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Discrete Mathematics 284 (2004) 181–188
DISCRETE MATHEMATICS
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The Hamilton–Waterloo problem: the case of Hamilton cycles and triangle-factors

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Received 6 January 2003; received in revised form 3 March 2003; accepted 21 November 2003

Dedicated to Curt Lindner on the occasion of his 65th birthday

Abstract

We discuss a special case of the Hamilton–Waterloo problem in which a 2-factorization of K_n is sought consisting of 2-factors of two kinds: Hamiltonian cycles, and triangle-factors. We determine completely the spectrum of solutions for several infinite classes of orders n.

Keywords: 2-Factorizations; Hamiltonian cycle; Triangle-factor

1. Introduction

The well-known *Oberwolfach problem* first formulated by Ringel at a meeting in Oberwolfach in 1967 (and motivated by seating arrangements there) asks for a 2-factorization of the complete graph K_{2n+1} into 2-factors each of which is isomorphic to a given 2-factor Q. If the components of Q are cycles of length $c_1, \ldots, c_s$ (with $\Sigma c_i = 2n+1$) then the corresponding instance of the Oberwolfach problem is denoted by OP$(2n+1; c_1, \ldots, c_s)$. The Oberwolfach problem has been completely settled in the case when $c_1 = \cdots = c_s$, i.e. when all components of Q are cycles of the same (necessarily odd) length ([3]; see also [2]); a solution has been shown to exist in all such cases. However, the Oberwolfach problem remains open in general. It has been conjectured that a solution to OP$(2n+1; c_1, \ldots, c_s)$ exists always apart from two exceptional cases of OP$(9; 4, 5)$ and OP$(11; 3, 3, 5)$ when the solution is known not to exist.

There are many known generalizations of the Oberwolfach problem: the spouse-avoiding variant [10], the bipartite analogue [12], and its extension to multigraphs [9], to name just a few. In this article, we deal with a special case of another extension of the Oberwolfach problem, the so-called *Hamilton–Waterloo problem*. This problem asks for a 2-factorization of the complete graph K_{2n+1} in which r of its 2-factors are isomorphic to a given 2-factor Q, and s of its 2-factors are isomorphic to a given 2-factor R, with $r+s=n$. If the components of Q are cycles of length $c_1, \ldots, c_q$ and the components of R are cycles of length $d_1, \ldots, d_t$ (with $\Sigma c_i = \Sigma d_j = 2n+1$), then the corresponding instance of the Hamilton–Waterloo problem is denoted by HW$(2n+1; r, s; c_1, \ldots, c_q; d_1, \ldots, d_t)$ or briefly by HW$(2n+1; r, s; Q, R)$ or just by HW$(r, s; Q, R)$.

The Hamilton–Waterloo problem was first mentioned in [7] where, as well as in [8], solutions for many small cases are given. First substantial contribution to its solution is in [1] which deals with the existence of solutions to the problem HW(r, s, Q, R) where Q and R are two of the following: a triangle-factor, a pentagon-factor, or a 15-gon-factor. Here, an s-gon-factor is a 2-factor whose all components are cycles of length s.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.11.031

The special case of the Hamilton–Waterloo problem that we deal with in this paper is the case where Q is a Hamiltonian cycle, and R is a triangle-factor (HC, and a Δ-factor, for brevity). We will use the asterisk to indicate this special case by HW*.

The Hamilton–Waterloo problem HW*, as described above, contains as a special case the Oberwolfach problem where Q is an HC, or where Q is a Δ-factor; in each of these cases a solution is well known to exist [2,6]. Thus we may assume $s > 0$, that is, we may restrict ourselves to instances where at least one Δ-factor is present which in turn implies that the number of vertices must be congruent to $3 \pmod 6$.

This paper is a contribution towards determining completely the spectrum HW* of solutions to this special case of the Hamilton–Waterloo problem. After a preliminary Section 2, in Sections 3 and 4 we present a direct construction, and a recursive construction, respectively, dealing with the spectrum HW*. In Section 5 this spectrum is determined for several small values of n while Section 6 contains our main result.

2. Preliminaries

Let $I(n) = \{0, 1, \ldots, (n-1)/2\}$. Let HW$^*(6k+3)$ be the set of all integers r such that there exists a 2-factorization of K_{6k+3} in which r of the 2-factors are Hamiltonian cycles, and the remaining $s = 3k+1-r$ two-factors are Δ-factors. Since the total number of 2-factors in any 2-factorization of K_{6k+3} equals $3k+1$, we have clearly $\mathrm{HW}^*(6k+3) \subseteq I(6k+3) = \{0, 1, \ldots, 3k+1\}$. Our comment in the last paragraph of the preceding section shows $0 \in \mathrm{HW}^*(6k+3)$ as well as $3k+1 \in \mathrm{HW}^*(6k+3)$ for all $k \geqslant 0$. It appears that, apart from certain small exceptions, $\mathrm{HW}^*(n) = I(n)$. In Section 6 we show this equality indeed holds for several infinite classes of orders $n = 6k+3$.

In what follows, the vertex-set of our complete graph K_{6k+3} will be the set $Z_{2k+1} \times \{1, 2, 3\}$ where $k \geqslant 1$. For simplicity of notation (especially, to avoid double subscripts), we will write $V \times \{1\} = A = \{a_i : i = 0, \ldots, 2k\}$, $V \times \{2\} = B = \{b_i : i = 0, \ldots, 2k\}$, $V \times \{3\} = C = \{c_i : i = 0, \ldots, 2k\}$. All indices will be taken modulo $2k+1$. Further, for $0 \leqslant d \leqslant 2k$, define sets of edges

$$(AB)_d = \{\{a_i, b_{i+d}\} : i = 0, \ldots, 2k\},$$

$$(BC)_d = \{\{b_i, c_{i+d}\} : i = 0, \ldots, 2k\},$$

$$(CA)_d = \{\{c_i, a_{i+d}\} : i = 0, \ldots, 2k\}.$$

Thus we can view the edge set E of our complete graph K_{6k+3} as

$$E = E([A]) \cup E([B]) \cup E([C]) \cup \bigcup_{d=0}^{2k} \{(AB)_d \cup (BC)_d \cup (CA)_d\}.$$

Here, as usual, $[X]$ is the complete graph induced by the set of vertices X.

Finally, for $d = 0, \ldots, 2k$, let F_d be the subgraph induced by the set of edges $(AB)_d \cup (BC)_d \cup (CA)_{-2d}$. Clearly, for each $d = 0, \ldots, 2k$, F_d is a Δ-factor of our K_{6k+3}. Indeed, the above definition implies that the edges of F_d form the triangles $(a_i b_{i+d} c_{i+2d})$ for $i = 0, \ldots, 2k$.

3. A direct construction

We start with a simple lemma.

Lemma 1. *Let $-2k \leqslant p, q, r \leqslant 2k$ be integers such that $p+q+r$ and $2k+1$ are relatively prime. Then the set of edges $(AB)_p \cup (BC)_q \cup (CA)_r$ induces a Hamiltonian cycle of K_{6k+3}.*

Proof. Set $t = p+q+r$. The edges of $(AB)_p \cup (BC)_q \cup (CA)_r$ may be arranged to form a walk $W = (a_0 b_p c_{p+q} a_t b_{t+p} c_{t+p+q} a_{2t} \cdots a_{it} b_{it+p} c_{it+p+q} a_{(i+1)t} \cdots a_{2kt} b_{2kt+p}\ c_{2kt+p+q} a_{(2k+1)t} (= a_0))$. As $(t, 2k+1) = 1$, all vertices $a_0, a_t, a_{2t}, \ldots, a_{2kt}$ are mutually distinct, thus $\{a_0, a_t, a_{2t}, \ldots, a_{2kt}\} = A$. Similar holds for B and C. Thus W passes through each vertex of K_{6k+3} exactly once, and hence W is a Hamiltonian cycle. $\square$

Corollary 2. *The edges of $F_0 \cup F_2 \cup F_{2k}$ can be decomposed into three HCs.*

Proof. The edge-set of $F_0 \cup F_2 \cup F_{2k}$ can also be viewed as the union of sets $G_1 \cup G_2 \cup G_3$ where $G_1 = (AB)_0 \cup (BC)_0 \cup (CA)_{-4}$, $G_2 = (AB)_2 \cup (BC)_{2k} \cup (CA)_{-4k}$, $G_3 = (AB)_{2k} \cup (BC)_2 \cup (CA)_0$. As all three integers $0+0-4$, $2+2k-4k$, and $2k+2+0$ are relatively prime to $2k+1$, by Lemma 1, each of the three graphs induced by G_1, G_2 and G_3 forms an HC. □

Corollary 3. *For each $d = 0, \ldots, 2k-1$, the edges of $F_d \cup F_{d+1}$ can be decomposed into two HCs.*

Proof. The edge-set of $F_d \cup F_{d+1}$ can also be viewed as the union of $H_1 \cup H_2$ where $H_1 = (AB)_d \cup (BC)_{d+1} \cup (CA)_{-2d}$, $H_2 = (AB)_{d+1} \cup (BC)_d \cup (CA)_{-2d-2}$. Since both $d + (d+1) - 2d = 1$ and $d + 1 + d - (2d+2) = -1$ are relatively prime to $2k+1$, Lemma 1 completes the proof. □

Corollary 4. *The edges of $F_1 \cup F_3$ can be decomposed into two HCs.*

Proof. Similar to that of Corollary 3—it suffices to note that both $1+1-6=-4$ and $3+3-2=4$ are relatively prime to $2k+1$. □

Lemma 5. *The complete graph K_{2k+1}, $k \geqslant 2$, can be decomposed into k Hamiltonian paths $P_0, \ldots, P_{k-1}$ and a set of k edges $H = \{e_0, \ldots, e_{k-1}\}$ such that for $i = 0, \ldots, k-1$, the endvertices of P_i are the same as the two vertices of the edge e_i, and any of the following can be made to hold*:

(i) *the edges of H are independent*, *i.e. H is a matching with k edges*;
(ii) *$k-1$ of the edges of H form a path P, and the remaining kth edge is vertex-disjoint from P*;
(iii) *H consists of a matching M with $k-2$ edges and the remaining two edges of H form a path of length two which is vertex-disjoint from M.*

Proof. We will show that each of conditions (i)–(iii), respectively, can be made to hold for each integer $k \geqslant 2$. Consider the well-known "Walecki" decomposition of K_{2k+1} into k Hamiltonian cycles $C_0, \ldots, C_{k-1}$ (see, e.g. [5]) where

$$C_0 = (v_0 v_1 v_{2k-1} v_2 v_{2k-2} v_3 v_{2k-3} \cdots v_{k+2} v_{k-1} v_{k+1} v_k v_\infty v_0),$$

and C_i is obtained from C_0 by increasing all subscripts by i modulo $2k$. In order to obtain k Hamiltonian paths P_i, $i = 0, \ldots, k-1$ and the desired set of edges H, we remove from C_i an edge e_i, $i = 0, \ldots, k-1$; this will guarantee that each path P_i has the same endvertices as the edge e_i. For case (i), i.e. when H is to be a matching with k edges, we take $e_i = (v_{\lceil k/2 \rceil + i}, v_{\lceil k/2 \rceil + k + i})$. For case (ii) (when H is to be a path of length $k-1$ and a disjoint edge), take $e_i = (v_i, v_{i+1})$ for $i = 1, \ldots, k-1$, and $e_0 = (v_0, v_\infty)$. Finally, for case (iii) (when H consists of a matching with $k-2$ edges and a disjoint path of length 2), put $e_i = (v_{\lceil k/2 \rceil + i}, v_{\lceil k/2 \rceil + k + i})$ for $i = 1, \ldots, k-1$, and $e_0 = (v_0, v_\infty)$. □

Lemma 6. *The edge-set $[A] \cup [B] \cup [C] \cup F_0$ can be decomposed into $k+1$ Hamiltonian cycles.*

Proof. Clearly, we may assume $k \geqslant 2$. By Lemma 5(ii), we can decompose the edge-set of $[A]$ into k Hamiltonian paths $P_0, \ldots, P_{k-1}$ and a subgraph H which consists of a path of length $k-1$ and a disjoint edge; we do it so that $e_0 = (a_1, a_2)$, $e_1 = (a_0, a_4)$, and $e_i = (a_{2i}, a_{2i+2})$ for $i = 2, \ldots, k-1$. Similarly, by Lemma 5(i), we can decompose the edge-set of $[B]$ into k Hamiltonian paths $P'_0, \ldots, P'_{k-1}$ and a subgraph H' consisting of a matching with k edges; we do it so that $e'_0 = (b_0, b_1)$ and $e'_i = (b_{2i+1}, b_{2i+2})$ for $i = 1, \ldots, k-1$. Finally, by Lemma 5(iii) we can decompose the edge-set of $[C]$ into k Hamiltonian paths $P''_0, \ldots, P''_{k-1}$ and a subgraph H'' consisting of a matching with $k-2$ edges and a disjoint path of length two; we do it so that $e''_0 = (c_0, c_2)$, $e''_1 = (c_0, c_3)$, and $e''_i = (c_{2i}, c_{2i+1})$ for $i = 2, \ldots, k-1$.

We take as one of the Hamiltonian cycles the following:
$C = (a_0 b_0 b_1 c_1 a_1 a_2 b_2 c_2 c_0 c_3 a_3 b_3 b_4 c_4 c_5 a_5 b_5 b_6 c_6 \ldots c_{2i} c_{2i+1} a_{2i+1} b_{2i+1} b_{2i+2} c_{2i+2} \ldots\ c_{2k-2} c_{2k-1} a_{2k-1} b_{2k-1} b_{2k} c_{2k} a_{2k} a_{2k-2} a_{2k-4} \ldots a_{2i} a_{2i-2} \ldots a_6 a_4 a_0)$. Note that C contains all edges of $H \cup H' \cup H''$.

The remaining k Hamiltonian cycles of G are given next. Here the symbol $P_t[u \cdots v]$ indicates that the Hamiltonian path P_t has the vertices u and v as its endvertices.

$$H_0 = (P_0[a_1 \ldots a_2] c_2 P''_0[c_2 \ldots c_0] b_0 P'_0[b_0 \ldots b_1] a_1),$$

$$H_1 = (P_1[a_0 \ldots a_4] b_4 P'_1[b_4 \ldots b_3] c_3 P''_1[c_3 \ldots c_0] a_0),$$

$$H_i = (P_i[a_{2i} \ldots a_{2i+2}] b_{2i+2} P'_i[b_{2i+2} \ldots b_{2i+1}] c_{2i+1} P''_i[c_{2i+1} \ldots c_{2i}] a_{2i})$$

for $i = 2, \ldots, k-1$.

It is straightforward to verify that the Hamiltonian cycles $C, H_0, \dots, H_{2k-1}$ are edge-disjoint, and that their union equals $E(G)$, the edge-set of G. □

We also record the following obvious corollary.

Corollary 7. *For any* $i \in \{0, \dots, 2k\}$, *the edge-set of* $[A] \cup [B] \cup [C] \cup F_i$ *can be decomposed into* $k+1$ *Hamiltonian cycles.*

We conclude this section with two theorems concerning the sets $\mathrm{HW}^*(6k+3)$.

Theorem 8. *Let* $n = 6k+3$, *and assume* $k \equiv 1 \pmod 3$. *Then* $I(n) \backslash \{1\} \subset \mathrm{HW}^*(n)$.

Proof. Recall that $I(n) = \{0, 1, \dots, (n-1)/2\}$. Then the statement of the theorem is that when $k \equiv 1 \pmod 3$ and $t \in I(n), t \neq 1$, there exists a 2-factorization of the complete graph K_n (where $n \equiv 9 \pmod{18}$) whose t two-factors are Hamiltonian cycles and the remaining $(n-1)/2 - t = 3k + 1 - t$ two-factors are Δ-factors. Since $n \equiv 9 \pmod{18}$, we have $|A| = |B| = |C| \equiv 3 \pmod 6$, and thus there exists a Kirkman triple system (KTS) (cf. [6]) of order $|A|$. Let T_i, $i = 1, \dots, k$ be the Δ-factors obtained as the union of respective Δ-factors, i.e., the parallel classes of the KTSs on $[A]$, $[B]$, and $[C]$, respectively; we have $\bigcup_{i=1}^{k} T_i = E([A]) \cup E([B]) \cup E([C])$.

To show $2m \in \mathrm{HW}^*(n)$ for $2m \leqslant 2k$, we apply Corollary 3 to $\{F_{2i}, F_{2i+1}\}$ for $i = 0, \dots, m-1$. The Δ-factors are formed by F_j, $j = 2m, \dots, 2k$, and by T_j, $j = 1, \dots, k$. To show $2m+1 \in \mathrm{HW}^*(n)$ for $3 \leqslant 2m+1 \leqslant 2k+1$, we apply Corollary 2 to $\{F_0, F_2, F_{2k}\}$, and if $2m+1 \geqslant 5$, we apply Corollary 3 to $\{F_{2i}, F_{2i+1}\}$ for $i = 2, \dots, m$. When $2m+1 = 2k+1$, we also apply Corollary 4 to $\{F_1, F_3\}$. The remaining F_i's as well as all T_i's form the Δ-factors of the 2-factorization.

Finally, to show $m \in \mathrm{HW}^*(n)$ for $m \geqslant 2k+1$, we apply first Lemma 6 to obtain $k+1$ Hamiltonian cycles. The remaining $m-k-1$ Hamiltonian cycles are then obtained by applying Corollary 3 suitably many times, and/or Corollary 2, taking also into account Corollary 7 (in which we put $i = 1$). □

Theorem 9. *Let* $n = 6k+3$, *and assume* $k \equiv 0, 2 \pmod 3$. *Then* $\{(n+3)/6, (n+3)/6+2, (n+3)/6+3, \dots, (n-1)/2\} \subset \mathrm{HW}^*(n)$.

Proof. To show $(n+3)/6 = k+1 \in \mathrm{HW}^*(n)$, we apply Lemma 6; the Δ-factors are formed by the F_i's, $i = 1, \dots, 2k$. Applying then Corollary 3 a suitable number of times and/or Corollary 2 completes the proof. □

4. A triplicating lemma

In this section we deal with the case of exactly one Hamiltonian cycle. The construction of the following lemma was obtained by using the properties of graph coverings and voltage assignments from topological graph theory; however, below we provide a "voltage-assignments-free" proof.

Recall that a *Mendelsohn triple system* MTS(n) (cf. [6]) is a system of cyclically oriented triples on n points such that every ordered pair of points appears in exactly one triple.

First, we need an auxiliary result.

Lemma 10. *For every* $n \equiv 3 \pmod 6$, $n \geqslant 9$, *there exists two* Δ*-factorizations* (*i.e. Kirkman triple systems*) $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *of* K_n *on the set* V *such that*

(i) $\{1,2,3\}, \{5,6,7\} \in \mathscr{F}_1$, $\{2,3,4\}, \{6,7,8\} \in \mathscr{F}_2$ *where* $\{1,2,\dots,8\} \subset V$ *are* 8 *mutually distinct vertices*, *and*
(ii) *the union of all triples of* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *is orientable*, *i.e. it underlies a Mendelsohn triple system MTS*(n) (*cf.* [6, 25.2]).

Proof. For $n = 9$, take as triples of $\mathscr{F}_1$ the triples $\{1,2,3\}, \{5,6,7\}, \{4,8,9\}, \{1,5,9\}, \{2,6,8\}, \{3,4,7\}, \{1,4,6\}, \{3,5,8\}, \{2,7,9\}, \{1,7,8\}, \{2,4,5\}, \{3,6,9\}$, and take $\mathscr{F}_2$ as $\alpha\mathscr{F}_1$ where $\alpha = (1\ 4)(5\ 8)$ is a permutation. We can orient the triples e.g. as follows: $(1,2,3), (2,5,4), (3,4,7), (1,7,8), (5,6,7), (2,8,6), (2,4,3), (1,3,7), (2,1,8), (4,5,7), (2,6,5), (6,8,7)$; the first six are oriented triples of $\mathscr{F}_1$, the last six of $\mathscr{F}_2$. The remaining six triples occur in both $\mathscr{F}_1$ and $\mathscr{F}_2$, and so can clearly be oriented (with opposite orientation, e.g. once as, say, (a,b,c), and once as (a,c,b)).

For an example when $n = 15$, see [4].

For $n=21$, let $V(K_{21})=Z_7\times\{1,2,3\}$, and consider the KTS(21) whose 10 two-factors are given by the base Δ-factor $F=\{01,12,23\},\{21,42,63\},\{51,52,53\},\{61,32,03\},\{11,31,41\},\{02,22,62\},\ \{13,33,43\}$
yielding 7 Δ-factors, while the remaining three Δ-factors are

$$G_1=\{11,62,43\}\ \mathrm{mod}\,(7,-),$$

$$G_2=\{31,22,13\}\ \mathrm{mod}\,(7,-),$$

$$G_3=\{41,02,33\}\ \mathrm{mod}\,(7,-).$$

Take this KTS to be our $\mathscr{F}_1$. Let α be the mapping of the set of points of $\mathscr{F}_1$ into itself given by $\alpha(ij)=-ij,\ i\in Z_7$, $j\in\{1,2,3\}$, and let $\mathscr{F}_2$ be the KTS obtained from $\mathscr{F}_1$ by applying α to the triples of $\mathscr{F}_1$ (strictly speaking, by applying the mapping α^* induced by α). It is readily seen that the union of all triples of $\mathscr{F}_1$ and $\mathscr{F}_2$ is orientable: the only triples that are *not* repeated are those on the sets $Z_7\times\{j\},\ j=1,2,3$. But for each j, these triples are those of the (unique up to an isomorphism) twofold triple system of order 7 which is well known (and easily seen) to be orientable. Moreover, the collection of four triples (say) $\{11,31,41\},\{02,22,62\}\in\mathscr{F}_1,\ \{11,21,41\},\{12,22,62\}\in\mathscr{F}_2$ is isomorphic to the collection of four triples $\{1,2,3\},\{5,6,7\},\{2,3,4\},\{6,7,8\}$ in the statement of the lemma.

Let now $n\equiv 3\pmod 6), n\geq 27$. By a theorem of [13] (cf. also [6, 19.7]), there exists a KTS(n) with a sub-KTS of order 9. Taking all triples of the KTS(n) but those of the sub-KTS(9) twice and orienting each of these oppositely, together with the orientation of the triples of the sub-KTS(9) as in the example above completes the proof. □

Lemma 11. *Let* $n\equiv 3\,(\mathrm{mod}\,6),\ n\geqslant 15$, *and suppose* $1\in\ \mathrm{HW}^*(n)$. *Then* $1\in\mathrm{HW}^*(3n)$.

Proof. Let us take two Δ-factorizations (i.e. Kirkman triple systems) $\mathscr{F}_1$ and $\mathscr{F}_2$ of K_n as given in Lemma 10. It follows that there exists a cyclic orientation of triples in $\mathscr{F}_1$ and $\mathscr{F}_2$ such that for each couple of oppositely directed pairs of points $xy,\ yx$, one belongs to a triple in $\mathscr{F}_1$ while the other is contained in a triple of $\mathscr{F}_2$. Further, we may assume that the four triples $\{1,2,3\},\{5,6,7\},\{2,3,4\}$ and $\{6,7,8\}$ are oriented as $(1,2,3),(5,6,7),(3,2,4)$ and $(7,6,8)$.

By our hypothesis, $1\in\mathrm{HW}^*(n)$, thus let $\mathscr{F}_0$ be a 2-factorization of K_n such that all its 2-factors but one are Δ-factors while the remaining 2-factor is an HC. We assume that this Hamiltonian cycle passes through the vertices $1,2,\ldots,8,\ldots,n$ as follows: $(3,1,5,7,4,2,8,6,9,10,11,\ldots,n-1,n)$. We now identify the vertices of K_{3n} with the elements of $\{1,2,\ldots,n\}\times Z_3$. Define now a 2-factorization of K_{3n} as follows:

(1) for every Δ-factor F in $\mathscr{F}_1$ or in $\mathscr{F}_2$ and for every oriented triple $(x,y,z)\in F$ different from $\{1,2,3\},\{2,3,4\},\{5,6,7\}$, $\{6,7,8\}$, let $\{x_i,y_{i+1},z_{i+2}\}\in F',\ i\in Z_3$;
(2) for every Δ-factor F in $\mathscr{F}_0$ and for every triple $\{x,y,z\}\in F$, let $\{x_i,y_i,z_i\}\in F',\ i\in Z_3$;
(3) triples $\{x,y,z\}\in\{\{1,2,3\},\{5,6,7\},\{2,3,4\},\{6,7,8\}\}$ give rise to triples $\{x_i,y_{i+1},z_i\}$ for all $i\in Z_3$;
(4) the Hamiltonian cycle $C=(3,1,5,7,4,2,8,6,9,10,11,\ldots,n-1,n)$ gives rise to the cycle

$$C'=(3_0,1_1,5_1,7_0,4_0,2_2,8_2,6_1,9_1,10_1,11_1,\ldots,(n-1)_1,n_1,$$

$$3_1,1_2,5_2,7_1,4_1,2_0,8_0,6_2,9_2,10_2,11_2,\ldots,(n-1)_2,n_2,$$

$$3_2,1_0,5_0,7_2,4_2,2_1,8_1,6_0,9_0,10_0,11_0,\ldots,(n-1)_0,n_0);$$

(5) an additional Δ-factor G is formed by triples $\{x_0,x_1,x_2\}$ where x ranges over $\{1,2,\ldots,n\}$.

We claim that the above defined sets of triples F' where F is a Δ-factor in $\mathscr{F}_i,\ i\in Z_3$, form a family of $(3n-5)/2$ pairwise edge-disjoint Δ-factors which together with G and C' determine a 2-factorization $\mathscr{F}'$ of K_{3n} into one HC and a set of $(3n-3)/2$ Δ-factors. The proof consists of checking that any pair a_i,b_j occurs in precisely one cycle of a factor of $\mathscr{F}'$. One needs to distinguish three cases: (1) $a=b$; (2) $a\neq b$ and at least one of a,b is not in $\{1,2,\ldots,8\}$; and (3) $a\neq b$, and both a,b are in $\{1,2,\ldots,8\}$. This checking is straightforward, and is omitted. □

5. Small cases

In this section we deal with the set $\mathrm{HW}^*(n)$ for small values of $n=6k+3$.

Lemma 12. (i) $\mathrm{HW}^*(9)=\{0,2,3,4\}$; (ii) $\mathrm{HW}^*(15)=\{0,1,2,3,4,5,6,7\}$.

Proof. For (i), see [8]. For (ii), in view of Theorem 9 one only has to show $1,2,4 \in \mathrm{HW}^*(15)$. In [7] it is shown that there are exactly 10 nonisomorphic 2-factorizations of K_{15} having one HC and six Δ-factors, thus $1 \in \mathrm{HW}^*(15)$. It is easily seen that the union of any two edge-disjoint Δ-factors in K_{15} (there are only two nonisomorphic cases) can be decomposed into two HCs. So we may take, in any of the (7 nonisomorphic) Kirkman triple systems of order 15, two distinct parallel classes (four distinct parallel classes, respectively), and decompose these into two or four HCs, respectively. Thus $2,4 \in \mathrm{HW}^*(15)$. □

Lemma 13. $\mathrm{HW}^*(21) = \{0,1,\ldots,10\}$.

Proof. By Theorem 9, for $n=21$ we only have to show $\{1,2,3,5\} \subset \mathrm{HW}^*(21)$. Consider the KTS(21) $\mathscr{F}_1$ from the proof of Lemma 10 above. It is an easy exercise to see that $G_1 \cup G_2$, and $G_1 \cup G_2 \cup G_3$ can be decomposed into two, and into three HCs, respectively. Thus $2,3 \in \mathrm{HW}^*(21)$.

To show $5 \in \mathrm{HW}^*(21)$, we first observe that the circulant graph with edge-distances $1,4,5$ or with edge-distances $2,8,10$ can each be decomposed into three Δ-factors F_j, and G_j, respectively, e.g. as follows:

$$F_j = \{3i+j, 3i+4+j, 3i+5+j\}, \quad i=0,1,\ldots,6, \ \ j=0,1,2,$$

$$G_j = \{3i+j, 3i+2+j, 3i+10+j\}, \quad i=0,1,\ldots,6, \ \ j=0,1,2.$$

Furthermore, if $C(3)$ is the circulant graph with edge-distance 3 then the graph $C(3) \cup F_1$ can be decomposed into two HCs e.g. as follows: one Hamiltonian cycle is

$$H = (1,5,8,4,7,11,14,10,13,17,20,16,21,18,15,12,9,6,3,2,19,1)$$

while the other is

$$H' = (1,6,5,2,20,21,3,19,16,13,18,17,14,15,10,7,12,11,8,9,4,1).$$

Finally, the circulant graph with edge-distances $6,7,9$ can be decomposed into three HCs J_0, J_1, J_2 where

$$J_0 = (0,6,20,8,14,7,19,13,4,10,17,2,16,1,15,3,12,5,11,18,9,0)$$

and J_1 and J_2 are obtained from J_0 by adding 8 and 15 modulo 21, respectively, to each vertex of J_0.

In the 2-factorization of K_{21} with 2-factors $F_0, F_2, G_0, G_1, G_2, H, H', J_0, J_1, J_2$, the first five are Δ-factors and the last five are HCS, thus $5 \in \mathrm{HW}^*(21)$.

Finally, the following 2-factorization of K_{21} found by Meszka [11] shows $1 \in \mathrm{HW}^*(21)$:

(0, 1, 2), (3, 5, 7), (4, 6, 8), (9, 11, 14), (10, 12, 13), (15, 17, 20), (16, 18, 19)
(0, 3, 4), (1, 5, 8), (2, 6, 7), (9, 12, 15), (10, 11, 16), (13, 17, 19), (14, 18, 20)
(0, 5, 6), (1, 3, 9), (2, 4, 10), (7, 11, 20), (8, 14, 19), (12, 16, 17), (13, 15, 18)
(0, 7, 8), (1, 11, 15), (2, 19, 20), (3, 10, 17), (4, 12, 18), (5, 14, 16), (6, 9, 13)
(0, 9, 10), (1, 13, 16), (2, 12, 14), (3, 6, 18), (4, 5, 20), (7, 15, 19), (8, 11, 17)
(0, 11, 12), (1, 6, 17), (2, 8, 16), (3, 13, 20), (4, 14, 15), (5, 9, 19), (7, 10, 18)
(0, 13, 14), (1, 4, 19), (2, 11, 18), (3, 8, 12), (5, 10, 15), (6, 16, 20), (7, 9, 17)
(0, 15, 16), (1, 12, 20), (2, 5, 17), (3, 11, 19), (4, 7, 13), (6, 10, 14), (8, 9, 18)
(0, 17, 18), (1, 7, 14), (2, 3, 15), (4, 9, 16), (5, 11, 13), (6, 12, 19), (8, 10, 20)
(0, 19, 10, 1, 18, 5, 12, 7, 16, 3, 14, 17, 4, 11, 6, 15, 8, 13, 2, 9, 20). □

Lemma 14. *Let* $n \in \{39, 57\}$. *Then* $\mathrm{HW}^*(n) = I(n)$.

Proof. In view of Theorem 9, we only need to show $\{1,2,3,4,5,6,8\} \subset \mathrm{HW}^*(39)$, and $\{1,2,3,4,5,6,7,8,9,11\} \subset \mathrm{HW}^*(57)$.

For $n=39$, take $V(K_{39}) = Z_{39}$, and consider the following Δ-factors:

F : $\{0,7,16\}, \{13,20,29\}, \{3,26,33\}, \{6,11,21\}, \{19,24,34\}, \{8,32,37\}, \{1,4,18\}, \{14,17,31\}, \{5,27,30\}, \{10,22,28\}, \{2,23,35\}, \{9,15,36\}, \{12,25,38\}$;

G : $\{3i, 3i+8, 3i+19\}$, $i=0,1,\ldots,12$.

Developing F and G, respectively, modulo 39 gives an orbit of Δ-factors of length 13, and 3, respectively, for a total of 16 Δ-factors. The union of these Δ-factors uses up all pairs with differences in Z_{39} other than the differences $1,2$ and 4. On the other hand, the circulant graph with edge distances $1,2,4$ (i.e. the graph whose edge-set is the set

$\{\{x,y\}:|x-y|\in\{1,2,4\}\}$) can be decomposed into two Δ-factors and one HC. Indeed, the two Δ-factors can be taken to be

$\{3i,3i+1,3i+2\},\ i=0,1,\dots,12$ and $\{6i+2,6i+4,6i+6\},\ i=0,1,\dots,12$ (modulo 39); the unused edges are easily seen to form a Hamiltonian cycle. Thus $1\in \mathrm{HW}^*(39)$.

Next, we note that the circulant graph with edge-distances 1, 2, 4 can also be decomposed into one Δ-factor and two HCs, or into three HCs, and that the circulant with edge-distances 8, 11, 19 (the same distances as those used by the orbit determined by G above) can be decomposed into three HCs, as all of 1, 2, 4, 8, 11, 19 are relatively prime to 39. This immediately yields $\{2,3,4,5,6\}\subset \mathrm{HW}^*(39)$. Finally, to show $8\in \mathrm{HW}^*(39)$, it suffices to observe that the graph $F\cup F'$ (where F' is obtained from F by adding 1 modulo 39 to each vertex) can be decomposed into two HCs.

Similarly, for $n=57$ take $V(K_{57})=Z_{57}$, and Δ-factors given by

$\{0,1,21\},\{19,20,40\},\{2,38,39\},\{11,13,29\},\{30,32,48\},\{10,49,51\},$
$\{9,12,36\},\{28,31,55\},\{17,47,50\},\{3,7,15\},\{22,26,34\},\{41,45,53\},\{8,14,23\},$
$\{27,33,42\},\{4,46,52\},\{18,25,35\},\{37,44,54\},\{6,16,56\},\{5,24,43\};$

$\{3i,3i+5,3i+28\},\ i=0,1,\dots,18$ (modulo 57), and

$\{3i,3i+11,3i+25\},\ i=0,1,\dots,18$ (modulo 57).

Upon developing these Δ-factors modulo 57, the first of these yields an orbit of Δ-factors of length 19, while the remaining two yield an orbit of Δ-factors of length 3 each, for a total of 25 Δ-factors. The only unused distances between elements of Z_{57} are distances 13, 22, 26; but the circulant graph with distances 13, 22, 26 is isomorphic to the circulant with distances 1, 2, 4, and thus can be decomposed into two Δ-factors and one HC, as above. Thus $1\in \mathrm{HW}^*(57)$. The remainder of the proof is similar to that in the case $n=39$, and so is omitted. □

6. Main result

Theorem 15. *Let $n=a.3^m$ where $a\in\{5,7,13,19\}$ and $m\geqslant 1$. Then* $\mathrm{HW}^*(n)=I(n)$.

Proof. For $m=1$ and $a=5$, the statement follows from Lemma 12(ii). For $m=1$ and $a=7$, the statement follows from Lemma 13. For $m=1$ and $a=13$ or $a=19$, the statement follows from Lemma 14. For $m\geqslant 2$, the statement follows from Theorem 8 and a repeated application of Theorem 11. □

7. Conclusion

We conjecture that for $n\geqslant 15$, $\mathrm{HW}^*(n)=I(n)$. While in this paper we prove that this is indeed so for several infinite classes of orders, we are far away from a complete proof. The main difficulty appears to be the lack of a direct (or otherwise) construction showing $1\in \mathrm{HW}^*(n)$.

Acknowledgements

We thank Mariusz Meszka for providing us with the specific 2-factorization of K_{21} given in Lemma 13. Research of the second author was partially supported by Slovak Academy of Sciences and Ministry of Education of the Slovak Republic, Grants APVT-51-012502 and VEGA 2/2060/22. Research of the third author was supported by NSERC of Canada Grant No. OGP007268.

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