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The homotopy perturbation method for nonlinear oscillators

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ABSTRACT

The homotopy perturbation method is applied to the nonlinear oscillators. Only one iteration results in high accuracy of the solutions.

1. Introduction

The study of nonlinear oscillators is of great importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are nonlinear and are described by nonlinear equations. Recently, considerable attention has been directed towards the analytical solutions for nonlinear oscillators, for example, variational iteration method [1–7], parameter-expanding method [8–11], variational methods [12,13], and Exp-function method [14,15]. Surveys of the literature with multitudinous references and useful bibliographies have been given in [16,17]. In this paper, we will show how to solve nonlinear oscillators quickly by using the homotopy perturbation method [18–20].

2. Solution procedures

This paper considers the following two nonlinear oscillators.

Case 1: An important and interesting nonlinear differential equation is the following one

$$u'' + \frac{u}{1+u^2} = 0, \qquad u'(0) = A, \qquad u(0) = 0. \tag{1}$$

Re-write Eq. (1) in the form

$$u'' + \frac{u}{1+\left(p^{\frac{1}{2}}u\right)^2} = u'' + \frac{1\cdot u}{1+\left(p^{\frac{1}{2}}u\right)^2} = 0, \quad u'(0) = A, \qquad u(0) = 0, \tag{2}$$

where $p \in [0, 1]$ and is an imbedding parameter. As in He's homotopy perturbation method [18–20], it is obvious that when $p = 0$, Eq. (2) becomes a linear equation; when $p = 1$, it becomes the original nonlinear one. Applying the perturbation technique, the solution of Eq. (2) and the coefficient 1 can be expressed as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots, \tag{3}$$

$$1 = \omega^2 + p\omega_1 + p^2\omega_2 + p^3\omega_3 + \cdots. \tag{4}$$

Setting $p = 1$ leads to the approximate solution of the problem:

$$u_{app} = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots. \tag{5}$$

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Substituting (3) and (4) into (2) and equating the coefficients of like powers of p, we expand $\frac{1\cdot u}{1+\left(p^{1/2}u\right)^2}$ into Taylor series

$$\frac{1\cdot u}{1+\left(p^{\frac{1}{2}}u\right)^2}=\frac{1\cdot u}{1-\left(-pu^2\right)}=u\left(1-pu^2+p^2u^4-p^3u^6+\cdots\right)\left(\omega^2+p\omega_1+p^2\omega_2+p^3\omega_3+\cdots\right). \tag{6}$$

We can obtain series of inhomogeneous linear differential equations

$$p^0: u_0''=-\omega^2u_0,\quad u_0'(0)=A,\qquad u_0(0)=0, \tag{7}$$

$$p^1: u_1''=-\omega^2u_1+\omega^2u_0^3-\omega_1u_0,\quad u_1'(0)=0,\qquad u_1(0)=0 \tag{8}$$

$$\vdots$$

Thus, by solving the equations above, we obtain

$$u_0=A\cos\omega t. \tag{9}$$

If the first-order approximation is enough, then setting $p=1$, we have

$$1=\omega^2+\omega_1. \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) yields

$$u_1''=-\omega^2u_1+\omega^2\left(A\cos\omega t\right)^3-\left(1-\omega^2\right)\left(A\cos\omega t\right), \tag{11}$$

or

$$u_1''+\omega^2u_1+A\left(1-\omega^2-\frac{3A^2\omega^2}{4}\right)\cos\omega t-\frac{A^3\omega^2}{4}\cos 3\omega t=0. \tag{12}$$

No secular terms requires

$$1-\omega^2-\frac{3A^2\omega^2}{4}=0. \tag{13}$$

Thus, we obtain the relation between the frequency and amplitude, which reads

$$\omega=\frac{1}{\sqrt{1+\frac{3A^2}{4}}}. \tag{14}$$

Solving the following equation

$$u_1''+\omega^2u_1-\frac{A^3\omega^2}{4}\cos 3\omega t=0, \tag{15}$$

we have

$$u_1=-\frac{A^3\omega^2}{4\left(9\omega^2-1\right)}\left(\cos 3\omega t-\cos\omega t\right). \tag{16}$$

Consequently, the first-order approximate solution can be written as follows

$$u=u_0+u_1=A\cos\omega t-\frac{A^3\omega^2}{4\left(9\omega^2-1\right)}\left(\cos 3\omega t-\cos\omega t\right). \tag{17}$$

Its periodic solution is generally expressed in the form

$$u(t)=A\cos\left[\left(1+\frac{3A^2}{4}\right)^{-1/2}t\right]. \tag{18}$$

Case 2: Mickens recently analyzed the nonlinear differential equation [21]

$$u''+\frac{1}{u}=0,\quad u'(0)=A,\qquad u(0)=0. \tag{19}$$

Re-writing Eq. (19), we have

$$uu''+1=0, \tag{20}$$

or

$$u'' + u\left(u''\right)^2 = 0. \tag{21}$$

Then we establish the following homotopy

$$u'' + \omega^2 \cdot u + p\left[u\left(u''\right)^2 - \omega^2 u\right] = 0, \quad p \in [0, 1]. \tag{22}$$

It is obvious that when $p = 0$, Eq. (22) becomes a linear equation; when $p = 1$, it becomes the original nonlinear one.

By the homotopy perturbation method [18–20], we can obtain a series of linear equations, and we write only the first two linear equations:

$$p^0 : u_0'' + \omega^2 u_0 = 0, \quad u_0'(0) = A, \qquad u_0(0) = 0, \tag{23}$$

$$p^1 : u_1'' + \omega^2 u_1 + u_0\left(u_0''\right)^2 - \omega^2 u_0 = 0, \quad u_1'(0) = 0, \qquad u_1(0) = 0 \tag{24}$$

$$\vdots$$

From Eq. (23), we obtain

$$u_0 = A\cos\omega t. \tag{25}$$

Substituting Eq. (25) into Eq. (24) leads to

$$u_1'' + \omega^2 u_1 + A\cos\omega t\left[\left(-A\omega^2\cos\omega t\right)^2 - \omega^2\right] = 0, \qquad u_1'(0) = 0, \qquad u_1(0) = 0, \tag{26}$$

or

$$u_1'' + \omega^2 u_1 + \frac{A\omega^2}{4}\left(4 - 3A^2\omega^2\right)\cos\omega t - \frac{A^3\omega^4}{4}\cos 3\omega t = 0. \tag{27}$$

Eliminating the secular term, we have

$$\frac{A\omega^2}{4}\left(4 - 3A^2\omega^2\right) = 0. \tag{28}$$

From the above equation, we can easily find that

$$\omega = \frac{2}{\sqrt{3}A}, \tag{29}$$

which reduces to that in Ref. [21]

According to Eqs. (27) and (28), the solution reads

$$u_1 = \frac{A^3\omega^4}{32}\left(\cos 3\omega t - \cos\omega t\right). \tag{30}$$

We, therefore, obtain the first-order approximation by setting $p = 1$

$$u = u_0 + u_1 = A\cos\omega t + \frac{A^3\omega^4}{32}\left(\cos 3\omega t - \cos\omega t\right). \tag{31}$$

Its periodic solution is generally expressed in the form

$$u(t) = A\cos\left(\frac{2}{\sqrt{3}}A^{-1}t\right). \tag{32}$$

3. Conclusion

The homotopy perturbation method is proved to be a useful mathematical tool to nonlinear oscillators and the present short note can be used as a paradigm for many other applications in searching for period or frequency of nonlinear oscillators.

References

[1] J.H. He, Variational iteration method—a kind of nonlinear analytical technique: Some examples, Internat. J. Nonlinear. Mech. 34 (4) (1999) 699–708.
[2] J.H. He, A review on some new recently developed nonlinear analytical techniques, Int. J. Nonliear. Sci. 1 (1) (2000) 51–70.
[3] J.H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B 20 (10) (2006) 1141–1199.

[4] J.H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, Chaos. Soliton. Fract. 29 (2006) 108–111.
[5] J.H. He, Variational iteration method- some recent results and new interpretations, J. Comput. Appl. Math. 207 (1) (2007) 3–17.
[6] D.H. Shou, J.H. He, Beyond Adomian method: The variational iteration method for solving heat-like and wave-like equations with variable coefficients, Phys. Lett. A 372 (3) (2008) 233–237.
[7] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, Int. J. Nonliear. Sci. 7 (1) (2006) 65–70.
[8] J.H. He, Non-Perturbative Methods for Strongly Nonlinear Problems, Dissertation de-Verlag im Internet GmbH, Berlin, 2006.
[9] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (10) (2006) 1141–1199.
[10] J.H. He, Bookkeeping parameter in perturbation methods, Int. J. Nonlinear Sci. Numer. Simul. 2 (3) (2001) 257–264.
[11] D.H. Shou, J.H. He, Application of parameter-expanding method to strongly nonlinear oscillators, Int. J. Nonliear. Sci. 8 (1) (2007) 121–124.
[12] J.H. He, Variational approach for nonlinear oscillators, Chaos Solitons Fractals 34 (2007) 1430–1439.
[13] D.H. Shou, Variational approach to the nonlinear oscillator of a mass attached to a stretched wire, Phys. Scripta 77 (2008) 045006.
[14] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos. Soliton. Fract. 30 (2006) 700–708.
[15] J.H. He, M.A. Abdou, New periodic solutions for nonlinear evolution equations using exp-function method, Chaos. Soliton. Fract. 34 (2007) 1421–1429.
[16] J.H. He, Non-Perturbative Methods for Strongly Nonlinear Problems, Dissertation de-Verlag im Internet GmbH, Berlin, 2006.
[17] J.H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B 20 (2006) 1141–1199.
[18] J.H. He, Homotopy perturbation technique, Comput. Method. Appl. Math. 178 (1999) 257–262.
[19] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Internat. J. Nolinear. Mech. 35 (2000) 37–43.
[20] J.H. He, New interpretation of homotopy perturbation method, Int. J. Mod. Phys. B 20 (2006) 2561–2568.
[21] R.E. Mickens, Harmonic balance and iteration calculations of periodic solutions to $\ddot{y} + y^{-1} = 0$, J. Sound Vibration 306 (2007) 968–972.