma(\mathfrak{a}_{0\mathbb{C}}^*)$ such that the function $\lambda \mapsto q(\lambda)\,{}^\circ C_{Q|P}(1 : -\bar{\lambda})^*$ is polynomial on $\mathfrak{a}_{0\mathbb{C}}^*$. This in turn implies that the map $q \circ \gamma_{P|Q}$ maps ${}^u\mathrm{PW}_R(G, \tau, Q)$ continuously into $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$.

Let $\alpha \in \Sigma$ and $c \in \mathbb{C}$ be such that $l : \lambda \mapsto \langle \lambda, \alpha \rangle - c$ is a factor of q. Let d be the highest integer such that l^d is still a factor of q. Let H_α denote the element of $\mathfrak{a}_0$ determined by $H_\alpha \perp \ker \alpha$ and $\alpha(H_\alpha) = 2$. Fix $0 \leqslant k < d$. Then the element $H_\alpha^k \in S(\mathfrak{a}_0)$, viewed as a constant coefficient differential operator on $\mathfrak{a}_{0\mathbb{C}}^*$ satisfies $H_\alpha^k q = 0$ on $l^{-1}(0)$. Fix $\lambda_0 \in l^{-1}(0)$ and consider the functional $\mathcal{L} \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)_{\mathrm{tayl}}^* \otimes \mathcal{A}_2^*$ defined by

$$\mathcal{L}(\varphi) := \mathrm{ev}_{\lambda_0} \circ H_\alpha^k [q \gamma_{P|Q} \varphi], \tag{6.5}$$

for $\varphi \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$. It follows from (6.2) that, for all $x \in G$ and $v \in V_\tau$, the function $\gamma_{P|Q}{}^uE^*(Q : \cdot : x)v$ equals ${}^uE^*(P : \cdot : x)v$, hence is holomorphic on $\mathfrak{a}_{0\mathbb{C}}^*$. By application of the Leibniz rule we now see that $\mathcal{L}({}^uE^*(Q : \cdot : x)v) = 0$ for all $x \in G$ and $v \in V_\tau$. Hence, $\mathcal{L}$ belongs to ${}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, Q)$. Let now $\varphi \in {}^u\mathrm{PW}_R(G, \tau, Q)$. Then it follows that (6.5) equals zero. As this is valid for every $\lambda_0 \in l^{-1}(0)$ and all $0 \leqslant k < d$, it follows that l^d divides $q(\lambda)\gamma_{P|Q}\varphi$. Treating all factors of q in this fashion, we see that $\gamma_{P|Q}\varphi$ is holomorphic on $\mathfrak{a}_{0\mathbb{C}}^*$ outside a subset of complex codimension 2. It follows that $\gamma_{P|Q}$ maps ${}^u\mathrm{PW}_R(G, \tau, Q)$ into $\mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$. Since $q\gamma_{P|Q}$ maps ${}^u\mathrm{PW}_R(G, \tau, Q)$ continuous linearly into $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$, it follows by a repeated application of Cauchy's integral formula, treating the linear factors of q one at a time, that $\gamma_{P|Q}$ is a continuous linear map ${}^u\mathrm{PW}_R(G, \tau, Q) \to \mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$. $\square$

If $\mathcal{L} \in {}^u\mathrm{AC}(G, \tau, P)$, then the family of Taylor functionals $(\mathcal{L}'_Q)_{Q \in \mathcal{P}_0}$ defined by $\mathcal{L}'_P = \mathcal{L}$ and by $\mathcal{L}'_Q = 0$ for $Q \neq P$ belongs to ${}^u\mathrm{AC}(G, \tau, \mathcal{P}_0)$. Accordingly, we may view ${}^u\mathrm{AC}(G, \tau, P)$ as a subspace of ${}^u\mathrm{AC}(G, \tau, \mathcal{P}_0)$. It follows that ${}^u\mathrm{PW}_R(G, \tau, \mathcal{P}_0)$, for $R > 0$, is a subspace of the direct sum of the spaces ${}^u\mathrm{PW}_R(G, \tau, P)$, for $P \in \mathcal{P}_0$. Moreover, by continuity of Taylor functionals, this subspace is closed.

Proposition 6.4. *Let $Q \in \mathcal{P}_0$. Then, for each $R > 0$, the projection onto the component with index Q induces a topological linear isomorphism*

$$(6.6)\qquad {}^u\mathrm{PW}_R(G,\tau,\mathcal{P}_0) \to {}^u\mathrm{PW}_R(G,\tau,Q).$$

Proof. Let E denote the direct sum of a finite number of copies of $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$, labeled by the elements of $\mathcal{P}_0$. For each such element P let $\mathrm{pr}_P : E \to \mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ denote the projection onto the component of label P. We define the map $\gamma_Q : {}^u\mathrm{PW}_R(G,\tau,Q) \to E$ by

$$\mathrm{pr}_P \circ \gamma_Q = \gamma_{P|Q} \quad (\forall P \in \mathcal{P}_0).$$

Then γ_Q is continuous linear; we will show that it maps into the subspace ${}^u\mathrm{PW}_R(G,\tau,\mathcal{P}_0)$ of E. Let $(\mathcal{L}_P)_{P\in\mathcal{P}_0}$ belong to ${}^u\mathrm{AC}_{\mathrm{hol}}(G,\tau,\mathcal{P}_0)$. For each $P \in \mathcal{P}_0$ we select $q_P \in \Pi_\Sigma(\mathfrak{a}_{0\mathbb{C}}^*)$ such that $\lambda \mapsto q_P(\lambda)^\circ C_{Q|P}(1 : -\bar{\lambda})^*$ is a polynomial function. By Lemma 6.5 below, there exists a $\mathcal{L}'_P \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)^*_{\mathrm{tayl}} \otimes \mathcal{A}_2^*$ such that $\mathcal{L}_P = \mathcal{L}'_P \circ q_P$ on $\mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$. By application of the Leibniz rule, we see that $\mathcal{L}'' = \sum_P \mathcal{L}'_P \circ q_P \circ \gamma_{P|Q}$ defines an element of $\mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)^*_{\mathrm{tayl}} \otimes \mathcal{A}_2^*$. It follows from the functional equations for the Eisenstein integral that, for every $x \in G$ and $v_Q \in V_\tau$,

$$\begin{aligned}\mathcal{L}''\big[{}^uE^*(Q : \cdot : x)v_Q\big] &= \sum_P \mathcal{L}'_P \circ q_P\big[{}^uE^*(P : \cdot : x)v_Q\big]\\ &= \sum_P \mathcal{L}_P\big[{}^uE^*(P : \cdot : x)v_Q\big] = 0.\end{aligned}$$

Hence, $\mathcal{L}'' \in {}^u\mathrm{AC}_{\mathrm{hol}}(G,\tau,Q)$. It follows that for $\varphi \in {}^u\mathrm{PW}_R(G,\tau,Q)$ we have

$$0 = \mathcal{L}''(\varphi) = \sum_P \mathcal{L}'_P\big[q_P\gamma_{P|Q}(\varphi)\big].$$

Moreover, since $\gamma_{P|Q}(\varphi)$ is holomorphic for each $P \in \mathcal{P}_0$, it follows that the latter expression equals $\sum_P \mathcal{L}_P\gamma_Q(\varphi)_P$. Hence, γ_Q maps ${}^u\mathrm{PW}_R(G,\tau,Q)$ into ${}^u\mathrm{PW}_R(G,\tau,\mathcal{P}_0)$. Moreover, it does so continuously, as the latter space carries the relative topology from E.

From the definition of γ_Q we see that $\mathrm{pr}_Q \circ \gamma_Q = \gamma_{Q|Q} = I$. Moreover, if $\varphi \in {}^u\mathrm{PW}_R(G,\tau,\mathcal{P}_0)$, then by Lemma 6.2 with $s = 1$, it follows that $\varphi_P = \gamma_{P|Q}(\varphi_Q)$, for all $P \in \mathcal{P}_0$. Hence, $\gamma_Q \circ \mathrm{pr}_Q = I$ on ${}^u\mathrm{PW}_R(G,\tau,\mathcal{P}_0)$. It follows that pr_Q restricts to a topological linear isomorphism (6.6) with inverse γ_Q. □

Lemma 6.5. *Let $q \in \Pi_\Sigma(\mathfrak{a}_{0\mathbb{C}}^*)$. Then for every $\mathcal{L} \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)^*_{\mathrm{tayl}}$ there exists a $\mathcal{L}' \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)^*_{\mathrm{tayl}}$ such that $\mathcal{L} = \mathcal{L}' \circ q$ on $\mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*)$.*

Proof. This may be proved in the same fashion as [7, Lemma 10.5]. □

It is an immediate consequence of Proposition 6.4 that Theorems 3.3 and 3.6 are equivalent.

The purpose of this section is to give equivalent versions of the Paley–Wiener theorems discussed in the previous sections, Theorems 3.6 and 4.2. The new versions are formulated in terms of a suitably normalized Fourier transform, which in the final section will be shown to coincide with the analogous Fourier transform for the group viewed as a symmetric space. The normalized Fourier transform is defined as in Section 2, but with a differently normalized Eisenstein integral.

Let $P \in \mathcal{P}_0$. We define the normalized Eisenstein integral by

$$(7.1) \qquad E^\circ(P : \lambda : x) = E(P : \lambda : x) C_{P|P}(1 : \lambda)^{-1},$$

for generic $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ and for $x \in G$. Then $\lambda \mapsto E^\circ(P : \lambda)$ is a meromorphic $C^\infty(G) \otimes \mathrm{Hom}(\mathcal{A}_2, V_\tau)$-valued function on $\mathfrak{a}_{0\mathbb{C}}^*$. It is not entire holomorphic anymore, but its singular set is of a simple nature. Indeed, by Lemma 5.2 the singular set is a locally finite union of hyperplanes of the form $\langle \lambda, \alpha \rangle = c$, with $\alpha \in \Sigma^+$, $c \in \mathbb{R}$. Moreover, the occurring constants c are bounded from below. It is known that the singular set is disjoint from the imaginary space $i\mathfrak{a}_0^*$, but we shall not need this here.

As before we define (normalized) dual Eisenstein integrals by

$$(7.2) \qquad E^*(P : \lambda : x) = E^\circ(P : -\bar{\lambda} : x)^* \in \mathrm{Hom}(V_\tau, \mathcal{A}_2)$$

for generic $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ and for $x \in G$. In terms of these we define the normalized Fourier transform $\mathcal{F}_P : C_c^{-\infty}(G : \tau) \to \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ by

$$\mathcal{F}_P f(\lambda) = \int_G E^*(P : \lambda : x) f(x)\, dx.$$

Lemma 7.1. *Let $f \in C_c^{-\infty}(G, \tau)$. The unnormalized and normalized Fourier transforms are related by*

$$(7.3) \qquad C_{P|P}(1 : -\bar{\lambda})^* \mathcal{F}_P f(\lambda) = {}^u\mathcal{F}_P f(\lambda),$$

as an identity of meromorphic functions of the variable $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$.

Proof. Replacing λ by $-\bar{\lambda}$ in both sides of (7.1), then multiplying with the C-function and taking conjugates, we obtain, in view of (3.1) and (7.2),

$$C_{P|P}(1 : -\bar{\lambda})^* E^*(P : \lambda : x) = {}^uE^*(P : \lambda : x),$$

as meromorphic functions of $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ with values in $C^\infty(G) \otimes \mathrm{Hom}(V_\tau, \mathcal{A}_2)$. The result follows by testing with $f\, dx$. $\square$

The singular nature of the normalized Eisenstein integral does not allow us to define Arthur–Campoli functionals in terms of Taylor functionals as we did in Section 3. Instead we need the concept of Laurent functional introduced in [6, Section 12]. We briefly recall its definition.

Let V be a finite-dimensional real linear space and let $X \subset V^* \setminus \{0\}$ be a finite subset. For a point $a \in V_{\mathbb{C}}$, we define the polynomial function $\pi_a : V_{\mathbb{C}} \to \mathbb{C}$ by

$$\pi_a := \prod_{\xi \in X} (\xi - \xi(a)).$$

The ring of germs of meromorphic functions at a is denoted by $\mathcal{M}(V_{\mathbb{C}}, a)$. We define the subring

$$\mathcal{M}(V_{\mathbb{C}}, a, X) := \bigcup_{N \in \mathbb{N}} \pi_a^{-N} \mathcal{O}_a.$$

Let ev_a denote the linear functional on $\mathcal{O}_a$ that assigns to a germ $f \in \mathcal{O}_a$ its value $f(a)$ at a.

An X-Laurent functional at $a \in V_{\mathbb{C}}$ is a linear functional $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, a, X)^*$ such that for every $N \in \mathbb{N}$ there exists a $u_N \in S(V)$ such that

$$\mathcal{L} = \mathrm{ev}_a \circ u_N \circ \pi_a^N \quad \text{on } \pi_a^{-N} \mathcal{O}_a. \tag{7.4}$$

The space of all Laurent functionals on $V_{\mathbb{C}}$, relative to X, is defined as the algebraic direct sum of linear spaces

$$\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}} := \bigoplus_{a \in V_{\mathbb{C}}} \mathcal{M}(V_{\mathbb{C}}, X, a)^*_{\mathrm{laur}}. \tag{7.5}$$

For $\mathcal{L}$ in the space (7.5), the finite set of $a \in V_{\mathbb{C}}$ for which the component $\mathcal{L}_a$ is nonzero is called the support of $\mathcal{L}$ and denoted by $\operatorname{supp} \mathcal{L}$.

According to the above definition, any $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}}$ may be decomposed as

$$\mathcal{L} = \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a.$$

Let $\mathcal{M}(V_{\mathbb{C}}, X)$ denote the space of meromorphic functions φ on $V_{\mathbb{C}}$ with the property that the germ φ_a at any point $a \in V_{\mathbb{C}}$ belongs to $\mathcal{M}(V_{\mathbb{C}}, a, X)$. Then the natural bilinear map $\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}} \times \mathcal{M}(V_{\mathbb{C}}, X) \to \mathbb{C}$, given by

$$(\mathcal{L}, \varphi) \mapsto \mathcal{L}\varphi := \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a \varphi_a$$

induces an embedding of $\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}}$ onto a linear subspace of the dual space $\mathcal{M}(V_{\mathbb{C}}, X)^*$. For more details concerning these definitions, we refer the reader to [6, Section 12].

Lemma 7.2. *Let $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}}$ and let $\psi \in \mathcal{M}(\Omega, X)$, for Ω an open neighborhood of* $\operatorname{supp} \mathcal{L}$. *Then $\mathcal{L} \circ \psi$ belongs to $\mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}}$.*

Proof. Without loss of generality we may assume that $\mathcal{L}$ is supported by a single point $a \in V_{\mathbb{C}}$. First assume that $\psi_a \in \mathcal{O}_a$. Then the result follows by a straightforward application of the definition containing (7.4) combined with the Leibniz rule. It remains to establish the result for $\psi = \pi_a^{-k}$, with $k \in \mathbb{N}$. In this case the result is an immediate consequence of the mentioned definition. □

The following result relates the Laurent functionals to the Taylor functionals defined in Section 3. The inclusion map $\mathcal{O}(V_\mathbb{C}) \subset \mathcal{M}(V_\mathbb{C}, X)$ induces a surjection $\mathcal{M}(V_\mathbb{C}, X)^* \to \mathcal{O}(V_\mathbb{C})^*$. This property of surjectivity also holds on the level of Laurent functionals. The space $\mathcal{O}(V_\mathbb{C})^*_{\mathrm{tayl}}$ may naturally be viewed as a subspace of $\mathcal{O}(V_\mathbb{C})^*$. The natural map $\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}} \to \mathcal{O}(V_\mathbb{C})^*$ maps into this subspace.

Lemma 7.3. *The natural map $\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}} \to \mathcal{O}(V_\mathbb{C})^*_{\mathrm{tayl}}$ is surjective.*

Proof. Let $U \in \mathcal{O}(V_\mathbb{C})^*_{\mathrm{tayl}}$. Without loss of generality we may assume that U is supported by a single point $a \in V_\mathbb{C}$. Then $U = \mathrm{ev}_a \circ u$ for some $u \in S(V)$. By [4, Lemma 1.7] with $d' = 0$, there exists a $\mathcal{L} \in \mathcal{M}(V_\mathbb{C}, a, X)^*_{\mathrm{laur}}$ determined by a sequence $(u_n)_{n\in\mathbb{N}} \subset S(V)$, such that $u_0 = u$. Thus, $\mathcal{L}$ restricts to U on $\mathcal{O}(V_\mathbb{C})$. □

Before proceeding, we formulate a result concerning division that will be frequently used in the sequel.

Lemma 7.4. *Let E be a finite-dimensional complex linear space. Let S be a linear subspace of $\mathcal{M}(V_\mathbb{C}, X) \otimes E$, let S° be the annihilator of S in $\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}} \otimes E^*$ and let $S^{\circ\circ}$ be the space of functions $\varphi \in \mathcal{M}(V_\mathbb{C}, X) \otimes E$ such that $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in S^\circ$.*

Let ψ be a nonzero $\mathrm{End}(E)$*-valued meromorphic function on $V_\mathbb{C}$ such that both ψ and ψ^{-1} belong to $\mathcal{M}(V_\mathbb{C}, X) \otimes \mathrm{End}(E)$. Let $\Omega \subset V_\mathbb{C}$ be an open subset such that $\psi\varphi$ is regular on Ω for all $\varphi \in S$. Then $\psi\varphi$ is regular on Ω for all $\varphi \in S^{\circ\circ}$.*

Proof. We first assume that $\psi = I$. Then every $\varphi \in S$ is regular on Ω. Let now $\varphi \in S^{\circ\circ}$ and consider a point $a \in \Omega$. Then it suffices to show that the germ φ_a is regular at a. As $\varphi_a \in \mathcal{M}(V_\mathbb{C}, a) \otimes E$, there exists a product q of factors of the form $\xi - \xi(a)$, with $\xi \in X$, such that $q\varphi_a$ is regular at a. We fix q of minimal degree. Then $q\varphi_a$ has a non-trivial value at a. Hence, there exists a linear functional $\eta \in E^*$ such that $\mathrm{ev}_a \circ q\langle\varphi_a, \eta\rangle \neq 0$. There exists a Laurent functional $\mathcal{L} \in \mathcal{M}(V_\mathbb{C}, a)^*_{\mathrm{laur}}$ such that $\mathcal{L} = \mathrm{ev}_a$ on $\mathcal{O}_a(V_\mathbb{C})$. Now $\mathcal{L} \circ q$ is a Laurent functional, and it follows from the above that $\mathcal{L}_1 := [\mathcal{L} \circ q] \otimes \eta$ is nonzero on φ_a. Hence $\mathcal{L}_1 \notin S^\circ$. It follows that there exists a function $\varphi_1 \in S$ such that $q(a)\eta(\varphi_1(a)) = \mathcal{L}_1\varphi_1$ is nonzero. This implies that q is nonzero at a, hence constant. We conclude that φ_a is regular at a.

We now turn to the case with ψ general. Then multiplication by ψ induces a linear automorphism of $\mathcal{M}(V_\mathbb{C}, X) \otimes E$, whose inverse is multiplication by ψ^{-1}. In view of Lemma 7.2, the map $\psi^* : \mathcal{L} \mapsto \mathcal{L} \circ \psi$ is a linear automorphism of $\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}} \otimes E$ with inverse $(\psi^{-1})^*$. Put $S_1 := \psi S$. Then $S_1^\circ = \psi^{*-1}(S^\circ)$ and $S_1^{\circ\circ} = \psi S^{\circ\circ}$. By the first part of the proof it follows that all elements of $S_1^{\circ\circ}$ are regular on Ω. The result follows. □

We use the non-degenerate bilinear map $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}_0^*$ to identify this space with its real linear dual. Accordingly we view Σ as a finite subset of $\mathfrak{a}_0^{**}\setminus\{0\}$ and invoke the space of Σ-Laurent functionals on $\mathfrak{a}_{0\mathbb{C}}^*$ in the following definition.

Definition 7.5. A (normalized) Arthur–Campoli functional for (G,τ,P) is a Laurent functional $\mathcal{L}\in\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\Sigma)_{\mathrm{laur}}^*\otimes\mathcal{A}_2^*$ such that

$$\mathcal{L}\big[E^*(P:\cdot:x)v\big]=0$$

for all $x\in G$ and $v\in V_\tau$. The space of such functionals is denoted by $\mathrm{AC}(G,\tau,P)$.

Our next objective is to define suitable spaces of meromorphic functions with controlled singular behavior.

A Σ-hyperplane in $\mathfrak{a}_{0\mathbb{C}}^*$ is defined to be a hyperplane of the form $l^{-1}(0)$, where $l:\lambda\mapsto\langle\lambda,\alpha\rangle-c$ with $\alpha\in\Sigma$ and $c\in\mathbb{C}$. The hyperplane is said to be real if $c\in\mathbb{R}$. A Σ-configuration in $\mathfrak{a}_{0\mathbb{C}}^*$ is a locally finite collection of Σ-hyperplanes. The configuration is said to be real if all its hyperplanes are real. Let now $\mathcal{H}$ be a real Σ-configuration. For each $H\in\mathcal{H}$ we fix $\alpha_H\in\Sigma$ and $s_H\in\mathbb{R}$ such that H equals the zero locus of $l_H:\lambda\mapsto\langle\lambda,\alpha\rangle-s_H$.

Let $d:\mathcal{H}\to\mathbb{N}$ be a map. For ω a subset of $\mathfrak{a}_0^*$ whose closure intersects only finitely many hyperplanes from $\mathcal{H}$, we define the polynomial function $\pi_{\omega,d}$ on $\mathfrak{a}_{0\mathbb{C}}^*$ by

$$\pi_{\omega,d}=\prod_{\substack{H\in\mathcal{H}\\ H\cap\mathrm{cl}\,\omega\neq\emptyset}} l_H^{d(H)}. \tag{7.6}$$

Moreover, we define $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\mathcal{H},d)$ to be the space of meromorphic functions $\varphi\in\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*)$ such that for every bounded open subset ω of $\mathfrak{a}_0^*$, the function $\pi_{\omega,d}\varphi$ is regular on $\omega+i\mathfrak{a}_0^*$.

For ω a bounded subset of $\mathfrak{a}_0^*$ and $n\in\mathbb{Z}$ we define the $[0,\infty]$-valued seminorm $\nu_{\omega,d,n}$ on $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\mathcal{H},d)$ by

$$\nu_{\omega,d,n}(\varphi):=\sup_{\lambda\in\omega+i\mathfrak{a}_0^*}\big(1+|\lambda|\big)^n\big|\pi_{\omega,d}(\lambda)\varphi(\lambda)\big|. \tag{7.7}$$

We define $\mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*,\mathcal{H},d)$ to be the space of functions $\varphi\in\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\mathcal{H},d)$ such that $\nu_{\omega,d,n}(\varphi)<\infty$ for every compact set $\omega\subset\mathfrak{a}_0^*$ and all $n\in\mathbb{N}$. This space is a Fréchet space with topology induced by the collection of seminorms $\nu_{\omega,d,n}$, for ω compact and $n\in\mathbb{N}$.

We denote by $\mathcal{N}=\mathcal{N}(\mathfrak{a}_0^*)$ the collection of maps $n:\mathcal{C}\to\mathbb{N}$, with $\mathcal{C}$ the collection of compact subsets of $\mathfrak{a}_0^*$. On $\mathcal{N}$ we define the partial ordering $\leqslant$ by $n\leqslant m$ if and only if $n(\omega)\leqslant m(\omega)$ for all $\omega\in\mathcal{C}$. For $n\in\mathcal{N}$ we define $\mathcal{P}_n^*(\mathfrak{a}_{0\mathbb{C}}^*,\mathcal{H},d)$ to be the space of functions $\varphi\in\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\mathcal{H},d)$ such that

$$\nu_{\omega,d,-n(\omega)}(\varphi)=\sup_{\lambda\in\omega+i\mathfrak{a}_0^*}\big(1+|\lambda|\big)^{-n(\omega)}\big|\pi_{\omega,d}(\lambda)\varphi(\lambda)\big|<\infty$$

for every compact subset $\omega \subset \mathfrak{a}_0^*$. Equipped with the seminorms $\nu_{\omega,d,-n(\omega)}$ the space $\mathcal{P}_n^*(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d)$ is a complete locally convex space. If $m \leqslant n$ then clearly $\mathcal{P}_m^* \subset \mathcal{P}_n^*$, with continuous linear inclusion map.

We now define

$$\mathcal{P}^*(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) := \bigcup_{n \in \mathcal{N}(\mathfrak{a}_0^*)} \mathcal{P}_n^*(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d),$$

and equip this space with the inductive limit locally convex topology.

In particular, these definitions may be interpreted for $\mathcal{H} = \emptyset$ and $d = \emptyset$. In this case we have $\pi_{\omega,d} = 1$ for every $\omega \subset \mathfrak{a}_0^*$, so that

$$\mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset) \quad \text{and} \quad \mathcal{P}^*(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset)$$

are just spaces of holomorphic functions φ on $\mathfrak{a}_{0\mathbb{C}}^*$ determined in the above fashion by seminorms of the form

$$\nu_{\omega,n}(\varphi) := \sup_{\lambda \in \omega + i\mathfrak{a}_0^*} (1 + |\lambda|)^n |\varphi(\lambda)|.$$

For the rest of this section, let $P \in \mathcal{P}_0$ be fixed and let $\mathcal{H} = \mathcal{H}_{G,\tau,P}$ be the smallest collection of Σ-hyperplanes in $\mathfrak{a}_{0\mathbb{C}}^*$ such that the singular locus of $\lambda \mapsto E^*(P : \lambda : \cdot)$ is contained in the union $\bigcup \mathcal{H}$. In view of Lemma 5.2 the collection $\mathcal{H}$ is locally finite and consists of real Σ-hyperplanes. Moreover, by the same lemma, the set of $H \in \mathcal{H}$ with $H \cap \mathfrak{a}_0^*(P, R) \neq \emptyset$ is finite, for every $R \in \mathbb{R}$.

We define the map $d = d_{G,\tau,P} : \mathcal{H} \to \mathbb{N}$ as follows. For each $H \in \mathcal{H}$ we fix $l_H : \mathfrak{a}_{0\mathbb{C}}^* \to \mathbb{C}$ as in (7.6) and define $d(H)$ as the smallest integer $k \geqslant 0$ such that the $C^\infty(G) \otimes \mathrm{Hom}(V_\tau, \mathcal{A}_2)$-valued meromorphic function $l_H^k E^*(P : \cdot)$ extends regularly over $H \setminus \bigcup\{H' \in \mathcal{H} \mid H' \neq H\}$.

Given a subset $\omega \subset \mathfrak{a}_0^*$ whose closure meets only finitely many hyperplanes from $\mathcal{H}$, we define the polynomial function $\pi_{\omega,d}$ as in (7.6). In particular, we write $\pi = \pi_P$ for this polynomial with $\omega = \mathfrak{a}_0^* \cap \bar{\mathfrak{a}}_0^*(P, 0)$, where the second set in the intersection denotes the closure of the set (5.2) with $R = 0$. Thus, the $C^\infty(G) \otimes \mathrm{Hom}(V_\tau, \mathcal{A}_2)$-valued function $\lambda \mapsto \pi(\lambda) E^*(P : \lambda)$ is holomorphic on a neighborhood of $\bar{\mathfrak{a}}_0^*(P, R)$ and $\pi \in \Pi_{\Sigma,\mathbb{R}}(\mathfrak{a}_0^*)$ is minimal with this property.

We define the following closed subspace of $\mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) \otimes \mathcal{A}_2$:

$$\begin{aligned} (7.8) \qquad & \mathcal{P}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P) \\ & := \{\varphi \in \mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) \otimes \mathcal{A}_2 \mid \mathcal{L}\varphi = 0, \ \forall \mathcal{L} \in \mathrm{AC}(G, \tau, P)\}. \end{aligned}$$

Finally, we define the space $\mathcal{P}_{\mathrm{AC}}^*(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$ in a similar fashion, but with $\mathcal{P}$ replaced by $\mathcal{P}^*$.

Definition 7.6. (a) Let $R > 0$. We define the *Paley–Wiener space* $\mathrm{PW}_R(G, \tau, P)$ to be the subspace of $\mathcal{P}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$ consisting of functions φ such that, for all $n \in \mathbb{N}$,

$$\sup_{\lambda \in \bar{\mathfrak{a}}_0^*(P,0)} \big(1 + \|\lambda\|\big)^n e^{-R|\mathrm{Re}\,\lambda|} \big\|\pi(\lambda)\varphi(\lambda)\big\| < \infty. \tag{7.9}$$

The space is equipped with the relative topology.

(b) For $R > 0$ we define the *distributional Paley–Wiener space* $\mathrm{PW}_R^*(G, \tau, P)$ to be the subspace of $\mathcal{P}_{\mathrm{AC}}^*(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$ consisting of functions φ for which there exists a constant $n \in \mathbb{N}$ such that

$$\sup_{\lambda \in \bar{\mathfrak{a}}_0^*(P,0)} \big(1 + \|\lambda\|\big)^{-n} e^{-R|\mathrm{Re}\,\lambda|} \big\|\pi(\lambda)\varphi(\lambda)\big\| < \infty. \tag{7.10}$$

This space is also equipped with the relative topology.

We will finish this section by discussing the relation of these Paley–Wiener spaces with the unnormalized Paley–Wiener spaces introduced in Definitions 3.5 and 4.1. As a preparation, we first give another characterization of the unnormalized Paley–Wiener spaces.

We define $\mathcal{P}_{^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ and $\mathcal{P}_{^u\mathrm{AC}}^*(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ as the closed subspaces of the spaces $\mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset) \otimes \mathcal{A}_2$ and $\mathcal{P}^*(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset) \otimes \mathcal{A}_2$, respectively, consisting of the functions φ satisfying the relations $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in {^u\mathrm{AC}}_{\mathrm{hol}}(G, \tau, P)$.

Proposition 7.7. *Let $R > 0$.*

(a) *The space ${^u\mathrm{PW}}_R(G, \tau, P)$ consists of the functions $\varphi \in \mathcal{P}_{^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ with the property that, for every $n \in \mathbb{N}$,*

$$\sup_{\lambda \in \bar{\mathfrak{a}}_0^*(P,0)} \big(1 + |\lambda|\big)^n e^{-R|\mathrm{Re}\,\lambda|} \big\|\varphi(\lambda)\big\| < \infty. \tag{7.11}$$

Moreover, the topology of ${^u\mathrm{PW}}_R(G, \tau, P)$ coincides with the relative topology from $\mathcal{P}_{^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$.

(b) *The space ${^u\mathrm{PW}}_R^*(G, \tau, P)$ consists of the functions $\varphi \in \mathcal{P}_{^u\mathrm{AC}}^*(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ with the property that there exists a number $n \in \mathbb{N}$ such that*

$$\sup_{\lambda \in \bar{\mathfrak{a}}_0^*(P,0)} \big(1 + |\lambda|\big)^{-n} e^{-R|\mathrm{Re}\,\lambda|} \big\|\varphi(\lambda)\big\| < \infty. \tag{7.12}$$

Moreover, the topology of ${^u\mathrm{PW}}_R^*(G, \tau, P)$ coincides with the relative topology from $\mathcal{P}_{^u\mathrm{AC}}^*(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$.

Proof. We start by making some remarks on Euclidean Paley–Wiener spaces. From the text preceding Definition 3.2 we recall the definition of the space $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*)$, equipped with the Fréchet topology $\mathcal{T}$ induced by the seminorms $\nu_{R,n}$, for $n \in \mathbb{N}$.

Clearly, $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \subset \mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \varnothing)$, with continuous inclusion map. We denote by $\mathcal{T}_r$ the associated relative topology on $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*)$. Then $\mathcal{T}$ is finer than $\mathcal{T}_r$. We will show that both topologies are in fact equal.

By the Euclidean Paley–Wiener theorem, Euclidean Fourier transform $\mathcal{F}_{\text{eucl}}$ defines a continuous linear isomorphism from $C_R^\infty(\mathfrak{a}_0)$ onto $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*)$. From a straightforward estimation it follows that the inverse Fourier transform $\mathcal{F}_{\text{eucl}}^{-1}$ is continuous from $\mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \varnothing)$ to $C^\infty(\mathfrak{a}_0)$. It follows from the above that the identity map $\mathcal{F}_{\text{eucl}} \circ \mathcal{F}_{\text{eucl}}^{-1}$ is continuous from $(\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*), \mathcal{T}_r)$ to $(\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*), \mathcal{T})$. Hence $\mathcal{T} = \mathcal{T}_r$.

From the text preceding Definition 4.1 we recall the definition of $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$, equipped with the inductive limit locally convex topology denoted $\mathcal{T}^*$. Clearly, $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*) \subset \mathcal{P}^*(\mathfrak{a}_{0\mathbb{C}}^*, \varnothing)$, with continuous inclusion map. Let $\mathcal{T}_r^*$ denote the associated relative topology on $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$. Then $\mathcal{T}^*$ is finer than $\mathcal{T}_r^*$. We will show that both topologies are equal.

By the distributional Euclidean Paley–Wiener theorem, $\mathcal{F}_{\text{eucl}}$ maps $C_R^{-\infty}(\mathfrak{a}_0)$ bijectively onto $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$. Fix $R' > R$ and let k be an arbitrary positive integer. Then $C_R^{-\infty}(\mathfrak{a}_0)_k$, the subspace of generalized functions of order at most k, naturally embeds into the continuous linear dual of the Banach space $C_{R'}^k(\mathfrak{a}_0)$, equipped with the C^k norm $\|\cdot\|_{C^k}$. Accordingly, we equip $C_R^{-\infty}(\mathfrak{a}_0)_k$ with the restriction of the dual norm. By a straightforward estimation, there exists a $C_k > 0$ such that

$$\nu_{R,k}\big(\mathcal{F}_{\text{eucl}}(f)\big) \leqslant C_k \|f\|_{C^k}$$

for all $f \in C_{R'}^k(\mathfrak{a}_0)$. Let $n \in \mathcal{N}(\mathfrak{a}_0^*)$. Then by transposition it readily follows that $\mathcal{F}_{\text{eucl}}^{-1}$ maps $\mathcal{P}_n^*(\mathfrak{a}_{0\mathbb{C}}^*, \varnothing) \cap \mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$ into $C_R^{-\infty}(\mathfrak{a}_0)_k$, with $k = n(\{0\}) + \dim \mathfrak{a}_0 + 1$. Moreover, this map is continuous with respect to the relative topology from $\mathcal{P}_n^*(\mathfrak{a}_{0\mathbb{C}}^*, \varnothing)$ on the first of these spaces. Now $\mathcal{F}_{\text{eucl}}$ maps $C_R^{-\infty}(\mathfrak{a}_0)_k$ continuously into $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$. It follows that the identity map $\mathcal{F}_{\text{eucl}} \circ \mathcal{F}_{\text{eucl}}^{-1}$ is continuous from $\mathcal{P}_n^*(\mathfrak{a}_{0\mathbb{C}}^*, \varnothing) \cap \mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$ to $\mathrm{H}_R^*(\mathfrak{a}_{0\mathbb{C}}^*)$. By the universal property of the inductive limit, it follows that $\mathcal{T}_r^*$ is finer than $\mathcal{T}^*$. Hence $\mathcal{T}^* = \mathcal{T}_r^*$.

We proceed with the actual proof. We denote the subspace of $\mathcal{P}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ defined in (a) by $\mathcal{PW}$ and the similar subspace defined in (b) by $\mathcal{PW}^*$. These subspaces are equipped with the relative topologies. Clearly, ${}^u\mathrm{PW}_R(G, \tau, P)$ is a subspace of $\mathcal{PW}$, and ${}^u\mathrm{PW}_R^*(G, \tau, P)$ a subspace of $\mathcal{PW}^*$, with continuous inclusion maps. To conclude the proof we must establish the converse inclusions, also with continuous inclusion maps.

A holomorphic function $\varphi \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ that is annihilated by ${}^u\mathrm{AC}_{\text{hol}}(G, \tau, P)$, satisfies the functional equations

$$(7.13) \qquad \varphi(\lambda) = {}^\circ C_{P|P}(s : -\bar{\lambda})^* \varphi(s\lambda),$$

for $s \in W$ and generic $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$; in view of Proposition 6.4 this follows from Lemma 6.2 with $P = Q$.

In view of Lemma 5.3, there exists a product q_s of linear factors of the form $\langle\cdot,\alpha\rangle - c$, with $\alpha \in \Sigma$ and $c \in \mathbb{C}$, such that $\lambda \mapsto q_s(\lambda)^\circ C_{P|P}(s : -s^{-1}\bar{\lambda})^*$ is polynomial. We define

$$q(\lambda) := \prod_{s\in W} q_s(s\lambda) \quad (\lambda \in \mathfrak{a}_{0\mathbb{C}}^*).$$

Then there exist constants $C > 0$ and $N \in \mathbb{N}$, such that, for all $s \in W$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$,

$$\left\| q(s^{-1}\lambda)^\circ C_{P|P}(s : -s^{-1}\bar{\lambda}) \right\| \leqslant C\big(1 + |\lambda|\big)^N.$$

If we combine this with the functional equation (7.13), we see that, for each $s \in W$, every $n \in \mathbb{Z}$ and all $\lambda \in \bar{\mathfrak{a}}_0^*(P,0)$,

$$\begin{aligned}&\big(1 + |s^{-1}\lambda|\big)^n e^{-R|\mathrm{Re}\, s^{-1}\lambda|} \left\| q(s^{-1}\lambda)\varphi(s^{-1}\lambda) \right\| \\ &\quad\leqslant C\big(1 + |\lambda|\big)^{n+N} e^{-R|\mathrm{Re}\,\lambda|} \left\| \varphi(\lambda) \right\|.\end{aligned}$$

Combining these estimates for $s \in W$, we obtain

$$\nu_{R,n}(q\varphi) \leqslant C \sup_{\lambda\in\bar{\mathfrak{a}}_0^*(P,0)} \big(1 + |\lambda|\big)^{n+N} e^{-R|\mathrm{Re}\,\lambda|} \left\| \varphi(\lambda) \right\|,$$

where $\nu_{R,n}$ is defined as in the first part of the proof. On the other hand, by an easy application of Cauchy's integral formula it follows that for every $n \in \mathbb{Z}$ there exists a constant $C_n > 0$ such that

$$\nu_{R,n}(\varphi) \leqslant C_n \nu_{R,n}(q\varphi),$$

for all $\varphi \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$. It follows from these estimates that $\mathcal{PW}$ equals the intersection of $\mathcal{P}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ with the Euclidean Paley–Wiener space $\mathrm{H}_R(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$. By definition, the topology of $\mathcal{PW}$ equals the relative topology from the first of these spaces. By the first part of the proof, the topology also coincides with the relative topology from the second of these spaces. It follows that $\mathcal{PW} \subset {}^u\mathrm{PW}_R(G,\tau,P)$ with continuous inclusion map. This establishes (a). Assertion (b) follows by a similar argument. □

We define the map $\mathcal{U}_P \in \mathrm{Aut}(\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2)$ by

$$\mathcal{U}_P\varphi(\lambda) = C_{P|P}(1 : -\bar{\lambda})^*\varphi(\lambda).$$

Then (7.3) may be rephrased as

$${}^u\mathcal{F}_P = \mathcal{U}_P \circ \mathcal{F}_P.$$

Theorem 7.8. *Let $R > 0$. The map $\mathcal{U}_P \in \mathrm{Aut}(\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2)$ restricts to a topological linear isomorphism*

$$\mathrm{PW}_R^*(G,\tau,P) \xrightarrow{\simeq} {}^u\mathrm{PW}_R^*(G,\tau,P),$$

and similarly to a topological linear isomorphism

$$\mathrm{PW}_R(G, \tau, P) \xrightarrow{\simeq} {}^u\mathrm{PW}_R(G, \tau, P).$$

Proof. It follows from Lemma 5.3 that the functions $\lambda \mapsto C_{P|P}(1 : -\bar{\lambda})^{*\pm 1}$ belong to $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma) \otimes \mathrm{End}(\mathcal{A}_2)$, so that the map $\mathcal{U}_P$ restricts to a linear automorphism of the space $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma) \otimes \mathcal{A}_2$. It follows from Lemma 7.2 that transposition induces an automorphism $\mathcal{U}_P^t$ of $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes \mathcal{A}_2^*$.

Let ${}^u\mathrm{AC}(G, \tau, P)$ denote the space of Laurent functionals $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes \mathcal{A}_2^*$ such that (3.3) holds for all $x \in G$ and $v \in V_\tau$. Then it follows from Lemma 7.3 that the natural map

$$(7.14) \qquad {}^u\mathrm{AC}(G, \tau, P) \to {}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, P)$$

is surjective.

If $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes \mathcal{A}_2^*$, then

$$\mathcal{L}\big[{}^uE^*(P : \cdot : x)v\big] = \mathcal{L} \circ \mathcal{U}_P\big[E^*(P : \cdot : x)v\big],$$

for all $x \in G$ and $v \in V_\tau$. It follows that $\mathcal{U}_P^t$ restricts to a linear isomorphism from ${}^u\mathrm{AC}(G, \tau, P)$ onto the space $\mathrm{AC}(G, \tau, P)$.

Let $\mathcal{M}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$ denote the space of $\varphi \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma) \otimes \mathcal{A}_2$ such that $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in \mathrm{AC}(G, \tau, P)$. Similarly, let $\mathcal{M}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$ denote the space of $\varphi \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma) \otimes \mathcal{A}_2$ such that $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in {}^u\mathrm{AC}(G, \tau, P)$. Then it follows from the above that $\mathcal{U}_P$ defines a linear isomorphism from $\mathcal{M}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$ onto $\mathcal{M}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$. The latter of the two spaces consists of holomorphic functions, by Lemma 7.4, applied with S consisting of the functions ${}^uE^*(P : \cdot : x)v$, for $x \in G$ and $v \in V_\tau$. By surjectivity of the map (7.14) it follows that $\mathcal{M}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$ equals the space $\mathcal{O}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, P)$ of $\varphi \in \mathcal{O}(\mathfrak{a}_{0\mathbb{C}}^*) \otimes \mathcal{A}_2$ with $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in {}^u\mathrm{AC}_{\mathrm{hol}}(G, \tau, P)$.

We conclude that $\mathcal{U}_P$ defines a linear isomorphism from $\mathcal{M}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$ onto $\mathcal{O}_{{}^u\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, P)$. Only the estimates remain to be taken care of.

Let $\omega \subset \mathfrak{a}_0^*$ be a bounded open subset. Applying Lemma 7.4 with S consisting of all functions $E^*(P : \cdot : x)$, for $x \in G$ and with $\Omega = \omega + i\mathfrak{a}_0^*$ and $\psi = \pi_{\omega,d} \otimes I$, we see that for every $\varphi \in \mathcal{M}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P)$ the function $\pi_{\omega,d}\varphi$ is holomorphic on $\omega + i\mathfrak{a}_0^*$. This implies that $\mathcal{M}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \Sigma, P) \subset \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) \otimes \mathcal{A}_2$.

Again, let $\omega \subset \mathfrak{a}_0^*$ be a bounded open subset. Then $\omega + i\mathfrak{a}_0^* \subset \mathfrak{a}_0^*(P, r)$ for a suitable real number r. By Lemma 5.2, there exists a polynomial $p \in \Pi_\Sigma(\mathfrak{a}_0^*)$ such that $\lambda \mapsto p(\lambda)C_{P|P}(1 : -\bar{\lambda})^*$ is holomorphic on $\mathfrak{a}_0^*(P, r)$. Moreover, by application of the same lemma, there exist $N \in \mathbb{N}$ and $C > 0$ such that, for every $n \in \mathbb{Z}$,

$$\nu_{\omega,n}(p\pi_{\omega,d}\mathcal{U}_P\varphi) \leqslant C\nu_{\omega,n+N}(\pi_{\omega,d}\varphi)$$

for all $\varphi \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) \otimes \mathcal{A}_2$. On the other hand, let ω_0 be a relatively compact subset of ω. Then, by an easy application of Cauchy's integral formula, there exists for every $n \in \mathbb{Z}$ a constant $C_n > 0$, such that

$$\nu_{\omega_0,n}(\psi) \leqslant C_n \nu_{\omega,n}(p\pi_{\omega,d}\psi)$$

for all $\psi \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) \otimes \mathcal{A}_2$ that are regular on $\omega + i\mathfrak{a}_0^*$. It follows that

$$\nu_{\omega_0,n}(\mathcal{U}_P\varphi) \leqslant C_n C\, \nu_{\omega,d,n+N}(\varphi),$$

for all $\varphi \in \mathcal{P}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$. This implies that $\mathcal{U}_P$ maps $\mathcal{P}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$ continuous linearly into $\mathcal{P}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset) \otimes \mathcal{A}_2$, hence also continuously into $\mathcal{P}^u{}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$. Moreover, the same statement holds for the spaces with superscript $*$. By a similar argument, involving Lemma 5.2 for the inverse of the C-function, it follows that $\mathcal{U}_P^{-1}$ maps $\mathcal{P}^u{}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \emptyset, P)$ continuous linearly into $\mathcal{P}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$. A similar statement is true for the spaces with the superscript $*$.

Finally, using Lemma 5.2 once more in the above fashion, it follows that a function $\varphi \in \mathcal{P}_{\mathrm{AC}}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d, P)$ satisfies the estimate (7.9) for all $n \in \mathbb{N}$ if and only if $\mathcal{U}_P\varphi$ satisfies the estimate (7.11) for all $n \in \mathbb{N}$. If we combine this with Proposition 7.7 (a), we see that $\mathcal{U}_P$ restricts to a topological linear isomorphism from $\mathrm{PW}_R(G, \tau, P)$ onto ${}^u\mathrm{PW}_R(G, \tau, P)$. The analogous statements for the spaces with the superscript $*$ are proved in a similar fashion. $\square$

8. THE GROUP AS A SYMMETRIC SPACE

We retain the notation of the previous sections. In this section we will view the group G as a symmetric space, and compare the Fourier transforms and Paley–Wiener spaces for G with those for the associated symmetric space. This will allow us to deduce the Paley–Wiener theorems for the group from the analogous theorems for symmetric spaces.

As G is of the Harish-Chandra class, the group ${}_*G := G \times G$ is of this class as well. We consider the involution ${}_*\sigma$ of ${}_*G$ defined by ${}_*\sigma(x, y) = (y, x)$. Its group of fixed points, the diagonal subgroup, is denoted by ${}_*H$. The space ${}_*\mathrm{X} := {}_*G/{}_*H$ is a reductive symmetric space of the Harish-Chandra class. The map ${}_*G \to G$ given by $(x, y) \mapsto xy^{-1}$ induces a diffeomorphism

$$(8.1) \qquad p : {}_*\mathrm{X} := {}_*G/{}_*H \to G,$$

intertwining the natural left action of ${}_*G$ with the action of ${}_*G$ on G given by $(x, y)g = xgy^{-1}$. Accordingly, G becomes a reductive symmetric space of the Harish-Chandra class. We fix a choice of Haar measure dg on G; then $dx = p^*(dg)$ is a choice of ${}_*G$-invariant measure on ${}_*\mathrm{X}$.

The map ${}_*\theta := (\theta, \theta)$ is a Cartan involution of ${}_*G$ which commutes with ${}_*\sigma$. The associated maximal compact subgroup equals ${}_*K := K \times K$. We recall that $\tau = (\tau_1, \tau_2)$ is a double unitary representation of K in V_τ and define the unitary representation of ${}_*K$ in V_τ by ${}_*\tau(k_1, k_2)v = \tau(k_1)v\tau(k_2)^{-1}$. Then pull-back by p induces a topological linear isomorphism

$$(8.2) \qquad p^* : C^{-\infty}(G : \tau) \xrightarrow{\simeq} C^{-\infty}({}_*\mathrm{X} : {}_*\tau),$$

which we shall also denote by $f \mapsto {}_*f$. Clearly this isomorphism restricts to an isomorphism between the subspaces indicated by $C_c^{-\infty}$, C^∞ and C_c^∞.

The -1 eigenspaces of ${}_*\theta$ and ${}_*\sigma$ in ${}_*\mathfrak{g}$ equal ${}_*\mathfrak{p} := \mathfrak{p} \times \mathfrak{p}$ and ${}_*\mathfrak{q} := \{(X, -X) \mid X \in \mathfrak{g}\}$, respectively. It follows that a maximal abelian subspace ${}_*\mathfrak{a}_\mathrm{q}$ of ${}_*\mathfrak{p} \cap {}_*\mathfrak{q}$ is given by

$${}_*\mathfrak{a}_\mathrm{q} := \big\{(X, -X) \mid X \in \mathfrak{a}_0\big\}.$$

The derivative of p equals the isomorphism ${}_*\mathfrak{g}/{}_*\mathfrak{h} \to \mathfrak{g}$ induced by the map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto X - Y$; we will denote this derivative by p as well. The map p restricts to the isomorphism from ${}_*\mathfrak{a}_\mathrm{q}$ onto $\mathfrak{a}_0$ given by $(X, -X) \mapsto 2X$. Via pull-back under the isomorphism p, we transfer the given inner product on $\mathfrak{a}_0$ to an inner product on ${}_*\mathfrak{a}_\mathrm{q}$. Accordingly, for every $R > 0$ the closed ball ${}_*\bar{B}_R$ of center 0 and radius R in ${}_*\mathfrak{a}_\mathrm{q}$ is mapped onto the similar ball $\bar{B}_R$ in $\mathfrak{a}_0$. It follows that $p : {}_*\mathrm{X} \to G$ maps ${}_*\mathrm{X}_R := {}_*K \exp {}_*\bar{B}_R {}_*H$ onto $G_R = K\bar{B}_R K$. The following result is now obvious.

Lemma 8.1. *Let $R > 0$. The map p^*, defined in* (8.2), *restricts to a topological linear isomorphism from $C_R^{-\infty}(G : \tau)$ onto $C_R^{-\infty}({}_*\mathrm{X} : {}_*\tau)$, and, similarly, to a topological linear isomorphism from $C_R^\infty(G : \tau)$ onto $C_R^\infty({}_*\mathrm{X} : {}_*\tau)$.*

We will now compare the definition of the normalized Eisenstein integral for $({}_*\mathrm{X}, {}_*\tau)$ given in [3, Section 2], with the one for (G, τ) given in the present paper. The isometry $p : {}_*\mathfrak{a}_\mathrm{q} \to \mathfrak{a}_0$ induces an isometry $p^* : \mathfrak{a}_0^* \to {}_*\mathfrak{a}_\mathrm{q}^*$ (for the dual inner products on these spaces). The complex linear extension of this map is denoted by $p^* : \lambda \mapsto {}_*\lambda$, $\mathfrak{a}_{0\mathbb{C}}^* \to {}_*\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

The system ${}_*\Sigma$ of restricted roots of ${}_*\mathfrak{a}_\mathrm{q}$ in ${}_*\mathfrak{g}$ consists of the roots $\frac{1}{2}{}_*\alpha$, for $\alpha \in \Sigma$. The root space for the root $\frac{1}{2}{}_*\alpha$ is given by $\mathfrak{g}_\alpha \times \{0\} \oplus \{0\} \times \mathfrak{g}_{-\alpha}$. Thus, if $P \in \mathcal{P}_0$ then ${}_*P := P \times \bar{P}$ belongs to the set ${}_*\mathcal{P}_\sigma^{\min}$ of minimal ${}_*\sigma{}_*\theta$-stable parabolic subgroups of ${}_*G$ containing ${}_*A_\mathrm{q}$; the associated system of positive roots is ${}_*\Sigma({}_*P) = \{\frac{1}{2}{}_*\alpha \mid \alpha \in \Sigma(P)\}$. As usual, let $\rho_{{}_*P} \in {}_*\mathfrak{a}_\mathrm{q}^*$ be defined by

$$\rho_{{}_*P}(\cdot) := \frac{1}{2}\,\mathrm{tr}\big(\mathrm{ad}(\cdot)|\mathfrak{n}_{{}_*P}\big).$$

Then $\rho_{{}_*P} = {}_*(\rho_P)$. Thus, without ambiguity, we may use the notation ${}_*\rho_P$ for this functional.

Via the isometry ${}_*\mathfrak{a}_\mathrm{q} \simeq \mathfrak{a}_0$ we see that every element of the Weyl group of ${}_*\Sigma$ can be realized by an element of ${}_*K \cap {}_*H = \mathrm{diag}(K)$. It follows that the coset space ${}_*W/{}_*W_{{}_*K \cap {}_*H}$ consists of one element. Thus as a set of representatives for this coset space in the normalizer of ${}_*\mathfrak{a}_q$ in ${}_*K$ we may fix ${}_*\mathcal{W} = \{e\}$. Accordingly, the space ${}^\circ C({}_*\tau)$ of [3, Eq. (17)], now denoted by ${}_*\mathcal{A}_2$, is given by

$$ {}_*\mathcal{A}_2 := C^\infty({}_*M/{}_*M \cap {}_*H : {}_*\tau) = L^2({}_*M/{}_*M \cap {}_*H : {}_*\tau). $$

We equip the space ${}_*M/{}_*M \cap {}_*H$ with the pull-back of the invariant measure on M_0 under the analogue of the map (8.1) for the tuple (M_0, τ_0), and the space ${}_*\mathcal{A}_2$ with the associated L^2-type inner product. Then the analogue of the isomorphism (8.2) for the tuple (M_0, τ_0) gives a unitary isomorphism

$$ p^* : \psi \mapsto {}_*\psi, \quad \mathcal{A}_2 \to {}_*\mathcal{A}_2. $$

Given $\psi \in \mathcal{A}_2$, we define the Eisenstein integral $E({}_*P : {}_*\psi : {}_*\lambda)$ as in [3, Eq. (20)]. Then we have the following relation with the Eisenstein integral defined in (2.3).

Lemma 8.2. *Let $P \in \mathcal{P}_0$, $\psi \in \mathcal{A}_2$. Then for every $(x, y) \in {}_*G$,*

$$ E({}_*P : {}_*\psi : {}_*\lambda)(x, y) = E\big(P : C_{\bar{P}|P}(1 : -\bar{\lambda})^* \psi : \lambda\big)(xy^{-1}), \tag{8.3} $$

as an identity of meromorphic functions in the variable $\lambda \in \mathfrak{a}^*_{0\mathbb{C}}$.

Proof. We briefly write $N = N_P$. Let $\lambda \in \mathfrak{a}^*_{0\mathbb{C}}$ be such that $\mathrm{Re}\,\lambda + \rho_P$ is $\bar{P}$-dominant. Then $\mathrm{Re}\,{}_*\lambda + {}_*\rho_P$ is ${}_*\bar{P}$-dominant. Let ${}_*\tilde{\psi}({}_*\lambda) : {}_*G \to V_\tau$ be defined as in [3, Eq. (17)], for the situation at hand. Then ${}_*\tilde{\psi}({}_*\lambda) = 0$ outside ${}_*P{}_*H$ and

$$ {}_*\tilde{\psi}({}_*\lambda : nam_1 g, \bar{n}a^{-1}m_2 g) = a^{2\lambda + 2\rho_P} \psi(m_1 m_2^{-1}), $$

for $n \in N, \bar{n} \in \bar{N}, a \in A_0, m_1, m_2 \in M_0$ and $g \in G$. It follows that ${}_*\tilde{\psi}({}_*\lambda) = {}_*[\tilde{\psi}(\lambda)]$, where $\tilde{\psi}(\lambda) : G \to V_\tau$ is defined to be zero outside $NA_0M_0\bar{N}$ and

$$ \tilde{\psi}(\lambda : nam\bar{n}) = a^{\lambda + \rho_P} \psi(m) $$

for $(n, a, m, \bar{n}) \in N \times A_0 \times M_0 \times \bar{N}$. In view of [3, Eq. (20)], we now infer that

$$ \begin{aligned} E\big({}_*P : {}_*\psi : {}_*\lambda : (x_1, x_2)\big) &= \int_{K \times K} {}_*\tau(k_1, k_2)^{-1} {}_*\tilde{\psi}({}_*\lambda : k_1 x_1, k_2 x_2)\, dk_1\, dk_2 \\ &= \int_{K \times K} \tau(k_1)^{-1} \tilde{\psi}\big(\lambda : k_1 x_1 x_2^{-1} k_2^{-1}\big) \tau(k_2)\, dk_1\, dk_2 \\ &= E\big(P : \Psi(\lambda) : \lambda : x_1 x_2^{-1}\big), \end{aligned} \tag{8.4} $$

see (2.3), where the function $\Psi(\lambda) : M_0 \to V_\tau$ is defined by

$$ \Psi(\lambda : m) = \int_K \tilde{\psi}\big(\lambda : mk^{-1}\big) \tau(k)\, dk. \tag{8.5} $$

As M_0 normalizes $\bar{N}$, the function $\tilde{\psi}(\lambda)$ transforms according to $\tilde{\psi}(\lambda : xm) = \tilde{\psi}(\lambda : x)\tau(m)$, for $x \in G$ and $m \in M_0$. It follows that the integrand in (8.5) is a left M_0-invariant measurable function on K. We now consider the real analytic map $(\kappa, H, \nu) : G \to K \times \mathfrak{a}_0 \times N$ determined by

$$x = \kappa(x) \exp H(x) \nu(x) \quad (x \in G). \tag{8.6}$$

Then the Haar measure $d\bar{n}$ may be normalized such that for every $\varphi \in C(K/M_0)$ we have

$$\int_K \varphi(k)\, dk = \int_{\bar{N}} \varphi\big(\kappa(\bar{n})\big)\, e^{-2\rho_P H(\bar{n})}\, d\bar{n}.$$

We apply this substitution of variables to the integral (8.5). Since $\kappa(\bar{n}) = \bar{n}\nu(\bar{n})^{-1} \times \exp[-H(\bar{n})]$, whereas $\tilde{\psi}(\lambda)$ is right $\bar{N}$-invariant, it follows that

$$\Psi(\lambda, m) = \int_{\bar{N}} e^{\langle \lambda - \rho_P, H(\bar{n}) \rangle} \psi(m) \tau\big(\kappa(\bar{n})\big)\, d\bar{n} = \big[C_{\bar{P}|P}(1 : -\bar{\lambda})^* \psi\big](m). \tag{8.7}$$

The last equality follows from [14, §19, Theorem 1], since τ is unitary (take Remark 5.1 into account). In the notation of [14], we have $\mu_P = 0$ since P is minimal.

Combining (8.4) with (8.7) we obtain the desired identity for $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ such that $\operatorname{Re}\lambda + \rho_P$ is $\bar{P}$-dominant. Now apply analytic continuation. $\square$

Remark 8.3. With the same method of proof it can be shown that Lemma 8.2 generalizes to arbitrary parabolic subgroups of G. In the more general lemma, the expression on the left-hand side is defined as in Harish-Chandra's work, taking account of Remark 2.1. Moreover, the Eisenstein integral on the right-hand side is defined as in [10, p. 61], with λ in place of $-\lambda$. In the proof one has to replace the decomposition (8.6) by the decomposition induced by $G = K \exp(\mathfrak{m}_P \cap \mathfrak{p}) A_P N_P$.

Remark 8.4. Lemma 8.2 can also be derived from [2, Lemma 1], by expressing both Eisenstein integrals as matrix coefficients of representations of the principal series, see [3, Eq. (25)] and [13, Theorem 7.1].

Corollary 8.5. *Let $P \in \mathcal{P}_0$, $\psi \in \mathcal{A}_2$. Then*

$$E^\circ({}_*P : p^*\psi : {}_*\lambda) = p^*\big(E^\circ(P : \psi : \lambda)\big), \tag{8.8}$$

as an identity of meromorphic functions in the variable $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$.

Proof. In (8.3) we substitute $x_1 = m_1 a$ and $x_2 = m_2 a^{-1}$ for $m_1, m_2 \in M_0 A_0$ and $a \in A_0$. Comparing coefficients in the asymptotic expansions of type (5.1) for both sides, as $a \to \infty$ in A_P^+, we obtain that $p^{*-1} \circ C_{*P|*P}(1 : {}_*\lambda) \circ p^* =$

$C_{P|P}(1:\lambda)C_{\bar{P}|P}(1:-\bar{\lambda})^*$. The result now follows from Lemma 8.2 if we apply the definitions of the normalized Eisenstein integrals, see (7.1) and [3, Eq. (49)]. $\square$

We can now formulate the relation between the Fourier transforms for G and those for the associated symmetric space ${}_*\mathrm{X}$; for the definition of the latter, we refer to [3, Eq. (59)]. We define the linear isomorphism

$$(8.9)\qquad p^*: \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*)\otimes\mathcal{A}_2 \xrightarrow{\simeq} \mathcal{M}({}_*\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*)\otimes{}_*\mathcal{A}_2$$

by $p^*(\psi)({}_*\lambda)=p^*[\psi(\lambda)]$, for $\psi\in\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*)\otimes\mathcal{A}_2$ and for generic $\lambda\in\mathfrak{a}_{0\mathbb{C}}^*$.

Lemma 8.6. *Let $P\in\mathcal{P}_0$. Then the following diagram commutes*:

$$\begin{tikzcd} C_c^{-\infty}(G:\tau) \arrow[r, "p^*"] \arrow[d, "\mathcal{F}_P"'] & C_c^{-\infty}({}_*\mathrm{X}:{}_*\tau) \arrow[d, "\mathcal{F}_{*P}"] \\ \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*)\otimes\mathcal{A}_2 \arrow[r, "p^*"] & \mathcal{M}({}_*\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*)\otimes{}_*\mathcal{A}_2. \end{tikzcd}$$

Proof. Let $f\in C_c^{-\infty}(G:\tau)$ and put ${}_*f:=p^*(f)\in C_c^{-\infty}({}_*\mathrm{X}:{}_*\tau)$. Let $\psi\in\mathcal{A}_2$. Then it follows by application of Corollary 8.5 and the fact that $dx=p^*(dg)$ that

$$\begin{aligned}\langle\mathcal{F}_{*P}({}_*f)({}_*\lambda),{}_*\psi\rangle &= \int_{{}_*\mathrm{X}}\langle{}_*f(x),E^\circ({}_*P:{}_*\psi:-{}_*\bar{\lambda}:x)\rangle\,dx\\ &=\int_G\langle f(g),E^\circ(P:\psi:-\bar{\lambda}:g)\rangle\,dg\\ &=\langle\mathcal{F}_Pf(\lambda),\psi\rangle=\langle{}_*[\mathcal{F}_Pf(\lambda)],{}_*\psi\rangle.\end{aligned}$$

In the last equality we have used that $p^*:\psi\mapsto{}_*\psi$ is a unitary isomorphism from $\mathcal{A}_2$ onto ${}_*\mathcal{A}_2$. Using the definition of the map (8.9) we conclude that $\mathcal{F}_{[*P]}\circ p^*(f)=p^*\circ\mathcal{F}_Pf$. $\square$

The map $p^*:\lambda\mapsto{}_*\lambda$ is a linear isomorphism from $\mathfrak{a}_0^*$ onto ${}_*\mathfrak{a}_\mathrm{q}^*$, mapping the set Σ onto $2{}_*\Sigma$. It follows that the map p^* in (8.9) maps $\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\Sigma)\otimes\mathcal{A}_2$ isomorphically onto $\mathcal{M}({}_*\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,{}_*\Sigma)\otimes{}_*\mathcal{A}_2$. Moreover, the transpose of its inverse restricts to a linear isomorphism

$$(8.10)\qquad p^*:\mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*,\Sigma)_{\mathrm{laur}}^*\otimes\mathcal{A}_2^*\xrightarrow{\simeq}\mathcal{M}({}_*\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*,{}_*\Sigma)_{\mathrm{laur}}^*\otimes{}_*\mathcal{A}_2^*,$$

which we shall also denote by $\mathcal{L}\mapsto{}_*\mathcal{L}$.

Lemma 8.7. *The isomorphism* (8.10) *maps* $\mathrm{AC}(G,\tau,P)$ *onto* $\mathrm{AC}({}_*\mathrm{X},{}_*\tau,{}_*P)$.

Proof. Let $(x,y)\in G\times G$ and $v\in V_\tau$. We consider the functions $f:\mu\mapsto E^*({}_*P:\mu:(x,y))v$ and $g:\lambda\mapsto E^*(P:\lambda:xy^{-1})v$, where the first dual Eisenstein integral is defined in a fashion analogous to (7.2), see [5, Eq. (2.3)].

It follows from Corollary 8.5 that $f = p^* g$. Thus, for every $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{0\mathbb{C}}^*)_{\mathrm{laur}}^* \otimes \mathcal{A}_2^*$ we have that $p^*(\mathcal{L}) f = {}_*\mathcal{L}_* g = \mathcal{L} g$. It follows from this that $\mathcal{L} \in \mathrm{AC}(G, \tau, P)$ if and only if $p^*(\mathcal{L}) \in \mathrm{AC}({}_*\mathrm{X} : {}_*\tau : {}_*P)$. The result follows. □

Let $P \in \mathcal{P}_0$ and $R > 0$. We define the Paley–Wiener space $\mathrm{PW}_R({}_*\mathrm{X} : {}_*\tau : {}_*P)$ as in [7, Definition 3.4]. The mentioned definition depends on a choice of positive roots, which we take to be ${}_*\Sigma({}_*P)$. We enlarge this space to a distributional Paley–Wiener space $\mathrm{PW}_R^*({}_*\mathrm{X} : {}_*\tau : {}_*P)$ in complete analogy with the way in which (b) enlarges (a) in Definition 7.6.

Theorem 8.8. *Let $P \in \mathcal{P}_0$ and $R > 0$. The map* (8.9) *restricts to a topological linear isomorphism*

$$(8.11) \qquad p^* : \mathrm{PW}_R^*(G, \tau, P) \xrightarrow{\simeq} \mathrm{PW}_R^*({}_*\mathrm{X}, {}_*\tau, {}_*P)$$

and to a similar isomorphism between the spaces without the superscript $*$.

Remark 8.9. The definition of $\mathrm{PW}_R({}_*\mathrm{X}, {}_*\tau, {}_*P)$ in [7, Definition 3.4], is not completely analogous to Definition 7.6 (a), as the definition in [7] invokes only the relations determined by the space $\mathrm{AC}_{\mathbb{R}}({}_*\mathrm{X}, \tau, P)$ of Arthur–Campoli functionals with real support; see also [7, Definition 3.2]. However, it follows by application of [7, Theorem 3.6], that the functions in the Paley–Wiener space thus defined satisfy all remaining Arthur–Campoli relations as well. Consequently, [7, Definition 3.4], determines the same Paley–Wiener space as the analogue of Definition 7.6 for the triple $({}_*\mathrm{X}, {}_*\tau, {}_*P)$. A similar remark can be made for the distributional Paley–Wiener space.

It follows from these observations, combined with the results of this paper, that the Paley–Wiener spaces introduced in Definitions 3.2, 3.5 and 7.6 remain unaltered if only the Arthur–Campoli functionals with real support are invoked.

Proof. We define the hyperplane configuration ${}_*\mathcal{H} = {}_*\mathcal{H}_{{}_*\mathrm{X}, {}_*\tau, {}_*P}$ and the map ${}_*d = d_{{}_*\mathrm{X}, {}_*\tau, {}_*P}$ as in [7, text following Lemma 2.1]. In view of the relation between the dual Eisenstein integrals, it follows that $p^*(\mathcal{H}) = \mathcal{H}$ and that $d_* = d \circ p^*$. This implies that the map p^* introduced in (8.9) restricts to a topological linear isomorphism

$$\mathcal{P}^*(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d) \to \mathcal{P}^*({}_*\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, {}_*\mathcal{H}, {}_*d),$$

and to a similar isomorphism between the spaces without the superscript $*$. In view of Lemma 8.7 these isomorphisms restrict to isomorphisms of the closed subspaces with index AC.

Let ${}_*\pi$ be defined as π in [7], for the tuple $({}_*\mathrm{X}, {}_*\tau)$ and the positive system ${}_*\Sigma^+ = {}_*\Sigma({}_*P)$. Then it follows from the relation between the dual Eisenstein integrals that ${}_*\pi = p^*(\pi)$, possibly up to a nonzero constant factor, which we may ignore here. As $p^* : \mathfrak{a}_0^* \to {}_*\mathfrak{a}_{\mathrm{q}}^*$, $\lambda \mapsto {}_*\lambda$, is an isometry, it follows that

$$\pi(\lambda) e^{R|\mathrm{Re}\,\lambda|} = {}_*\pi({}_*\lambda) e^{R|\mathrm{Re}\,{}_*\lambda|}.$$

Moreover, from $p^*(\Sigma(P)) = 2_*\Sigma({}_*P)$ it follows that ${}_*\mathfrak{a}_\mathrm{q}^*({}_*P, 0) = p^*(\mathfrak{a}_0^*(P, 0))$. Thus, a function $\varphi \in \mathcal{P}_{\mathrm{AC}}^{(*)}(\mathfrak{a}_{0\mathbb{C}}^*, \mathcal{H}, d)$ satisfies an estimate of type (7.9) (or of type (7.10)) if and only if the function $p^*(\varphi)$ satisfies the analogous estimate for the triple $({}_*\mathrm{X}, {}_*\tau, {}_*P)$. The result now follows in view of Remark 8.9. $\square$

It follows from Lemma 8.6 combined with Lemma 8.1 and Theorem 8.8 that $\mathcal{F}_P$ is a topological linear isomorphism $C_R^\infty(G : \tau) \to \mathrm{PW}_R(G, \tau, P)$ if and only if $\mathcal{F}_{*P}$ is a topological linear isomorphism $C_R^\infty({}_*\mathrm{X} : {}_*\tau) \to \mathrm{PW}_R({}_*\mathrm{X}, {}_*\tau, {}_*P)$. In view of the results of Section 7, it now follows that Theorem 3.6, hence Arthur's Paley–Wiener theorem, is a consequence of [7, Theorem 3.6]. Similarly, it follows from the Paley–Wiener theorem proved in [8] that $\mathcal{F}_P$ is a topological linear isomorphism from $C_R^{-\infty}(G : \tau)$ onto $\mathrm{PW}_R^*(G, \tau, P)$. Thus, the validity of Theorem 4.2 follows from the main result of [8].

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