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Control of stochastic chaos using sliding mode method

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ABSTRACT

Stabilizing unstable periodic orbits of a deterministic chaotic system which is perturbed by a stochastic process is studied in this paper. The stochastic chaos is modeled by exciting a deterministic chaotic system with a white noise obtained from derivative of a Wiener process which eventually generates an Ito differential equation. It is also assumed that the chaotic system being studied has some model uncertainties which are not random. The sliding mode controller with some modifications is used for stochastic chaos suppression. It is shown that the system states converge to the desired orbit in such a way that the error covariance converges to an arbitrarily small bound around zero. As some case studies, the stabilization of 1-cycle and 2-cycle orbits of chaotic Duffing and Φ^6 Van der Pol systems is investigated by applying the proposed method to their corresponding stochastically perturbed systems. Simulation results show the effectiveness of the method and the accuracy of the statements proved in the paper.

1. Introduction

Chaos control with the aim of stabilizing the unstable periodic orbits of a chaotic system has received a great deal of interest in the last two decades. The first documented research in this field goes back to 1990s, when Ott, Grebgi and York proposed a novel method for chaos control by linearization of Poicare map around the fixed points of a chaotic map [1–3]. In 1992 another approach, based on delayed feedback control, was presented for stabilizing unstable periodic orbits of continuous time chaotic systems, called Pyragas method [4]. In recent years, many nonlinear techniques for chaos control were used, such as feedback linearization [5–7], sliding mode control [8–10], Lyapunov based control [11–13] and fuzzy system based control [14–16]. In all of the mentioned methods the chaotic system has a deterministic differential equation; there is no random parameter or random excitation on the system governing equation. In many practical situations, the chaotic system may be affected by random inputs due to system uncertainties and environmental noise. Controlling chaos in such systems needs to implement stochastic tools and concepts. For modeling stochastic chaos in continuous time systems, an Ito stochastic differential form [17] is utilized by using the derivative of a Wiener process which creates a white Gaussian noise. This provides a standard stochastic differential equation. Any other stochastic chaotic system should be expressed by this standard form.

There are some important articles on the stochastic chaos dynamics, the effects of noise on chaotic systems, and the ergodic theory in stochastic chaotic systems [18,19]. Stochastic chaotic systems appear in many fields of science and engineering such as mechanical engineering [20], biology [21–23], chemistry [24], physics and laser science [25] and economics [26]. In most of these works, the stochastic chaos has been modeled by adding a bounded random process to some systems parameters or by applying a bounded random excitation to the input signal. These models do not possess the special properties of standard stochastic differential equations, because they have utilized bounded random processes [27].

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In [20] the stochastic chaos of a Duffing system is studied where a bounded random process is added to one of the system parameters, and by the use of a non-feedback strategy the chaos has been quenched in the system. A general way to classify stochastic chaos is presented and is applied to population dynamic models in [23]. In [24] the relative intensity noise in a laser diode is considered as a result of the feedback-induced deterministic chaos and the intensity noise suppression is treated from the view point of chaos control. Controlling the Van der Pol oscillator under the influence of white noise in the regime below the Hopf bifurcation where the deterministic system has a stable fixed point and is not chaotic is investigated in [28]. Stabilizing unstable periodic solutions of chaotic systems which have been perturbed by white Gaussian noise is investigated in [29]. In [29] stochastic chaos has been modeled through Ito differential form and a control algorithm has been proposed based on adaptive control.

In the present work, the Ito form of differential equation is used to model the stochastic chaotic systems. The derivative of a standard Wienner process is used to generate a continuous time Gaussian white noise, and this random process is used to model the stochastic excitation or stochastically time varying parameters of the system. The stochastic chaotic system is produced by applying the stochastic excitation or stochastically time varying parameters to a deterministic chaotic system. In this paper the unstable periodic orbit of a deterministic chaotic system is targeted for stabilization when the system is excited by a white Gaussian noise. It is assumed that the system dynamic equation has also some uncertainties besides the random excitations. The sliding mode control with some modification is used for controlling stochastic chaos toward desired unstable periodic orbits of the deterministic chaotic system. It must be noted that the conventional techniques of chaos control which are based on the sliding mode method [8–10,30,31] are commonly used only for deterministic systems. It is shown that the convergence of the stochastic states to the desired periodic orbit can not be completely achieved but tracking error variance will converge to an arbitrarily small bound around zero by applying the proposed sliding mode control. The stochastic Duffing and Φ^6-Van der Pol chaotic systems are used for examining the proposed control scheme. Simulation results show the correctness and accuracy of the statements presented in this paper.

2. Problem statement

Consider the following ordinary differential equation as a chaotic system:

$$x^{(n)} = f(\underline{x}, t) + b(\underline{x}, t)u \tag{1}$$

where $\underline{x} = (x, \dot{x}, \ldots, x^{(n-1)}) = (x_1, x_2, \ldots, x_n) \in \Re^n$ is the state vector, $f : \Re^n \times \Re^+ \to \Re$ and $b : \Re^n \times \Re^+ \to \Re$ are two nonlinear and sufficiently smooth functions and u is the control variable. It is assumed that for $u = 0$ Eq. (1) shows chaotic behavior. A stochastic chaotic system which is modeled by an Ito stochastic differential equation is obtained by adding a white noise process to the right hand side of Eq. (1):

$$x^{(n)} = f(\underline{x}, t) + b(\underline{x}, t)u + h(\underline{x}, t)\dot{v} \tag{2}$$

where $h : \Re^n \times \Re^+ \to \Re$ is a nonlinear, sufficiently smooth and bounded function, $|h(x, t)| \leq H$. v is a standard Wiener process and $\dot{v} = \mathrm{d}v/\mathrm{d}t$ produces white Gaussian noise process. Due to some technicalities and restrictions used in definition of $\dot{v} = \mathrm{d}v/\mathrm{d}t$ [17], Eq. (2) must be rewritten in the following differential form:

$$\mathrm{d}x^{(n-1)} = [f(\underline{x}, t) + b(\underline{x}, t)u]\mathrm{d}t + h(\underline{x}, t)\mathrm{d}v. \tag{3}$$

It is assumed that $\underline{x}_d(t)$ is one of unstable periodic orbits of the deterministic chaotic system of Eq. (1), so:

$$x_d^{(n)} = f(\underline{x}_d, t). \tag{4}$$

It is supposed that $h(\underline{x}_d, t) \neq 0$. In addition it is assumed that the functions f and b have some uncertainties with known upper and lower bounds, and their nominal values are denoted by $\hat{f}$ and $\hat{b}$, respectively. Also the function b is a positive definite function which has a strictly positive lower bound b_m, and upper bound b_M:

$$|f(\underline{x}, t) - \hat{f}(\underline{x}, t)| < F(\underline{x}, t) \tag{5}$$

$$\begin{aligned} 0 < b_m(\underline{x}, t) < b(\underline{x}, t) < b_M(\underline{x}, t) \\ 0 < b_m(\underline{x}, t) < \hat{b}(\underline{x}, t) < b_M(\underline{x}, t). \end{aligned} \tag{6}$$

Uncertainties on function $h(\underline{x}, t)$ are modeled through Wiener process $\dot{v}$, so there is no need to set any assumption on the uncertainties of $h(\underline{x}, t)$. The main goal is to design a feedback Markov controller u which stabilizes the unstable periodic orbits of the deterministic chaotic system (1), in the stochastic chaotic system (2).

3. Stochastic chaos control

It is assumed that all the states of system (2) are measurable. Setting $e = x - x_d$, $\underline{e} = \underline{x} - \underline{x}_d$ and subtracting (4) from (2) one obtains the error dynamics as:

$$\mathrm{e}^{(n)} = f(\underline{x}_d + \underline{e}, t) - f(\underline{x}_d, t) + b(\underline{x}_d + \underline{e}, t)u + h(\underline{x}_d + \underline{e}, t)\dot{v}. \tag{7}$$

The above equation is an Ito stochastic differential equation that must be rewritten in the following differential form:

$$de^{(n-1)} = [f(\underline{x}_d + \underline{e}, t) - f(\underline{x}_d, t) + b(\underline{x}_d + \underline{e}, t)u]\mathrm{d}t + h(\underline{x}_d + \underline{e}, t)\mathrm{d}v. \tag{8}$$

The above equation implies that $\underline{e}(t)$ is a stochastic process so it is preferred to use the mean square convergence in designing an appropriate controller which asymptotically stabilizes the unstable periodic orbit, $\underline{x}_d$. The mean square norm is defined as:

$$\|\underline{e}\| \triangleq \left(E[\underline{e}^{\mathrm{T}}\underline{e}]\right)^{1/2} \tag{9}$$

where $E[.]$ is the expected value function.

Prior to move into controller design procedure, three important remarks should be stated.

Remark 1. Regarding the definition of the Wiener process we have [17]:

$$E\left[h(x(t), t)\mathrm{d}v(t)\right] = 0, \qquad E\left[k(\underline{x}, t)\mathrm{d}v(t)\right]^2 = k^2(\underline{x}, t)\mathrm{d}t. \quad ♣ \tag{10}$$

Remark 2. There is not any feedback Markov control such that complete tracking achieves, i.e. $\|x(t) - x_d(t)\|_2 \to 0$, where $\|.\|_2$ is the standard Euclidian norm. Otherwise if we assume the achievement of complete tracking, it means that $\underline{x}_d$ must satisfy Eq. (2), so:

$$x_d^{(n)} = f(\underline{x}_d, t) + b(\underline{x}_d, t)u + h(\underline{x}_d, t)\dot{v}. \tag{11}$$

Then, comparing with Eq. (4) it is obtained that:

$$b(\underline{x}_d, t)u + h(\underline{x}_d, t)\dot{v} = 0. \tag{12}$$

This implies that u is not a Markov process, so Eq. (11) can not be achieved. ♣

Remark 3. If $y = g(\underline{x}, t)$ is a scalar value, i.e. $g : \Re^n \times \Re^+ \to \Re$, and g is a twice differentiable function, where $\underline{x}$ satisfies Eq. (3), then:

$$\mathrm{d}y = \frac{\partial g}{\partial t}\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\mathrm{d}x_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}\mathrm{d}x_i\mathrm{d}x_j \tag{13}$$

where x_i is the ith element of the $\underline{x}$ vector. For expanding Eq. (13) it must be noted that [17]:

$$\mathrm{d}t \cdot \mathrm{d}t = 0, \qquad \mathrm{d}t \cdot \mathrm{d}v = 0, \qquad \mathrm{d}v \cdot \mathrm{d}v = \mathrm{d}t \tag{14}$$

where v is a Weiner process. ♣

Design procedure of the sliding mode control starts with definition of a sliding surface as:

$$S(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right)^{n-1} e(t) = \sum_{m=0}^{n-1} \binom{n-1}{m} \mathrm{e}^{(n-1-m)}\lambda^m, \quad \binom{n-1}{m} = \frac{(n-1)!}{m!(n-1-m)!} \tag{15}$$

where λ is an arbitrary positive constant. Note that S is a stochastic process which satisfies the following Ito differential form:

$$\mathrm{d}S(t) = \sum_{m=1}^{n-1} \binom{n-1}{m} \lambda^m \mathrm{e}^{(n-1-m)}\mathrm{d}t + \left[f(\underline{x}_d + e, t) - f(\underline{x}_d, t) + b(\underline{x}_d + \underline{e}, t)\right]\mathrm{d}t + h(\underline{x}_d + e, t)\mathrm{d}v. \tag{16}$$

To design the control law, first consider the following theorem.

Theorem 1. *Let Δ be a set as:*

$$\Delta = \left\{S \in \Re | E\left[S^2\right] \le \frac{H^2}{\theta}\right\} \tag{17}$$

where θ is an arbitrary positive constant. Define a control law as:

$$u = -\frac{1}{b_m(\underline{x}_d + \underline{e}, t)}\left[\hat{f}(\underline{x}_d + \underline{e}, t) - \hat{f}(\underline{x}_d, t) + \sum_{m=1}^{n-1} \binom{n-1}{m} \mathrm{e}^{(n-1-m)}\lambda^m + K\mathrm{sign}(S(t)) + \theta S(t)\right] \tag{18}$$

where

$$K \geq \left\{ F(\underline{x}_d + \underline{e}, t) + F(\underline{x}_d, t) + \left(\frac{b_M}{b_m} - 1\right) \left(|\hat{f}(\underline{x}_d + \underline{e}, t) - \hat{f}(\underline{x}_d, t)| + \left| \sum_{m=1}^{n-2} \binom{n-2}{m} e^{(n-m)} \lambda^m \right| \right) \right\}. \tag{19}$$

Applying the control action of Eq. (18) to the stochastic chaotic system of Eq. (8), make the Δ set to be an attracting set, i.e.

$$E\left[S^2(t)\right] \leq \frac{H^2}{\theta} \quad \text{as } t \to \infty. \tag{20}$$

In other words, the control law of Eq. (18) implies that, an exponentially stable form of the tracking errors converge, in the mean square norm, toward an arbitrarily small bound around zero.

Proof. Consider a Lyapunov function constructed by the mean square norm of $S(t)$ and its differential form:

$$V(t) = \frac{1}{2} E\left[S^2(t)\right] \tag{21}$$

$$\mathrm{d}V = \frac{1}{2} E\left[\mathrm{d}(S^2(t))\right]. \tag{22}$$

Using Eq. (13) it is obtained that:

$$\mathrm{d}(S^2(t)) = 2S(t)\mathrm{d}S(t) + \mathrm{d}S(t)\mathrm{d}S(t). \tag{23}$$

Substituting Eq. (16) into Eq. (23) and using Eq. (14) result in:

$$\mathrm{d}(S^2(t)) = 2S(t) \left[\sum_{m=1}^{n-1} \binom{n-1}{m} \lambda^m e^{(n-1-m)} \mathrm{d}t + \left[f(\underline{x}_d + e, t) - f(\underline{x}_d, t) + b(\underline{x}_d + \underline{e}, t) \right] \mathrm{d}t + h(\underline{x}_d + e, t) u \mathrm{d}v \right] + h^2(\underline{x}_d + e, t)\mathrm{d}t. \tag{24}$$

Applying the expected value function to Eq. (24) and using the properties of Weiner process mentioned in Remark 1, it is obtained that:

$$\mathrm{d}V = E\left(S(t) \left[\sum_{m=1}^{n-1} \binom{n-1}{m} \lambda^m e^{(n-1-m)} \mathrm{d}t + \left[f(\underline{x}_d + e, t) - f(\underline{x}_d, t) + b(\underline{x}_d + \underline{e}, t) u \right] \mathrm{d}t \right] + \frac{1}{2} h^2(\underline{x}_d + e, t)\mathrm{d}t \right). \tag{25}$$

So:

$$\dot{V} = E\left(S(t) \left[\sum_{m=1}^{n-1} \binom{n-1}{m} \lambda^m e^{(n-1-m)} + f(\underline{x}_d + e, t) - f(\underline{x}_d, t) + b(\underline{x}_d + \underline{e}, t) u \right] + \frac{1}{2} h^2(\underline{x}_d + e, t) \right). \tag{26}$$

Substituting control law (18) into the right hand side of Eq. (26) and regarding Eq. (19) result in:

$$\dot{V} \leq -\theta E\left[S^2(t)\right] + \frac{1}{2} E\left[h^2(\underline{x}_d + e, t)\right]. \tag{27}$$

Eq. (27) implies that the set defined in Eq. (17) is an attracting set of the $S(t)$ trajectories. To show this, first assume that in $t = t_0$, $S(t)$ is outside the Δ set, so:

$$E\left[S^2(t_0)\right] > \frac{H^2}{\theta}. \tag{28}$$

Since H is the upper bound of $|h(x, t)|$, then:

$$\dot{V}(t_0) \leq -\theta E\left[S^2(t_0)\right] + \frac{1}{2} E\left[h^2(\underline{x}_d + e, t)\right] < 0. \tag{29}$$

This implies that V decreases along $S(t)$ trajectory until $S(t)$ enters into the Δ set.

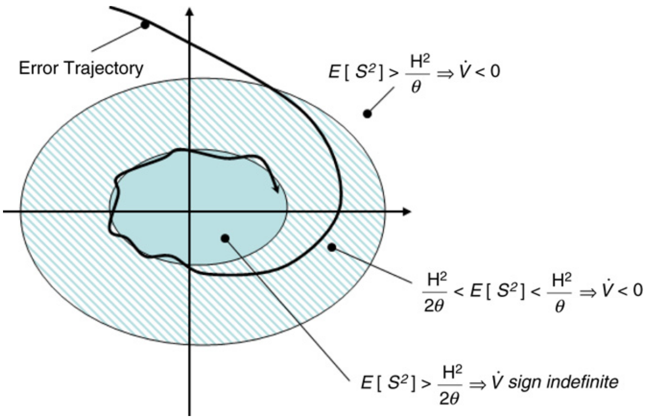

Fig. 1. Schematic diagram of the convergence in Theorem 1.

On the other hand if in $t = t_0$, $S(t) \in \Delta$ then for all $t > t_0$, $S(t)$ will remain in Δ. If $S(t)$ tends to exit the Δ set, it must pass the following region:

$$\frac{H^2}{2\theta} < E\left[S^2(t)\right] < \frac{H^2}{\theta}. \tag{30}$$

In this region, regarding Eq. (29) it is obvious that $\dot{V} < 0$ hence V is a decreasing function, and then $S(t)$ can not exist the Δ set. Therefore, the statement of Eq. (20) is proved (see Fig. 1). ♣

Note that θ is an arbitrary constant, so for large values of θ, the mean square of $S(t)$ is arbitrarily small.

Theorem 2. *Let:*

$$\underline{\eta} = \begin{bmatrix} e & \dot{e} & \cdots & e^{(n-2)} \end{bmatrix}^{\mathrm{T}}. \tag{31}$$

Under the assumptions of Theorem 1 it is obtained that:

$$\begin{aligned} &E\left[\eta^{\mathrm{T}}(t)\eta(t)\right] \le \frac{H^2}{\lambda^{2n-2}\theta}, \\ &E\left[\left(\mathrm{e}^{(n-1)}(t)\right)^2\right] \le 2\left(\frac{1}{\lambda^{2n-2}}\sum_{m=1}^{n-1}\left[\binom{n-1}{m}\lambda^m\right]^2 + 1\right)\frac{H^2}{\theta}. \end{aligned} \tag{32}$$

In other words, as the switching function $S(t)$ converges to the Δ set of Eq. (17), the error trajectories converge, in the mean square norm, to an arbitrarily small region around zero.

Proof. Considering the definition of $S(t)$, one can write:

$$\sum_{m=0}^{n-1}\binom{n-1}{m}\lambda^m \mathrm{e}^{(n-1-m)} = \delta(t) \Rightarrow \mathrm{e}^{(n-1)} = -\sum_{m=1}^{n-1}\binom{n-1}{m}\lambda^m \mathrm{e}^{(n-1-m)} + \delta(t) \tag{33}$$

where $E\left[\delta^2(t)\right] < \frac{H^2}{\theta}$ for sufficiently large t. Now let:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\lambda^{n-1}\binom{n-1}{n-1} & -\lambda^{n-2}\binom{n-1}{n-2} & \cdots & -\lambda^2\binom{n-1}{2} & -\lambda\binom{n-1}{1} \end{bmatrix}_{(n-1)\times(n-1)}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n-1)\times 1} \tag{34}$$

then Eq. (33) can be rewritten as:

$$\dot{\underline{\eta}}(t) = A\underline{\eta}(t) + B\delta(t), \qquad \underline{\eta} = \begin{bmatrix} e & \dot{e} & \cdots & \mathrm{e}^{(n-2)} \end{bmatrix}^{\mathrm{T}}. \tag{35}$$

Solving the above equation for $\underline{\eta}$ it is obtained that:

$$\underline{\eta}(t) = \exp\left(-A(t-t_0)\right)\underline{\eta}(t_0) + \int_{t_0}^{t} \exp\left(-A(t-\tau)\right) B\delta(\tau)\mathrm{d}\tau. \tag{36}$$

To calculate the mean square of error we have:

$$\begin{aligned} E\left[\underline{\eta}^{\mathrm{T}}(t)\underline{\eta}(t)\right] &= E\left[\left(\exp\left(-A(t-t_0)\right)\underline{\eta}(t_0)\right)^{\mathrm{T}}\left(\exp\left(-A(t-t_0)\right)\underline{\eta}(t_0)\right)\right] \\ &\quad + 2E\left[\left(\exp\left(-A(t-t_0)\right)\underline{\eta}(t_0)\right)^{\mathrm{T}}\int_{t_0}^{t}\exp\left(-A(t-\tau)\right)B\delta(\tau)\mathrm{d}\tau\right] \\ &\quad + E\left[\left(\int_{t_0}^{t}\exp\left(-A(t-\tau)\right)B\delta(\tau)\mathrm{d}\tau\right)^{\mathrm{T}}\int_{t_0}^{t}\exp\left(-A(t-\tau)\right)B\delta(\tau)\mathrm{d}\tau\right]. \end{aligned} \tag{37}$$

On the sliding surface, the system dynamics is chosen to be exponentially stable. Therefore two first terms of the above equation converge to zero for large t, so:

$$\lim_{t\to\infty} E\left[\underline{\eta}^{\mathrm{T}}(t)\underline{\eta}(t)\right] = \lim_{t\to\infty} E\left[\left(\int_{t_0}^{t}\exp\left(-A(t-\tau)\right)B\delta(\tau)\mathrm{d}\tau\right)^{\mathrm{T}}\int_{t_0}^{t}\exp\left(-A(t-\tau)\right)B\delta(\tau)\mathrm{d}\tau\right]. \tag{38}$$

From linear system and real analysis theories [32], it is concluded that:

$$\begin{aligned} \lim_{t\to\infty} E\left[\underline{\eta}^{\mathrm{T}}(t)\underline{\eta}(t)\right] &\le \left(B^{\mathrm{T}}\left(A^{-1}\right)^{\mathrm{T}}A^{-1}B\right)\limsup_{t\to\infty}\left(E\left[\delta^2(t)\right]\right) \\ &\le \left(B^{\mathrm{T}}\left(A^{-1}\right)^{\mathrm{T}}A^{-1}B\right)\frac{H^2}{\theta} = \frac{H^2}{\lambda^{2n-2}\theta}. \end{aligned} \tag{39}$$

From Eq. (33) it is obtained that:

$$\left(\mathrm{e}^{(n-1)}\right)^2 \le 2\left\{\sum_{m=1}^{n-1}\left[\binom{n-1}{m}\lambda^m\right]^2\left(\mathrm{e}^{(n-1-m)}\right)^2 + (\delta(t))^2\right\}. \tag{40}$$

Hence,

$$E\left[\left(\mathrm{e}^{(n-1)}\right)^2\right] \le 2\left\{\sum_{m=1}^{n-1}\left[\binom{n-1}{m}\lambda^m\right]^2 E\left[\left(\mathrm{e}^{(n-1-m)}\right)^2\right] + E\left[(\delta(t))^2\right]\right\}. \tag{41}$$

Substituting Eq. (39) into Eq. (41) yields:

$$E\left[\left(\mathrm{e}^{(n-1)}\right)^2\right] \le 2\left\{\frac{1}{\lambda^{2n-2}}\sum_{m=1}^{n-1}\left[\binom{n-1}{m}\lambda^m\right]^2 + 1\right\}\frac{H^2}{\theta} \tag{42}$$

and this completes the proof. ♣

Therefore, the mean square error can also be arbitrarily small for sufficiently large θ.

Remark 4 (*UPO Calculation*). There are some standard techniques to extract UPO in deterministic chaotic systems, e.g. References [33,34] introduce some of these techniques. For the stochastic case since $\underline{x}_d$ is the UPO of the deterministic system, we have

$$\underline{x}_d = E\left[\underline{x}_d\right]. \tag{43}$$

So the desired orbit satisfies the following equation:

$$\begin{aligned} &x_d^{(n)} = f(\underline{x}_d, t) \Rightarrow \\ &x_d^{(n)} = E\left[x_d^{(n)}\right] = E\left[f(\underline{x}_d, t) + h(\underline{x}_d, t)\dot{\upsilon}\right] = E\left[f(\underline{x}_d, t)\right]. \end{aligned} \tag{44}$$

Note that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}E[x] = E\left[x^{(n)}\right] = E\left[f(\underline{x}, t) + h(\underline{x}, t)\dot{\upsilon}\right] \tag{45}$$

hence $\underline{x}_d = E\left[\underline{x}_d\right]$ is a UPO of the above equation. Therefore one can use the conventional methods of finding the UPO of deterministic systems by applying them to $E\left(\underline{x}\right)$ which is calculated from the observation, i.e. the measurement of $\underline{x}$. ♣

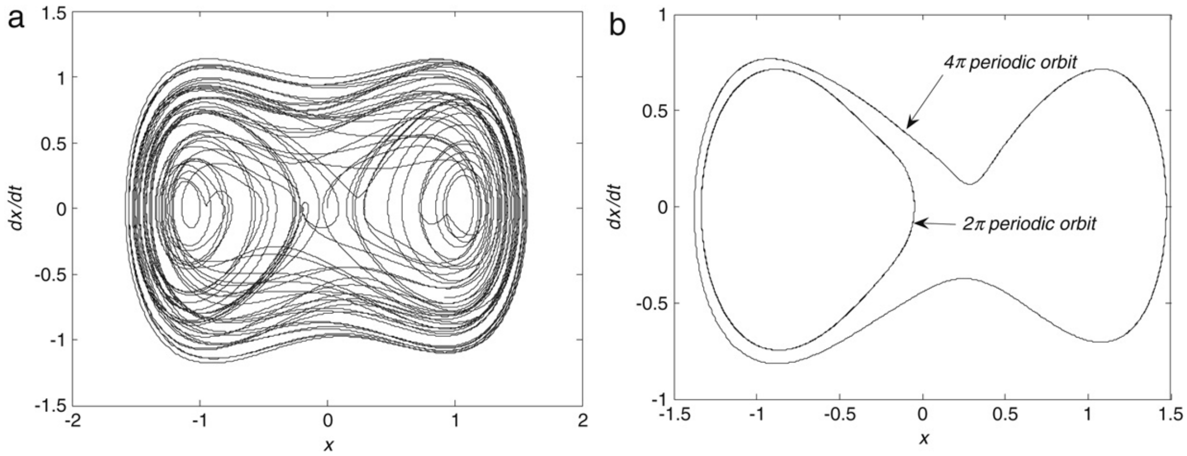

Fig. 2. (a) Chaotic attractor of the Duffing system, (b) Two unstable periodic orbits of the Duffing system.

4. Simulation results

In this section the suggested controller is applied to two different systems and the control performance will be examined.

4.1. Duffing system

The following differential equation represents the chaotic Duffing system excited by a white noise:

$$\ddot{x} = \alpha x + \beta x^3 + \gamma \dot{x} + f_0 \cos t + (1 + x^2)u + a\dot{v}. \tag{46}$$

For $\alpha = 1, \beta = -1, \gamma = -0.15, f_0 = 0.3, u = 0$ and $a = 0$ a deterministic chaotic Duffing system is achieved. u is the control action and v is a standard Wiener process. So the functions defined in Eqs. (1) and (2) are:

$$\begin{aligned} f(x, \dot{x}, t) &= \alpha x + \beta x^3 + \gamma \dot{x} + f_0 \cos t \\ b(x, \dot{x}, t) &= 1 + x^2 \\ h(x, \dot{x}, t) &= a. \end{aligned} \tag{47}$$

It is assumed that the system parameters and functions have some uncertainties and their nominal values are, $\hat{\alpha} = 1.2$, $\hat{\beta} = -1.5, \hat{\gamma} = -0.1, \hat{f}_0 = 0.5, \hat{b}(x, \dot{x}, t) = 1.5 + x^2$. The upper and lower bounds of the uncertain functions are assumed to be known and selected as:

$$F(x, \dot{x}, t) = |x| + |x|^3 + |\dot{x}| + 0.3, \qquad b_m(x, \dot{x}, t) = 1, \qquad b_M(x, \dot{x}, t) = 2 + x^2. \tag{48}$$

Here the unstable periodic orbits of periods 2π and 4π are chosen for stabilization. Fig. 2(a), (b) show, respectively, the phase portrait, the 2π-periodic (1-cycle) and the 4π-periodic (2-cycle) orbits of the deterministic chaotic Duffing system. The presented control scheme in Eqs. (18) and (19) with $\theta = 1$ and $\lambda = 1$ is used for stabilizing the mentioned periodic orbits in the phase space of the stochastic Duffing equation with $h(x, \dot{x}, t) = a = 0.5$. The results of simulation using a fourth-order Runge–Kuta–Maruyama algorithm with time step of 0.001 are shown in Figs. 3 and 4. In each simulation for time less than 20 s, the control action is not applied and for $t \geq 20$ the control algorithm is executed. The initial conditions are for both cases are $x(0) = 1, \dot{x}(0) = 2$. Figs. 3(a) and 4(a) illustrate the results for 2π and 4π-periodic orbits in the time domain. As it is seen the system states converge to the desired UPO and deviates randomly around it with a fixed bound. However both periodic orbits are indeed two solutions of the deterministic chaotic system, but due to random inputs, i.e. the white noise, to the deterministic Duffing equation, the control action does not converge to zero. Figs. 3(b) and 4(b) show the same results in the phase plane. Comparing to deterministic case, the simulation is repeated for the 4π-periodic orbit when the white Gaussian noise omitted from the system. The results are shown in Fig. 5. It is obvious that the control action converges to zero since there is not any stochastic perturbing signal and the desired periodic orbit satisfies the system equation.

4.2. Φ^6-Van der Pol system

The stochastically perturbed Φ^6-Van der Pol system has the following equation:

$$\ddot{x} = \mu(1 - x^2)\dot{x} + \zeta x + \delta x^3 + \rho x^5 + f_0 \cos(\omega t) + (2 + \cos t)u + a \cos(2t)\dot{v} \tag{49}$$

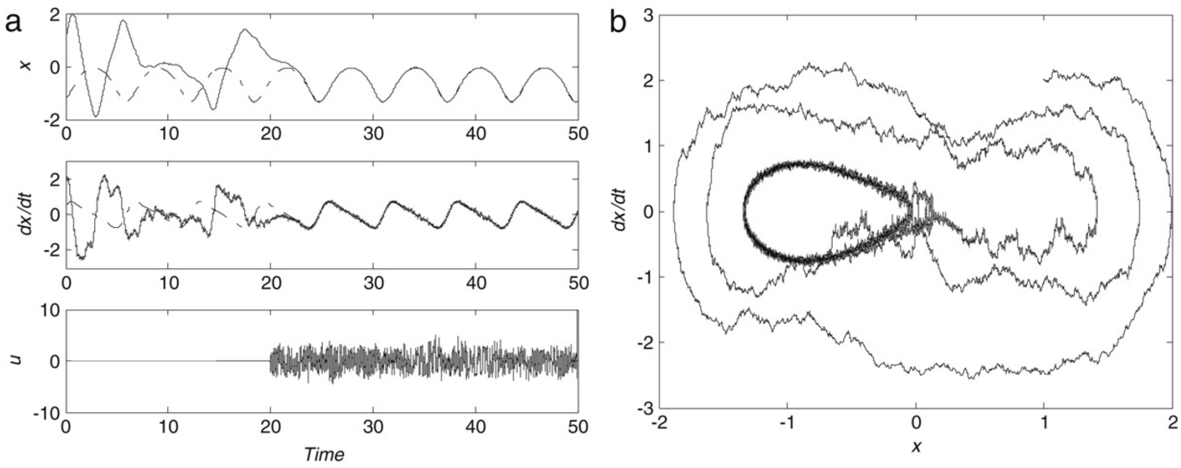

Fig. 3. Stabilizing 1-cycle orbit in stochastic Duffing system. (a) the results in time series, (b) the results in phase space.

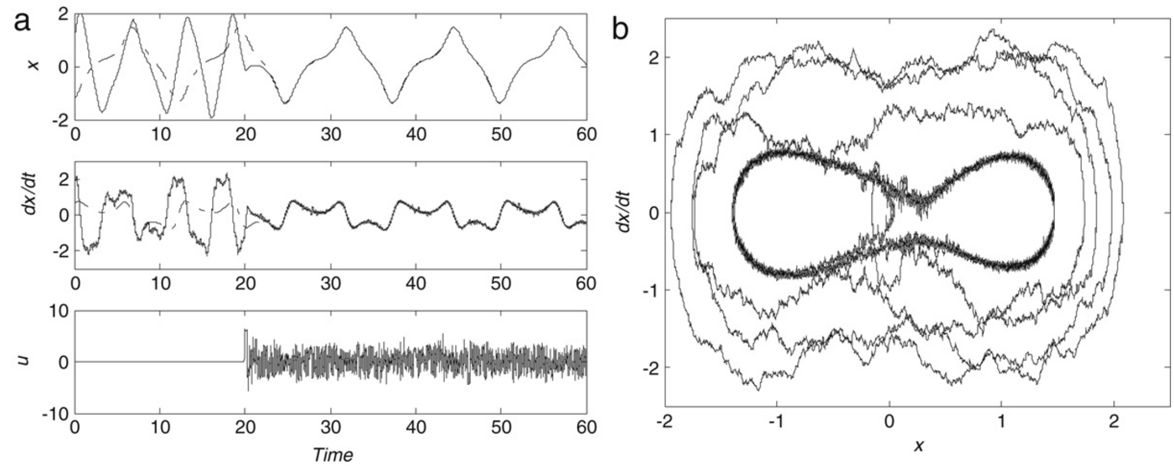

Fig. 4. Stabilizing 2-cycle orbit in stochastic Duffing system. (a) the results in time series, (b) the results in phase space.

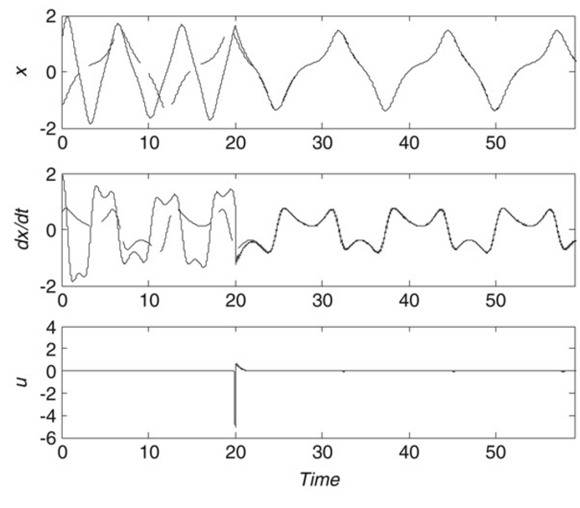

Fig. 5. Stabilizing 2-cycle orbit in deterministic Duffing system using proposed method.

where for $\mu = 0.4$, $\zeta = -0.26$, $\delta = -1$, $\rho = -0.1$, $\omega = 0.86$, $f_0 = 4.5$, $u = 0$ and $a = 0$ the behavior of the system is chaotic which is shown in Fig. 6(a). In Eq. (49), u is the control action and υ is a standard Wiener process. Here the functions

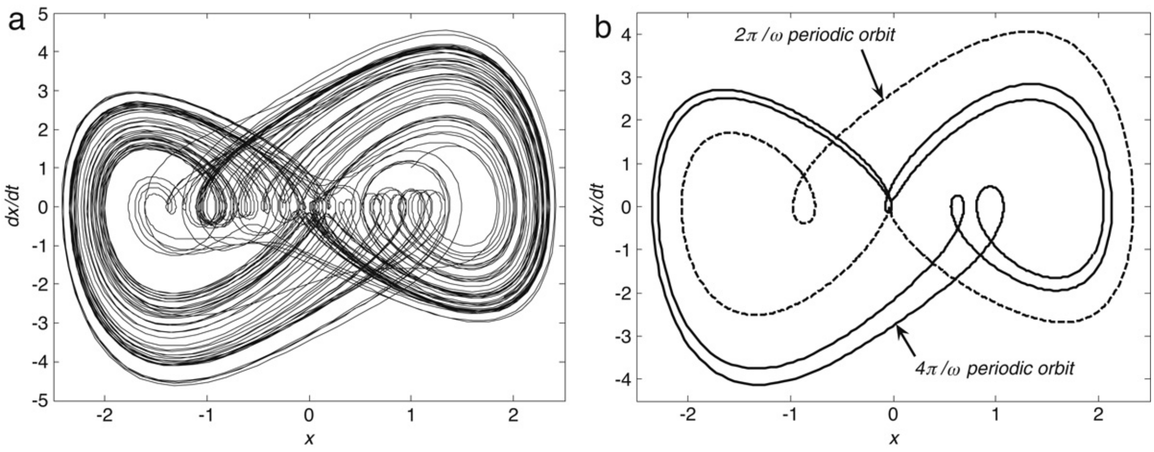

Fig. 6. (a) Chaotic attractor of the Φ^6-Van der Pol system, (b) Two unstable periodic orbits of the Φ^6-Van der Pol system.

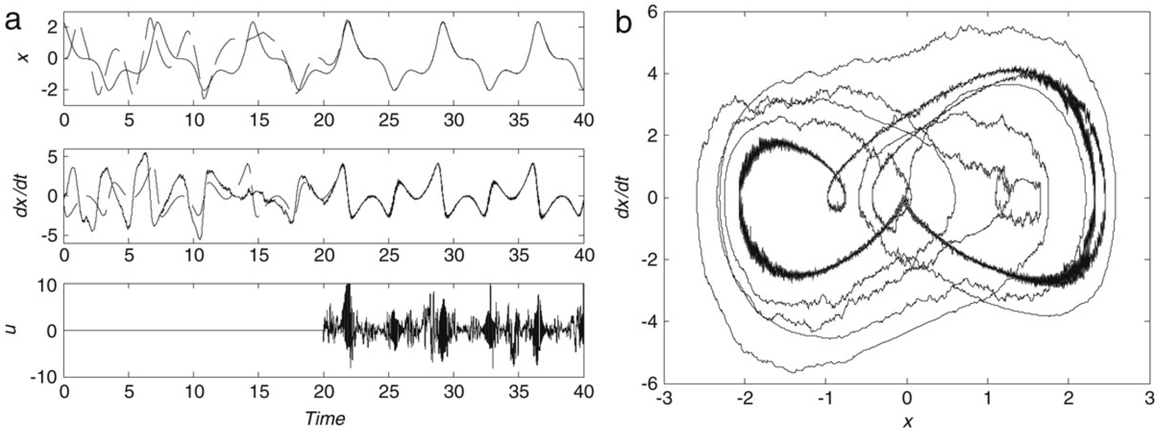

Fig. 7. Stabilizing 1-cycle orbit in stochastic Φ^6-Van der Pol system. (a) the results in time series, (b) the results in phase space.

defined in Eqs. (1) and (2) are:

$$\begin{aligned} f(x, \dot{x}, t) &= \mu(1 - x^2)\dot{x} + \zeta x + \delta x^3 + \rho x^5 + f_0 \cos(\omega t) \\ b(x, \dot{x}, t) &= 2 + \cos t \\ h(x, \dot{x}, t) &= a \cos(2t). \end{aligned} \tag{50}$$

The nominal values of the system coefficients are $\hat{\mu} = 0.6$, $\hat{\zeta} = -0.36$, $\hat{\delta} = -0.5$, $\hat{\rho} = -0.3$, $\hat{f}_0 = 5.5$, $\hat{b}(x, \dot{x}, t) = 1.5$. So $F(x, \dot{x}, t)$, $b_m(x, \dot{x}, t)$ and $b_M(x, \dot{x}, t)$ in Eq. (7) are set to:

$$F(x, \dot{x}, t) = |(1 - x^2)\dot{x}| + |x| + |x|^3 + |x|^5 + 1, \qquad b_m(x, \dot{x}, t) = 1, \qquad b_M(x, \dot{x}, t) = 3. \tag{51}$$

Fig. 6(b) shows the unstable periodic orbits of periods $2\pi/\omega$ (1-cycle) and $4\pi/\omega$ (2-cycle) when $a = 0$. The presented control scheme in Eqs. (18) and (19) with $\theta = 1$ and $\lambda = 1$ is used for stabilizing the mentioned periodic orbits in the phase space of the stochastic Φ^6-Van der Pol equation when $a = 1$. Again the fourth-order Runge–Kuta–Maruyama algorithm with time step size 0.001 is used for numerical simulation. Simulation results with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ are shown in Figs. 7 and 8. The control action applies to the system for $t \geq 20$. Figs. 7(a) and 8(a) illustrate the system time series which converge to the desired UPOs in a way that the tracking error lies in a small neighborhood around zero. Figs. 7(b) and 8(b) show the results in the phase space. Again the simulation for $4\pi/\omega$-periodic solution is repeated when the perturbing stochastic input is omitted from the system. Since the desired orbit is a solution of the deterministic system, the control action will settle on zero as shown in Fig. 9.

Remark 5. As it is observed from simulation results, in the case of stochastic chaos the control signal u is larger than the deterministic chaos case. It is due to white noise which exerts to the stochastic system. In deterministic cases when the stochastic input is zero, the control action should converge to zero, so in compare to the standard methods used for

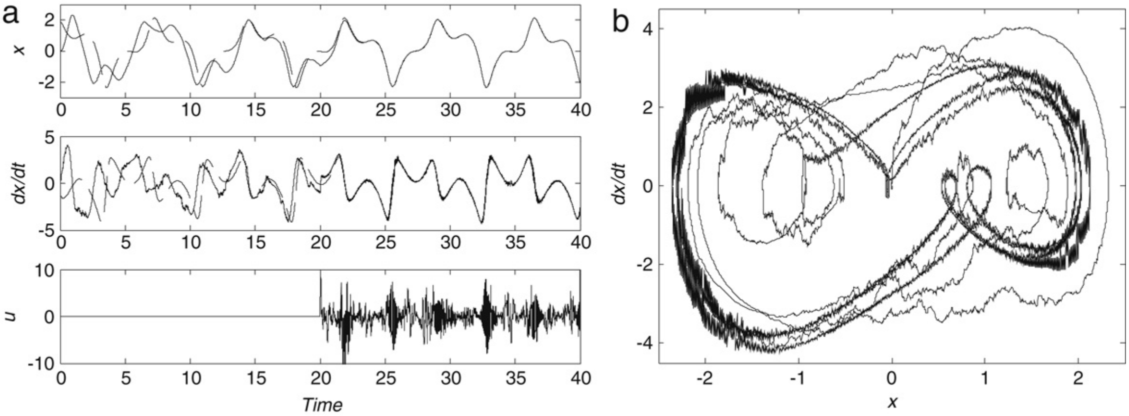

Fig. 8. Stabilizing 2-cycle orbits in stochastic Φ^6-Van der Pol system. (a) the results in time series, (b) the results in phase space.

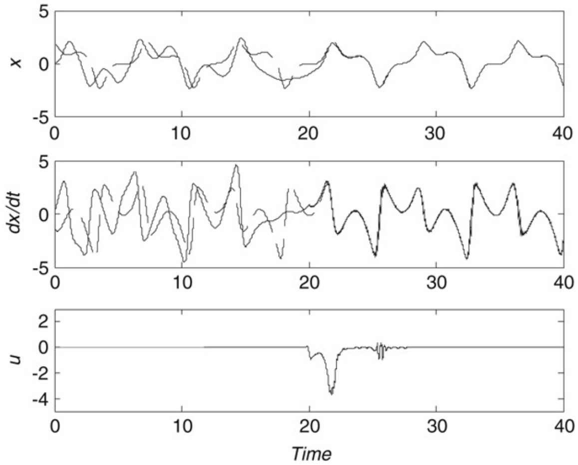

Fig. 9. Stabilizing 2-cycle orbits in deterministic Φ^6-Van der Pol system using the proposed method.

deterministic systems, the proposed control signal is large. There exist some techniques, such as using saturation function at the output of the control signal, which can result in reduction of the control action value. However, it needs further studies to investigate its nonlinear effects on stabilization of the controlled system. ♣

5. Conclusion

In this paper the problem of chaos control in stochastic chaotic systems has been investigated. For modeling the stochastic chaos, an Ito differential equation produced by applying a white noise to a deterministic chaotic system, has been considered. Then one of the unstable periodic orbits of the deterministic part of the stochastic differential equation has been chosen for stabilization. Using the concept of the sliding mode control a controller has been designed for stabilizing the unstable periodic orbit. It has been shown that, the system trajectories converge to the desired UPO and deviate randomly around it in a way that the mean square norms of all error states lie in an arbitrarily small neighborhood of zero. The stochastic Duffing and Φ^6-Van der Pol systems as two stochastic chaotic systems have been used for applying the proposed control scheme. The simulation results verify the performance and feasibility of the control algorithm.

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