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The Triangulations of the 3-Sphere with up to 8 Vertices

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The different combinatorial types of triangulations of the 3-sphere with up to 8 vertices are determined. Using similar methods we show that one cannot always preassign the shape of a facet of a 4-polytope.

1. Introduction

The problem of enumerating the different combinatorial types of d-polytopes with a given number of vertices (or facets) seems to be extremely difficult. Even for $d = 3$ the problem is unsolved for polytopes with more than 8 vertices. For simplicial 4-polytopes an enumeration of the combinatorial types with at most 8 vertices was made by Grünbaum and Sreedharan [5]. This corrected and completed work done by Brückner in 1909 [4]. Brückner enumerated 3-dimensional structures which he assumed could be realized by projections of 4-polytopes into one of their facets. All that can be said is that they were isomorphic to duals of triangulations of the 3-sphere.

The work of Grünbaum and Sreedharan lead to the discovery of a triangulation of the 3-sphere with 8 vertices that is not isomorphic to the boundary of any 4-polytope. This leads to the question: What are the different combinatorial types of triangulations of the 3-sphere with up to 8 vertices? In this paper we shall answer the question by constructing all of the duals of these triangulations.

2. Definitions

A *d-cell complex* is a collection $\mathscr{C}$ of k-cells, $0 \leqslant k \leqslant d$, such that

(i) each d-cell $\mathscr{F}$ is associated with a convex d-polytope $P(\mathscr{F})$ and with a homeomorphism $h(\mathscr{F})$ between $\mathscr{F}$ and $P(\mathscr{F})$;

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(ii) each face of a d-cell in $\mathscr{C}$ is a member of $\mathscr{C}$, where *a face* of a d-cell $\mathscr{F}$ is the inverse image of a face of $P(\mathscr{F})$ under $h(\mathscr{F})$;

(iii) the non-empty intersection of any two members of $\mathscr{C}$ is a face of both members (we shall say that two faces *meet properly* if their intersection is a face of both).

Two members of $\mathscr{C}$ are *incident* if one contains the other. Two cell complexes $\mathscr{C}_1$ and $\mathscr{C}_2$ are isomorphic if there is a 1-1, dimension preserving, incidence preserving correspondence between the members (hereafter called *faces*) of $\mathscr{C}_1$ and $\mathscr{C}_2$. Two d-cell complexes $\mathscr{C}_1$ and $\mathscr{C}_2$ are *dual* if there is a 1-1 incidence reversing correspondence taking faces of dimension k onto faces of dimension $d - k - 1$.

A *combinatorial* d-sphere is a d-cell complex whose body (i.e. the union of its members) is homeomorphic to S^d. A *triangulation* of S^d is a combinatorial d-sphere all of whose cells are simplices.

A theorem of Steinitz [7] says that a graph without loops or multiple edges is the graphs of a 3-polytope provided it is planar and 3-connected. This implies that any combinatorial 2-sphere is isomorphic to the boundary of some 3-polytopes. This guarantees that when we take the dual of a triangulation of the 3-sphere, condition (i) will be satisfied. Since conditions (ii) and (iii) are easily seen to be satisfied we have that the dual of a triangulation of the 3-sphere is a 3-cell complex. The duals of the triangulation of S^3 will be called *simple combinatorial* 3-*spheres*, which we shall abbreviate SC3S.

A *facet* of a 3-cell complex $\mathscr{C}$ is a 3-dimensional face of $\mathscr{C}$, a *subfacet* is a 2-dimensional face of $\mathscr{C}$, an *edge* is a 1-dimensional face of $\mathscr{C}$ and a *vertex* is a 0-dimensional *face* of $\mathscr{C}$.

We shall use a process called *removing subfacets*, which we now describe. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two facets of an SC3S $\mathscr{S}$ that meet on a subfacet α. Let $\mathscr{C} = \{F_i^k \cup F_j^k \mid F_i^k$ and F_j^k are k-faces of $\mathscr{S}$ such that $F_i^k \cap F_j^k \cap \alpha$ is a $k - 1$ face of $\mathscr{S}$, $1 \leqslant k \leqslant 3\} \cup \{F_i^k \mid F_i^k$ is a k-face of $\mathscr{S}$ and $F_i^k \cap \alpha = \varnothing\}$. If $\mathscr{C}$ forms a cell complex we say that α is *removable* and that $\mathscr{C}$ is obtained from $\mathscr{S}$ by *removing* α. Note that this new cell complex is also a SC3S. It can be seen that the inverse operation to removing subfacets is *facet splitting* which consists of taking a simple closed curve Γ on the boundary of a facet $\mathscr{F}$ of an SC3S, where Γ misses every vertex, intersects an edge at most once, and crosses a subfacet at most once, and then spanning Γ by a 2-cell whose relative interior is in the relative interior of $\mathscr{F}$.

The main theorem of this paper is that the combinatorial types of SC3S's with at most 8 facets can be generated from the boundary of the 4 simplex, $\beta(T^4)$, by facet splitting. That is, given any SC3S, $\mathscr{S}$, with at most 8 facets there is a sequence of SC3S's $\beta(T^4) = \mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_n$ such that $\mathscr{S}_i$ is obtained from $\mathscr{S}_{i-1}$ by facet splitting and $\mathscr{S}_n$ is isomorphic to $\mathscr{S}$. Clearly, proving this is equivalent to proving that any SC3S with at most 8 facets has a removable subfacet.

3. Sufficient Conditions For α to Be Removable

When discussing removable facets we shall always assume that our SC3S is not isomorphic to the boundary of the 4-simplex.

Lemma 1. *If G is a graph without multiple edges embedded in a 2-sphere, breaking the sphere into cells, no two of which have a multiple connected union, then G is the graph of a 3-polytope.*

Proof. By Steinitz's theorem, we need to show that G is 3-connected. Let V be a minimal set of vertices that disconnects G and let $v \in V$. Consider the set of 2-cells that contain v. Since no two have a multiply connected union, the union of these 2-cells is again a 2-cell. The boundary of this 2-cell is a circuit Γ in G. By the minimality of V, each component of the separated graph will be joined to v by an edge, thus Γ has vertices in every component. Since Γ is a circuit, it requires at least two vertices to separate it, thus V has at least three vertices and G is 3-connected.

Lemma 2. *A sufficient condition for $\mathscr{C}$ to be a 3-cell complex is that every face of $\mathscr{S}$ that meets $\mathscr{F}_1$ and $\mathscr{F}_2$ meets α.*

Proof. First we show that $\mathscr{F}_1 \cup \mathscr{F}_2$ is isomorphic to a 3-polytope.

Suppose $\mathscr{F}_1 \cup \mathscr{F}_2$ has a double edge. In this case $\mathscr{F}_1$ has a triangular 2-face F_1 meeting a triangular 2-face F_2 of $\mathscr{F}_2$. In $\mathscr{F}_1$ there will be two 2-faces meeting F_1 on edges of $\mathscr{F}_1$ and in $\mathscr{F}_2$ there will be two 2-faces meeting F_2 on edges of $\mathscr{F}_2$. These four 2-faces will form two 2-faces, F_3 and F_4 of $\mathscr{F}_1 \cup \mathscr{F}_2$, with a multiply connected union (unless $\mathscr{F}_1$ and $\mathscr{F}_2$ are both tetrahedra, in which case $\mathscr{S}$ is a simplex). This implies that there will be facets $\mathscr{F}_5$ meeting $\mathscr{F}_1 \cup \mathscr{F}_2$ on F_3, and $\mathscr{F}_6$ meeting $\mathscr{F}_1 \cup \mathscr{F}_2$ on F_4. The intersection of these two facets will be a 2-face meeting $\mathscr{F}_1$ and $\mathscr{F}_2$ but not α, which is a contradiction.

Suppose two-faces of $\mathscr{F}_1 \cup \mathscr{F}_2$ have a multiply connected union. In this case, as we have just seen, this gives us a 2-face of $\mathscr{S}$ meeting $\mathscr{F}_1$ and $\mathscr{F}_2$ but not α.

Next we show that any other cell, $\mathscr{F}_3$ of $\mathscr{C}$, meeting α, is isomorphic to a 3-polytope.

Suppose $\mathscr{F}_3$ has a double edge. We have already taken care of the case in which this double edge lies in $\mathscr{F}_1 \cup \mathscr{F}_2$. The only other possibility is that one edge e lies in $\mathscr{F}_3$ but not $\mathscr{F}_1 \cup \mathscr{F}_2$ and the other edge lies in $\mathscr{F}_1 \cup \mathscr{F}_2$. In this case e meets $\mathscr{F}_1 \cup \mathscr{F}_2$ but not α.

Suppose two 2-cells of $\mathscr{F}_3$ have a multiply connected union. If both 2-cells lie in $\mathscr{F}_1 \cup \mathscr{F}_2$ then a previous argument applies. If one of the 2-cells lies in $\mathscr{F}_3$ but not $\mathscr{F}_1 \cup \mathscr{F}_2$ then this 2-cell meets $\mathscr{F}_1 \cup \mathscr{F}_2$ but not α.

Finally, we show that pairs of faces of $\mathscr{C}$ intersect properly. The only way a pair of faces G_1 and G_2 can intersect improperly is if they intersect on a face G_3 of $\mathscr{F}_1$ and a face G_4 of $\mathscr{F}_2$. We have already seen that if G_1 and G_2 belong to $\mathscr{F}_1 \cup \mathscr{F}_2$ then we reach a contradiction, so we shall examine the case in which G_1 does not lie on $\mathscr{F}_1 \cup \mathscr{F}_2$. If G_1 misses α we are done. If G_1 meets α, say at a vertex v, then v and G_3 belong to a face of $\mathscr{F}_1$ and G_1; and v and G_4 lie on a face of $\mathscr{F}_2$ and G_1. Thus G_3 and G_4 lie on a face G_5 of $\mathscr{F}_1 \cup \mathscr{F}_2$ and of G_3. This means that G_2 and G_5 are two faces of $\mathscr{F}_1 \cup \mathscr{F}_2$ that do not meet properly, which is a case we have already taken care of.

3. The Main Theorem

If $\mathscr{S}$ is an SC3S with at most 8 facets it follows that each facet of $\mathscr{S}$ is isomorphic to a simple (i.e., 3-valent) 3-polytope with at most 7 facets. Figure 6 gives the Schlegel diagrams of the 9 combinatorial types of such polytopes.

Theorem 1. *If an* SC3S $\mathscr{S}$ *is not isomorphic to the boundary of the simplex and if it has* 8 *or fewer facets then it has a removable subfacet.*

Proof. We shall examine several cases:

Case I. Two triangular subfacets of $\mathscr{S}$ meet on an edge. These two subfacets belong to a common facet which must be a tetrahedron. If the tetrahedron meets another tetrahedron on a subfacet then $\mathscr{S}$ is the simplex. The reader may easily verify that if the tetrahedron does not meet any other tetrahedron on a subfacet then each subfacet on the tetrahedron is removable.

In the following cases we shall assume that no two triangular subfacets meet on an edge.

Case II. Some pair of facets $\mathscr{F}_1$ and $\mathscr{F}_2$, meet on a subfacet α, and some subfacet β meets $\mathscr{F}_1$ and $\mathscr{F}_2$ on an edge of $\mathscr{F}_1$ and an edge of $\mathscr{F}_2$ but does not meet α; and every facet meeting $\mathscr{F}_1$ and $\mathscr{F}_2$ meets α. Let $\mathscr{F}_3$ and $\mathscr{F}_4$ be the two facets containing β. The facets $\mathscr{F}_3$ and $\mathscr{F}_4$ will both meet $\mathscr{F}_1 \cup \mathscr{F}_2$ on a 2-cell of $\mathscr{F}_1 \cup \mathscr{F}_2$. Since β does not meet α, we have that $(\mathscr{F}_1 \cup \mathscr{F}_2) \cap \mathscr{F}_3$ and $(\mathscr{F}_1 \cup \mathscr{F}_2) \cap \mathscr{F}_4$ form an annulus. Thus

$\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3 \cup \mathscr{F}_4$ is homeomorphic to a 3-dimensional hollow ball and it separates the remaining facets of $\mathscr{S}$ into two sets, each homeomorphic to a 3-cell.

Since $\mathscr{S}$ has at most 8 facets, one of these two sets of facets must contain at most two facets. Each facet in such a set will meet at most 5 other facets, thus each is either (isomorphic to) a tetrahedron or a triangular prism. Since no two triangular subfacets meet on an edge the only possibilities are that they are both triangular prisms meeting as in Fig. 1. It is easily verified that in Fig. 1a the common subfacet is removable.

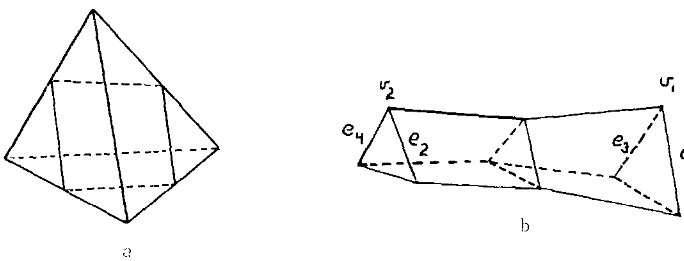

FIGURE 1

Suppose that in Fig. 1b the common subfacet is not removable. Then some face F of $\mathscr{S}$ meets both facets but not the common facet. But F will belong to a facet $\mathscr{F}_i$ ($i = 1, 2, 3$, or 4) that contains the remaining two triangular subfacets. This implies that the edges e_1 and e_2 (see Fig. 1) lie on a common subfacet, and e_3 and e_4 also lie on a common subfacet. The intersection of these two subfacets will be an edge joining v_1 and v_2. Similarly we see that the other pairs of vertices will be joined by edges. Now, however, we have an SC3S that can be obtained from the boundary of the 4-simplex by splitting a facet as in Fig. 2, which has a removable subfacet.

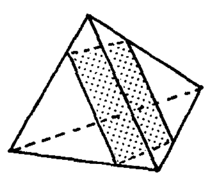

FIGURE 2

Case III. Some pair of facets $\mathscr{F}_1$ and $\mathscr{F}_2$ meet on a subfacet α and some edge e meets $\mathscr{F}_1$ and $\mathscr{F}_2$ but misses α; and no facet meets $\mathscr{F}_1$ and $\mathscr{F}_2$ and misses α. Every subfacet meeting $\mathscr{F}_1$ and $\mathscr{F}_2$ also meets α.

Let $\mathscr{F}_5$, $\mathscr{F}_6$, and $\mathscr{F}_7$ be the three facets of $\mathscr{S}$ that contain e. Let $F_i = \mathscr{F}_i \cap \mathscr{F}_1$, $G_i = \mathscr{F}_i \cap \mathscr{F}_2$ for $i = 5, 6$ or 7. For each i, F_i and G_i meet on an edge of α. Since no subfacet meets $\mathscr{F}_1$ and $\mathscr{F}_2$ and misses α, we see, as before, that no two 2-cells in $\mathscr{F}_1 \cup \mathscr{F}_2$ have a multiply connected union, thus $\{F_i \cap G_i \mid i = 5, 6 \text{ or } 7\}$ is the set of all edges of α. We also know that F_5, F_6, and F_7 meet at a common vertex of $\mathscr{F}_1$, and G_5, G_6, and G_7 meet at a common vertex of $\mathscr{F}_2$. This implies that the F_i's, the G_i's, and α are the only facets of $\mathscr{F}_1$ and $\mathscr{F}_2$. This, however, is a contradiction because this means that $\mathscr{F}_1$ and $\mathscr{F}_2$ are tetrahedra.

Case IV. Every pair $\mathscr{F}_1$, $\mathscr{F}_2$ of facets that meet on a subfacet α also meet a facet $\mathscr{F}_3$ on both $\mathscr{F}_1$ and $\mathscr{F}_2$ but not on α. In this case the only possible combinatorial types of facets are c_2, d_2, and d_5.

First we observe that due to the restriction on the types of facets no triangular subfacet meets a 4-sided subfacet in $\mathscr{C}$. This implies that no facet of type d_2 meets a facet of type d_5 on a 5-sided subfacet, and two d_5's can meet in only one way as illustrated in Fig. 3 (the edges common to both facets are emphasized).

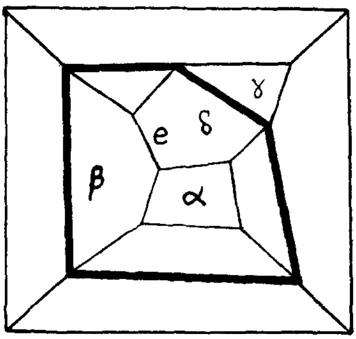

FIGURE 3

A d_5 must meet another d_5 as in Fig. 3. The subfacet β (see Fig. 3) belongs to another d_5 and they meet in such a way that the triangular subfacet of that d_5 has e as an edge. But now some facet of $\mathscr{C}$ contains γ, δ, and the triangular subfacet containing e, which is impossible because of the restrictions on the combinatorial types of facets. We may now conclude that $\mathscr{S}$ contains no d_5's.

Suppose two d_2's meet on a 5-sided subfacet. Since some facet $\mathscr{F}_3$ meets each of these two facets but not their common subfacet, $\mathscr{F}_3$ must also be a d_2. Now, however, the 4-sided subfacets of each of these facets belong to facets of type b, a contradiction.

We may now assume that all facets are of type c_2. The reader may verify that in this case the SC3S must be isomorphic to the boundary of the 4-cube and thus has removable facets.

4. Generating the SC3S's

At this point we could attempt to generate all of the SC3S's with eight facets; however, this would be a long, tedious job because, in general, there are many ways of splitting a facet of an SC3S. Instead we shall show that almost all of the SC3S's created are polyhedral, that is, isomorphic to the boundary complex of some 4-polytope, and since the simple 4-polytopes with at most eight facets have already been enumerated, these need not be considered. We shall show that most of the splittings applied to polyhedral 3-spheres can be accomplished by means of a *geometric facet splitting*, which we now define.

Let P be a simple 4-dimensional polytope with a specified facet F, and let H be a 2-dimensional plane in the affine hull of F, which contains a relative interior point of F. Let H' be a hyperplane (3-dimensional) which intersects F on H, and which is close enough to the affine hull of F so that all vertices of P that are not on F lie on one side of H'. Let S be the half-space determined by H' that contains all vertices that are not on F and let $P' = S \cap P$. It is easy to see that H separates F into two 3-polytopes, P_1 and P_2, and that intersecting P with S destroys F and replaces it with two facets combinatorially equivalent to P_1 and P_2. We shall say that P' was obtained from P by *geometric facet splitting*.

Some facet splittings can be accomplished geometrically regardless of the 4-polytopes or the shape of the facet. We shall consider these now.

The *length* of a facet splitting is the number of edges on the subfacet created in the splitting. In order to show that a splitting can be done geometrically it is sufficient to show that one can find the plane H which intersects the relative interior of the facet. We shall examine the several cases. These cases are simplified by the fact that we do not have to split a facet of any type other than types a, b, c_1, and c_2.

A. Splittings of length 3. These can be accomplished by taking a plane close enough to a vertex or triangular subfacet of the facet.

B. Splittings of length 4. These can be accomplished by taking a plane close to an edge or 4-sided subfacet.

C. Splittings of length 5:

Case I facets of type b. Combinatorially, there is only one such splitting, and it can be accomplished by first taking a plane close to a 4-sided subfacet and rotating the plane until it passes one vertex of that subfacet (see Fig. 4).

FIGURE 4

Case II facets of type c_1. Now, there are several ways of splitting the facet. We observe that, if we take the sequence of subfacets of the facet to be split as they appear along the new subfacet that is created, and look at the corresponding vertices in the dual of the facet, these vertices will form a circuit. Thus to enumerate the ways of splitting the facet we can enumerate the circuits of length five in the dual of the facet. These can be found by inspection, and are illustrated in the first four figures in column 1 of Table 2. The geometric splittings will again be done by first taking a plane close to and parallel to a subfacet (column 2) and moving it past a vertex or an edge (column 3).

Case III facets of type c_2. There is only one type of splitting and the same type of construction as in Case II will work (see Table 2, line 4).

D. Splittings of length 6:

Case I facets of type c_1. There are two splittings that can always be realized geometrically (Table 2, lines 6 and 7). For the type in line 7 a different construction is used; we take the plane determined by the points p_1, p_2, and p_3 in the figure. There is one splitting that cannot always be done geometrically (see Fig. 8).

Case II facets of type c_2. There is one splitting that can always be done geometrically (Table 2, line 8). In this splitting we choose a vertex v and points p and q on opposite edges as indicated in the table. There are three ways that the plane through v, p and q can intersect the cube, as illustrated in Fig. 5. In each case the plane that gives the splitting can be found by rotating the plane about the line through p and q. There is one splitting that cannot always be done geometrically (Fig. 8).

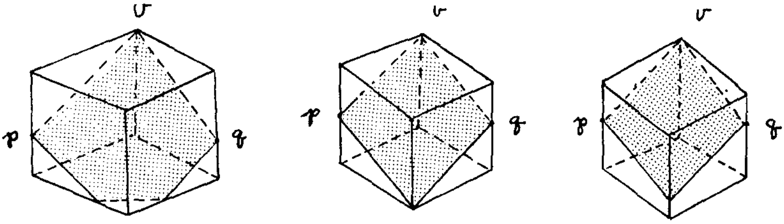

FIGURE 5

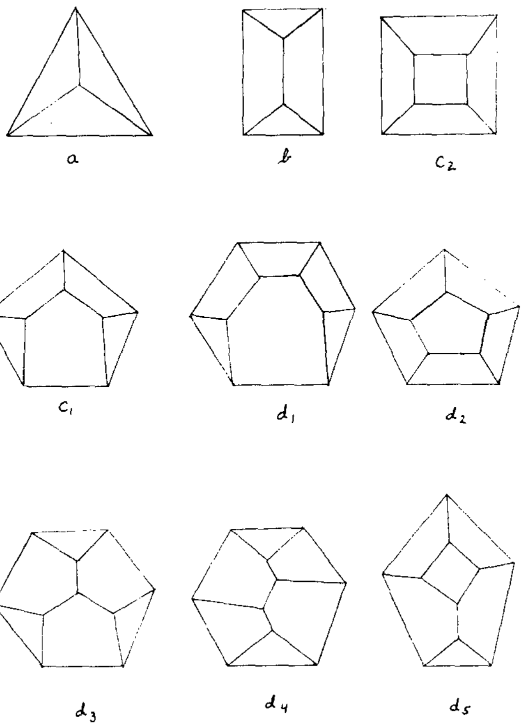

FIGURE 6

We shall not, at this point, prove that the splittings S_1 and S_2 (Fig. 8) are not always geometrically realizable, as this is not necessary to get our results.

Since any splitting applied to a facet with 5 or fewer subfacets can always be done geometrically we have

THEOREM 2. *All* SC3S's *with* 7 *or fewer facets are polyhedral.*

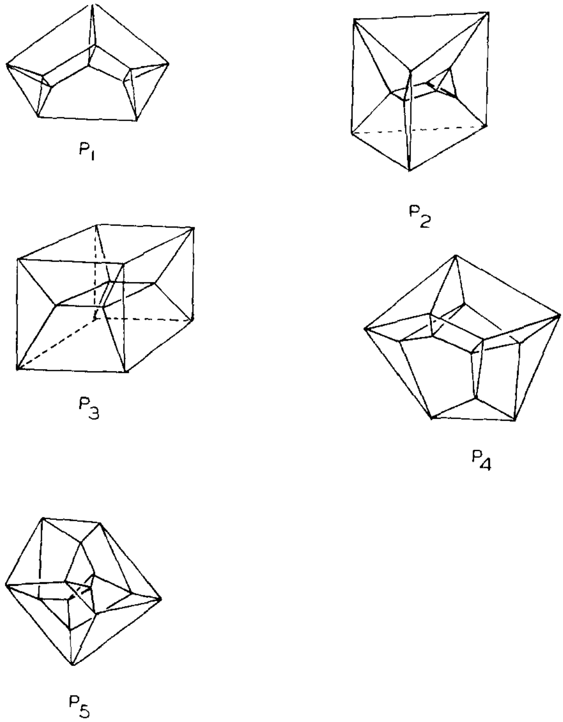

FIGURE 7

Proof. They can be obtained by starting with the 4-simplex and applying geometric facet splittings.

To obtain the SC3S's with 8 facets we now will apply facet splittings to the simple 4-polytopes with 7 facets. The Schlegel diagrams of these polytopes are given in Fig. 7. A complete combinatorial description is given in Table 1 (see [5]). We shall consider only splittings S_1 and S_2 since we have seen that the other splittings can always be done geometrically and thus will yield polytopes.

If we examine the proof of Theorem 1 we see that Cases I, II, and III give us the existence of a removable facet with 4 or fewer edges. Thus if two facets $\mathscr{F}_1$ and $\mathscr{F}_2$ meet on a subfacet α we know that one of three things will be true:

(a) α is removable,

(b) there is a removable subfacet with at most 4 edges, or

(c) some facet $\mathscr{F}_3$ meets $\mathscr{F}_1$ and $\mathscr{F}_2$ and misses α.

TABLE 1

Polytope	Vertices and facets incident to them		Facets	Combinatorial type of facet
P_1	*A*: 1256	*H*: 1367	1: *ABCDEFGH*	c_1
	B: 1245	*J*: 2367	2: *ABCDGJKL*	c_1
	C: 1234	*K*: 2345	3: *CDEFHJKL*	c_1
	D: 1237	*L*: 2356	4: *BCEK*	*a*
	E: 1345		5: *ABEFKL*	*b*
	F: 1356		6: *AFGHJL*	*b*
	G: 1267		7: *DGHJ*	*a*
P_2	*A*: 1245	*H*: 2356	1: *ABCDEF*	*b*
	B: 1246	*J*: 2347	2: *ABCGHJKL*	c_1
	C: 1256	*K*: 2367	3: *DEFGHJKM*	c_1
	D: 1345	*L*: 2467	4: *ABDEGJLM*	c_1
	E: 1346	*M*: 3467	5: *ACDFGH*	*b*
	F: 1356		6: *BCEFHKLM*	c_1
	G: 2345		7: *JKLM*	*a*
P_3	*A*: 1246	*H*: 1347	1: *ABCDFGH*	c_2
	B: 1256	*J*: 2346	2: *ABCDJKLM*	c_2
	C: 1257	*K*: 2356	3: *EFGHJKLM*	c_2
	D: 1247	*L*: 2357	4: *ADEHJM*	*b*
	E: 1346	*M*: 2347	5: *BCFGKL*	*b*
	F: 1356		6: *ABEFJK*	*b*
	G: 1357		7: *CDGHLM*	*b*
P_4	*A*: 2467	*H*: 1456	1: *CDGHJKLN*	c_1
	B: 2367	*J*: 1247	2: *ABEFJKMN*	c_1
	C: 1367	*K*: 1237	3: *BCFGKLMN*	c_1
	D: 1467	*L*: 1345	4: *ADEHJLMN*	c_1
	E: 2456	*M*: 2345	5: *EFGHLM*	*b*
	F: 2356	*N*: 1234	6: *ABCDEFGH*	c_2
	G: 1356		7: *ABCDJK*	
P_5	*A*: 1234	*H*: 1567	1: *ABCDEFGH*	c_1
	B: 1237	*J*: 2345	2: *ABCDEJKL*	c_1
	C: 1267	*K*: 2356	3: *ABFJKLMN*	c_1
	D: 1256	*L*: 2367	4: *AEFGJMNO*	c_1
	E: 1245	*M*: 3467	5: *DEGHJKNO*	c_1
	F: 1347	*N*: 3456	6: *CDHKLMNO*	c_1
	G: 1457	*O*: 4567	7: *BCFGHLMO*	c_1

TABLE 2

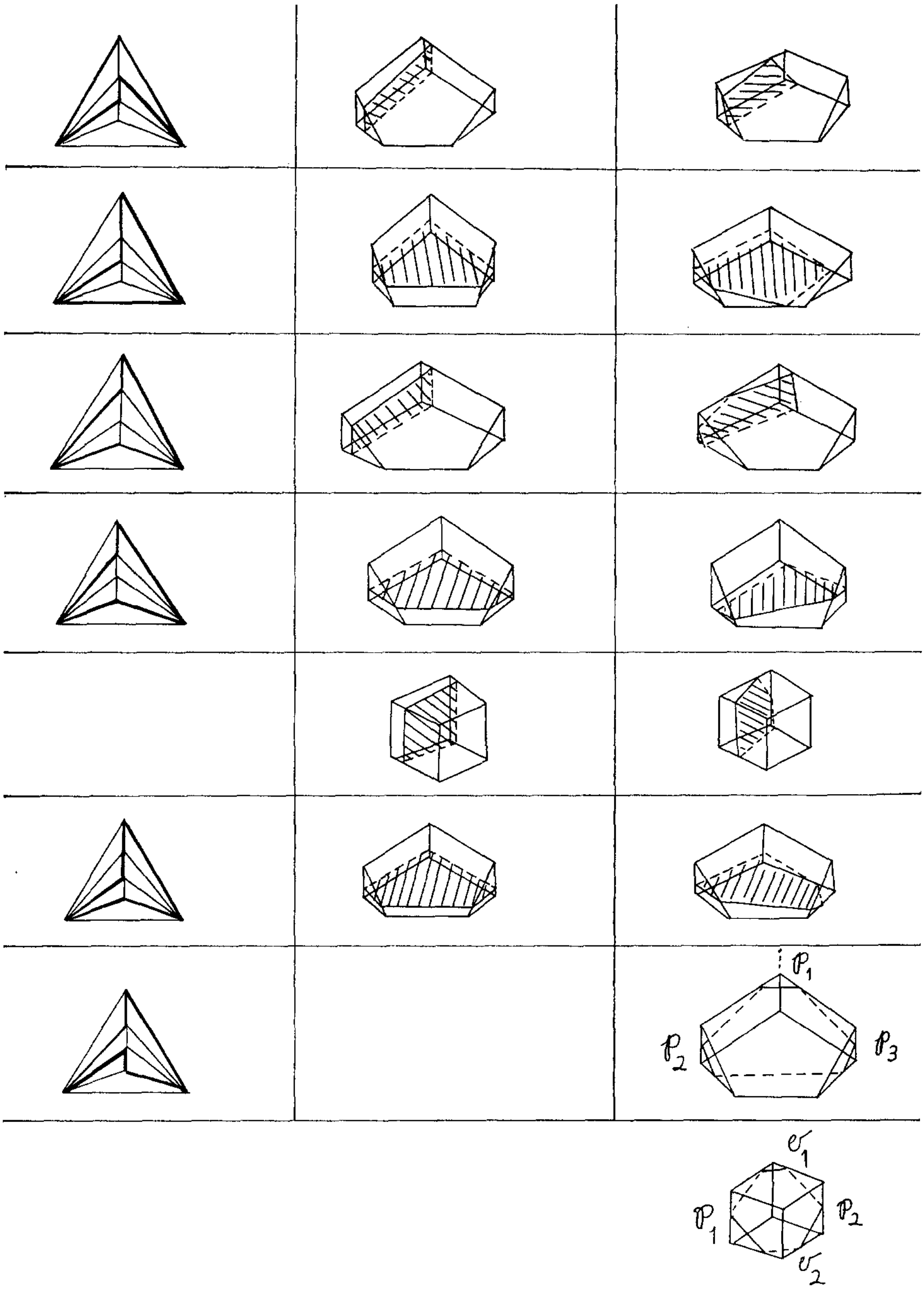

TABLE 3

Polytope	Facet split	Sequence of subfacets in splitting	Combinatorial types of resulting facets
P_1	3	*CKE, KEFL* *CEFHD, HDJ* *FHLJ, CKLJD*	d_4, d_4, b, b, c_1, c_1 d_5, d_5
P_2	2	*JKL, JGHK* *CBHKL, ABC* *ACGH, ABGLJ*	$d_4, d_4, b, d_4, c_1, c_1$ d_4, d_2
P_3	2	*ABCD, ADEF* *ABEH, EFGH* *DCFG, BCGH*	$d_4, d_4, c_1, c_1, c_1, c_1$ d_2, d_2
P_3	2	*ABCD, ADEF* *EFGH, ABEH* *BCGH, DCFG*	d_4, d_4, c_1, c_2 c_1, c_1, d_5, d_5
P_4	4	*LMN, EHLM* *AEJMN, ADJ* *ADEH, DHJLN*	d_4, d_4, d_4, c_1, d_5 c_1, d_5, d_4
P_4	6	*EFGH, DCHG* *ADEH, ABCD* *BCGF, ABEF*	d_4, d_4, d_4, c_2, d_4 d_4, c_2, d_4
P_5	1	*CDH, DEGH* *ABCDE, ABF* *AEFG, BCFGH*	d_4, d_4, d_1, d_3, d_4 d_1, d_3, d_4
P_5	1	*ABF, AEFG* *ABCDE, CDH* *DEGH, BCFGH*	d_4, d_4, d_4, d_4, d_5 d_4, d_4, d_5

From this we can conclude that, if any facet is of type a, b, or c, then there is a removable subfacet with 5 or fewer edges. With this in mind we examine the various splittings:

Case I splittings applied to P_1. By the symmetry of P_1 we need only split facet 3, and we need split in only one way. As we see in Table 3, the sphere produced contains a facet of type b and thus the sphere could be produced by a splitting of length at most 5. This implies that we could have obtained this sphere by applying a geometric facet splitting to a polytope and thus the new SC3S is polyhedral.

Case II splittings applied to P_2. The same argument as in Case I applies.

Case III splittings applied to P_3. By the symmetry of P_3 we need only split facet 2, but there are two different splittings that must be considered (see Table 3, lines 3 and 4).

In both cases, we have facets of type c_1 and thus the SC3S's are polyhedral.

Case IV splittings applied to P_4. By the symmetry of P_4 we need consider only one facet of type c_1. No matter how we split one, say facet 4, we get a facet of type c_1, thus we produce a polyhedral SC3S.

We do not need to consider all splittings of facet 6 because in all but two cases we produce a facet of type c_1, thus those SC3S's could be obtained by geometric facet splittings. In the other two cases (and by symmetry we need only consider one, see Table 3, line 4) we get an SC3S that is not in the list of 4-polytopes with 8 facets. It is the dual of the triangulation $\mathscr{M}'$ of the 3-sphere discovered recently by the author [1].

Case V splittings applied to P_5. By symmetry we need consider only one facet, but there are two different ways of splitting the facet (Table 3, lines 7 and 8).

The SC3S in line 7 is the dual of the polytope P_{37}^8 in [5]. The SC3S in line 8 does not appear in the list of 4-polytopes with 8 facets. It is the dual of the sphere $\mathscr{M}$ first discovered by Grünbaum and Sreedharan [5].

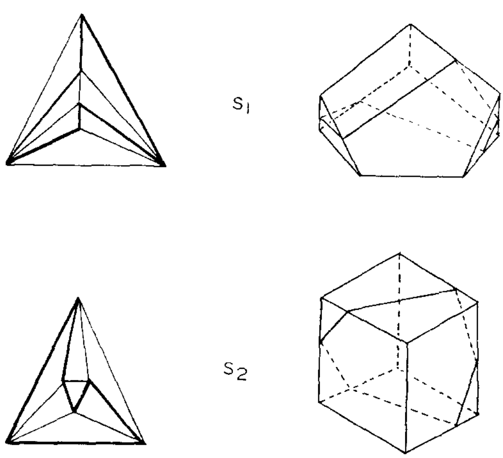

FIGURE 8

We now have

THEOREM 3. *The set of* SC3S'*s with eight facets consists of the boundary complexes of* 4*-polytopes with eight facets and the duals of the spheres* $\mathcal{M}$ *and* $\mathcal{M}'$.

Now that we have the SC3S's we have all the triangulations of S^3 with 8 vertices since they are just the duals of the SC3S's.

5. PREASSIGNING THE SHAPE OF A FACET

We are now in a position to answer the following question (see [2]): Given a 4-polytope P and a facet Q is there a 4-polytope combinatorially equivalent to P for which the facet corresponding to Q has a certain prescribed shape and position?

The answer is "no." Consider the 4-polytope P_4 (Fig. 7). We shall describe a shape that the facet $ABCDEFGH$ cannot have.

Let P be a cube obtained by truncating opposite edges of a tetrahedron (see Fig. 9). This 3-polytope will admit a geometric splitting of type S_2 (see Fig. 9).

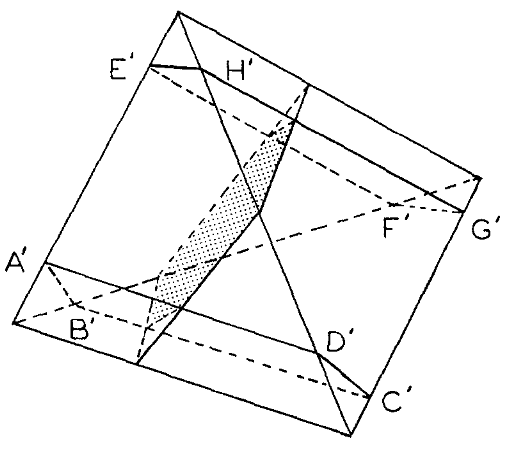

FIGURE 9

If $ABCDEFGH$ could be congruent to P with $A \leftrightarrow A'$, $B \leftrightarrow B'$ etc., (see Fig. 9) then we could split P_4 geometrically and obtain a 4-polytope combinatorially equivalent to the dual of $\mathcal{M}'$, which is impossible.

REMARKS

(i) The construction of $\mathscr{M}$ can be found in [1]. The construction $\mathscr{M}'$ can be found in [5] and [6]. Other results concerning the dual of $\mathscr{M}$ can be found in [3].

(ii) *Conjecture*: The combinatorial types of all SC3S's can be generated by facet splitting.

(iii) A complete list of all simplicial 4-polytopes with 8 vertices can be found in [5].

(iv) Theorem 2 has been generalized by Mani [7]. He has shown that every triangulation of the d-sphere with at most $d + 4$ vertices is polyhedral.

REFERENCES

1. D. BARNETTE, Diagrams and Schlegel diagrams, "Combinatorial Structures and Their Applications," Gordon & Breach, New York, 1970.
2. D. BARNETTE AND B. GRÜNBAUM, Preassigning the shape of a face, *Pacific J. Math.* **32** (1970), 299–306.
3. D. BARNETTE AND G. WEGNER, A 3-sphere that is not 4-polyhedral, *Studia Sci. Math. Hungar.* **6** (1971), 341–346.
4. M. BRUCKNER, Über die Ableitung der allgemeinen Polytope und die nach Isomorphismus Verschiedenen Typen der allgemeinen Achtzell (Oktatope), *Verh. Nederl. Akad. Wetensch. Afd. Natuurk Sect. I* **10**, No. 1 (1909).
5. B. GRÜNBAUM AND V. P. SREEDHARAN, An enumeration of simplicial 4-polytopes with 8 vertices, *J. Combinatorial Theory* **2** (1967), 437–465.
6. B. GRÜNBAUM, "Convex Polytopes," Wiley, New York, 1967.
7. P. MANI, On spheres with few vertices, *J. Combinatorial Theory*, to appear.
8. E. STEINITZ AND H. RADEMACHER, "Vorlesungen über die Theorie der Polyeder," Springer, Berlin, 1934.