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Note

On the wavelet transform in the field $\mathbb{Q}_p$ of p-adic numbers

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Abstract

In the present article we shall define the notion of the wavelet transform on $\mathbb{Q}_p$ and we shall show that, for any given admissible function $h \in L^2(\mathbb{Q}_p)$, satisfying (15), which is a step function, the wavelet transform of a step function f be a function of norms, and moreover be expressible to a summation form.

Keywords: Wavelet transform; p-adic number field; p-adic integral

1. Introduction

The field $\mathbb{Q}_p$ of the p-adic numbers is defined as the completion of the field $\mathbb{Q}$ of rational numbers with respect to the p-adic metric induced by the p-adic norm $|\cdot|_p$ [1]. A p-adic number $x \neq 0$ is uniquely represented in the canonical form

$$x = p^{-\gamma} \sum_{k=0}^{\infty} x_k p^k, \qquad |x|_p \stackrel{\text{def}}{=} p^{\gamma}, \tag{1}$$

where $\gamma \in \mathbb{Z}$ and $x_k \in \mathbb{Z}$ such that $0 \leqslant x_k \leqslant p-1$, $x_0 \neq 0$.

A well-known fact is that wavelet transform has been used as a real analysis tool for the signal processing [2,3]. In the p-adic analysis, wavelet transform in the field $\mathbb{Q}_p$, which will be defined, may be one of the most important parts in the field of application.

Throughout the article we shall deal with a complex valued function of p-adic argument and we shall also call it a step function if it has finite range on each circle $|x|_p = \text{const}$ of $\mathbb{Q}_p$. In the present article we shall define the notion of the wavelet transform on $\mathbb{Q}_p$ and we shall show that, for any given admissible function $h \in L^2(\mathbb{Q}_p)$, satisfying (15), which is a step function, the wavelet transform of a step function f be a function of norms, and moreover be expressible to a summation form. The results obtained in the present article will be usable to the field of research in data compression for signal processing according to the following scheme. Let a signal $f(t)$ be given, where t denotes the time variable.

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1. A positive real number $x \in \mathbb{R}_+$ can be uniquely represented in the form
$$x = p^{\gamma} \sum_{k=0}^{\infty} x_k p^{-k},$$
where $\gamma \in \mathbb{Z}$ and $x_k \in \mathbb{Z}$ such that $0 \leqslant x_k \leqslant p-1$, $x_0 \neq 0$ provided that we exclude the cases that all except finitely many x_k are $p-1$. Hence we may define a mapping $P : \mathbb{R}_+ \to \mathbb{Q}_p$ by
$$P(0) = 0, \qquad P\left(p^{\gamma} \sum_{k=0}^{\infty} x_k p^{-k}\right) = p^{-\gamma} \sum_{k=0}^{\infty} x_k p^{k}.$$
P is clearly 1-1 but not onto. Hence we can define the left inverse $P_* : \mathbb{Q}_p \to \mathbb{R}_+$ of P by
$$P_*(0) = 0, \qquad P_*\left(p^{-\gamma} \sum_{k=0}^{n} x_k p^{k}\right) = p^{\gamma} \sum_{k=0}^{\infty} x_k p^{-k}.$$
It is noteworthy that the set of p-adic numbers not in the range of P is countable and consists of the p-adic numbers of the form
$$x = p^{-\gamma}\left(\sum_{k=0}^{n} x_k p^{k} + (p-1) \sum_{k=n+1}^{\infty} p^{k}\right), \quad x_n \neq p-1,$$
for some integer $n \geqslant 0$, and that the range of P is dense in $\mathbb{Q}_p$.
2. For a given signal $f(t)$, $t \in \mathbb{R}_+$, we consider a function of p-adic variable $f_p : \mathbb{Q}_p \to \mathbb{R}$ by means of $f_p = f \circ P_*$.
3. We could obtain much information about f_p for the data compression by using the wavelet transform in $\mathbb{Q}_p$ and then transmit and receive it, and do inverse wavelet transform of it in $\mathbb{Q}_p$. Finally we would obtain desirable information about the original signal f by virtue of $f = f_p \circ P$.

2. Main theorems

Definition 2.1. Let α be a given real number. For a given $h \in L(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$, the mapping $f \mapsto \psi_h f$ from $L^2(\mathbb{Q}_p)$ to $L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$, defined by
$$(\psi_h f)(a,b) = \frac{1}{\sqrt{c}|a|_p^{\alpha}} \left\langle f(\cdot), h\left(\frac{\cdot - b}{a}\right)\right\rangle, \quad a, b \in \mathbb{Q}_p,\ a \neq 0 \tag{2}$$
is called a wavelet transform in $\mathbb{Q}_p$, where the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{Q}_p)$.

In this article, symbols $\hat{f}$ and $\bar{f}$ denote the Fourier transform of f defined by
$$\hat{f}(\xi) = \int_{\mathbb{Q}_p} f(x) \chi_p(\xi x)\, \mathrm{d}x, \quad \chi_p(\xi x) \stackrel{\text{def}}{=} \exp\bigl(2\pi i \{\xi x\}\bigr)$$
and the complex conjugate of f, respectively.

Theorem 2.2. *Let h be a real valued function such that*
$$c \stackrel{\text{def}}{=} \int_{\mathbb{Q}_p} \frac{|\hat{h}(a)|^2}{|a|_p}\, \mathrm{d}a < +\infty \tag{3}$$
exists, then we have, for any real α and β such that $2\alpha + \beta = 3$,
$$f(x) = \frac{1}{\sqrt{c}} \int_{\mathbb{Q}_p} \frac{\mathrm{d}a}{|a|_p^{\alpha+\beta}} \int_{\mathbb{Q}_p} (\psi_h f)(a,b) h\left(\frac{x-b}{a}\right) \mathrm{d}b. \tag{4}$$
A function satisfying (3) *will be called an admissible function.*

Proof. By using the Fourier transform properties

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \qquad \left[h\left(\frac{\cdot - b}{a}\right)\right]^{\wedge} = |a|_p \hat{h}(a\xi)\chi_p(-\xi b),$$

we have

$$\begin{aligned}(\psi_h f)(a,b) &= \frac{1}{\sqrt{c}|a|_p^{\alpha}}\left\langle f(\cdot), h\left(\frac{\cdot - b}{a}\right)\right\rangle = \frac{1}{\sqrt{c}}|a|_p^{-\alpha+1}\big\langle \hat{f}(\cdot), \hat{h}(a\xi)\chi_p(\xi b)\big\rangle \\ &= \frac{|a|_p^{-\alpha+1}}{\sqrt{c}} \int\limits_{\mathbb{Q}_p} \hat{f}(\xi)\bar{\hat{h}}(a\xi)\chi_p(-b\xi)\,\mathrm{d}\xi. \end{aligned} \tag{5}$$

Since $(\psi_h f)(a,b)$ can be regarded as the inverse Fourier transform of

$$\frac{|a|_p^{-\alpha+1}}{\sqrt{c}}\hat{f}(\xi)\bar{\hat{h}}(a\xi) \tag{6}$$

as a function of ξ, we have, by virtue of Fourier transform,

$$\frac{|a|_p^{-\alpha+1}}{\sqrt{c}}\hat{f}(\xi)\bar{\hat{h}}(a\xi) = \int\limits_{\mathbb{Q}_p} (\psi_h f)(a,b)\chi_p(\xi b)\,\mathrm{d}b. \tag{7}$$

Hence we have

$$\begin{aligned} I &\stackrel{\text{def}}{=} \frac{1}{\sqrt{c}} \int\limits_{\mathbb{Q}_p}\left(\int\limits_{\mathbb{Q}_p} (\psi_h f)(a,b) h\left(\frac{b-x}{-a}\right)\mathrm{d}b\right)\frac{\mathrm{d}a}{|a|_p^{\alpha+\beta}} \\ &= \frac{1}{\sqrt{c}} \int\limits_{\mathbb{Q}_p}\left(\int\limits_{\mathbb{Q}_p} \big[(\psi_h f)(a,\cdot)\big]\hat{f}(\xi)\bar{\hat{h}}(-a\xi)\chi_p(-x\xi)\,\mathrm{d}\xi\right)\frac{\mathrm{d}a}{|a|_p^{\alpha+\beta-1}} \\ &= \frac{1}{c} \int\limits_{\mathbb{Q}_p}\left(\int\limits_{\mathbb{Q}_p} \hat{f}(\xi)\bar{\hat{h}}(a\xi)\hat{h}(a\xi)\chi_p(-\xi x)\,\mathrm{d}\xi\right)\frac{\mathrm{d}a}{|a|_p^{2\alpha+\beta-2}} \\ &= \frac{1}{c} \int\limits_{\mathbb{Q}_p} \hat{f}(\xi)\chi_p(-\xi x)\,\mathrm{d}\xi \int\limits_{\mathbb{Q}_p} \frac{|\hat{h}(a\xi)|^2}{|a|_p}\,\mathrm{d}a \quad \big(2\alpha+\beta-2 \stackrel{\text{def}}{=} 1\big) \\ &= \frac{1}{c} f(x) \int\limits_{\mathbb{Q}_p} \frac{|\bar{\hat{h}}(a')|^2}{|a'|_p}\,\mathrm{d}a' \\ &= f(x), \end{aligned} \tag{8}$$

where we used the fact that

$$\int\limits_{\mathbb{Q}_p} f(ax)\,\mathrm{d}x = \int\limits_{\mathbb{Q}_p} f(x)|a|_p^{-1}\,\mathrm{d}x. \qquad \square$$

Theorem 2.3. *Let $f \in L^2(\mathbb{Q}_p)$ and let h be be an admissible function defined by, for $x = p^{-\gamma}(x_0 + x_1 p^2 + \cdots) \in \mathbb{Q}_p$,*

$$f(x) = f\big(p^{-\gamma}\big), \qquad h(x) = h\big(p^{-\gamma}\big), \tag{9}$$

respectively. Then we have

$$\begin{aligned}(\psi_h f)(a,b) = \frac{1}{\sqrt{c}|a|_p^{\alpha}}\Bigg\{&\left(1-\frac{1}{p}\right)\sum_{\gamma>\gamma_b} f\big(p^{-\gamma}\big)\bar{h}\big(|a|_p p^{-\gamma}\big)p^{\gamma} \\ &+ \left(1-\frac{1}{p}\right)\bar{h}\left(\frac{|a|_p}{|b|_p}\right)\sum_{\gamma<\gamma_b} f\big(p^{-\gamma}\big)p^{\gamma}\end{aligned}$$

$$+\left(1-\frac{1}{p}\right)|b|_p f\left(|b|_p^{-1}\right)\sum_{k=1}^{\infty}\bar{h}\left(\left|\frac{a}{b}\right|_p p^k\right)p^{-k}\Bigg\}$$
$$+\left(1-\frac{2}{p}\right)|b|_p f\left(|b|_p^{-1}\right)\bar{h}\left(\left|\frac{a}{b}\right|_p\right). \tag{10}$$

Proof. We have

$$\begin{aligned}(\psi_h f)(a,b) &= \frac{1}{\sqrt{c}|a|_p^{\alpha}}\int_{\mathbb{Q}_p} f(x)\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x\\ &= \frac{1}{\sqrt{c}|a|_p^{\alpha}}\left(\int_{|x|_p>|b|_p}+\int_{|x|_p<|b|_p}+\int_{|x|_p=|b|_p}\right)f(x)\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x.\end{aligned} \tag{11}$$

For the first part of the above integral, we have

$$\begin{aligned}I_1 &\stackrel{\text{def}}{=} \sum_{\gamma>\gamma_b}\int_{S_\gamma} f\left(|x|_p^{-1}\right)\bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right)\mathrm{d}x\\ &= \left(1-\frac{1}{p}\right)\sum_{\gamma>\gamma_b} f\left(p^{-\gamma}\right)\bar{h}\left(|a|_p p^{-\gamma}\right)p^{\gamma},\end{aligned} \tag{12}$$

where $|b|_p = p^{\gamma_b}$. For the second part of the integral, we have

$$\begin{aligned}I_2 &\stackrel{\text{def}}{=} \sum_{\gamma<\gamma_b}\int_{S_\gamma} f\left(|x|_p^{-1}\right)\bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right)\mathrm{d}x\\ &= \left(1-\frac{1}{p}\right)\bar{h}\left(\frac{|a|_p}{|b|_p}\right)\sum_{\gamma<\gamma_b} f\left(p^{-\gamma}\right)p^{\gamma}.\end{aligned} \tag{13}$$

For the third part of the integral, we have

$$\begin{aligned}I_3 &\stackrel{\text{def}}{=} f\left(|b|_p^{-1}\right)\left(\int_{S_{\gamma_b},x_0\neq b_0}+\int_{S_{\gamma_b},x_0=b_0}\right)\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x\\ &= f\left(|b|_p^{-1}\right)\left\{\int_{S_{\gamma_b},x_0\neq b_0}+\int_{S_{\gamma_b},x_0=b_0,\ x_1\neq b_1}+\int_{S_{\gamma_b},x_0=b_0,\ x_1=b_1}\right\}\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x\\ &\ \ \vdots\\ &= f\left(|b|_p^{-1}\right)\sum_{k=0}^{n}\int_{S_{\gamma_b},x_0=b_0,\ \ldots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k}\bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right)\mathrm{d}x\\ &\quad+\int_{S_{\gamma_b},x_0=b_0,\ \ldots,\ x_{n-1}=b_{n-1},\ x_n=b_n}\bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right)\mathrm{d}x\\ &= f\left(|b|_p^{-1}\right)\sum_{k=1}^{\infty}\bar{h}\left(\frac{|a|_p}{|b|_p}p^k\right)\int_{S_{\gamma_b},x_0=b_0,\ \ldots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k}\mathrm{d}x\\ &\quad+f\left(|b|_p^{-1}\right)\bar{h}\left(\left|\frac{a}{b}\right|_p\right)\int_{S_{\gamma_b},x_0\neq b_0}\mathrm{d}x\\ &= \left(1-\frac{1}{p}\right)|b|_p\sum_{k=1}^{\infty}\bar{h}\left(\left|\frac{a}{b}\right|_p p^k\right)p^{-k}+\left(1-\frac{2}{p}\right)|b|_p f\left(|b|_p^{-1}\right)\bar{h}\left(\left|\frac{a}{b}\right|_p\right).\end{aligned} \tag{14}$$

Substituting (12), (13), and (14) into (11) completes our assertion. □

Theorem 2.4. *Let h be an admissible function and let $f \in L^2(\mathbb{Q}_p)$ be a step function given by, for each $x = p^{-\gamma}(x_0 + x_1 p + \cdots)$,*

$$h(x) \overset{\text{def}}{=} h\big(p^{-\gamma}\big), \qquad f(x) \overset{\text{def}}{=} f\big(kp^{-\gamma}\big), \quad \text{if } x_0 = k,\ 1 \leqslant k \leqslant p-1. \tag{15}$$

Then we have

$$\begin{aligned}(\psi_h f)(a,b) = \frac{1}{\sqrt{c}|a|_p^{\alpha}} \Bigg\{ & \sum_{\gamma > \gamma_b} \bar{h}\big(|a|_p p^{-\gamma}\big) p^{\gamma-1} \sum_{k=1}^{p-1} f\big(kp^{-\gamma}\big) \\ & + \bar{h}\left(\frac{|a|_p}{|b|_p}\right) \sum_{\gamma < \gamma_b} p^{\gamma-1} \sum_{k=1}^{p-1} f\big(kp^{-\gamma}\big) \\ & + \left(1 - \frac{1}{p}\right) |b|_p f\big(b_0 |b|_p^{-1}\big) \sum_{k=1}^{\infty} \bar{h}\left(\left|\frac{a}{b}\right|_p \cdot p^k\right) p^{-k} \\ & + p^{\gamma_b - 1} \bar{h}\left(\left|\frac{a}{b}\right|_p\right) \sum_{k=1,\ k \neq b_0}^{p-1} f\big(k|b|_p^{-1}\big) \Bigg\}. \end{aligned} \tag{16}$$

Proof. We have

$$\begin{aligned}(\psi_h f)(a,b) &= \frac{1}{\sqrt{c}|a|_p^{\alpha}} \int_{\mathbb{Q}_p} f(x) \bar{h}\left(\frac{x-b}{a}\right) \mathrm{d}x \\ &= \frac{1}{\sqrt{c}|a|_p^{\alpha}} \left(\int_{|x|_p > |b|_p} + \int_{|x|_p < |b|_p} + \int_{|x|_p = |b|_p} \right) f(x) \bar{h}\left(\frac{x-b}{a}\right) \mathrm{d}x \\ &\overset{\text{def}}{=} \frac{1}{\sqrt{c}|a|_p^{\alpha}} (I_1 + I_2 + I_3). \end{aligned} \tag{17}$$

For I_1, we have

$$\begin{aligned} I_1 &= \int_{|x|_p > |b|_p} f(x) \bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right) \mathrm{d}x = \int_{|x|_p > |b|_p} f(x) \bar{h}\left(\frac{|a|_p}{|x|_p}\right) \mathrm{d}x \\ &= \sum_{\gamma > \gamma_b} \bar{h}\left(\frac{|a|_p}{p^{\gamma}}\right) \int_{S_\gamma} f(x)\, \mathrm{d}x = \sum_{\gamma > \gamma_b} \bar{h}\left(\frac{|a|_p}{p^{\gamma}}\right) \sum_{k=1}^{p-1} \int_{S_\gamma, x_0 = k} f(x)\, \mathrm{d}x \\ &= \sum_{\gamma > \gamma_b} \bar{h}\left(\frac{|a|_p}{p^{\gamma}}\right) \sum_{k=1}^{p-1} f\big(kp^{-\gamma}\big) \int_{S_\gamma, x_0 = k} \mathrm{d}x \\ &= \sum_{\gamma > \gamma_b} \bar{h}\left(\frac{|a|_p}{p^{\gamma}}\right) p^{\gamma-1} \sum_{k=1}^{p-1} f\big(kp^{-\gamma}\big). \end{aligned} \tag{18}$$

For I_2, we have

$$\begin{aligned} I_2 &= \int_{|x|_p < |b|_p} f(x) \bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right) \mathrm{d}x = \bar{h}\left(\frac{|a|_p}{|b|_p}\right) \int_{|x|_p < |b|_p} f(x)\, \mathrm{d}x \\ &= \bar{h}\left(\frac{|a|_p}{|b|_p}\right) \sum_{\gamma < \gamma_b} \int_{S_\gamma} f(x)\, \mathrm{d}x = \bar{h}\left(\frac{|a|_p}{|b|_p}\right) \sum_{\gamma < \gamma_b} \sum_{k=1}^{p-1} f\big(kp^{-\gamma}\big) \int_{S_\gamma, x_0 = k} \mathrm{d}x \\ &= \bar{h}\left(\frac{|a|_p}{|b|_p}\right) \sum_{\gamma < \gamma_b} p^{\gamma-1} \sum_{k=1}^{p-1} f\big(kp^{-\gamma}\big). \end{aligned} \tag{19}$$

For I_3, we have

$$\begin{aligned}
I_3 &= \sum_{k=0}^{n} \int\limits_{S_{\gamma_b}, x_0=b_0,\ \ldots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k} f(x)\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x \\
&\quad + \int\limits_{S_{\gamma_b}, x_0=b_0,\ \ldots,\ x_{n-1}=b_{n-1},\ x_n=b_n} f(x)\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \int\limits_{S_{\gamma_b}, x_0=b_0,\ \ldots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k} f(x)\bar{h}\left(\frac{x-b}{a}\right)\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \int\limits_{S_{\gamma_b}, x_0=b_0,\ \cdots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k} f(x)\bar{h}\left(\left|\frac{x-b}{a}\right|_p^{-1}\right)\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \int\limits_{S_{\gamma_b}, x_0=b_0,\ \cdots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k} f(x)\bar{h}\left(\frac{|a|_p}{|p^{-\gamma_b}[(x_k-b_k)p^k+\cdots]|_p}\right)\mathrm{d}x \\
&= \sum_{k=1}^{\infty} \bar{h}\left(\left|\frac{a}{b}\right|_p \cdot p^k\right) \int\limits_{S_{\gamma_b}, x_0=b_0,\ \ldots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k} f(x)\,\mathrm{d}x \\
&\quad + \bar{h}\left(\left|\frac{a}{b}\right|_p\right) \int\limits_{S_{\gamma_b}, x_0\neq b_0} f(x)\,\mathrm{d}x \\
&= f\left(b_0|b|_p^{-1}\right) \sum_{k=1}^{\infty} \bar{h}\left(\left|\frac{a}{b}\right|_p \cdot p^k\right) \int\limits_{S_{\gamma_b}, x_0=b_0,\ \ldots,\ x_{k-1}=b_{k-1},\ x_k\neq b_k} \mathrm{d}x \\
&\quad + \bar{h}\left(\left|\frac{a}{b}\right|_p\right) \int\limits_{S_{\gamma_b}, x_0\neq b_0} f(x)\,\mathrm{d}x \\
&= \left(1-\frac{1}{p}\right)|b|_p f\left(b_0|b|_p^{-1}\right) \sum_{k=1}^{\infty} \bar{h}\left(\left|\frac{a}{b}\right|_p \cdot p^k\right) p^{-k} \\
&\quad + p^{\gamma_b-1}\bar{h}\left(\left|\frac{a}{b}\right|_p\right) \sum_{k=1,\ k\neq b_0}^{p-1} f\left(k|b|_p^{-1}\right).
\end{aligned} \tag{20}$$

Substituting (18), (19), and (20) into (17) completes our assertion. □

Changing variables by $(x-b)/a = x'$ in (2), we have

$$\int\limits_{\mathbb{Q}_p} f\left(\frac{x-(-b/a)}{1/a}\right)\bar{h}(x)|a|_p\,\mathrm{d}x,$$

where let $\bar{h}(x)$ be a function such that

$$\bar{h}(x) = \bar{h}\left(kp^{-\gamma}\right), \qquad \text{if } x \in S_\gamma,\ x_0 = k$$

and let $f(x) = f(|x|_p^{-1})$, then we have the following theorem by virtue of Theorem 2.4.

Theorem 2.5. *We have*

$$\begin{aligned}
(\psi_h f)(a,b) = \frac{|a|_p^\alpha}{\sqrt{c}} \Bigg\{ &\sum_{\gamma>\gamma_{b/a}} f\left(|a|_p^{-1}p^{-\gamma}\right)p^{\gamma-1}\sum_{k=1}^{p-1}\bar{h}\left(kp^{-\gamma}\right) \\
&+ f\left(\frac{1}{|b|_p}\right)\sum_{\gamma<\gamma_{b/a}} p^{\gamma-1}\sum_{k=1}^{p-1}\bar{h}\left(kp^{-\gamma}\right)
\end{aligned}$$

$$+\left(1-\frac{1}{p}\right)\left|\frac{b}{a}\right|_p \bar{h}\left(\alpha_0\left|\frac{a}{b}\right|_p\right)\sum_{k=1}^{\infty} f\left(\frac{p^k}{|b|_p}\right)p^{-k}$$
$$\left.+\, p^{\gamma_{b/a}-1} f\big(|b|_p\big)\sum_{k=1,\ k\neq b_0}^{p-1}\bar{h}\big(k|b|_p^{-1}\big)\right\}, \tag{21}$$

where α_0 *denotes integer such that* $a/b = p^{-\gamma_{a/b}}(\alpha_0+\alpha_1 p+\cdots)$ *and* $|a/b|_p = p^{\gamma_{a/b}}$.

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