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A Fixed Point Theorem for Mixed Monotone Operators with Applications

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1. INTRODUCTION

In [1], we considered the existence of minimal and maximal fixed point to discontinuous increasing operators. In this paper, we investigate the existence of coupled minimal and maximal fixed points for mixed monotone operators. Our results include several results concerning fixed point theorems of increasing operators. We also give some applications to differential equations with discontinuous right hand side.

2. COUPLED MINIMAL AND MAXIMAL FIXED POINT

In this section we always assume that E is a real Banach space and P a normal cone in E. The order "$\leqslant$" is introduced by cone P, i.e., $x, y \in E$, $x \leqslant y$ if and only if $y - x \in P$. Therefore E becomes a partially ordered real Banach space. For convenience some definitions are recalled.

DEFINITION 1. *Let D be a set of E. Operator $A: D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x for each fixed $y \in D$ and nonincreasing in y for each fixed $x \in D$.*

DEFINITION 2. Let D be a set of E and $A: D \times D \to E$ an operator.

(a) If $x, y \in D$ with $x \leqslant y$ can be found such that

$$x \leqslant A(x, y) \qquad \text{and} \qquad A(x, y) \leqslant y$$

then (x, y) is called a coupled lower and upper fixed point of A.

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(b) If $x, y \in D$ with $x \leqslant y$ can be found such that

$$x = A(x, y) \quad \text{and} \quad A(x, y) = y$$

then (x, y) is called a coupled fixed point of A. If a coupled fixed coupled point (x^*, y^*) can be found such that

$$x^* \leqslant x \quad \text{and} \quad y \leqslant y^*$$

for every coupled fixed point (x, y) of A, then (x^*, y^*) is called the minimal and maximal fixed point of A.

(c) x^* is a fixed point of A if $A(x^*, x^*) = x^*$.

The main theorem of this paper is

THEOREM 1. *Let $u, v \in E$ with $u \leqslant v$ and $D = [u, v]$. Suppose that $A: D \times D \to E$ is a mixed monotone operator and the following conditions hold*:

(i) *(u, v) is a coupled lower and upper fixed point of A;*

(ii) *$A(D, D)$ is separable and weakly sequentially compact in E.*

Then A has the coupled minimal and maximal fixed point in D.

Proof. Let $u_1 = A(u, v)$ and $v_1 = A(v, u)$. It follows from condition (i) that

$$u \leqslant u_1 \quad \text{and} \quad v_1 \leqslant v.$$

And therefore

$$u_1 = A(u, v) \leqslant A(u_1, v) \leqslant A(u_1, v_1) \leqslant A(v, u) = v_1, \tag{1}$$

$$v_1 = A(v, u) \geqslant A(v_1, u) \geqslant A(v_1, u_1); \tag{2}$$

and

$$u_1 \leqslant A(x, v) \leqslant A(x, y) = x, \tag{3}$$

$$v_1 \geqslant A(y, u) \geqslant A(y, x) = y, \tag{4}$$

whenever $(x, y) \in D \times D$ is a coupled fixed point of A.

Let $\text{CFix}(A) = \{(x, y) \in D \times D : (x, y) \text{ is a coupled fixed point of } A\}$ and $M = \{(x, y) \in D \times D : (x, y) \text{ is a coupled lower and upper fixed point of } A$ with $x, y \in A(D, D)$ and $\text{CFix}(A) \subset [x, y]\}$. From (1)–(4) it follows that M is not empty and we shall show that $\text{CFix}(A)$ is not empty later. A partial order is defined in M as follows: for $(x_1, y_1), (x_2, y_2) \in M$,

$$(x_1, y_1) \leqslant (x_2, y_2) \quad \text{if and only if} \quad x_2 \leqslant x_1 \leqslant y_1 \leqslant y_2.$$

We are going to show that M has a minimal element. In order to do this we first show that each totally ordered subset of M has a lower bound. In fact, let N be a totally ordered subset of M. Since $N \subset A(D, D) \times A(D, D)$ and $A(D, D)$ is separable in E, sequence $\{(x'_n, y'_n)\}_{n=1}^{\infty}$ can be chosen from N such that $\{(x'_n, y'_n)\}_{n=1}^{\infty}$ is dense in N. Set, for each n,

$$(x_n, y_n) = \min\{(x'_1, y'_1), (x'_2, y'_2), ..., (x'_n, y'_n)\}.$$

It makes sense because $\{(x'_n, y'_n)\}_{n=1}^{\infty}$ is a totally ordered set. It follows from $A(D, D)$ being weakly sequentially compact that a subsequence $\{(x_{n_i}, y_{n_i})\}_{i=1}^{\infty}$ of $\{(x_n, y_n)\}_{n=1}^{\infty}$ and x'_0, y'_0 of E can be found such that

$$x_{n_i}(w) \to x'_0 \qquad \text{and} \qquad y_{n_i}(w) \to y'_0 \quad \text{as} \quad i \to +\infty. \tag{5}$$

Obviously, x'_0, y'_0 are elements of D. We now show that (x_0, y_0) is an element of M, where $x_0 = A(x'_0, y'_0)$ and $y_0 = A(y'_0, x'_0)$. In fact, we have for any positive integers n_i and p

$$x_{n_i} \leqslant x_{n_i+p} \leqslant y_{n_i+p} \leqslant y_{n_i}. \tag{6}$$

Let p go to infinity in (6), then

$$x_{n_i} \leqslant x'_0 \leqslant y'_0 \leqslant y_{n_i}.$$

By virtue of the mixed monotone property of A and (x_{n_i}, y_{n_i}) being coupled lower and upper fixed point of A the following must hold:

$$x_{n_i} \leqslant A(x'_0, y_{n_i}) \leqslant A(x'_0, y'_0) \tag{7}$$

$$y_{n_i} \leqslant A(y'_0, x_{n_i}) \geqslant (y'_0, x'_0). \tag{8}$$

Let i go to infinity in (7) and (8), then

$$x'_0 \leqslant A(x'_0, y'_0) \qquad \text{and} \qquad A(y'_0, x'_0) \leqslant y'_0 \tag{9}$$

and it is easy to show $\mathrm{CFix}(A) \subset [x'_0, y'_0]$ because of $\mathrm{CFix}(A) \subset [x_n, y_n]$ for each n. From (9) and the similar arguments to (1)–(4) it follows that (x_0, y_0) is in M. Since $\{(x_n, y_n)\}_{n=1}^{\infty} \subset N$ is dense in N, (x_0, y_0) is really lower bound of N. Hence M has a minimal element $(x^*, y^*) \in M$ by virtue of Zorn's lemma. By the definition of M we know that (x^*, y^*) is a coupled lower and upper fixed point of A and $\mathrm{CFix}(A) \subset [x^*, y^*]$.

We are now in the position to show that (x^*, y^*) is the coupled minimal and maximal fixed point of A. In fact, it follows from (x^*, y^*) being a coupled lower and upper fixed point that

$$x^* \leqslant A(x^*, y^*) \qquad \text{and} \qquad A(y^*, x^*) \leqslant y^*.$$

Therefore, if (x^*, y^*) is not a coupled fixed point of A we must have either

$$x^* < A(x^*, y^*) \quad \text{and} \quad A(y^*, x^*) \leqslant y^* \tag{10}$$

or

$$x^* \leqslant A(x^*, y^*) \quad \text{and} \quad A(y^*, x^*) < y^*. \tag{11}$$

Without loss of generality, we assume that (10) holds. Let $u' = A(x^*, y^*)$ and $v' = A(y^*, x^*)$. By the similar arguments to (1)–(4) we obtain (u', v') is in M. This means that (x^*, y^*) is not the minimal element of M. We have arrived at a contradiction. Hence, (x^*, y^*) must be a coupled fixed point of A. On the other hand, we know that $\mathrm{CFix}(A) \subset [x^*, y^*]$, so (x^*, y^*) is really the coupled minimal and maximal fixed point of A in D.

Remark 1. If $A(x, y)$ is independent to y, i.e., $A(x, y) = F(x)$, then $F: D \to D$ is increasing and the coupled fixed point of A is really the fixed point of F. Therefore, Theorem 1 includes many known results about fixed point theorems to increasing operators.

From Remark 1 it follows that the problem mentioned at the beginning of this paper is solved as follows:

THEOREM 2. *Let $u, v \in E$ with $u \leqslant v$ and $D = [u, v]$. Suppose that $F: D \to D$ is an increasing operator and the following conditions hold*

(i) *u, v are lower and upper fixed points of F;*

(ii) *$F(D)$ is separable and weakly sequentially compact in E.*

Then F has minimal and maximal fixed point in D.

Remark 2. If $A(x, y)$ is independent to x, i.e., $A(x, y) = G(y)$, then $G: D \to D$ is decreasing. Hence some results about decreasing operators can be derived from Theorem 1 immediately.

Theorem 1 gives a positive answer to the existence of coupled fixed point of $A(\ , \ )$. But sometimes we should know if the coupled fixed point of A is really a fixed point of A.

THEOREM 2. *Let $u, v \in E$ with $u \leqslant v$ and $D = [u, v]$. Suppose that A: $D \times D \to E$ is an operator satisfying all assumptions in Theorem* 1. *Suppose further that*

(iii) *For any fixed $x \in D$*

$$\|A(u, x) - A(v, x)\| \leqslant L \, \|u - v\|, \qquad \forall u, v \in D;$$

(iv) *For any fixed* $y \in D$,

$$\|A(y, u) - A(y, v)\| \leqslant L\, \|u - v\|, \qquad \forall u, v \in D,$$

where $0 \leqslant L < \frac{1}{2}$. *Then* A *has a unique fixed point* u^* *in* D *and* $x^* = u^* = y^*$, *where* (x^*, y^*) *is the coupled minimal and maximal fixed point of* A *in* D.

Proof. From Theorem 1 it follows that A has the coupled minimal and maximal fixed point (x^*, y^*) in D. We are going to show $x^* = y^*$.

We assume the contrary, i.e., $x^* \neq y^*$. By virtue of conditions (iii) and (iv) we obtain

$$\begin{aligned}
\|x^* - y^*\| &= \|A(x^*, y^*) - A(y^*, x^*)\| \\
&\leqslant \|A(x^*, y^*) - A(x^*, x^*)\| + \|A(x^*, x^*) - A(y^*, x^*)\| \\
&\leqslant L\, \|x^* - y^*\| + L\, \|x^* - y^*\| \\
&< \|x^* - y^*\|.
\end{aligned}$$

It is impossible. So x^* must be y^* and $u^* = x^* = y^*$ is a fixed point of A. From $\mathrm{CFix}(A) \subset [x^*, y^*]$ it follows that u^* is the unique fixed point of A in D.

3. Initial Value Problems

In this section the following initial value problem will be considered

$$\begin{aligned}
u' &= f(t, u) \qquad \text{a.e. } J, \\
u(0) &= u_0,
\end{aligned} \tag{$*$}$$

where $J = [0, T]$ with $T > 0$, $f = (f_i, f_2, ..., f_n)$, $f_i: J \times R^n \to R$ such that $f_i(t, u_1(t), ..., u_n(t)) \in L(J, R)$ for any $u_j(t) \in C(J)$ $(j = 1, 2, ..., n)$. In order to use Theorem 1 easily some definitions and concepts are introduced here. Suppose that p_i and q_i are two nonnegative integers with $p_i + q_i = n - 1$, so vector u can be rewritten as $u = (u_i, [u]_{p_i}, [u]_{q_i})$ and problem $(*)$ can be rewritten as

$$\begin{aligned}
u_i' &= f_i(t, u_i, [u]_{p_i}, [u]_{q_i}) \qquad \text{a.e. } J \quad (i = 1, 2, ..., n) \\
u(0) &= u_0.
\end{aligned} \tag{$*$}'$$

Let $\mathrm{AC}(J, R^n)$ be the space of absolutely continuous vector functions on

J. Then $(v, w) \in \mathrm{AC}(J, R^n) \times \mathrm{AC}(J, R^n)$ with $v \leqslant w$ is called a coupled lower and upper solution to $(*)'$ if

$$\begin{aligned} v_i' &\leqslant f_i(t, v_i, [v]_{p_i}, [w]_{q_i}) \qquad \text{a.e. } J, \\ v(0) &\leqslant u_0, \end{aligned} \tag{$*$)''$$

and

$$\begin{aligned} w_i' &\geqslant f_i(t, w_i, [w]_{p_i}, [v]_{q_i}) \qquad \text{a.e. } J, \\ w(0) &\geqslant u_0. \end{aligned}$$

And $(v, w) \in AC(J, R^n) \times AC(J, R^n)$ with $v \leqslant w$ is called a coupled solution to $(*)'$ if

$$\begin{aligned} v_i' &= f_i(t, v_i, [v]_{p_i}, [w]_{q_i}) \qquad \text{a.e. } J, \\ v(0) &= u_0, \end{aligned}$$

and

$$\begin{aligned} w_i' &= f_i(t, w_i, [w]_{p_i}, [v]_{q_i}) \qquad \text{a.e. } J, \\ w(0) &= u_0. \end{aligned}$$

Specially, if $(v^*, w^*) \in \mathrm{AC}(J, R^n) \times \mathrm{AC}(J, R^n)$ is a coupled solution to $(*)'$ such that $v^* \leqslant v \leqslant w \leqslant w^*$ for every coupled solution (v, w) to $(*)'$, then (v^*, w^*) is called the coupled minimal and maximal solution to $(*)'$.

Function $f\colon J \times R^n \to R^n$ is said to be mixed monotone if $f_i(t, u_i, [u]_{p_i}, [u]_{q_i})$ is increasing in $[u]_{p_i}$ and decreasing in $[u]_{q_i}$ for each $i = 1, 2, ..., n$.

In the sequel, $L(J, R^n)$ denotes the space of Lebesgue integrable vector functions on J.

THEOREM 3. *Let function* $f\colon J \times R^n \to R^n$ *be mixed monotone and* $f\colon C(J, R^n) \to L(J, R^n)$. *Suppose that a coupled lower and upper solution* (v, w) *to* $(*)'$ *can be found and* f *satisfies further*

$$f_i(t, u_i, [u]_{p_i}, [u]_{q_i}) - f_i(t, \bar{u}_i, [u]_{p_i}, [u]_{q_i}) \geqslant 0, \qquad i = 1, 2, ..., n, \tag{i}$$

where $v \leqslant u \leqslant w$, $v_i \leqslant \bar{u}_i \leqslant u_i \leqslant w_i$. *Then, for* $v(0) \leqslant u_0 \leqslant w(0)$, *there exists the coupled minimal and maximal solution* $(v^*, w^*) \in [v, w] \times [v, w]$ *to problem* $(*)'$, *and* $v^* \leqslant u \leqslant w^*$ *holds for each solution* u *to problems* $(*)'$ *in* $[v, w]$.

Proof. Let real Banach space E be $C(J, R^n)$, $P = \{x \in E : x_i(t) \geqslant 0,\ t \in J,\ i = 1, 2, ..., n\}$. Then P is a normal cone in E. From $v, w \in E$ and the assumptions about f it follows that f is an operator from $[v, w]$ into $L(J, R^n)$. This indicates that

$$\int_0^t f_i(s, x_i(s), [x(s)]_{p_i}, [y(x)]_{q_i})\, ds$$

makes sense for any $v \leqslant x$, $y \leqslant w$. Therefore we can define an operator $A(\ ,\ )$ as follows: for any $v \leqslant x$, $y \leqslant w$,

$$A(x, y)(t) = (A(x, y)(t), ..., A(x, y)(t)), \qquad t \in J,$$

where, for $i = 1, 2, ..., n$,

$$A_i(x, y)(t) = u_{0_i} + \int_0^t f_i(s, x_i(s), [x(s)]_{p_i}, [y(s)]_{q_i})\, ds, \qquad t \in J.$$

From the properties of Lebesgue integral it follows that $A(x, y) \in AC(J, R^n)$ for any $x, y \in [v, w]$. Hence, by the knowledge of differential equations we know that finding the solutions to problem $(*)'$ is equivalent to finding the fixed points of A and finding the coupled solutions to problem $(*)'$ is equivalent to finding the coupled fixed points of A.

Since (v, w) is a coupled lower and upper solutions to problem $(*)'$ we obtain for each $i = 1, 2, ..., n$,

$$\begin{aligned} A_i(v, w)(t) &= u_{0_i} + \int_0^t f_i(s, v_i(s), [v(s)]_{p_i}, [w(s)]_{q_i})\, ds \\ &\geqslant v_i(0) + \int_0^t v_i'(s)\, ds = v_i(t), \qquad t \in J; \end{aligned}$$

and

$$\begin{aligned} A_i(w, v)(t) &= u_{0_i} + \int_0^t f_i(s, w_i(s), [w(s)]_{p_i}, [v(s)]_{q_i})\, ds \\ &\leqslant w_i(0) + \int_0^t w_i'(s)\, ds = w_i(t), \qquad t \in J. \end{aligned}$$

This indicates

$$A(v, w) \geqslant v \qquad \text{and} \qquad A(w, v) \leqslant w.$$

On the other hand, $A(\ ,\ )$ is a mixed monotone operator from $[v, w]$ into

$[v, w]$. In fact, by virtue of (i) we obtain for any $x_1, x_2, y \in [v, w]$ with $x_1 \leqslant x_2$,

$$\begin{aligned}
&A_i(x_1, y)(t) - A_i(x_2, y)(t) \\
&\quad = \int_0^t \{f_i(s, x_{1_i}(s), [x_1(s)]_{p_i}, [y(s)]_{q_i}) \\
&\qquad - f_i(s, x_{2_i}(s), [x_2(s)]_{p_i}, [y(s)]_{q_i})\}\, ds \\
&\quad \leqslant \int_0^t \{f_i(s, x_{1_i}(s), [x_1(s)]_{p_i}, [y(s)]_{q_i}) \\
&\qquad - f_i(s, x_{2_i}(s), [x_2(s)]_{p_i}, [y(s)]_{q_i})\}\, ds \leqslant 0, \\
&\qquad\qquad t \in J, \qquad i = 1, 2, ..., n.
\end{aligned}$$

Therefore, $A(x, y)$ is increasing in x. By the same argument we know that $A(x, y)$ is decreasing in y. From the mixed monotone property of A it follows that, for any $x, y \in [v, w]$,

$$v \leqslant A(v, w) \leqslant A(x, w) \leqslant A(x, y)$$
$$w \geqslant A(w, v) \geqslant A(x, v) \geqslant A(x, y).$$

This means that $A(\ ,\ )$ is an operator from $[v, w]$ into $[v, w]$. Hence, it is necessary for us to show that $A([v, w], [v, w])$ is a relatively compact subset of E in order to use Theorem 1 to prove Theorem 3.

Obviously, $[v, w]$ is a bounded subset of E because P is a normal cone of E, and hence $A([v, w], [v, w])$ is bounded. We now show that $A([v, w], [v, w])$ is an equi-continuous subset of E also.

For any $v \leqslant x, y \leqslant w$ and $t, t' \in J$, we obtain

$$\begin{aligned}
&|A_i(x, y)(t) - A_i(x, y)(t')| \\
&\quad = \left| \int_t^{t'} f_i(s, x_i(s), [x(s)]_{p_i}, [y(s)]_{q_i})\, ds \right|, \qquad i = 1, 2, ..., n. \qquad (12)
\end{aligned}$$

From $v \leqslant A(x, y) \leqslant w$ it follows, without loss of generality we assume that $t' > t$, that

$$\begin{aligned}
\int_t^{t'} v_i(s)\, ds &\leqslant \int_t^{t'} f_i(s, x_i(s), [x(s)]_{p_i}, [y(s)]_{q_i})\, ds \\
&\leqslant \int_t^{t'} w_i(s)\, ds, \qquad i = 1, 2, ..., n.
\end{aligned}$$

And hence,

$$\left|\int_t^{t'} f_i(s, x_i(s), [x(s)]_{p_i}, [y(s)]_{q_i})\,ds\right|$$

$$\leqslant \left|\int_t^{t'} v_i(s)\,ds\right| + \left|\int_t^{t'} w(s)\,ds\right|, \qquad i=1, 2, ..., n. \tag{13}$$

Finally, from (12), (13) and the absolute continuity of Lebesgue integral it follows that $A([v, w], [v, w])$ is an equi-continuous subset of E, so it is relatively compact.

All conditions in Theorem 1 are satisfied by the operator $A(\ ,\ )$, and hence $A(\ ,\ )$ has the coupled minimal and maximal fixed point (v^*, w^*) in $[v, w]$ and $v^* \leqslant u \leqslant w^*$ holds for every fixed point u of $A(\ ,\ )$ in $[v, w]$. We finally know that Theorem 3 holds by the above statement.

Remark 3. Function f in Theorem 3 need not be continuous, so it is a generalization of [2, Theorem 1.4.1] and it is also a generalization of [1, Theorem 3]

The following theorem is about the relation between coupled solutions and solutions to problem $(*)'$.

THEOREM 4. *Let all assumptions in Theorem 3 hold here. Suppose further that for any $x, y \in R^n$ and $i=1, 2, ..., n$ we have*

$$|f_i(t, x) - f_i(t, y)| \leqslant L\,\|x-y\|,$$

where L is a positive constant. Then $v^* = w^*$ and $u^* = v^* = w^*$ is the unique solution to problem $(*)'$ in $[v, w]$ where (v^*, w^*) is the coupled minimal and maximal solution to problem $(*)'$ in $[v, w]$.

Proof. We suppose that v^* is not w^*. Let

$$m(t) = \max_{0 \leqslant s \leqslant t} \left\{\sum_{i=1}^{n} |v_i^*(s) - w_i^*(s)|^2\right\}^{1/2}.$$

Then $m(0)=0$, and hence there exists a positive number c such that $0 \leqslant t_0 < t_0 + c < T$, $Lc < \sqrt{1/n}$ and

$$0 \leqslant t \leqslant t_0 \Rightarrow m(t) = 0; \qquad t_0 < t \leqslant t_0 + c \Rightarrow m(t) > 0.$$

Since, for $t_0 < t \leqslant t_0 + c$ and $i=1, 2, ..., n$,

$$\begin{aligned}|(w_i^* - v_i^*)(t)| &\leqslant \int_{t_0}^{t} |f_i(s, w_i^*(s), [w^*(s)]_{p_i}, [v^*(s)]_{q_i}) \\ &\quad - f_i(s, v_i^*(s), [v^*(s)]_{p_i}, [w^*(s)]_{q_i})|\, ds \\ &\geqslant \int_{t_0}^{t} L \left\{ \sum_{i=1}^{n} |w_i^*(s) - v_i^*(s)|^2 \right\}^{1/2} ds \\ &\leqslant Lcm(t) < \sqrt{1/n}\, m(t),\end{aligned}$$

we obtain

$$m(t) < \left\{ \sum_{i=1}^{n} (\sqrt{1/n}\, m(t))^2 \right\}^{1/2} = m(t).$$

It is a contradiction. Hence v^* must be w^*. From (v^*, w^*) being the coupled minimal and maximal solution to problem $(*)'$ in $[v, w]$ it follows that $u^* = v^* = w^*$ is the unique solution to problem $(*)'$ in $[v, w]$.

4. Periodic Boundary Value Problems

In this section we use the same signs and definitions as those in Section 3. Consider the following differention equation with periodic boundary value:

$$\begin{aligned} u' &= f(t, u) \qquad \text{a.e. } J, \\ u(0) &= u(T). \end{aligned} \tag{$**$}$$

Equation $(**)$ can be rewritten as, by the same way as in Section 3,

$$\begin{aligned} u_i' &= f_i(t, u_i, [u]_{p_i}, [u]_{q_i}) \qquad \text{a.e. } J, \\ u_i(0) &= u_i(T), \end{aligned} \tag{$**$}'$$

where $i = 1, 2, ..., n$.

Let $v, w \in AC(J, R^n)$. Then (v, w) with $v \leqslant w$ is called a coupled periodic solution to $(**)'$ if, $i = 1, 2, ..., n$,

$$\begin{aligned} v_i' &= f_i(t, v_i, [v]_{p_i}, [w]_{q_i}) \qquad \text{a.e. } J; \qquad v_i(0) = v_i(T), \\ w_i' &= f_i(t, w_i, [w]_{p_i}, [v]_{q_i}) \qquad \text{a.e. } J; \qquad w_i(0) = w_i(T). \end{aligned} \tag{14}$$

And (v, w) with $v \leqslant w$ is called a coupled lower and upper periodic solution to $(**)'$ if, for $i = 1, 2, ..., n$,

$$\begin{aligned} v_i' &\leqslant f_i(t, v_i, [v]_{p_i}, [w]_{q_i}) \qquad \text{a.e. } J; \qquad v_i(0) \leqslant v_i(T), \\ w_i' &\geqslant f_i(t, w_i, [w]_{p_i}, [v]_{q_i}) \qquad \text{a.e. } J; \qquad w_i(0) \geqslant w_i(T). \end{aligned}$$

THEOREM 5. *Let function* $f: C(J, R^n) \to L(J, R^n)$ *be mixed monotone. Suppose that* $(v_0, w_0) \in AC(J, R^n) \times AC(J, R^n)$ *is a coupled lower and upper periodic solution to* $(**)'$. *Suppose further that*

$$\begin{aligned} &f_i(t, u_i, [u]_{p_i}, [u]_{q_i}) - f_i(t, \bar{u}_i, [u]_{p_i}, [u]_{q_i}) \\ &\quad \geqslant -h_i(t)(u_i - \bar{u}_i), \qquad t \in J \quad \text{and} \quad i = 1, 2, ..., n, \end{aligned} \tag{15}$$

where $h_i \in L(J, R^n)$ *and* $\int_0^T h_i(s)\, ds > 0$ $(i = 1, 2, ..., n)$. *Then there exists a coupled minimal and maximal periodic solution* (v^*, w^*) *to problem* $(**)'$ *in order section* $[v, w]$. *And* $v^*(t) \leqslant u(t) \leqslant w^*(t)$ $(\forall t \in J)$ *holds for every solution* $u(t)$ *to problem* $(**)'$ *in* $[v, w]$.

Proof. Let E be the real Banach space $C(J, R^n)$ and $P = \{x \in E : x_i(t) \geqslant 0,\ t \in J \text{ and } i = 1, 2, ..., n\}$. Then P is a normal cone in E. Set

$$\begin{aligned} &g_i(t, u_i, [u]_{p_i}, [u]_{q_i}) \\ &\quad = f_i(t, u_i e^{-\int_0^t h_i(s)\, ds}, [u e^{-\int_0^t h(s)\, ds}]_{p_i}, [u e^{-\int_0^t h(s)\, ds}]_{q_i})\, e^{\int_0^t h_i(s)\, ds} + h_i(t) u_i, \\ &\quad t \in J \quad u \in R^n, \quad \text{and} \quad i = 1, 2, ..., n, \end{aligned}$$

where $u e^{-\int_0^t h(s)\, ds} = (u_1 e^{-\int_0^t h_1(s)\, ds}, ..., u_n e^{-\int_0^t h_n(s)\, ds})$. In the sequel we always denote $\bar{u} = u e^{-\int_0^t h(s)\, ds} (u \in C(J, R^n))$ for any $u \in C(J, R^n)$.

Let $v = v_0 e^{\int_0^t h(s)\, ds}$ and $w = w_0 e^{\int_0^t h(s)\, ds}$. Operator $A(\ ,\ )$ from $[v, w]$ into E is defined as follows: for any $v \leqslant x$, $y \leqslant w$,

$$A(x, y)(t) = (A_1(x, y)(t), ..., A_n(x, y)(t)), \quad t \in J,$$

where for $i = 1, 2, ..., n$

$$\begin{aligned} A_i(x, y)(t) &= (e^{\int_0^t h_i(s)\, ds} - 1)^{-1} \cdot \int_0^T g_i(s, x_i(s), [x(s)]_{p_i}, [y(s)]_{q_i})\, ds \\ &\quad + \int_0^t g_i(s, x_i(s), [x(s)]_{p_i}, [y(s)]_{q_i})\, ds, \qquad t \in J. \end{aligned}$$

From (v_0, w_0) is a coupled lower and upper periodic solution to $(**)'$ it follows that, for each $i = 1, 2, ..., n$,

$$\begin{aligned} (e^{\int_0^t h_i(s)\, ds} v_{0_i})' &\leqslant g_i(t, v_i, [v]_{p_i}, [w]_{q_i}), \\ (e^{\int_0^t h_i(s)\, ds} w_{0_i})' &\geqslant g_i(t, w_i, [w]_{p_i}, [v]_{q_i}). \end{aligned}$$

Hence, for each $i = 1, 2, ..., n$ and

$$\begin{aligned} v &= (v_{0_i} \cdot e^{\int_0^t h_i(s)\, ds}, ..., v_{0_n} \cdot e^{\int_0^t h_n(s)\, ds}), \\ w &= (w_{0_i} \cdot e^{\int_0^t h_1(s)\, ds}, ..., w_{0_n} \cdot e^{\int_0^t h_n(s)\, ds}), \end{aligned}$$

we obtain

$$\begin{aligned} A_i(v, w)(t) &\geqslant (e^{\int_0^t h_i(s)\,ds} - 1)^{-1} \cdot (v_{0_i}(T) e^{\int_0^t h_i(s)\,ds} - v_{0_i}(0)) \\ &\quad + v_{0_i}(t) e^{\int_0^t h_i(s)\,ds} - v_{0_i}(0) \geqslant v_{0_i}(t) e^{\int_0^t h_i(s)\,ds} \\ &= v_i(t) \end{aligned}$$

and, in the same way,

$$A_i(w, v)(t) \leqslant w_i(t).$$

This implies that $(v, w) \in AC(J, R^n) \times AC(J, R^n)$ with $v \leqslant w$ is a coupled lower and upper fixed point of $A(\ ,\ )$. By the same argument as in Theorem 3 we can show that $A(\ ,\ )\colon [v, w] \to E$ is a mixed monotone operator from $[v, w]$ into $[v, w]$ and $A([v, w], [v, w])$ is also a relatively compact subset of E. Therefore, from Theorem 1 it follows that operator $A(\ ,\ )$ has the coupled minimal and maximal fixed point (x^*, y^*) in $[v, w]$ and, for every $i = 1, 2, ..., n$,

$$\begin{aligned} x^*(T) &= A_i(x^*, y^*)(T) \\ &= (e^{\int_0^T h_i(s)\,ds} - 1)^{-1} \cdot \int_0^T g_i(s, x_i^*(s), [x^*(s)]_{p_i}, [y^*(s)]_{q_i})\, ds \\ &\quad + \int_0^T g_i(s, x_i^*(s), [x^*(s)]_{p_i}, [y^*(s)]_{q_i})\, ds \\ &= A_i(x^*, y^*)(0) e^{\int_0^T h_i(s)\,ds} = x_i^*(0) e^{\int_0^T h_i(s)\,ds}, \\ y_i^*(T) &= y_i^*(0) \cdot e^{\int_0^T h_i(s)\,ds}. \end{aligned}$$

Let

$$\begin{aligned} v^*(t) &= x^*(t) e^{-\int_0^t h(s)\,ds} = (x_i^*(t) e^{-\int_0^t h_1(s)\,ds}, ..., x_n^*(t) e^{-\int_0^t h_n(s)\,ds}), \\ w^*(t) &= y^*(t) e^{-\int_0^t h(s)\,ds} = (y_1^*(t) e^{-\int_0^t h_i(s)\,ds}, ..., y_n^*(t) e^{-\int_0^t h_n(s)\,ds}), \end{aligned}$$

Then from the above statement, the definition of A, and simple calculation it follows that $(v^*, w^*) \in AC(J, R^n) \times AC(J, R^n)$ is the coupled minimal and maximal periodic solution to $(**)'$ and Theorem 5 is true.

We surely can discuss the relation between the solution and the coupled solution to $(**)'$. But we do not state it here since the method is similar to that in Section 3.

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