Fibonacci morphisms and Sturmian words*

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Abstract


Márton Kősa (1987) has stated five conjectures devoted to the study of some binary morphisms and the factors of infinite words these morphisms generate. Positive answers to the first four of these conjectures are given in this paper, the main result being the characterization of a class of morphisms whose stationary words are Sturmian.

1. Introduction

Kősa [7] has stated a series of interesting conjectures devoted to the study of some particular morphisms and the infinite words these morphisms generate. The aim of this paper is to give answers to the four conjectures of Kősa. In particular, a large class of morphisms closely related to the Fibonacci morphisms, and whose stationary words are Sturmian, is defined in Section 4. This result is very interesting because the Fibonacci sequence has been widely studied [1, 3, 4, 6, 9, 10, 11, 12, 14] and is an example of Sturmian word. But less is known about other Sturmian words (which are of minimal complexity for the number of factors of any length) [2, 5, 9] and the result given here is, to the best of my knowledge, the first known characterization of a simple way to obtain, automatically, Sturmian words.

After some definitions and notations (Section 2), the first conjecture of Kősa is proved in Proposition 3.3. The second conjecture of Kősa is solved in Proposition 4.7

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and, in the last section, using deep results on the Fibonacci numbers, the third and fourth conjectures of Kősa are solved (Propositions 5.11 and 5.12).

2. Definitions and notations

Let $A$ be a finite set called alphabet and $A^*$, the free monoid generated by $A$. The elements of $A$ are called letters and the elements of $A^*$ are called words. The neutral element of $A^*$, also called the empty word, is denoted by $\varepsilon$, and $A^* = A^* - \{\varepsilon\}$.

Let $u \in A^*$. $|u|$ denotes the length of $u$, i.e. the number of letters of $u$. In particular, $|\varepsilon| = 0$. If $a \in A$, $|u|_a$ denotes the number of occurrences of the letter $a$ in the word $u$.

The mirror image of a word $u \in A^*$ is denoted by $\bar{u}$ and is defined as follows: $\bar{\varepsilon} = \varepsilon$ and if $u = x_1 \ldots x_n$, $x_i \in A$, $1 \leq i \leq n$ then $\bar{u} = x_n \ldots x_1$.

Let $u$ and $v$ be two words of $A^*$. $v$ is said to be a factor (left factor, right factor, proper factor) of $u$ if there exist $u_1 \in A^*$ and $u_2 \in A^*$ such that $u = u_1 v u_2$ ($u_1 = \varepsilon$, $u_2 = \varepsilon$, $u_1 \neq \varepsilon$ or $u_2 \neq \varepsilon$).

For any $a \in A$ and $k \in \mathbb{N}$, $a^k$ represents the word composed of $k$ occurrences of the single letter $a$. An infinite word on $A$ is a function $x : \mathbb{N} \rightarrow A$. It is written as $x = x_0 x_1 \ldots x_n \ldots$, $x_i \in A$. The set of infinite words on $A$ is denoted by $A^\omega$ and $\hat{A} = A^* \cup A^\omega$. The notions of factor, left factor, proper factor are extended in a natural way to $A$. An infinite word $x$ on $A$ is said to be ultimately periodic if there exist $u \in A^*$ and $v \in A^*$ such that $x = u^v$.

Let $f_1, f_2, \ldots, f_p$, $p \in \mathbb{N}$, be endomorphisms on $A$. If $\{f_1, f_2, \ldots, f_p\}$ is considered as a $p$ letter alphabet, any morphism $g = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$ ($k \in \mathbb{N}$ and $i_j \in \{1, 2, \ldots, p\}$ for any $j \in \mathbb{N}$, $1 \leq j \leq k$), where $\circ$ represents the function composition, can be seen as a word on the alphabet $\{f_1, f_2, \ldots, f_p\}$.

If $\Theta$ denotes the relation which associates with any such $g$ the corresponding word on $\{f_1, f_2, \ldots, f_p\}$, the length of $g$ (denoted as $\|g\|$) is the number of letters of $\{f_1, f_2, \ldots, f_p\}$ in the word $\Theta(g)$. In the same manner, for any $i \in \mathbb{N}$, $1 \leq i \leq p$, $\|g\|_{f_i}$ denotes the number of occurrences of the letter $f_i$ in the word $\Theta(g)$.

Two morphisms $f$ and $g$ on $A$ are said to be equal if, for any $a \in A$, $f(a) = g(a)$. Let $f$ be a morphism on $A$ and $x \in A^\omega$. $x$ is said to be generated by $f$ if there exist $i \in \mathbb{N}$ and $a \in A$ such that, for any $n \in \mathbb{N}$, $(f^i)^n(a)$ is a left factor of $x$ and $|(f^i)^n(a)| < |(f^i)^{n+1}(a)|$ (where $(f^i)^n$ represents the morphism $f^i \circ f^i \circ \cdots \circ f^i$ $n$ times). In this case the usual notation is: $x = (f^i)^n(a)$.

3. Conjecture 1 of Kősa [7]

Let $A = \{0, 1\}$ be an alphabet and $\text{Em}(A)$, the monoid obtained by composing endomorphisms on $\hat{A}$. Let

$$
X : A^* \rightarrow A^*, \quad L : A^* \rightarrow A^*, \quad R : A^* \rightarrow A^*,
$$

$$
0 \rightarrow 1, \quad 0 \rightarrow 0, \quad 0 \rightarrow 0,
$$

$$
1 \rightarrow 0, \quad 1 \rightarrow 01, \quad 1 \rightarrow 10.
$$
be endomorphisms. Let \( \text{St}(A) \) be the submonoid of \( \text{Em}(A) \) generated by \( \{ X, L, R \} \) and \( \text{Id} \), the identical transformation on \( A \) (\( \text{Id}(0)=0 \), \( \text{Id}(1)=1 \)), which is the unit element of \( \text{Em}(A) \) and \( \text{St}(A) \).

The two following equalities are obviously true in \( \text{St}(A) \):

(1) \( X^2 = \text{Id} \);

(2) \( LR = RL \).

Furthermore, \( \text{St}(A) \) has also the following property.

**Property 3.1.** For any \( k \in \mathbb{N} \),

(3) \( R X R^k X L = L X L^k X R \).

Before proving this property, it must be noted that in the original paper of Kősa [7], composition was made from left to right (which means, for example, that the image of 0 by \( L X R \) was calculated as follows: \( (0)LXR=(0)XR=(1)R=10 \)).

From now on, composition from right to left (\( LXR(0)=LX(0)=L(1)=01 \)) is used; this explains the slight differences from Kősa’s notations.

**Proof of the property 3.1.** For any \( k \in \mathbb{N} \), one has

\[
L^k : A^* \rightarrow A^*, \\
R^k : A^* \rightarrow A^*.
\]

Thus, \( RXR^k XL(0)=RXR^k X(0)=RXR^k(1)=RX(10^k)=0(10^k) \) and, in the same way, \( RXR^k XL(1)=0(10^k+1) \), \( LXL^k XR(0)=(01)^k 0 \), and \( LXL^k XR(1)=(01)^k+1 0 \). Consequently, for any \( k \in \mathbb{N} \), \( RXR^k XL = LXL^k XR \). \( \square \)

One can note that equality (2) is a particular case of (3) when \( k=0 \). The first conjecture of Kősa [7] is the following.

**Conjecture 3.2 (Kősa [7, Problem 149]).** (1), (2) and (3) form a complete set of generating relations of \( \text{St}(A) \).

Before proving this conjecture here is a useful definition: let \( G \in \text{St}(A) \). \( G \) is called reduced if \( \| G \|_x=0 \) and \( \Theta(g) \) does not contain any occurrence of \( RL \) (note that any \( G \in \text{St}(A) \) can be reduced in applying, as much as possible, equalities (1) and (2). Furthermore, if \( G \) is reduced, then \( \| G \| \) is a well-defined unique number).

**Proposition 3.3.** Let \( G \) and \( G' \) be reduced elements of \( \text{St}(A) \) such that \( \| G \| \geq \| G' \| \).

Then, the two following conditions are equivalent:

(a) \( G = G' \);

(b) \( G' \) can be obtained from \( G \) by using only equalities (1), (2) and (3).
**Proof.** The implication (b)⇒(a) being self evident, one has just to prove (a)⇒(b). This is proved by induction on \(|G|\). Let \(|G|=m\). The relation can be easily established for \(m<5\); so, let us suppose that it is true for any \(m\in\mathbb{N}, m \geq 5\), and show that it is then true for \(p=m+1\).

So, let \(G\in \text{St}(A)\) be such that \(|G|=p, p \geq 6\) and \(F\in \text{St}(A)\) such that \(G=XF\) or \(LF\) or \(RF\). Let \(G'\in \text{St}(A)\) be such that \(|G'| \leq p\) and \(F'\in \text{St}(A)\) such that \(G'=XF'\) or \(LF'\) or \(RF'\). Furthermore, let us suppose that \(G\) and \(G'\) are reduced and \(G=G'\).

**Case 1:** If \(G=XF\), \(\mathcal{O}(F)\) starts with \(L\) or \(R\) (since \(G\) is reduced). By definition of \(L\) and \(R\), and since \(|F|>5\), \(F(0)\) or \(F(1)\) contains 00 as a factor; thus, \(G(0)\) or \(G(1)\) contains 11 as a factor. But, for any \(x\in[0,1]^*, L(x)\) and \(R(x)\) do not contain 11 as a factor. Thus, if \(G'=LF'\) or \(RF'\), one has \(G' \neq G\); this contradicts the hypothesis. Consequently, \(G'=XF'\) and, in this case, \(F=F'\) and the implication is true by induction.

**Case 2:** Now, let us suppose that \(G=LF\) and \(G'=RF'\) (the case \(G=RF\) and \(G'=LF'\) would be similar). Since \(G\) is reduced, there exists \(\{s_1, r_2, s_2, \ldots, r_q, s_q\} \subset \mathbb{N}\) such that \(G=R^{s_1}X^{r_2}X^{s_2}X \ldots X^{r_q}X^{s_q}\) and \(s_i \geq 1; s_i \geq 0, 2 \leq i \leq q; r_i \geq 0, 2 \leq i \leq q; r_i \neq 0\) or \(s_i \neq 0, 2 \leq i \leq q-1\).

If \(s_i \neq 0\) then, applying equality (2), one has \(G'=R^{s_i}L^{r_i}X \ldots X^{r_i}R^{s_i}\) and \(F=F'\). In this case the implication is true by induction. So let us suppose \(G'=L^{r_i}X^{s_i}X \ldots X^{r_i}R^{s_i}\). If the least \(j\in \mathbb{N}\) such that \(s_j \neq 0 (r_j \neq 0)\) is odd then, applying equalities (2) and (3), one has that \(G'=RF' (G=LF)\), and by induction the implication is true. On the other hand, if the least \(j\in \mathbb{N}\) such that \(r_j \neq 0 (s_j \neq 0)\) is even, then \(G (G')\) starts with \(R^{s_i}XL (L^rXR)\).

Thus one has to examine the following cases:

(i) \(G\) starting with \(R^{s_i}XL\) or \(G=R^{s_i}XR^{s_i}X \ldots X^{r_i}R^{s_i}\) or \(G=R^{s_i}\), and

(ii) \(G'\) starting with \(L^{r_i}XR\) or \(G'=L^{r_i}XL^{r_i}X \ldots X^{r_i}L^{r_i}\) or \(G'=L^{r_i}\).

One has, for any \(n\in \mathbb{N}\):

\[
\begin{align*}
L^n(0) &= 0, & L^n(1) &= 0^n1; \\
L^n XR(0) &= 0^n1, & L^n XR(1) &= 0^{n+1}1; \\
L^n XL(0) &= 0^n1, & L^n XL(1) &= 0^n10.
\end{align*}
\]

Thus, \(G(0)\) and \(G'(1)\) both start with 0.

Now, for any \(n\in \mathbb{N}, R^nXL(0)=0^n\) and \(R^nXL(1)=10^n+1\). Thus, if \(G\) starts with \(R^{s_i}XL\), one cannot have \(G=G'\), which contradicts the hypothesis. In the same way, since \(R(1)=10, G=R^{s_i}\), one has a \(G(1)\) starting with 1; thus, \(G(1) \neq G'(1)\), which contradicts the hypothesis. Now, let us suppose that \(G=R^{s_i}XR^{s_i}X \ldots X^{r_i}R^{s_i}\). Since, for any \(n\in \mathbb{N}, XR^n(0)=1, XR^n(1)=0^n1, R(0)=0, R(1)=10, \) one has a \(G(0)\) or \(G(1)\) starting with 1. Thus, \(G(0) \neq G'(0)\) or \((G(1) \neq G'(1)\), which contradicts the hypothesis.  □

Before stating conjecture 2 of Kósa [7], the following definition is stated.

**Definition 4.1** (Kósa [7]). $G \in \text{Em}(A)$ is *regular* if and only if the three following conditions hold:

1. $G(0)$ contains letter 1 at least once;
2. $G(1)$ contains letter 0 at least once;
3. At least one of the two following conditions holds:
   a. $G(0)$ contains letter 0 at least once,
   b. $G(1)$ contains letter 1 at least once.

This means that $G$ is strictly growing on the two letters 0 and 1, which assures that any stationary word $W$ of $G$ (word such that there exists $n \in \mathbb{N}$ such that $G^n(W) = W$) is in $A^\omega$. The set of such morphisms on $A$ is denoted by $\text{Reg}(A)$.

**Conjecture 4.2** (Kósa [7, Problem 150]). If $G \in \text{St}(A) \cap \text{Reg}(A)$, then its stationary words are Sturmian.

Let us recall a characteristic property of the Sturmian words (see, e.g. [9]):

**Definition 4.3.** Let $B$ be a two letter alphabet. For any word $U$ in $B^\omega$ and any $n \in \mathbb{N}$, $p(n)$ denotes the number of factors of $U$ of length $n$. Then $U$ is Sturmian if and only if $p(n) = n + 1$, for any $n \in \mathbb{N}$.

The best known example of Sturmian word is the Fibonacci word: let $B = \{a, b\}$ and $\varphi : B^* \to B^*$,

$$a \to ab,$$

$$b \to a.$$

The Fibonacci word is the fixed point of $\varphi$, i.e. $\varphi^\omega(a) \neq \varphi^\omega(b)$.

It is well known (see, e.g. [11]) that the morphism $\tilde{\varphi} : B^* \to B^*$,

$$a_1 \to ba,$$

$$b \to a$$

generates an infinite word which has the same factors as the Fibonacci word and which is, thus, also Sturmian.

Let us now give a property of the Fibonacci morphisms $\varphi$ and $\tilde{\varphi}$.

**Property 4.4.** The image of any Sturmian word of $B^\omega$ by $\varphi$ or $\tilde{\varphi}$ is also a Sturmian word.
**Proof.** Let \( W \in B^\omega \) be a Sturmian infinite word. Since \( W \) is Sturmian, \( W \) is not ultimately periodic; thus, \( \varphi(W) \) is not ultimately periodic (because \( \varphi \) is injective). Consequently, if \( p(n) \) denotes, for any \( n \in \mathbb{N} \), the number of factors of length \( n \) of \( \varphi(W) \), one has \( p(n) \geq n + 1 \).

Now, let us call, after Berstel [1], special factor a factor \( u \) of an infinite word on \( B \) such that \( ua \) and \( ub \) are also factors of this word. Since \( W \) is Sturmian, \( W \) contains exactly one special factor of length \( n \), for any \( n \in \mathbb{N} \). But, since \( W \) is not ultimately periodic, \( \varphi(W) \) contains at least one special factor of length \( n \), for any \( n \in \mathbb{N} \). Let \( v \) be such a factor. By definition of \( \varphi, v \) ends necessarily with \( a \). So, let \( v' \in B^+ \) be such that \( v = v'a \). If \( v' \) starts with \( a \) (\( b \)), there exists \( u \), a factor of \( W \), such that \( v' = \varphi(u) \) (\( uv' = \varphi(u) \) and \( av' \) is necessarily a factor of \( \varphi(W) \)). But, since \( v \) is a special factor of \( \varphi(W) \), \( va \) and \( vb \) are factors of \( \varphi(W) \); thus, \( v'aab \) and \( v'ab \) are factors of \( \varphi(W) \). \( v'aab = \varphi(uba) \) and \( v'ab = \varphi(ua) \). Thus, \( ua \) and \( ub \) are factors of \( W \). Consequently, \( u \) is the unique special factor of \( W \) and, so, \( v \) is also unique. Therefore, for any \( n \in \mathbb{N} \), \( \varphi(W) \) contains a unique special factor of length \( n \), which means that \( p(n) \leq n + 1 \). Since we saw above that \( p(n) \geq n + 1 \), one has \( p(n) = n + 1 \); \( \varphi(W) \) is, thus, Sturmian.

The demonstration would be exactly the same for \( \varphi(W) \) and the property is thus proved. \( \square \)

After the Sturmian words, let us consider morphisms \( G \in \text{St}(A) \cap \text{Reg}(A) \) and give two properties of these. It is very easy to see that, for any \( G \in \text{Em}(A) \), if \( W \) is a stationary word of \( G \), then \( G(W) = W \) or \( G^2(W) = W \) (this follows from the fact that \( A \) is a two letter alphabet). The property below follows immediately.

**Property 4.5.** Let \( G \in \text{St}(A) \cap \text{Reg}(A) \) and \( W \in A^\omega \). If \( W \) is a stationary word of \( G \), then \( W \) is generated by \( G \) or \( G^2 \) (this follows from the fact that if \( u \) is a left factor of \( W \), since \( G \) is necessarily strictly growing, then \( G(u) \) or \( G^2(u) \) is a left factor of \( W \) strictly longer than \( u \) and starting with \( u \)).

**Property 4.6.** Let \( G \in \text{St}(A) \cap \text{Reg}(A) \). Then \( O(G) \in \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+ \).

**Proof.** Since \( X^2 = \text{Id} \), one has: \( L = X^2L = LX^2 \) and \( R = X^2R = RX^2 \). Consequently, for any \( G \in \text{St}(A) \), \( \Theta(G) \in \{X, XL, LX, XR, RX\}^* \). If \( G \in \{X\}^* \), then \( G(0) = 0, G(1) = 1 \) (or \( G(0) = 1, G(1) = 0 \)) and, in both cases, \( G \notin \text{Reg}(A) \). Thus, if \( G \in \text{St}(A) \cap \text{Reg}(A), \Theta(G) \in \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+ \). \( \square \)

Now, we are ready to prove Conjecture 2 of Kősa [7].

**Proposition 4.7.** If \( G \in \text{St}(A) \cap \text{Reg}(A) \), then its stationary words are Sturmian.

**Proof.** One has

\[
\begin{align*}
LX(0) &= 01, & LX(1) &= 0; & XL(0) &= 1, & XL(1) &= 0; \\
RX(0) &= 10, & RX(1) &= 0; & XR(0) &= 1, & XR(1) &= 01.
\end{align*}
\]
Thus, $LX$ ($RX$) is obtained from $\varphi (\tilde{\varphi})$ by replacing $a$ by $0$ and $b$ by $1$, and $XL$ ($XR$) is obtained from $\varphi (\tilde{\varphi})$ by replacing $a$ by $1$ and $b$ by $0$.

Now, let $G \in \text{St}(A) \cap \text{Reg}(A)$. By Property 4.6, $G$ is a composition of Fibonacci morphisms and $X$. Furthermore, by Property 4.5, the stationary words of $G$ are $G^u(u)$ or $(G^2)^u(u)$, $u \in \{0, 1\}$.

By Property 4.4, the images of any factor of a Sturmian word by the Fibonacci morphisms are factors of Sturmian words. Consequently, since, for any $n \in \mathbb{N}$, application of the morphism $X$ does not change the number of factors of length $n$ of a word, the image of a left factor of a Sturmian word by $G$ or $G^2$ is also a left factor of a Sturmian word.

Thus, if $W$ is a stationary word of $G$, then, for any $n \in \mathbb{N}$, $p(n) \leq n + 1$. But, $W$ cannot be ultimately periodic. (In fact, if $G$ generates an ultimately periodic word and $\Theta(G) \notin \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+$, then $G(0) = 0$ or $G(1) = 1$; thus, $G \notin \text{Reg}(A)$.)

(For a complete characterization of all the binary morphisms which generate ultimately periodic words, see [13].)

Thus, for any $n \in \mathbb{N}$, $p(n) \geq n + 1$. Consequently, if $W$ is a stationary word of $G$ then, for any $n \in \mathbb{N}$, $p(n) = n + 1$, which means that $W$ is Sturmian and completes the proof of Proposition 4.7.

Some remarks need to be made about this result. The first two ones imply that the two conditions of Proposition 4.7 are both necessary.

**Remark 4.8.** $G \in \text{Reg}(A)$ does not imply that all stationary words of $G$ are Sturmian.

**Example.** The Thue-Morse sequences (defined as the stationary words of the morphism $\mu: A^* \to A^*$, $\mu(0) = 01$, $\mu(1) = 10$) are a well-known example of non-Sturmian words. In fact, in these sequences $p(n) \geq 2n$, for any $n \in \mathbb{N}$ (see [15]). Thus, in Proposition 4.7 the condition $G \in \text{St}(A)$ is necessary.

**Remark 4.9.** $\Theta(G) \notin \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+$ does not imply that all stationary words of $G$ are Sturmian.

**Example.** Let us consider $XRX$ ($XRX(0) = 01$, $XRX(1) = 1$). $\Theta(XRX) \in \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+$. However, $XRX^u(0) (-01^u)$, which is a stationary word of $XRX$, is not a Sturmian word.

Thus, in Proposition 4.7 the condition $G \in \text{Reg}(A)$ is necessary. The following remark shows that the converse of Property 4.6 is false.

**Remark 4.10.** $\Theta(G) \notin \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+$ does not imply that $G \in \text{Reg}(A)$.

**Example.** $XRX(1) = 1$, thus $XRX$ (see the previous example) is an example of a morphism such that $\Theta(G) \notin \{X, XL, LX, XR, RX\}^+ \setminus \{X\}^+$ does not imply that $G \in \text{Reg}(A)$. 


5. Conjectures 3 and 4 of Kősa [7]

After combinatorial considerations on Sturmian words and morphisms, proofs of Conjectures 3 and 4 of Kősa [7] involve more arithmetic and need some preliminary definitions.

**Definition 5.1.** Let

\[ G : A^* \rightarrow A^* , \]

\[ 0 \mapsto 101 , \]

\[ 1 \mapsto 10101 , \]

\[ G = XRXRXRL \] and \( Q \) is its only fixed point (\( Q \) is an infinite word, \( Q = G^\omega(0) = G^\omega(1) \)). Let \( B_0 = \{1, 2\} \) and \( B_1 = \{2, 3\} \) be alphabets. Let

\[ H_0 : A^* \rightarrow B_0^* , \quad \text{and} \quad H_1 : A^* \rightarrow B_1^* , \]

\[ 0 \mapsto 2 , \quad 0 \mapsto 23 , \]

\[ 1 \mapsto 21 , \quad 1 \mapsto 233 . \]

Let \( t(j, i) \) be the \( i \)th letter of \( H_j(Q) \), \( j = 0, 1 \), \( i \in \mathbb{N} \). Finally, let \( f(r, j) = -1 + \sum_{i=1}^{r} t(j, i) \), \( j = 0, 1 \), \( r \in \mathbb{N} \). (Note that here the first letter of \( H(Q) \) has index 1 and not 0 as in the basic definition of infinite words.)

**Conjecture 5.2** (Kősa [7, Problem 151]). For any \( r \in \mathbb{N} \),

\[ f(r, 0) = \left\lfloor \frac{1 + \sqrt{5}}{2} r + \frac{1 - \sqrt{5}}{4} \right\rfloor , \]

\[ f(r, 1) = \left\lfloor \frac{3 + \sqrt{5}}{2} r - \frac{1 + \sqrt{5}}{4} \right\rfloor , \]

(where \( \lfloor x \rfloor \) denotes the integer part of \( x \)).

The proofs of these conjectures need auxiliary constructions and some properties of the Fibonacci numbers and the Fibonacci word.

5.1. Some properties of the Fibonacci numbers

(For details on these properties, see in particular [1].) Fibonacci numbers can be defined as follows:

\[ F_0 = 1 , \quad F_1 = 2 , \]

\[ F_{n+2} = F_{n+1} + F_n \quad \text{for any} \quad n \in \mathbb{N} . \]
Theorem 5.3 (Zeckendorf). Any integer \( n > 0 \) can be decomposed into a sum of distinct Fibonacci numbers:

\[
n = F_{k_1} + \cdots + F_{k_{r-1}} + F_{k_r} \quad (k_1 > \cdots > k_{r-1} > k_r).
\]

Furthermore, this decomposition is unique under the following condition:

\[ k_i \geq k_{i+1} + 2 \quad \text{for any } i < r \]

(which means that there are no consecutive indices).

Now, let \( \varphi = \frac{1 + \sqrt{5}}{2} = 1.618 \ldots \) and \( \hat{\varphi} = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\varphi} = -0.618 \ldots \)

One has then

(a) \( \varphi^2 = \varphi + 1 \)

(b) For any \( n \in \mathbb{N} \), \( \varphi \hat{\varphi}^n + \hat{\varphi}^{n+1} = \hat{\varphi}^n \);

(c) For any \( n \in \mathbb{N} \), \( F_n + \hat{\varphi}^{n+2} \).

5.2. Some properties of the Fibonacci word

As seen in Section 4 the Fibonacci word, which is denoted as \( F \), is the fixed point of the morphism

\[
\varphi : B^* \to B^*,
\]

\[
a \mapsto ab,
\]

\[
b \mapsto b,
\]

where \( B = \{ a, b \} \). Now, let \( u \) be a left factor of \( F \), \( |u| \geq 3 \). If there exists some \( p \in \mathbb{N} \), \( p \geq 2 \), such that \( |u| = F_p \), then it is well known that \( |u|_a = F_{p-1} \) and \( |u|_b = F_{p-2} \). Otherwise, there exists \( \{ k_1, k_2, \ldots, k_r \} \subset \mathbb{N} \) such that \( |u| = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \) and for which condition (z) is true. In this case \( u \) can be written as \( u = u_{k_1} u_{k_2} \cdots u_{k_r} \), with \( |u_{k_i}| = F_{k_i} \), for any \( i \in \mathbb{N} \), \( 1 \leq i \leq r \). By construction of \( F \) any of the \( u_{k_i} \)'s is a left factor of \( F \), which means that, for any \( i \in \mathbb{N} \), \( 1 \leq i \leq r \), if \( k_i \geq 2 \) one has \( |u_{k_i}|_a = F_{k_i-1} \) and \( |u_{k_i}|_b = F_{k_i-2} \).

Since condition (z) is true, \( k_{r-1} \geq 2 \). Therefore,

\[
|u_{k_1} u_{k_2} \cdots u_{k_{r-1}}|_a = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_{r-1}-1},
\]

\[
|u_{k_1} u_{k_2} \cdots u_{k_{r-1}}|_b = F_{k_1-2} + F_{k_2-2} + \cdots + F_{k_{r-1}-2}.
\]

Now, if \( k_r = 1 \) one has necessarily \( u_{k_r} = ab \), and if \( k_r = 0 \), one has \( u_{k_r} = a \).

Consequently, one has the following property.
Property 5.4. For any $u \in B^+$ such that $u$ is a left factor of $F$, let $\{k_1, \ldots, k_r\} \subset \mathbb{N}$, $r \geq 1$, be such that $|u| = F_{k_1} + \cdots + F_{k_r}$ (with condition (z)). Then,

if $k_r \geq 2$,

$$|u|_a = F_{k_1} + \cdots + F_{k_r}, \quad |u|_b = F_{k_1} + \cdots + F_{k_r-2};$$

if $k_r = 1$,

$$|u|_a = F_{k_1} + \cdots + F_{k_r-1} + F_0, \quad |u|_b = F_{k_1} + \cdots + F_{k_r-2} + F_0;$$

if $k_r = 0$,

$$|u|_a = F_{k_1} + \cdots + F_{k_r-1} + F_0, \quad |u|_b = F_{k_1} + \cdots + F_{k_r-2}.$$

Let us now establish two combinatorial properties of the Fibonacci word which are going to be very useful in the rest of this section.

Property 5.5. (a) $F = (\phi \circ \hat{\phi}^2)(a)(\phi \circ \hat{\phi}^2)^2(a) \cdots (\phi \circ \hat{\phi}^2)^i(a) \cdots$,

(b) $F = a(\hat{\phi} \circ \phi \circ \hat{\phi})(ba)(\hat{\phi} \circ \phi \circ \hat{\phi})^2(ba) \cdots (\hat{\phi} \circ \phi \circ \hat{\phi})^i(ba) \cdots$.

Proof. First observe that it is enough to prove (a) because (a) implies (b). Indeed,

$$F = (\phi \circ \hat{\phi}^2)(a)(\phi \circ \hat{\phi}^2)^2(a) \cdots (\phi \circ \hat{\phi}^2)^i(a) \cdots$$

$$\Rightarrow \hat{\phi}(F) = \hat{\phi} \cdot (\phi \circ \hat{\phi}^3)(a) \hat{\phi} \cdot (\phi \circ \hat{\phi}^3)^2(a) \cdots \hat{\phi} \cdot (\phi \circ \hat{\phi}^3)^i(a) \cdots$$

$$= (\hat{\phi} \circ \phi \circ \hat{\phi})(\hat{\phi}(a)) (\hat{\phi} \circ \phi \circ \hat{\phi})^2(\hat{\phi}(a)) \cdots (\hat{\phi} \circ \phi \circ \hat{\phi})^i(\hat{\phi}(a)) \cdots$$

$$= (\hat{\phi} \circ \phi \circ \hat{\phi})(ba) (\hat{\phi} \circ \phi \circ \hat{\phi})^2(ba) \cdots (\hat{\phi} \circ \phi \circ \hat{\phi})^i(ba) \cdots$$

But (see [12, p. 150]), $(\hat{\phi}^2)^n(a) = abF$ and $\hat{\phi}((\hat{\phi}^2)^n(a)) = baF$. Thus, $baF = \hat{\phi}(abF) = baa\hat{\phi}(F)$. Consequently, $F = a\hat{\phi}(F)$ and (b) follows immediately.

Now, in order to prove (a), let us suppose that the left factor $u$ of $F$, $|u| = F_3 + F_6 + \cdots + F_{3n}$, is such that $u = (\phi \circ \hat{\phi}^2)(a) \cdots (\phi \circ \hat{\phi}^2)^i(a)$ (this is true for $n = 1$ or 2) and show by induction that in this case the left factor $v$ of $F$, $|v| = F_3 + \cdots + F_{3(n+1)}$, is such that $v = (\phi \circ \hat{\phi}^2)(a) \cdots (\phi \circ \hat{\phi}^2)^i(a)(\phi \circ \hat{\phi}^2)^{i+1}(a)$.

Fact 5.6. $v = \phi^3(ua)$.

Proof. $\phi^3(a) = abaab$, $\phi^3(b) = abab$,

$$|ua| = |u| + 1 = F_3 + F_6 + \cdots + F_{3n} + 1$$

$$= F_{3n} + \cdots + F_6 + F_3 + F_0.$$
Therefore,

\[ |\varphi^3(ua)| = 5|ua|_a + 3|ua|_b \]

\[ = F_3 + F_6 + \cdots + F_{3n} + F_{3(n+1)} \]

\[ = |u|. \]

But \( u \) is a left factor of \( F \); thus, \( ua \) or \( ub \) is a left factor of \( F \).
If \( ua \) is a left factor of \( F \), \( \varphi^3(ua) \) is a left factor of \( F \).
If \( ub \) is a left factor of \( F \), \( uba \) is also a left factor of \( F \) and \( \varphi^3(uba) \) is a left factor of \( F \).

Thus, in both cases, \( \varphi^3(ua) \) is a left factor of \( F \) and, since \( v \) is also a left factor of \( F \) and \( |v| = |\varphi^3(ua)| \), \( v = \varphi^3(ua) \). This completes the proof of Fact 5.6.  \( \Box \)

**Fact 5.7.** For any \( i \in \mathbb{N} - \{0\} \), there exists \( x_i \in \{a, b\}^+ \) such that

\[ (\varphi \circ \tilde{\phi}^2)^i(a) = ax_i \quad \text{and} \quad \tilde{\phi}^3((\varphi \circ \tilde{\phi}^2)^{i-1}(a)) = x_i a. \]

**Proof.** Let \( y = \tilde{\phi}^2((\varphi \circ \tilde{\phi}^2)^{i-1}(a)). \) Then

\[ (\varphi \circ \tilde{\phi}^2)^i(a) = \varphi(y) \quad \text{and} \quad \tilde{\phi}^3((\varphi \circ \tilde{\phi}^2)^{i-1}(a)) = \tilde{\phi}(y). \]

But it is well known (see [1]) that, for any factor \( u \) of \( F \), \( a \tilde{\phi}(u) = \varphi(u) a. \) Thus, if \( \varphi(y) = ax_i, \) \( \varphi(y) a = ax_i a = a \tilde{\phi}(y) \) and, so, \( \tilde{\phi}(y) = x_i a. \) This completes the proof of Fact 5.7.  \( \Box \)

**Proof of Property 5.5** (continued). Now since, for any \( i \in \mathbb{N} \), \( (\varphi \circ \tilde{\phi}^2)^i(a) \) starts with \( a, \)

\[ ua = (\varphi \circ \tilde{\phi}^2)(a) \ldots (\varphi \circ \tilde{\phi}^2)^x(a) a \]

\[ = ax_1 ax_2 a \ldots ax_n a \]

\[ = a \tilde{\phi}^3(a)(\tilde{\phi}^3(\varphi \circ \tilde{\phi}^2)(a)) \ldots (\tilde{\phi}^3(\varphi \circ \tilde{\phi}^2)^{n-1}(a)). \]

Consequently, applying \( \tilde{\phi}^2 \circ \varphi = \varphi^2 \circ \tilde{\phi}, \)

\[ v = \varphi^3(ua) \]

\[ = \varphi^3(a \tilde{\phi}^3(a) \ldots (\tilde{\phi}^3(\varphi \circ \tilde{\phi}^2)^{n-1}(a))) \]

\[ = \varphi^3(a)(\varphi \circ \tilde{\phi}^2)^{2}(a) \ldots (\varphi \circ \tilde{\phi}^2)^{n+1}(a), \]

and, since \( \varphi^3(a) = (\varphi \circ \tilde{\phi}^2)(a) \), Property 5.5 is proved.  \( \Box \)

Now, before giving auxiliary constructions needed to show that conjectures 3 and 4 of Kása [7] are true, a useful property about the length of the left factors of \( F \) is stated.
Property 5.8. Let \( v \) be a left factor of \( F \) such that

\[
F_3 + F_6 + \cdots + F_{3p} + F_{3p} < |v| \leq F_3 + F_6 + \cdots + F_{3p} + F_{3(p+1)}, \quad p \in \mathbb{N}.
\]

Then, there exists \( \{i_1, \ldots, i_k\} \subseteq \mathbb{N} \) such that

\[
|v| = F_{i_1} + \cdots + F_{i_k} \quad \text{and} \quad k \leq p + 3.
\]

**Proof.** \( F_3 + F_6 + \cdots + F_{3p} + F_{3p} < |v| \leq F_3 + F_6 + \cdots + F_{3p} + F_{3(p+1)} \) implies, using Property 5.5(a), that there exist \( u \in \{a, b\}^* \) and \( x \in \{a, b\}^* \) such that

\[
F_{3p} < |u| \leq F_{3(p+1)}
\]

\[
u = (\varphi \circ \tilde{\varphi}^2)^0(a)x
\]

\[
v = (\varphi \circ \tilde{\varphi}^2)(a)(\varphi \circ \tilde{\varphi}^2)^0(a)u
\]

\[
= (\varphi \circ \tilde{\varphi}^2)(a)(\varphi \circ \tilde{\varphi}^2)^0(a)(\varphi \circ \tilde{\varphi}^2)^0(a)x.
\]

Thus,

\[
|v| = |u| + F_{3p} + \cdots + F_6 + F_3
\]

\[
= F_{3p} + F_{3p} + \cdots + F_6 + F_3 + |x|.
\]

But, since \( F_{3p} < |u| \leq F_{3(p+1)} \), \( 1 \leq |x| \leq F_{3(p+1)} - F_{3p} \) and, since \( F_{3(p+1)} - F_{3p} = F_{3p+2} + F_{3p-1} \), \( 1 \leq |x| \leq F_{3p+2} + F_{3p-1} \). Thus, there exists \( \{i_1, \ldots, i_r\} \subseteq \mathbb{N} \) such that

\[
i_1 \leq 3p + 2 \quad \text{and} \quad |x| = F_{i_1} + F_{i_2} + \cdots + F_{i_r} \quad \text{(with condition (z)).}
\]

Let \( v' \) be such that \( v = v'x \). One has \( |v'| = F_{3p} + F_{3p} + \cdots + F_6 + F_3 \) and it will be shown that \( |x| \) does not add more than 3 to the number of factors of the decomposition of \( |v'| \). Since \( i_2 \leq 3p \) and \( i_{r-2} \geq 4 \), first it is proved that the addition of \( F_t, 4 \leq t \leq 3p \), in \( |v'| \) does not increase the number of factors in the decomposition of \( |v'| \).

**Case 1:** \( F_t = F_{3s}, \ 2 \leq s \leq p \). In this case there exists \( i \in \mathbb{N}, \ 0 < i < p - 2 \), such that \( s = p - i \).

\[
|v'| + F_t = F_{3p} + \cdots + 2F_{3(p-i)} + F_{3(p-i-1)} + \cdots + F_3
\]

(with, eventually, \( F_3 = F_{3(p-i-1)} \)). But \( 2F_{3(p-i)} + F_{3(p-i-1)} = F_{3(p-i)+1} + F_{3(p-i)-1} \).

Thus,

\[
|v'| + F_t = F_{3p} + \cdots + F_{3(p-i)+1} + F_{3(p-i)-1} + \cdots + F_3.
\]

**Case 2:** \( F_t = F_{3s+1}, \ 1 \leq s \leq p - 1 \). In this case, there exists \( i \in \mathbb{N}, \ 1 < i < p - 1 \), such that \( s = p - i \).

\[
|v'| + F_t = F_{3p} + \cdots + F_{3(p-i)+1} + F_{3(p-i)+1} + \cdots + F_3
\]
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(with, eventually, \( F_3 = F_3(p-i) \)). But \( F_3(p-i)+1 + F_3(p-i) = F_3(p-i)+2 \). Thus,

\[ |v'| + F_i = F_{3p} + \cdots + F_{3(p-i)+2} + \cdots + F_3. \]

Case 3: \( F_i = F_{3s+2}, 1 \leq s \leq p-1 \). In this case, there exists \( i \in \mathbb{N}, 1 \leq i \leq p-1 \), such that \( s = p-i \).

\[ |v'| + F_i = F_{3p} + \cdots + F_{3(p-i)+1} + F_{3(p-i)+2} + \cdots + F_3. \]

But \( F_{3(p-i)+1} + F_{3(p-i)+2} = F_{3(p-i)+1}+1 \). Thus,

\[ |v'| + F_i = F_{3p} + \cdots + F_{3(p-i)+1} + \cdots + F_3. \]

Consequently, in the three cases, the number of factors in the decomposition of \( |v'| \) is not increased by the addition of \( F_i, 4 \leq i \leq 3p \), which means that the number of factors in the decomposition of \( |v'| \) is not increased by the addition of \( F_i + \cdots + F_{i-2} \). Thus, the number of factors in the decomposition of \( |v'| \) can be increased only by addition of \( F_{i}, F_{i-1} \), and \( F_{i+1} \). Therefore, the number of factors in the decomposition of \( |v'| \) is less than or equal to the number of factors in the decomposition of \( |v'| \) increased by 3. And, since

\[ |v'| = F_{3p} + F_{3p} + F_{3(p-1)} + F_{3(p-2)} + \cdots + F_6 + F_3 \]

\[ = F_{3p+1} + F_{3p-1} + F_{3(p-2)} + \cdots + F_6 + F_3, \]

\( |v'| \) is decomposed into \( p \) factors. Thus, there exists \( \{l_1, \ldots, l_k\} \subset \mathbb{N} \) such that \( |v| = F_{l_1} + \cdots + F_{l_k} \) and \( k \leq p + 3 \). \( \square \)

Using this property, it is not difficult to prove the following property.

**Property 5.9.** Let \( v \) be a left factor of \( F \) such that

\[ 1 + F_4 + F_7 + \cdots + F_{3p+1} + F_{3p+1} \]

\[ < |v| \leq 1 + F_4 + F_7 + \cdots + F_{3p+1} + F_{3(p+1)}, \quad p \in \mathbb{N}. \]

Then, there exists \( \{l_1, \ldots, l_k\} \subset \mathbb{N} \) such that \( |v| = F_{l_1} + \cdots + F_{l_k} \) and \( k \leq p + 4 \).

5.3. Auxiliary constructions

Let

\[ G_0 : B^*_0 \rightarrow B^*_0, \quad \text{and} \quad G_1 : B^*_1 \rightarrow B^*_1, \]

\[ 2 \mapsto 21221, \quad 2 \mapsto 233, \]

\[ 1 \mapsto 221, \quad 3 \mapsto 23233. \]
Property 5.10. (a) \( H_0(Q) = G_0^0(2) \); \hspace{1em} (b) \( H_1(Q) = G_1^1(2) \)

Proof. The proof is given here only for (a) ((b) can be proved with exactly the same arguments).

Since \( G(0) = 101 \) and \( G(1) = 10101 \), \( G \) is obtained from \( \tilde{\varphi}^2 \circ \varphi \) by replacing \( a \) by 1 and \( b \) by 0. Therefore, applying properties of the Fibonacci morphisms, if \( u \) is a left factor of \( H_0(Q) \) and \( u' \) is a left factor of \( Q = G^\omega(0) \) such that \( u = H_0(u') \), then \( |u| = F_n \implies |u'| = F_{n-1} \) (for some \( n \in \mathbb{N} - \{0\} \)).

Now, it is proved that, for any \( n \in \mathbb{N} \), if \( u \) is the left factor of length \( F_n \) of \( H_0(Q) \) and \( v \) the left factor of length \( F_p \) of \( G^p(2) \), then \( u = v \). The proof is by induction on \( n \) and the statement can be easily verified for \( n \leq 3 \). Consequently, let us suppose that it is true for any \( n \in \mathbb{N} \), \( n \geq 3 \), and show that it is then also true for \( n = n + 1 \).

Let \( u \) be the left factor of \( H_0(Q) \) of length \( F_n \) and \( v \) the left factor of \( G^p(2) \) of length \( F_{p+1} \). \( G_0 \) can be obtained from \( \varphi \circ \tilde{\varphi}^2 \) by replacing \( a \) by 2 and \( b \) by 1.

Thus, as earlier, \( v = G_0(v') \), where \( v' \) is the left factor of length \( F_{p+1} \) of \( G^p(2) \). By induction, \( v' \) is also a left factor of \( H_0(Q) \) and there exists \( u' \), a left factor of \( Q \), such that \( v = H_0(u') \) and \( |u'| = F_{p+2} \). Therefore, \( v = G_0(v') = G_0 \circ H_0(u') \). But \( G_0 \circ H_0 = H_0 \circ G \); thus, \( v = H_0 \circ G(u') \).

Now, since \( u' \) is a left factor of \( Q \), \( H_0 \circ G(u') \) is a left factor of \( H_0(Q) \) and, since \( |H_0 \circ G(u')| = F_{p+1} \), \( u = H_0 \circ G(u') = v \). This completes the proof. \( \square \)

Now, we are ready to prove that Conjectures 3 and 4 of Kós [7] are true.

Proposition 5.11. For any \( r \in \mathbb{N} \), \( f(r, 0) = \left\lfloor \frac{1 + \sqrt{5}}{2} r + \frac{1 - \sqrt{5}}{4} \right\rfloor \).

Proof. Let \( F' \) be obtained from \( F \) by replacing \( a \) by 2 and \( b \) by 1 (\( F' \) is the Fibonacci word on the alphabet \( B_0 \)). Since \( G_0 \) is obtained from \( \varphi \circ \tilde{\varphi}^2 \) by replacing \( a \) by 2 and \( b \) by 1 (see Property 5.10), by Property 5.5, \( F' = G_0(2)G_0^2(2) \cdots G_0(2) \). \( F_0 \) can be obtained from \( \varphi \circ \tilde{\varphi}^2 \) by replacing \( a \) by 2 and \( b \) by 1 (see Property 5.10), \( H_0(Q) = G^p(2) \), \( u \) is a left factor of \( G^p(2) \). If \( |u| \leq 89 \) \( (-F_9) \), it is a bit tedious but easy to verify the assertion of Proposition 5.11. Otherwise, since for any \( n \in \mathbb{N} \), \( G^p_0(2) \) is a left factor of \( G^p_0(2) \), there exists \( p \in \mathbb{N} \), \( p \geq 3 \), such that \( G^p_0(2) \) is a proper left factor of \( u \) and \( u \) is a left factor of \( G^p_0(2) \), which means that \( |u| \leq F_{3p} \). Thus, there exists \( v \in B_0^p \) such that \( v \) is a left factor of \( F' \) and \( v = G_0(2) \cdots G_0(2)u \).

Let \( \{l_1, \ldots, l_k\} \subset \mathbb{N} \) be such that \( |v| = F_{l_1} + \cdots + F_{l_k} \). One has then

\[
|u| = |v| - |G_0(2)| - \cdots - |G_0^k(2)| = F_{l_1} + \cdots + F_{l_k} + F_3 + \cdots + F_{3p}
\]

and, by Property 5.8, one can suppose that \( k \leq p + 3 \). By Property 5.4, there are three possible cases according as \( l_k = 0 \), \( l_k = 1 \) or \( l_k \geq 2 \).
Case 1: $l_k \geq 2$. In this case, by Property 5.4,
\[
|u|_2 = F_{l_k-1} + \cdots + F_{l_k-2} - F_2 - F_4 - \cdots - F_{3p-1},
\]
\[
|u|_1 = F_{l_k-2} + \cdots + F_{l_k-2} - F_1 - F_4 - \cdots - F_{3p-2}.
\]

Now, let $r \in \mathbb{N}$, $r = |u|$. Then $f(r, 0) = -1 + \sum_{i=1}^r t(0, i)$. But, for any $i \in \mathbb{N} - \{0\}$, $t(0, i)$ denotes the $i$th letter of $H_0(Q)$. Thus, for any $i \leq r$, $t(0, i) = 2$ if the $i$th letter of $u$ is a $2$ and $t(0, i) = 1$ if the $i$th letter of $u$ is a $1$. Consequently, $\sum_{i=1}^r t(0, i) = 2|u|_2 + |u|_1$ and
\[
f(r, 0) = -1 + 2(F_{l_k-1} + \cdots + F_{l_k-2} - F_2 - F_4 - \cdots - F_{3p-1})
\]
\[+(F_{l_k-2} + \cdots + F_{l_k-2} - F_1 - F_4 - \cdots - F_{3p-2})
\]
\[= -1 + F_{l_k+1} + \cdots + F_{l_k+1} - F_4 - F_7 - \cdots - F_{3p+1}.
\]

Case 2: $l_k = 1$. In this case, by Property 5.4,
\[
|u|_2 = F_{l_k-1} + \cdots + F_{l_k-2} - F_2 - F_4 - \cdots - F_{3p-1},
\]
\[
|u|_1 = F_{l_k-2} + \cdots + F_{l_k-2} - F_1 - F_4 - \cdots - F_{3p-2}.
\]

Let $r \in \mathbb{N}$, $r = |u|$. As in the previous case,
\[
f(r, 0) = -1 + 2(F_{l_k-1} + \cdots + F_{l_k-2} + F_0 - F_2 - F_4 - \cdots - F_{3p-1})
\]
\[+(F_{l_k-2} + \cdots + F_{l_k-2} + F_0 - F_1 - F_4 - \cdots - F_{3p-2})
\]
\[= -1 + F_{l_k+1} + \cdots + F_{l_k+1} + 3F_0 - F_4 - F_7 - \cdots - F_{3p+1}.
\]

But $3F_0 = 3 = F_2 = F_{l_k+1}$ (since $l_k = 1$). Thus,
\[
f(r, 0) = -1 + F_{l_k+1} + \cdots + F_{l_k+1} - F_4 - F_7 - \cdots - F_{3p+1}.
\]

Case 3: $l_k = 0$. In this case, by Property 5.4,
\[
|u|_2 = F_{l_k-1} + \cdots + F_{l_k-2} - F_2 - F_4 - \cdots - F_{3p-1},
\]
\[
|u|_1 = F_{l_k-2} + \cdots + F_{l_k-2} - F_1 - F_4 - \cdots - F_{3p-2}.
\]

Let $r \in \mathbb{N}$, $r = |u|$. As in Case 1,
\[
f(r, 0) = -1 + 2(F_{l_k-1} + \cdots + F_{l_k-2} + F_0 - F_2 - F_4 - \cdots - F_{3p-1})
\]
\[+(F_{l_k-2} + \cdots + F_{l_k-2} - F_1 - F_4 - \cdots - F_{3p-2})
\]
\[= -1 + F_{l_k+1} + \cdots + F_{l_k+1} + 2F_0 - F_4 - F_7 - \cdots - F_{3p+1}.
\]

But $2F_0 = 2 = F_1 - F_{l_k+1}$ (since $l_k = 0$). Thus,
\[
f(r, 0) = -1 + F_{l_k+1} + \cdots + F_{l_k+1} - F_4 - F_7 - \cdots - F_{3p+1}.
\]
So, in all the three cases
\[ f(r, 0) = -1 + F_{l_1+1} + \cdots + F_{l_k+1} - F_4 - F_7 - \cdots - F_{3p+1}. \]

Now, applying \( F_{n+1} = \Phi F_n + \hat{\Phi}^{n+2} \) for any \( n \in \mathbb{N} \) (see the properties of the Fibonacci numbers, Section 51(c)),
\[
\begin{align*}
\ f(r, 0) &= -1 + \Phi(F_{l_1} + \cdots + F_{l_k} - F_3 - F_6 - \cdots - F_{3p}) \\
& \quad + \Phi^{l_1+2} + \cdots + \Phi^{l_k+2} - \hat{\Phi}^5 - \hat{\Phi}^8 - \cdots - \hat{\Phi}^{3p+2}.
\end{align*}
\]

But,
\[
\frac{1}{2}(1 + \sqrt{5}) r + \frac{1}{2}(1 - \sqrt{5}) = \Phi |u| + \frac{1}{2} \hat{\Phi}
\]
\[
\begin{align*}
&= \Phi(F_{l_1} + \cdots + F_{l_k} - F_3 - F_6 - \cdots - F_{3p}) + \frac{1}{2} \hat{\Phi}.
\end{align*}
\]

Consequently, one has to show the two following inequalities:
\[
\begin{align*}
&\ -1 + \Phi^{l_1+2} + \cdots + \Phi^{l_k+2} - \hat{\Phi}^5 - \hat{\Phi}^8 - \cdots - \hat{\Phi}^{3p+2} \leq \frac{1}{2} \hat{\Phi} \\
&\ \frac{1}{2} \hat{\Phi} < \Phi^{l_1+2} + \cdots + \Phi^{l_k+2} - \hat{\Phi}^5 - \hat{\Phi}^8 - \cdots - \hat{\Phi}^{3p+2}
\end{align*}
\]

Since \( \hat{\Phi} < 0 \) and \( l_k > 0 \),
\[
\begin{align*}
\Phi^{l_1+2} + \cdots + \Phi^{l_k+2} &= \sum_{i=1}^{k} \Phi^{2i}.
\end{align*}
\]

(\( \Phi^2, \Phi^4, \ldots, \Phi^{2k} \)) is a geometrical progression whose first term and ratio are \( \Phi^2 \). Thus,
\[
\sum_{i=1}^{k} \Phi^{2i} = \Phi^2 \left( \frac{\Phi^{2k} - 1}{\Phi^2 - 1} \right) = \Phi(\Phi^{2k} - 1).
\]

(\( \Phi^5, \Phi^6, \ldots, \Phi^{3p+2} \)) is a geometrical progression whose first term is \( \Phi^5 \) and ratio is \( \Phi^3 \). Furthermore, this progression contains \( p \) terms; thus,
\[
\begin{align*}
&- \Phi^5 - \Phi^6 - \cdots - \Phi^{3p+2} = - \Phi^5 \left( \frac{\Phi^{3p} - 1}{\Phi^3 - 1} \right) = - \frac{1}{2} \Phi^4 (\Phi^{3p} - 1).
\end{align*}
\]

Consequently,
\[
\begin{align*}
&-1 + \Phi^{l_1+2} + \cdots + \Phi^{l_k+2} - \Phi^5 - \Phi^8 - \cdots - \Phi^{3p+2} \\
&\leq -1 + \Phi(\Phi^{2k} - 1) - \frac{1}{2} \Phi^4 (\Phi^{3p} - 1).
\end{align*}
\]

But
\[
\begin{align*}
-1 + \Phi(\Phi^{2k} - 1) - \frac{1}{2} \Phi^4 (\Phi^{3p} - 1) &= -1 - \hat{\Phi} + \frac{1}{2} \hat{\Phi}^4 + \Phi^{2k+1} - \frac{1}{2} \Phi^{3p+4}
\end{align*}
\]

and
\[
-1 - \hat{\Phi} + \frac{1}{2} \hat{\Phi}^4 = \frac{1}{2} \hat{\Phi}.
\]
Therefore,
\[-1 + \hat{\Phi}_1 + \hat{\Phi}_2 + \ldots + \hat{\Phi}_k + \hat{\Phi}_k = \hat{\Phi} - \hat{\Phi}^5 - \ldots - \hat{\Phi}^{3p+2} + \frac{1}{2} \hat{\Phi} + \hat{\Phi}^{2k+1} - \frac{1}{2} \hat{\Phi}^{3p+4}.\]

Since \(2k+1\) is odd and \(\hat{\Phi} < 0\), \(\hat{\Phi}^{2k+1} < 0\). If \(p\) is even, \(\frac{1}{2} \hat{\Phi}^{3p+4}\) is positive and, thus, \(\hat{\Phi} + \hat{\Phi}^{2k+1} - \frac{1}{2} \hat{\Phi}^{3p+4} < \hat{\Phi}\). Otherwise, since \(k \leq p+3\), \(2k+1 \leq 2p+7\) and \(p \geq 3\) implies \(2p+7 \leq 3p+4\). Consequently, since \(-1 < \hat{\Phi} < 0\), \(\hat{\Phi}^{2k+1} \leq \hat{\Phi}^{3p+4} < 0\). Thus, \(\hat{\Phi}^{2k+1} - \frac{1}{2} \hat{\Phi}^{3p+4} < 0\), which means that \(\frac{1}{2} \hat{\Phi} + \hat{\Phi}^{2k+1} - \frac{1}{2} \hat{\Phi}^{3p+4} < \frac{1}{2} \hat{\Phi}\) and completes the proof of the first inequality.

For the second inequality, since \(\hat{\Phi} < 0\) and \(l_k \geq 0\),
\[
\hat{\Phi}^{l_k} + \hat{\Phi}^{l_k+2} + \ldots + \hat{\Phi}^{l_k+2} \geq \sum_{i=1}^{k} \hat{\Phi}^{2l+1}.
\]

\((\hat{\Phi}^3, \hat{\Phi}^5, \ldots, \hat{\Phi}^{2k+1})\) is a geometrical progression whose first term is \(\hat{\Phi}^3\) and ratio is \(\hat{\Phi}^2\). Furthermore, this progression contains \(k\) terms. Thus,
\[
\hat{\Phi}^3 + \hat{\Phi}^5 + \ldots + \hat{\Phi}^{2k+1} = \hat{\Phi}^3 \left( \frac{\hat{\Phi}^{2k+1}}{\hat{\Phi}^2 - 1} \right) = \hat{\Phi}^2 \left( \hat{\Phi}^{2k} - 1 \right).
\]

Consequently,
\[
\hat{\Phi}^{l_k} + \hat{\Phi}^{l_k+2} + \ldots + \hat{\Phi}^{l_k+2} - \hat{\Phi}^3 - \hat{\Phi}^5 - \ldots - \hat{\Phi}^{3p+2} \geq \hat{\Phi}^3 \left( \hat{\Phi}^{2k} - 1 \right) - \frac{1}{2} \hat{\Phi}^{3p+4}.
\]

But
\[
\hat{\Phi}^2 \left( \hat{\Phi}^{2k} - 1 \right) - \frac{1}{2} \hat{\Phi}^{3p+4} = -\hat{\Phi}^2 + \frac{1}{2} \hat{\Phi}^4 + \hat{\Phi}^{2(k+1)} - \frac{1}{2} \hat{\Phi}^{3p+4} \quad \text{and}
\]
\[-\hat{\Phi}^2 + \frac{1}{2} \hat{\Phi}^4 = \frac{1}{2} \hat{\Phi}.
\]

Therefore,
\[
\hat{\Phi}^{l_k} + \hat{\Phi}^{l_k+2} + \ldots + \hat{\Phi}^{l_k+2} - \hat{\Phi}^3 - \hat{\Phi}^5 - \ldots - \hat{\Phi}^{3p+2} \geq \frac{1}{2} \hat{\Phi} + \hat{\Phi}^{2(k+1)} - \frac{1}{2} \hat{\Phi}^{3p+4}.
\]

Since \(2(k+1)\) is even, \(\hat{\Phi}^{2(k+1)} > 0\).

If \(p\) is odd, \(\frac{1}{2} \hat{\Phi}^{3p+4}\) is negative and, thus, \(\frac{1}{2} \hat{\Phi} + \hat{\Phi}^{2(k+1)} - \frac{1}{2} \hat{\Phi}^{3p+4} > \frac{1}{2} \hat{\Phi}\). Otherwise, since \(p \geq 3\), \(p \geq 4\). But, since \(k \leq p+3\), \(2(k+1) \leq 2p+8\), and \(p \geq 4\) implies \(2p+8 \leq 3p+4\). Consequently, since \(-1 < \hat{\Phi} < 0\), \(\hat{\Phi}^{2(k+1)} \geq \hat{\Phi}^{3p+4} > 0\). Thus, \(\hat{\Phi}^{2(k+1)} - \frac{1}{2} \hat{\Phi}^{3p+4} > 0\), which means that \(\frac{1}{2} \hat{\Phi} + \hat{\Phi}^{2(k+1)} - \frac{1}{2} \hat{\Phi}^{3p+4} > \frac{1}{2} \hat{\Phi}\) and completes the proof of the second inequality and, therefore, the proof of Proposition 5.11.

**Proposition 5.12.** For any \(r \in \mathbb{N}\),
\[
f(r, 1) = \left| \frac{3 + \sqrt{5}}{2} \cdot \frac{r - 1 + \sqrt{5}}{4} \right|
\]

**Proof.** Let \(F^r\) be obtained from \(F\) by replacing \(a\) by 3 and \(b\) by 2 (\(F^r\) is the Fibonacci
word on the alphabet \( B_1 \). \( G_1 \) is obtained from \( \tilde{\phi} \circ \varphi \circ \tilde{\phi} \) by replacing \( a \) by 3 and \( b \) by 2 (see Property 5.10) and, by Property 5.5,

\[
F'' = 3G_1(23)G_1^2(23) \cdots G_1^2(23) \cdots
\]

Now, let \( u \in B_1^+ \) be a left factor of \( H_1(Q) \). Since, by Property 5.10, \( H_1(Q) = G_1^p(2) \), \( u \) is a left factor of \( G_1^p(2) \). If \( |u| \leq 610 \) (\( = F_{13} \)), it can be verified (using a computer) that the assertion of Proposition 5.4 is true. Otherwise, there exists \( p \in \mathbb{N}, p \geq 4 \), such that \( G_1^p(23) \) is a proper left factor of \( u \) and \( u \) is a left factor of \( G_1^{p+1}(23) \), which means that \( F_{3p+1} < |u| \leq F_{3(p+1)+1} \). Thus, there exists \( v \in B_1^+ \) such that \( v \) is a left factor of \( F'' \) and \( v = 3G_1(23)G_1^2(23) \cdots G_1^2(23)u \).

Let \( \{l_1, \ldots, l_k\} \subset \mathbb{N} \) be such that \( |u| = |v| - 1 - |G_1(23)| - \cdots - |G_1^p(23)| \)

\[
= F_{l_1} + \cdots + F_{l_k} - 1 - F_{4} - \cdots - F_{3p+1}.
\]

and, by Property 5.9, one can suppose that \( k \leq p + 4 \).

By Property 5.4, there are three possible cases according as \( l_k = 0, l_k = 1 \) or \( l_k \geq 2 \). Let \( r \in \mathbb{N}, r = |u| \). As in the proof of Proposition 5.11, it is easy to show that, in these three cases,

\[
f(r, 1) = -1 + F_{l_1+2} + \cdots + F_{l_k+2} - F_6 - F_9 - \cdots - F_{3(p+1)} - 3.
\]

Now, applying the fact that, for any \( n \in \mathbb{N}, F_{n+1} = \Phi F_n + \tilde{\Phi}^{n+2} \) and \( \Phi \tilde{\Phi} + \tilde{\Phi}^{n+1} = \Phi^n \) (see the properties of the Fibonacci numbers, Section 5.1, (b) and (c)), for any \( n \in \mathbb{N}, F_{n+2} = \Phi^2 F_n + \Phi^{n+2} \). Thus,

\[
f(r, 1) = -1 + \Phi^2(F_{l_1} + \cdots + F_{l_k} - F_4 - F_7 - \cdots - F_{3p+1})
+ \tilde{\Phi}^{l_1+2} + \cdots + \tilde{\Phi}^{l_k+2} - \tilde{\Phi}^6 - \tilde{\Phi}^9 - \cdots - \tilde{\Phi}^{3(p+1)} - 3,
\]

and, since \( 3 = F_2 = \Phi^2 + \tilde{\Phi}^2 \) and \( \tilde{\Phi}^2 = \tilde{\Phi}^3 - \tilde{\Phi} \) (see the properties of the Fibonacci numbers, (Section 5.1, (a)),

\[
f(r, 1) = -1 + \Phi^2(F_{l_1} + \cdots + F_{l_k} - F_4 - F_7 - \cdots - F_{3p+1})
+ \tilde{\Phi}^{l_1+2} + \cdots + \tilde{\Phi}^{l_k+2} - \tilde{\Phi}^3 - \tilde{\Phi}^6 - \cdots - \tilde{\Phi}^{3(p+1)} + \tilde{\Phi} - \Phi^2.
\]

But,

\[
\frac{1}{2}(3 + \sqrt{5})r - \frac{1}{2}(1 + \sqrt{5}) = \Phi^2 |u| - \frac{1}{2} \Phi
\]

\[
= \Phi^2(F_{l_1} + \cdots + F_{l_k} - 1 - F_4 - \cdots - F_{3p+1}) - \frac{1}{2} \Phi
\]

\[
= \Phi^2(F_{l_1} + \cdots + F_{l_k} - F_4 - \cdots - F_{3p+1}) - \Phi^2 - \frac{1}{2} \Phi
\]

Consequently, one has to show the 2 following inequalities:

\[
-1 + \tilde{\Phi}^{l_1+2} + \cdots + \tilde{\Phi}^{l_k+2} - \tilde{\Phi}^3 - \tilde{\Phi}^6 - \cdots - \tilde{\Phi}^{3(p+1)} + \tilde{\Phi} - \frac{1}{2} \Phi < 0.
\]
and
\[-\frac{1}{2}\Phi < \hat{\Phi}^{l_1+2} + \cdots + \hat{\Phi}^{l_k+2} - \hat{\Phi}^3 - \hat{\Phi}^6 - \cdots - \hat{\Phi}^{3(p+1)} + \hat{\Phi}.\]

As in the proof of Proposition (5.11),
\[\hat{\Phi}^{l_1+2} + \cdots + \hat{\Phi}^{l_k+2} < \hat{\Phi}(\hat{\Phi}^{2k} - 1).\]

Furthermore, \((\hat{\Phi}^3, \hat{\Phi}^6, \ldots, \hat{\Phi}^{3(p+1)})\) is a geometrical progression whose first term and ratio are \(\Phi^3\) and which contains \((p+1)\) terms. Thus,
\[-\hat{\Phi}^3 - \hat{\Phi}^6 - \cdots - \hat{\Phi}^{3(p+1)} = -\hat{\Phi}^3\left(\frac{\hat{\Phi}^{3(p+1)} - 1}{\phi^3 - 1}\right) = -\frac{1}{2}\Phi^2(\hat{\Phi}^{3(p+1)} - 1).\]

Consequently,
\[-1 + \hat{\Phi}^{l_1+2} + \cdots + \hat{\Phi}^{l_k+2} - \hat{\Phi}^3 - \hat{\Phi}^6 - \cdots - \hat{\Phi}^{3(p+1)} + \hat{\Phi} \leq -1 + \hat{\Phi}(\hat{\Phi}^{2k} - 1) - \frac{1}{2}\Phi^2(\hat{\Phi}^{3(p+1)} - 1) + \hat{\Phi},\]

and, since \(-1 + \frac{1}{2}\Phi^2 = -\frac{1}{2}\Phi\),
\[-1 + \hat{\Phi}^{l_1+2} + \cdots + \hat{\Phi}^{l_k+2} - \hat{\Phi}^3 - \hat{\Phi}^6 - \cdots - \hat{\Phi}^{3(p+1)} + \hat{\Phi} \leq -\frac{1}{2}\Phi + \hat{\Phi}^{2k+1} - \frac{1}{2}\Phi^3p^5.\]

Since \(2k+1\) is odd and \(\hat{\Phi} < 0\), \(\hat{\Phi}^{2k+1} < 0\). If \(p\) is odd, \(\frac{1}{2}\Phi^3p^5 + 5\) is positive and, thus, \(-\frac{1}{2}\Phi + \hat{\Phi}^{2k+1} - \frac{1}{2}\Phi^3p^5 < -\frac{1}{2}\Phi\). Otherwise, since \(k \leq p + 4\), \(2k + 1 \leq 2p + 9\) and \(p \geq 4\) implies \(2p + 9 \leq 3p + 5\). Consequently, since \(-1 < \hat{\Phi} < 0\), \(\hat{\Phi}^{2k+1} < \hat{\Phi}^{3p+5} < 0\). So, \(\hat{\Phi}^{2k+1} < \hat{\Phi}^{3p+5} < 0\), which means that \(-\frac{1}{2}\Phi + \hat{\Phi}^{2k+1} - \frac{1}{2}\Phi^3p^5 < -\frac{1}{2}\Phi\) and completes the proof of the first inequality.

For the second inequality, it is easy to show, as in the proof of Proposition 5.11, that:
\[\hat{\Phi}^{l_1+2} + \cdots + \hat{\Phi}^{l_k+2} - \hat{\Phi}^3 - \hat{\Phi}^6 - \cdots - \hat{\Phi}^{3(p+1)} + \hat{\Phi} \geq -\frac{1}{2}\Phi + \hat{\Phi}^{2(k+1)} - \frac{1}{2}\Phi^3p^5.\]

Since \(2(k + 1)\) is even, \(\hat{\Phi}^{2(k+1)} > 0\). If \(p\) is even, \(\frac{1}{2}\Phi^3p^5 + 5\) is negative and, thus, \(-\frac{1}{2}\Phi + \hat{\Phi}^{2(k+1)} - \frac{1}{2}\Phi^3p^5 > -\frac{1}{2}\Phi\). Otherwise, since \(p \geq 4\), \(p \geq 5\). Consequently, as above, \(2(k + 1) \leq 3p + 5\) and, thus, \(-\frac{1}{2}\Phi + \hat{\Phi}^{2(k+1)} - \frac{1}{2}\Phi^3p^5 > -\frac{1}{2}\Phi\). This completes the proof of Proposition 5.12. □

Note added in proof

Recently, F. Mignosi and the author showed that the converse of Proposition 4.7 is true (see “Actes du Colloque Thématic” in a special issue of Séminaire de Théorie des Nombres Bordeaux. See also a recent work of Dulucq and Gouyou-Beauchamps [16].
References


References added in proof