Effective Hamiltonians for resonance interaction dynamics and interdisciplinary analogies

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Abstract

Resonance interactions of oscillators are responsible for fundamental effects in different areas of physics and classical mechanics. The resonance between any two oscillators/modes destroys their individuality by generating a new effective oscillator of energy flow between the two parent oscillators, which is known as beating. In particular, the fundamental character of such energy exchange oscillators is revealed by the fact of their exact integrability in many physically reasonable cases. The present work illustrates such a standpoint on elastic oscillators, discrete liquid sloshing models, and ‘macroscopic’ quantum dynamics related to Josephson's effects. In the case of elastic oscillations, a strongly nonlinear conservative oscillator describing the dynamics of energy partition between two identical linearly coupled oscillators with polynomial restoring force characteristics is analyzed.

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1. Introduction

Resonance interactions of oscillators are considered to be the cause of fundamental physical effects in different areas physics and classical mechanics. In stationary cases, there is certain carrier of the energy, which is either normal or local vibration mode. A normal mode usually involves the entire system whose inertial components show steady state coherent vibrations, whereas a local mode has its leading inertial element (mass) carrying most of the total energy. In both cases, the energy sticks with its carrier theoretically during the unlimited time assuming that

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dissipation and other external perturbations are absent. The so-called ‘internal resonance’ may occur if some of the natural frequencies of linearized model form a co-measurable combination. This may cause transient motions, in which the energy is drifting around the vibration modes. As a result, it becomes reasonable to switch the attention on the energy itself rather than its carriers/modes. However, in contrast to modal analysis typically using positional coordinates, descriptive quantities for the energy dynamics may be introduced in multiple ways yet with no established criteria regarding their natural choice. The present work is to address this situation based on the case of 1:1 internal resonance in perfectly symmetric two-degrees-of-freedom models. In particular, a set of new coordinates is suggested, in which the dynamics energy flow inside the system becomes Hamiltonian regardless physical contents of the problem. Generally, the Hamiltonian form of equations is viewed to be most natural in physical sciences. Since such form automatically points to the integral of motion, in our cases, it is sufficient for deriving exact analytical solutions describing the slow energy drift. The text below illustrates the corresponding technical details on elastic oscillators, discrete liquid sloshing models, and reveals the analogy with behaviors of quantum arrays of particles in symmetric double well potentials.

2. Illustrating model and descriptive quantities

2.1. Resonance effects in coupled oscillators

For preliminary illustrating purposes, let us consider a system of two identical quasi linear mechanical oscillators coupled by a relatively weak linear spring as shown in Fig. 1.

![Fig. 1. Nonlinear identical oscillators coupled by a weak linear spring.](image)

The corresponding differential equations of motion can be represented in the form

\[
\begin{align*}
\ddot{q}_1 + 2\zeta\Omega\dot{q}_1 + \Omega^2 q_1 + f(q_1, q_2) &= 0 \\
\ddot{q}_2 + 2\zeta\Omega\dot{q}_2 + \Omega^2 q_2 + f(q_2, q_1) &= 0
\end{align*}
\]

where \( \Omega \) is a ‘partial’ frequency of a linearized oscillator, which takes place when the position of another oscillator is fixed; note that using such a frequency effectively eliminates one of the two coordinates from the coupling \( \beta(x_2 - x_1) \) in both equations (1), therefore function \( f(q_1, q_2) \equiv -\beta q_2 + \alpha_5 q_1^3 + \alpha_5 q_1^5 + \alpha_7 q_1^7 \) describes both the interaction between oscillators and nonlinear terms of the restoring force of base springs, where all the coefficients are constant; and finally \( \zeta \) is a damping ratio, which is small as compared to unity.

System (1) is under the so-called 1:1 resonance that may lead to the intensive energy exchange between the oscillators. Typically such energy exchange represents an oscillating energy flow (beating) in the temporal scale, which is proportional to the strength of coupling \( \beta \). The amount of ‘mobile’ energy is determined by the initial distribution of the total energy as well as the initial phase difference between the oscillators. Note that the resonance condition itself does not guarantee that the energy exchange will actually happen. For some specific initial conditions, the mobile portion of the total energy may become zero. In such case the system appears to be in one of the stationary nonlinear normal mode regimes\(^1\). For some other initial conditions, the entire total energy becomes mobile. Such a limit case is accompanied by noticeable dynamic effects that can be qualified as extremely non-stationary alternatives to nonlinear normal modes\(^2\). However, the present study focuses on specifics of the energy
exchange flow regardless the above mentioned limit cases. In particular, we show that the corresponding effective oscillator describing the energy exchange appears to be Hamiltonian and therefore exactly solvable.

2.2. Describing the resonance energy exchange

In order to describe the dynamics of resonance interaction, we introduce the following ‘energy matrix’

$$
E = \begin{bmatrix}
E_{11} & E_{12} \\
E_{12} & E_{22}
\end{bmatrix}
$$

(2)

where all the matrix elements are measured in energy units $E_{ij} = (\dot{q}_i \dot{q}_j + \Omega^2 q_i q_j)/2$.

Note that $E_{11}$ and $E_{22}$ are typical energies per unit mass of harmonic oscillators excluding possible contributions of nonlinear stiffness components. Nevertheless, it will be shown that the elements $E_{ij}$ still can be used as descriptive variables characterizing the energy exchange dynamics even with the presence of nonlinearities. In particular, if $\zeta = 0$ and $f \equiv 0$, the matrix $E$ is constant, otherwise the leading order components of the elements $E_{ij}$ are slowly varying functions as compared to the fast phase $\Omega$. It is more convenient, however, to use some combinations of the matrix elements $E_{ij}$ characterizing the assembly of oscillators, rather than the oscillators themselves. Such combinations are designed as follows$^4$:

- Total energy excluding coupling

$$
E_{11} + E_{22} = E_0
$$

(3)

- Energy partitioning

$$
\frac{E_{11} - E_{22}}{E_{11} + E_{22}} = P = -\sin \theta
$$

(4)

- Coherency of vibrations

$$
\frac{E_{12}}{\sqrt{E_{11}E_{22}}} = Q = -\cos \Delta
$$

(5)

Note that the possibility of considering the angles $\theta$ and $\Delta$ is justified by the fact that $-1 \leq P \leq 1$ and $-1 \leq Q \leq 1$. Specific reasons for introducing both notations, the indexes $\{P, Q\}$ and the angles $\{\theta, \Delta\}$, are explained by the next two sections. The most important reason however is that ‘mixed combination’ $P$ and $\Delta$ represents canonical coordinates of an effective Hamiltonian of energy flow.

2.3. Physical meaning of the descriptive quantities

Physical interpretation of the indexes $\{P, Q\}$ and the angles $\{\theta, \Delta\}$ is given below. For instance the left and right boundaries and the center of the energy partitioning interval $-1 \leq P \leq 1$ are associated with specific cases of the energy distribution as follows:
\[ P = -\sin \theta = \begin{cases} 
-1: & E_{11} = 0, \quad E_{22} = E_0 \quad \text{localization on 2nd oscc.} \\
0: & E_{11} = E_{22} = \frac{1}{2} E_0 \quad \text{equipartition} \\
1: & E_{11} = E_0, \quad E_{22} = 0 \quad \text{localization on 1st oscc.} 
\end{cases} \tag{6} \]

The boundaries and the center of the ‘coherency’ interval \(-1 \leq Q \leq 1\) are interpreted from the standpoint of modal trajectories on the configuration plane \(q_1 q_2\) based on the expressions; see Fig. 2 for illustration.

\[ Q = -\cos \Delta = \begin{cases} 
-1: & q_2 = \left(\frac{1 - P}{1 + P}\right)^{1/2} q_1 \quad \text{anti-phase} \\
0: & \frac{q_1^2}{1 + P} + \frac{q_2^2}{1 - P} = \frac{E_0}{\Omega^2} \quad \text{elliptic (incoherent)} \\
1: & q_2 = \left(\frac{1 - P}{1 + P}\right)^{1/2} q_1 \quad \text{in-phase} 
\end{cases} \tag{7} \]

Fig. 2. Geometrical meaning of the coherency index: in-phase \((Q = 1)\), out-of-phase \((Q = -1)\), and ‘rotating’ \((Q = 0)\) modes depending on the energy partitioning index \(P\).

Note that implementation of particular cases (6) and (7) requires certain initial conditions and therefore unlikely to be captured perfectly in real experiments. However, calculating quantities (3) through (5) from numerical solutions or experimental data records and then considering their time histories provides an adequate qualitative information on non-stationary effects of modal transitions. Furthermore, we show that the above descriptive quantities are especially convenient for using the averaging tools. For that reason, the differential equations for quantities (6) – (7) are obtained from the original equations (1) through the coordinate transformation \(\{q_1, \dot{q}_1, q_2, \dot{q}_2\} \rightarrow \{E_0, P, \Delta, \delta\}\):
\[ q_1 = \frac{1}{\Omega} \sqrt{E_0(1 + P)} \cos \delta, \quad \dot{q}_1 = -\sqrt{E_0(1 + P)} \sin \delta \]
\[ q_2 = -\frac{1}{\Omega} \sqrt{E_0(1 - P)} \cos(\delta + \Delta), \quad \dot{q}_2 = \sqrt{E_0(1 - P)} \sin(\delta + \Delta) \]

where \( \delta \) is a fast phase, namely, if \( \zeta = 0 \) and \( f \equiv 0 \) then \( \delta = \Omega t \) while other three coordinates are constant, and (8) satisfies (1) exactly.

3. Effective Hamiltonians

3.1. Coupled oscillators

Substituting transformation (8) in equations (1), and applying averaging with respect to \( \delta \) reveals that the energy partitioning \( \mathcal{P} \) and phase \( \Delta \) are canonical Hamiltonian variables of the 'energy exchange oscillator'

\[ dE_0 / dt = -2\zeta \omega E_0 \]  
\[ \frac{d\mathcal{P}}{dt} = -\frac{\partial H(P, \Delta; E_0)}{\partial \Delta}, \quad \frac{d\Delta}{dt} = \frac{\partial H(P, \Delta; E_0)}{\partial \mathcal{P}} \]

where \( H(P, \Delta; E_0) = -2(\omega / E_0) \langle \mathcal{E}(q_1, \dot{q}_1, q_2, \dot{q}_2) \rangle_{\delta} \) is effective Hamiltonian obtained by taking into account (8) and averaging the total energy with respect the fast phase \( \delta \).

\[ E(q_1, q_2, \dot{q}_1, \dot{q}_2) \equiv \frac{1}{2} (q_1^2 + q_2^2) - \beta q_1 q_2 + \sum_{k=0}^{3} \frac{\alpha_{2k+1}}{2k+2} (q_1^{2k+2} + q_2^{2k+2}) \]

In the particular case \( \alpha_{2} = 0 \), equations (10) take the form, which is shown in the middle column of Fig. 4, where the new temporal scale \( s = \beta t / \omega \), associated with the relative strength of coupling, is introduced.

3.2. Liquid sloshing

Let us consider the liquid sloshing phenomenon in a tank partially filled by fluid; see Fig. 3. The model assumes irrotational flows of incompressible and originally inviscid fluid inside the rectangular tank with perfectly stiff walls. The tank has a square base whose side length is unity. The fluid depth \( h \) is measured in the units of side wall length. Note that, on one hand, the assumption of square base brings some formal simplifications to the governing differential equations of motion. However, on the other hand, the symmetry induced 1:1 resonance coupling between the first two dominating modes essentially complicates the system dynamics. The modal discretization can be conducted by representing the velocity potential and the free liquid surface in the form of modal expansions as, respectively,

\[ \phi(t, x, y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}(t) W_j(x, y, z) \]
and

\[
\eta(t, x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} U_{ij}(x, y, z)
\]  \hspace{1cm} (13)

It was shown\(^5\) that the original set of partial differential equations of fluid dynamics and boundary conditions can be eventually reduced to a set of second-order ordinary differential equations with respect to time dependent coefficients of expansion (13). Note that any two linear modes indexed as \(i,j\) and \(j,i\) should have the same natural frequencies due to the tank’s symmetry \(x \leftrightarrow y\). Therefore, at least two modes, say \(q_1 = b_{10}\) and \(q_2 = b_{01}\), must be taken into account when considering the modal reduction. Generally speaking, higher modes will be excited as well due to nonlinear coupling\(^5\), whose strength depends upon frequency ratios of different modes. If the low and high modes do not resonate then the energy outflow from low to high modes can be ignored. Since the sloshing frequencies are functions of the fluid depth, then the number of essential modes may be different for different fluid levels.\(^4\) In particular, the two-mode model appears to be justified within the depth range \(0.28 < h < 0.5\). In such a case, the two-mode free sloshing can be described by equations (1) after replacing the function \(f(q_1, q_2)\) with

\[
f(q_1, q_2, \dot{q}_1, \dot{q}_2) \equiv S_3 q_1^2 q_2^2 + S_4 q_1^2 \dot{q}_2 + S_5 q_2 \dot{q}_1 q_2 + S_8 q_1^3 + S_9 q_1 q_2^2
\]  \hspace{1cm} (14)

where \(S_k\) (\(k = 3, 4, 5, 8, 9\)) are constant coefficients.\(^5\)

![Fig. 3. Sloshing phenomenon in a square base tank partially filled by fluid.](image)

Applying the coordinate transformation (8) and then averaging the resultant equations with respect to fast phase \(\delta\) gives finally the Hamiltonian system for the energy partitioning \(P\) and phase \(\Delta\), which is shown in the first column of Fig. 4. Note that the only parameter of the corresponding effective Hamiltonian \(\kappa\) is constant despite of the presence of dissipation. This is achieved by introducing a variable temporal scale \(s(t)\), which is linked to the decaying energy of sloshing; see the bottom of the first column in Fig. 4. Therefore, in the present example, the dynamic properties of energy exchange between the first two modes remain the same at any total energy level, which is due to the homogeneous form of function (14). Namely, introducing the variable temporal scale is possible due the property \(f(\lambda q_1, \lambda q_2, \lambda \dot{q}_1, \lambda \dot{q}_2) \equiv \lambda^n f(q_1, q_2, \dot{q}_1, \dot{q}_2)\), where, in the present case, \(n = 3\). As a result, even in the presence of damping, the effective Hamiltonian system is integrated exactly as
3.3. Macroscopic quantum dynamics

Let us show that the dynamics of interaction between two quantum states in macroscopic quantum dynamics is quite similar to those of the resonance interactions of classical oscillators. The term ‘macroscopic’ actually means that huge arrays of quantum particles behaves coherently enough in such a way that the entire assembly can be represented by a single nonlinear Schrödinger equation also known as Gross-Pitaevskii equation \[ \psi_t = -\frac{i\hbar}{2m} \frac{\nabla^2 \psi(r,t) + [V(r) + g_0 |\psi(r,t)|^2] \psi(r,t)}{m} = 0 \]

where the nonlinearity is due to transition from the conventional multi-body to a ‘single-body’ modeling of quantum systems; \( m \) and \( r \) are the atomic mass and spatial coordinate, respectively, \( g_0 \) is a scattering wave length. The two states can be associated with two weakly coupled parts of Bose-Einstein condensate (BEC) accumulated in different traps of the double-well trap potential \( V(r) \), whose shape is shown at the top of third column in Fig. 4.

The temporal evolution of the wave function is represented by the superposition of two wave functions

\[ \psi(r,t) = \psi_1(t)\Phi_1(r) + \psi_2(t)\Phi_2(r) \]

where \( \psi_{1,2}(t) = \sqrt{N_{1,2}(t)} \exp[i\Theta_{1,2}(t)] \) are the modal time dependent amplitudes, \( N_i \) and \( \Theta_i \) \((i = 1,2)\) are the numbers of particles and phases of the condensates associated with two different traps, such that \( N_1 + N_2 = N_T \) is a fixed total number of particles; \( \Phi_1(r) \) and \( \Phi_2(r) \) can be interpreted as two different modes localized near the first and the second traps respectively, such that

\[ \int_{-\infty}^{\infty} \Phi_1(r)\Phi_2(r)dr \approx 0 \]

Note that the modes are not exactly orthogonal due to their ‘tails’ overlapping the opposite potential traps. Further analytical procedure is quite similar to the typical modal discretization of continuous elastic structures, except manipulations with complex numbers are involved. Namely, the modal expansion (17) is substituted in the partial differential equation (17). Then, taking into account condition (18), Galerkin’s method is applied in order to obtain two ordinary differential equations for the complex modal amplitudes. This eventually gives two first-order differential equations with respect to the real-valued quantities \( z = (N_1 - N_2)/(N_1 + N_2) \) and \( \phi = \Theta_2 - \Theta_1 \) characterizing the particle distribution and phase difference, respectively. Furthermore, after re-naming the unknowns as \( z = P \) and \( \phi = \pi - \Delta \) the system appears to have perfectly the same Hamiltonian form as that describing the energy exchange between the classical oscillators considered in Section 3.1; compare the second and third columns in Fig. 4.
4. Conclusions

The outcome of the present study is summarized in Fig. 4, which confirms that the variables (3) through (5) may provide a natural choice for the description of energy flows between resonating subcomponents in a wide range of systems. This is due to the fact the quantities \( P \) and \( \Delta \) represent a pair of canonical Hamiltonian coordinates, and such statement holds for dynamic models of quite different physical contents.

![Fig. 4. Physical analogies: same pair of canonical Hamiltonian variables describes the energy exchange between resonating liquid sloshing modes, the energy flow between weakly coupled elastic oscillators, and the transition of quantum particles between the wells of symmetric double well potential; there is almost perfect analogy between the last two cases, while some structural difference in the effective Hamiltonian of liquid sloshing reflects the essentially nonlinear nature of coupling (14).](image)

References