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Rational approximation rate for the Müntz system $\{x^{\lambda_n}\}$ with $\lambda_n \searrow 0$

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Abstract

The present paper establishes, by employing some new ideas, a nontrivial result of quantitative rational approximation rate for the Müntz system $\{x^{\lambda_n}\}$ in case $\lambda_n \searrow 0$ as $n \to \infty$.

Keywords: Rational approximation; Müntz system; Approximation rate

1. Introduction

From the Müntz theorem (cf. [4]), it is well known that the combinations of $\{x^{\lambda_n}\}$ for

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

are dense in the continuous function space on [0, 1] (which we denote by $C_{[0,1]}$) if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

As to the rational case, Newman [6] asked a natural question: What is the condition on the λ_n which makes the rational combinations of $\{x^{\lambda_n}\}$ (denoted by $R(\Lambda)$) dense in $C_{[0,1]}$? The correct necessary and sufficient condition is not simply that $\sum_{n=1}^{\infty}(1/\lambda_n) = \infty$, what is it?

In 1976, Somorjai's surprising result in [7] showed that for any sequence $\{\lambda_n\}$ of distinct nonnegative increasing numbers, $R(\Lambda)$ are always dense in $C_{[0,1]}$. In 1978, Bak and Newman [3] proved that if $\{\lambda_n\}$ is a sequence of distinct positive numbers, then $R(\Lambda)$ are dense in $C_{[0,1]}$ as well. Recently, in [8] we generalized the above results to include the case when $\{\lambda_n\}$ is a sequence of distinct negative numbers.

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Let

$$\omega(f, t)_{[a,b]} = \max\{|f(x+h) - f(x)|: x \in [a, b-h], 0 < h \leqslant t\},$$

for continuous functions f on $[a, b]$ (denoted $f \in C_{[a,b]}$),

$$\omega(f, t) := \omega(f, t)_{[0,1]},$$

and

$$\| f \|_{[a,b]} = \max_{x \in [a,b]} |f(x)|, \qquad \| f \| = \| f \|_{[0,1]}.$$

Denote

$$\Lambda_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \qquad R(\Lambda_n) = R(\operatorname{span}\{x^{\lambda_k}\}: \lambda_k \in \Lambda_n),$$

for $f \in C_{[a,b]}$,

$$R_n(f, \Lambda)_{[a,b]} = \min_{r \in R(\Lambda_n)} \| f - r \|_{[a,b]}, \quad a \geqslant 0, \qquad R_n(f, \Lambda) = R_n(f, \Lambda)_{[0,1]}.$$

Throughout the paper, C always indicates an absolute constant which may have different values in different places.

On quantitative Müntz rational approximation rate with respect to $\{x^{\lambda_n}\}$, one important untreated case is when $\lambda_n \searrow 0$ as $n \to \infty$ (which means that λ_n strictly decreases to 0 as $n \to \infty$). Simply following Bak and Newman's method for density in [3], one can obtain a trivial result as follows. For $f \in C_{[0,1]}$, find a polynomial

$$p(x) = \sum_{j=0}^{n} a_j x^j$$

such that (see [5])

$$\| f - p \| \leqslant C\omega(f, n^{-1}) \quad \text{and} \quad \| p^{(r)} \| \leqslant C^r n^r \omega(f, n^{-1}).$$

Then Bak and Newman's calculation leads to a rational function $r \in R(\Lambda_{2n})$ such that

$$\| p - r \| \leqslant C\omega(f, n^{-1}), \quad \text{whenever } \lambda_n \leqslant \frac{\omega(f, n^{-1})}{\sum_{j=0}^{n} |a_j|}.$$

Noting that from the above cited result [5] (it is unimprovable in general),

$$|a_j| = \left| \frac{p^{(j)}(0)}{j!} \right| \leqslant \frac{C^j}{j!} n^j \omega(f, n^{-1}),$$

we obtain from a rough calculation that

$$R_n(f, \Lambda) \leqslant C\omega(f, n^{-1}), \quad \text{whenever} \quad \lambda_n \leqslant Cn^{-n}.$$

Of course this trivial result does not much increase the present level of knowledge concerning this case. But the above observation reveals where the main difficulty lies in dealing with this: one cannot achieve better estimates for coefficients of polynomials!

The present paper establishes, by employing some new ideas, a nontrivial result of quantitative rational approximation rate for the Müntz system $\{x^{\lambda_n}\}$ in case $\lambda_n \searrow 0$ as $n \to \infty$, which, we wish, could prompt further research in this direction.

2. Results and proofs

We establish first the following lemma, which itself is of some independent interest.

Lemma 1. *Let $m_n = n^2$, $n = 1, 2, \ldots$. Fix $n \geqslant 2$. Then for $f \in C_{[0,1]}$ with $f(0) = 0$, there is a rational function with the following form:*

$$r(x) = \frac{\sum_{j=n_0+1}^{n-1} f(x_j) Q_j(x)}{\sum_{j=n_0+1}^{n-1} Q_j^*(x)}$$

such that

$$\| f - r \| \leqslant C\omega(f, n^{-1/2}),$$

where

$$x_j = \frac{j}{n}, \quad j = n_0 + 1 := [\sqrt{n}] + 1, n_0 + 2, \ldots, n-1,$$

and for $j = n_0 + 1, n_0 + 2, \ldots, n - 1$,

$$Q_j(x) = x^{m_j} \prod_{l=1}^{j} x_l^{-\Delta m_l}, \qquad Q_j^*(x) = x^{m_j - 1} x_j^{-\Delta m_j + 1} \prod_{l=1}^{j-1} x_l^{-\Delta m_l},$$

$$\Delta m_1 = m_1, \qquad \Delta m_n = m_n - m_{n-1}, \quad n \geqslant 2.$$

Proof. Since $\Delta m_n \geqslant n$ for $n \geqslant 2$, in a similar way to [1,9], we can prove that for $x_k - \frac{1}{2}\Delta^* x_k \leqslant x < x_k + \frac{1}{2}\Delta^* x_{k+1}$, $k = n_0 + 2, n_0 + 3, \ldots, n - 1$, and $j \in \{n_0 + 1, n_0 + 2, \ldots, n - 1\} \setminus \{k - 1, k\}$,

$$\left| \frac{Q_j^*(x)}{Q_{k^*}^*(x)} \right| \leqslant 2\mathrm{e}^{-\alpha|k-j|/8}, \tag{1}$$

where

$$\Delta^* x_k = \begin{cases} 2x_{n_0+2}, & k = n_0 + 2, \\ 1/n, & n_0 + 3 \leqslant k \leqslant n - 1, \\ 2/n, & k = n, \end{cases}$$

$$k^* := k^*(x) = \begin{cases} n_0 + 1, & 0 \leqslant x \leqslant x_{n_0+2}, \\ k - 1, & x_k - \frac{1}{2}\Delta^* x_k \leqslant x \leqslant x_k,\ n_0 + 3 \leqslant k \leqslant n - 1, \\ k, & x_k < x < x_k + \frac{1}{2}\Delta^* x_{k+1},\ n_0 + 2 \leqslant k \leqslant n - 1. \end{cases}$$

Write

$$f(x)-r(x)=\frac{\sum_{j=n_0+1}^{n-1}\left(f(x)-f(x_j)\right)Q_j^*(x)}{\sum_{j=n_0+1}^{n-1}Q_j^*(x)}+\frac{\sum_{j=n_0+1}^{n-1}f(x_j)\left(Q_j^*(x)-Q_j(x)\right)}{\sum_{j=n_0+1}^{n-1}Q_j^*(x)}$$

$$:=\Sigma_1+\Sigma_2.$$

By (1), for $x_k-\frac{1}{2}\Delta^*x_k\leqslant x<x_k+\frac{1}{2}\Delta^*x_{k+1}$, $k=n_0+3,\ n_0+4,\dots,n-1$,

$$|\Sigma_1|\leqslant 2\sum_{j=n_0+1}^{k-2}\omega\left(f,\ \frac{k-j+1}{n}\right)\mathrm{e}^{-(k-j)/8}+2\sum_{j=k+1}^{n-1}\omega\left(f,\ \frac{j-k+1}{n}\right)\mathrm{e}^{-(k-j)/8}+2\omega\left(f,\ \frac{2}{n}\right)$$

$$\leqslant 8\omega(f,\ n^{-1})\sum_{j=1}^{\infty}j\mathrm{e}^{-j/8}+4\omega(f,\ n^{-1})\leqslant C\omega(f,\ n^{-1}). \tag{2}$$

Similarly, for $x\in[0,\ x_{n_0+2}+1/(2n)]$, by (1) again, we have

$$|\Sigma_1|\leqslant 4\omega(f,\ x_{n_0+3})\sum_{j=1}^{\infty}j\mathrm{e}^{-j/8}+4\omega(f,\ x_{n_0+3})\leqslant C\omega(f,\ n^{-1/2}). \tag{3}$$

On the other hand, we see that

$$Q_j^*(x)-Q_j(x)=Q_j^*(x)\left(1-\frac{x}{x_j}\right). \tag{4}$$

Let $x\in[x_k-\frac{1}{2}\Delta^*x_k,\ x_k+\frac{1}{2}\Delta^*x_{k+1})$, $k=n_0+3,\ n_0+4,\dots,n-1$; then,

$$|Q_{k-1}^*(x)-Q_{k-1}(x)|\leqslant Q_{k-1}^*(x)\frac{2}{k-1}\leqslant\frac{2}{\sqrt{n}+1}Q_{k-1}^*(x) \tag{5}$$

and

$$|Q_k^*(x)-Q_k(x)|\leqslant\frac{1}{\sqrt{n}+2}Q_k^*(x). \tag{6}$$

Meanwhile for $x\in[x_k-\frac{1}{2}\Delta^*x_k,\ x_k+\frac{1}{2}\Delta^*x_{k+1})$, $k=n_0+3,\ n_0+4,\dots,n-1$, by (4),

$$\sum_{j=n_0+1}^{k-2}|Q_j^*(x)-Q_j(x)|\leqslant Q_{k^*}^*(x)\sum_{j=n_0+1}^{k-2}\mathrm{e}^{-(k-j)/8}\frac{k-j+1}{j}$$

$$\leqslant\frac{2Q_{k^*}^*(x)}{\sqrt{n}}\sum_{j=n_0+1}^{\infty}j\mathrm{e}^{-j/8}\leqslant C\frac{Q_{k^*}^*(x)}{\sqrt{n}},$$

$$\sum_{j=k+1}^{n-1}|Q_j^*(x)-Q_j(x)|\leqslant C\frac{Q_{k^*}^*(x)}{\sqrt{n}}.$$

Combining these estimates with (5) and (6) yields that for $x_k - \frac{1}{2}\Delta^* x_k \leqslant x < x_k + \frac{1}{2}\Delta^* x_{k+1}$, $k = n_0+3, n_0+4, \ldots, n-1$,

$$|\Sigma_2| \leqslant Cn^{-1/2}\|f\| \leqslant C\omega(f, n^{-1/2}). \tag{7}$$

Finally, suppose $x \in [0, x_{n_0+2} + 1/(2n)]$. Obviously for $j = n_0+1, n_0+2, \ldots, n-1$,

$$Q_j^*(x) - Q_j(x) = \mathrm{O}(Q_j^*(x))$$

holds in the present case, and applying (1) and $f(0)=0$ leads to

$$\begin{aligned}
&\sum_{j=n_0+1}^{n-1} |f(x_j)||Q_j^*(x) - Q_j(x)| \\
&\quad\leqslant 2\omega(f, x_{n_0+3})\big(Q_{n_0+1}^*(x) + Q_{n_0+2}^*(x)\big) \\
&\qquad + CQ_{n_0+1}^*(x) \sum_{j=n_0+3}^{n-1} \left(\omega(f, x_{n_0+1}) + \omega\left(f, \frac{j-n_0-1}{n}\right)\right) \mathrm{e}^{-(j-n_0-1)/8} \\
&\quad\leqslant C\omega(f, n^{-1/2}) \sum_{j=n_0+1}^{n-1} Q_j^*(x).
\end{aligned} \tag{8}$$

Combining (2), (3), (7) and (8) completes the proof of Lemma 1. □

Lemma 2. *Let*

$$r^*(x) = \frac{\sum_{j=n_0+1}^{n-1} f(x_j) Q_j(-1/\log(x/\mathrm{e}))}{\sum_{j=n_0+1}^{n-1} Q_j^*(-1/\log(x/\mathrm{e}))}.$$

Then there is a rational function $R(x) \in R(\Lambda_{4m_n})$ such that

$$\|r^* - R\| \leqslant C\|f\|\lambda_{m_n}.$$

Proof. Let $P_k(x, a_0, a_1, \ldots, a_k)$ denote the kth divided difference of $(x/\mathrm{e})^\alpha$ at $\alpha = a_0, a_1, \ldots, a_k$ with respect to α, that is,

$$P_0(x, a_0) = \left(\frac{x}{\mathrm{e}}\right)^{a_0},$$

$$P_k(x, a_0, a_1, \ldots, a_k) = \frac{P_{k-1}(x, a_0, a_1, \ldots, a_{k-1}) - P_{k-1}(x, a_1, a_2, \ldots, a_k)}{a_0 - a_k}.$$

Write

$$P_0(x) = P_0(x, \lambda_{m_n}),$$

for $k = 1, 2, \ldots, m_{n-1}$,

$$P_k(x) = P_k(x, \lambda_{m_n}, \lambda_{m_n+1}, \ldots, \lambda_{m_n+k}),$$

for $k=1, 2, \ldots, m_{n-1}-1$,

$$P_k^*(x)=P_{k-1}(x, \lambda_{m_n+m_{n-1}+1}, \lambda_{m_n+m_{n-1}+2}, \ldots, \lambda_{m_n+m_{n-1}+k})$$

and

$$P_{m_{n-1}}^*(x)=P_{m_{n-1}-1}(x, \lambda_{m_n+2m_{n-1}}, \lambda_{m_n+2m_{n-1}+1}, \ldots, \lambda_{m_n+3m_{n-1}-1}).$$

By the mean value theorem,

$$P_k(x)=\frac{(x/\mathrm{e})^{\eta_k}\log^k(x/\mathrm{e})}{k!}, \tag{9}$$

$$\lambda_{m_n}\leqslant\eta_k\leqslant\lambda_{m_n+k}, \quad k=0, 1, \ldots, m_{n-1},$$

$$P_k^*(x)=\frac{(x/\mathrm{e})^{\eta_k^*}\log^{k-1}(x/\mathrm{e})}{(k-1)!}, \tag{10}$$

$$\lambda_{m_n+m_{n-1}+1}\leqslant\eta_k^*\leqslant\lambda_{m_n+m_{n-1}+k}, \quad k=0, 1, \ldots, m_{n-1}-1,$$

$$P_{m_{n-1}}^*(x)=\frac{(x/\mathrm{e})^{\eta_{m_{n-1}}^*}\log^{m_{n-1}-1}(x/\mathrm{e})}{(m_{n-1}-1)!}, \tag{11}$$

$$\lambda_{m_n+2m_{n-1}}\leqslant\eta_{m_{n-1}}^*\leqslant\lambda_{m_n+3m_{n-1}-1}\to\lambda_{m_n+3m_{n-1}-1}.$$

Define

$$R(x)=\frac{\sum_{j=n_0+1}^{n-1}(-1)^{m_j}(m_{n-1}-m_j)!\, f(x_j)P_{m_{n-1}-m_j}(x)}{\sum_{j=n_0+1}^{n-1}(-1)^{m_j-1}(m_{n-1}-m_j-1)!\, P_{m_{n-1}-m_j}^*(x)};$$

then $R(x)\in R(\Lambda_{4m_n})$. By (9)–(11),

$$R(x)=\frac{\sum_{j=n_0+1}^{n-1}f(x_j)Q_j(-1/\log(x/\mathrm{e}))(x/\mathrm{e})^{\alpha_j}}{\sum_{j=n_0+1}^{n-1}Q_j^*(-1/\log(x/\mathrm{e}))(x/\mathrm{e})^{\alpha_j^*}},$$

where for $j=n_0+1, n_0+2, \ldots, n-1$,

$$\alpha_j=\eta_{m_{n-1}-m_j}-\eta_{m_{n-1}-m_{n_0+1}}>0, \qquad 0\leqslant\alpha_j^*=\eta_{m_{n-1}-m_j}^*-\eta_{m_{n-1}-m_{n_0+1}}<\alpha_j.$$

We come to estimate $r^*(x)-R(x)$. Write

$$p(x)=\sum_{j=n_0+1}^{n-1}f(x_j)Q_j\left(\frac{-1}{\log(x/\mathrm{e})}\right), \qquad p_1(x)=\sum_{j=n_0+1}^{n-1}f(x_j)Q_j\left(\frac{-1}{\log(x/\mathrm{e})}\right)\left(\frac{x}{\mathrm{e}}\right)^{\alpha_j},$$

$$q(x)=\sum_{j=n_0+1}^{n-1}Q_j^*\left(\frac{-1}{\log(x/\mathrm{e})}\right), \qquad q_1(x)=\sum_{j=n_0+1}^{n-1}Q_j^*\left(\frac{-1}{\log(x/\mathrm{e})}\right)\left(\frac{x}{\mathrm{e}}\right)^{\alpha_j^*}.$$

Then,

$$r^*(x)-R(x)=\frac{p(x)-p_1(x)}{q(x)}+\frac{p_1(x)}{q_1(x)}\frac{q_1(x)-q(x)}{q(x)}. \tag{12}$$

It follows from

$$\left\|\frac{1-(x/\mathrm{e})^{\eta}}{\log(x/\mathrm{e})}\right\|\leqslant\eta,$$

for $\eta>0$ that

$$\left|\frac{p(x)-p_1(x)}{q(x)}\right|\leqslant\frac{\sum_{j=n_0+1}^{n-1}|f(x_j)\log^{-m_j}(x/\mathrm{e})|\,|(x/\mathrm{e})^{\alpha_j}-1|}{q(x)}\leqslant\|f\|\max_{n_0+1\leqslant j\leqslant n-1}\alpha_j.$$

Similarly,

$$\left|\frac{q_1(x)-q(x)}{q(x)}\right|\leqslant\left|\log\left(\frac{x}{\mathrm{e}}\right)\right|\max_{n_0+1\leqslant j\leqslant n-1}\alpha_j^* \quad\text{and}\quad \left|\frac{p_1(x)}{q_1(x)}\right|\leqslant\|f\|\left|\log^{-1}\left(\frac{x}{\mathrm{e}}\right)\right|,$$

since $\alpha_j^*<\alpha_j$ for all $j=n_0+1,\ n_0+2,\dots,n-1$. Combining these estimates with (12), we get for $x\in(0,1]$ that

$$|r^*(x)-R(x)|\leqslant\|f\|\max_{n_0+1\leqslant j\leqslant n-1}\{\alpha_j,\ \alpha_j^*\},$$

which is the required result. □

Theorem 3. *Let $\{\lambda_n\}$ be a sequence with $\lambda_n\searrow 0$ as $n\to\infty$. Suppose that $\{\sigma_n\}$ is a positive decreasing sequence satisfying*

$$\lambda_n\leqslant\sigma_n,$$

for $n=1,2,\dots,$ and

$$\frac{\sigma_n}{\sigma_{2n}}=\mathrm{O}(1).$$

Denote

$$s_n=\max\{n^{-1/4},\ \sigma_n\}.$$

Then for any $f\in C_{[0,1]}$,

$$R_n(f,\ \Lambda)\leqslant C\omega(f,\ s_n).$$

Corollary 4. *Let $\{\lambda_n\}$ be a sequence with $\lambda_n\searrow 0$ as $n\to\infty$. Suppose that*

$$\lambda_n\leqslant Cn^{-1/4},$$

for $n=1,2,\dots$. Then for any $f\in C_{[0,1]}$,

$$R_n(f,\ \Lambda)\leqslant C\omega(f,\ n^{-1/4}).$$

Proof of Theorem 3. For $f \in C_{[0,1]}$, set

$$g(t) = f(\mathrm{e}^{1-1/t}) - f(0);$$

then $g(t) \in C_{[0,1]}$ with $g(0) = 0$. Applying Lemma 1, we find a rational function r of degree m_n such that

$$\| g - r \| < C\omega(g, n^{-1/2}) \leqslant C\omega(f, n^{-1/2})$$

or

$$\| f(x) - f(0) - r^*(x) \| < C\omega(f, n^{-1/2}). \tag{13}$$

Let $R(x) \in R(\Lambda_{4m_n})$ be defined as in Lemma 2; then we have

$$\| r^* - R \| \leqslant C\lambda_{m_n} \| f \| \leqslant C\sigma_{m_n} \| f \| \leqslant C\omega(f, \sigma_{m_n}). \tag{14}$$

The estimates (13) and (14) then imply that

$$\| f(x) - f(0) - R(x) \| \leqslant C\omega(f, s_{m_n}),$$

or, in an equivalent form,

$$R_n(f, \Lambda) \leqslant C\omega(f, s_n),$$

since $f(0) + R(x) \in R(\Lambda_{4m_n})$. Theorem 3 is completed. □

With the same calculation we can establish better estimates in the interval $[a, 1]$ for $a > 0$.

Theorem 5. *Let $\{\lambda_n\}$ be a sequence with $\lambda_n \searrow 0$ as $n \to \infty$. Suppose that $\{\sigma_n\}$ is a positive decreasing sequence satisfying*

$$\lambda_n \leqslant \sigma_n,$$

for $n = 1, 2, \ldots,$ and

$$\frac{\sigma_n}{\sigma_{2n}} = \mathrm{O}(1).$$

Denote

$$s_n = \max\{n^{-1/2}, \sigma_n\}.$$

Then for any $f \in C_{[a,1]}$, $a > 0$,

$$R_n(f, \Lambda)_{[a,1]} \leqslant C_a\omega(f, s_n)_{[a,1]},$$

where C_a is a positive constant depending upon a only.

Corollary 6. *Let $\{\lambda_n\}$ be a sequence with $\lambda_n \searrow 0$ as $n \to \infty$. Suppose that*

$$\lambda_n \leqslant Cn^{-1/2},$$

for $n = 1, 2, \ldots$. Then for any $f \in C_{[a,1]}$, $a > 0$,

$$R_n(f, \Lambda)_{[a,1]} \leqslant C_a\omega(f, n^{-1/2})_{[a,1]}.$$

3. Remark

On the quantitative Müntz rational approximation rate, Bak [1] proved that if $f \in C_{[0,1]}$ and $\{\lambda_n\}$ is a sequence of distinct nonnegative increasing numbers with $\Delta\lambda_k \geqslant k$ for all $k \geqslant 2$, then [2]

$$R_n(f, \Lambda) \leqslant C\omega(f, n^{-1}).$$

In the case $\lambda_n \to l$ for some l, $0 < l < \infty$, the best nth Müntz polynomial approximation rate of $f \in C_{[0,1]}$ with respect to $\{x^{\lambda_n}\}$ is well known to be $C\omega(f, n^{-1/2})$ (see, for example, [2]); hence a trivial consequence is that the bound for $R_n(f, \Lambda)$ in this case is $C\omega(f, n^{-1/2})$ as well.

All these results together with Theorem 3 are still far away from confirmation of the following problem raised by Newman.

Problem (Newman [6, Problem 10.3]). *Is it true that for any $f \in C_{[0,1]}$ there exists $R(x) \in R(\Lambda_n)$ such that*

$$\| f - R \| \leqslant C\omega(f, n^{-1})?$$

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[2] Newman [6] announced that Bak's result also holds for $\{\lambda_n\}$ with $\Delta\lambda_k \geqslant C > 0$, but we found no reference about this.