Groups with Many FC-Subgroups

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Communicated by Gernot Stroth

Received February 2, 1998

DEDICATED TO DEREK J. S. ROBINSON ON THE OCCASION OF HIS 60TH BIRTHDAY

1. INTRODUCTION

A group $G$ is called an FC-group if every element $x$ of $G$ has only finitely many conjugates in $G$, that is, if the centralizer $C_G(x)$ has finite index in $G$. There exists a wide literature on this subject, and the monograph [18] can be used as a general reference. In the last few years many authors have studied the structure of minimal-non-FC groups, i.e., those groups which are not FC-groups while all their proper subgroups have the property FC (see, for instance, [2, 4, 5, 10] and the last section of [18]). Clearly Tarski groups are minimal-non-FC, and hence in this investigation it is necessary to impose some additional condition in order to avoid such groups. In the above mentioned articles, it has been proved that minimal-non-FC groups having proper commutator subgroup are Černikov groups, and that every perfect locally graded minimal-non-FC group is a

* This research was done while the last author was a visiting professor at the Università di Napoli “Federico II” supported by the “Istituto Nazionale di Alta Matematica.” He is grateful to the Department of Mathematics for its excellent hospitality.
countable $p$-group for some prime number $p$. Our aim here is to extend such results, considering groups that are rich in some sense of subgroups which are $FC$-groups.

Bruno and Phillips have proved in [6] that if a locally graded group $G$ satisfies the minimal condition on subgroups which have infinite commutator subgroup, then either $G$ is a Černikov group or its commutator subgroup $G'$ is finite (recall that a group $G$ is locally graded if every finitely generated non-trivial subgroup of $G$ contains a proper subgroup of finite index). On the other hand, it is well known that a group has finite commutator subgroup if and only if it has boundedly finite conjugacy classes (see [14, Part I, Theorem 4.35]), and hence the result of Bruno and Phillips suggests considering the minimal condition on non-$FC$ subgroups. This is the subject of the second section of this article, where it will be proved that a locally graded group with the minimal condition on non-$FC$ subgroups has an ascending normal series whose factors either are abelian or direct products of finite simple non-abelian groups; in particular, a simple locally graded group with this property must be finite. Moreover, if a group satisfies the minimal condition on non-$FC$ subgroups and has a descending series with finite or abelian factors, then it is either an $FC$-group or a Černikov group. It turns out that locally graded groups with finitely many conjugacy classes of non-$FC$ subgroups satisfy both the minimal and the maximal condition on non-$FC$ subgroups, and this fact will be used in the last section of the article to give a characterization of locally graded groups with such a property. It follows in particular that a locally graded group with finitely many conjugacy classes of non-$FC$ subgroups has only finitely many subgroups which are not $FC$-groups.

Most of our notation is standard and can be found in [14]. For the main properties of $FC$-groups we refer the reader to the monograph [18].

2. THE MINIMAL CONDITION ON NON-$FC$ SUBGROUPS

In order to study groups satisfying the minimal condition on non-$FC$ subgroups, we have first to consider the special case of groups whose proper subgroups either are $FC$-groups or Černikov groups.

**Lemma 2.1.** Let $G$ be a group having a descending series whose factors either are finite or abelian. If every proper subgroup of $G$ either is an $FC$-group or a Černikov group, then the group $G$ itself either is an $FC$-group or a Černikov group.

**Proof.** Assume that the statement is false. Then by a result of Belyaev and Sesekin [5] the group $G$ must contain a proper subgroup $H$ which is
not an $FC$-group, so that $H$ is a Černikov group, and hence it can be chosen minimal with respect to this condition. Clearly every proper subgroup of $G$ containing $H$ is a Černikov group. By hypothesis there exists a proper normal subgroup $N$ of $G$ such that $G/N$ either is finite or abelian. If $HN \neq G$, then $HN$ is a Černikov group and so $G/N$ cannot be finite. Thus $G/N$ is abelian and $HN$ is a normal subgroup of $G$. As the factor group $G/HN$ satisfies the minimal condition on subgroups, it follows that $G$ is a Černikov group. This contradiction shows that $G = HN$, so that $N$ is a non-Černikov $FC$-group. Let $J$ be the finite residual of the Černikov group $H$, and assume that $G = JN$. Then $H = J(N \cap H)$, and $N \cap H$ is a normal subgroup of $H$, so that $N \cap H$ is infinite as $H$ is not an $FC$-group. It follows that $N \cap J$ is also infinite. On the other hand, $H$ is minimal-$non-FC$, and hence $J$ does not contain infinite proper $H$-invariant subgroups (see [5]), so that $N \cap J = J$ and $J$ is a subgroup of $N$. This contradiction shows that $JN$ is a proper normal subgroup of $G$, and by replacing $N$ by $JN$ we may suppose that $J$ is contained in $N$. Let $E$ be a finite subgroup of $H$ such that $H = EJ$, so that $G = EN$. If $K$ is any $E$-invariant subgroup of $N$ such that $G = EK$, then $N = K(E \cap N)$ and the index $|N : K|$ is bounded by the order of $E$, so that there exists an $E$-invariant subgroup $L$ of $N$ which is minimal with respect to the condition $G = EL$. Assume that $L$ contains a proper subgroup of finite index $M$. Then also the core $M_G$ of $M$ has finite index in $G$, so that $EM_G$ is a proper subgroup of $G$ containing $J$, and hence $H$ is also contained in $EM_G$. It follows that $EM_G$ is a Černikov group, and $G$ itself has this property. This contradiction shows that the $FC$-group $L$ has no proper subgroups of finite index, and hence it is a radicable abelian group. Assume that $L$ contains an element $x$ of infinite order. Then $\langle x \rangle^E$ is a finitely generated normal subgroup of $G$, and $H\langle x \rangle^E$ is neither an $FC$-group nor a Černikov group, so that $G = H\langle x \rangle^E$. It follows that $L = \langle x \rangle^E(H \cap L)$, and $L/H \cap L$ is a non-trivial finitely generated abelian group, a contradiction. Thus $L$ must be periodic, and its socle $S$ is an infinite normal subgroup of $G$. Then $HS$ is not a Černikov group, so that $G = HS$ and $L = S(H \cap L)$. Therefore also in this case $L$ has a finite non-trivial homomorphic image, and this last contradiction completes the proof of the lemma.

**Theorem 2.2.** Let $G$ be a group satisfying the minimal condition on non-$FC$ subgroups. If $G$ has a descending series whose factors either are finite or abelian, then either $G$ is an $FC$-group or it is a Černikov group.

**Proof.** Assume that the theorem is false, so that the set $\mathcal{L}$ of all subgroups of $G$ which are neither $FC$-groups nor Černikov groups is not empty. Then $\mathcal{L}$ contains a minimal element $L$, and obviously every proper
subgroup of $L$ is either an FC-group or a Černikov group, contradicting Lemma 2.1. The theorem is proved.

It is an open question whether there exist perfect locally graded minimal-non-FC groups. Since such groups obviously satisfy the minimal condition on non-FC subgroups, a positive solution to this problem would also prove that in Theorem 2.2 the hypothesis that the group $G$ has a descending series with finite or abelian factors could not be weakened assuming that $G$ is locally graded. On the other hand, information on the structure of locally graded groups satisfying the minimal condition on non-FC subgroups can be obtained.

**Lemma 2.3.** Let $G$ be a locally graded group satisfying the minimal condition on non-FC subgroups. Then $G$ is locally (finite-by-abelian).

*Proof.* Let $E$ be any infinite finitely generated subgroup of $G$. Since $G$ is locally graded, there exists an infinite strictly descending sequence

$$E_1 > E_2 > \cdots > E_n > \cdots$$

of subgroups of finite index of $E$. Thus the subgroup $E_n$ is an FC-group for some positive integer $n$, so that $E_n/Z(E_n)$ is finite and $E$ is abelian-by-finite. In particular, $E$ satisfies the maximal condition on subgroups, so that all its FC-subgroups are central-by-finite. It follows that $E$ satisfies the minimal condition on subgroups which are not finite-by-abelian, and hence either $E$ is finite-by-abelian or it is a Černikov group (see [6, Theorem 1]). Then $E$ must be finite-by-abelian, and so $G$ is locally (finite-by-abelian).

**Theorem 2.4.** Let $G$ be a non-periodic locally graded group satisfying the minimal condition on non-FC subgroups. Then $G$ is an FC group.

*Proof.* The group $G$ is locally (finite-by-abelian) by Lemma 2.3, so that in particular its commutator subgroup is locally finite and the set $T$ of all elements of finite order of $G$ is a subgroup. If $x$ is an element of infinite order of $G$, the chain of subgroups

$$\langle T, x \rangle > \langle T, x^2 \rangle > \cdots > \langle T, x^n \rangle > \cdots$$

is infinite, and hence there exists a positive integer $n$ such that $\langle T, x^n \rangle$ is an FC-group. Assume that $G$ is not an FC-group. Then it contains a subgroup $H$ which is minimal non-FC, and $H$ must be locally finite (see [18, Lemma 8.14]). It follows that $H$ is contained in $T$, and this contradiction proves that $G$ is an FC-group.

It was mentioned in the Introduction that every locally graded minimal non-FC group is locally finite. As a consequence of Lemma 2.3 and
Theorem 2.4 it can be observed here that if a locally graded group $G$ satisfies the minimal condition on non-$FC$ subgroups, then either $G$ is an $FC$-group or it is locally finite.

**Lemma 2.5.** Let $G$ be a linear group over a field. If $G$ satisfies the minimal condition on non-$FC$ subgroups, then either $G$ is finite-by-abelian or it is a Černikov group.

**Proof.** It is well known that every linear group over a field is locally graded. Moreover, all linear $FC$-groups are central-by-finite (see [19, Corollary 5.6]) and so also finite-by-abelian. Therefore the group $G$ satisfies the minimal condition on subgroups that are not finite-by-abelian, and hence either $G$ itself is finite-by-abelian or it is a Černikov group (see [6, Theorem 1]).

It has been proved by Kuzucuoglu and Phillips [10] that a locally finite minimal non-$FC$-group cannot be simple. Our next results show in particular that every simple locally graded group satisfying the minimal condition on non-$FC$ subgroups is finite.

**Lemma 2.6.** Let $G$ be a locally graded group satisfying the minimal condition on non-$FC$ subgroups. Then every simple section of $G$ is finite.

**Proof.** Suppose first that $G$ is not periodic. Then $G$ is an $FC$-group by Theorem 2.4, and hence all its simple sections are finite. Assume now that $G$ is periodic, so that it follows from Lemma 2.3 that $G$ is locally finite. By contradiction let $H/K$ be an infinite simple section of $G$. Then $H$ is not an $FC$-group, and so it can be chosen minimal with respect to this condition. Since every infinite simple group contains countably infinite simple subgroups (see, for instance, [9, p. 114]), we obtain that $H/K$ must be countable. Moreover, $H/K$ obviously satisfies the minimal condition on non-$FC$ subgroups, and hence without loss of generality it can be assumed that $G$ is a countably infinite simple group and that all proper simple sections of $G$ are finite. There exists an ascending chain

$$H_1 < H_2 < \cdots < H_n < \cdots$$

of finite perfect subgroups of $G$ such that

$$G = \bigcup_{n \in \mathbb{N}} H_n,$$

and for each $n > 1$ there exists a maximal normal subgroup $M_n$ of $H_n$ such that $H_{n-1} \cap M_n = 1$ (see [13, Theorem 1] and the remark at the end of Section 1). Put $K_i = \langle M_i | i > t \rangle$ and $R_i = \langle H_i, K_i \rangle$ for every positive integer $t$. Since $H_i \cap K_i = 1$ for all $t$, the subgroup $R_i$ is residually finite,
and hence it is an FC-group by Theorem 2.2. If $M = K_r$, we obtain that the subgroup $[M, H_r]$ is finite. With this notation, the last part of the proof of Lemma 1 of [10] can be used to produce an ascending chain

$$G_1 < G_2 < \cdots < G_n < \cdots$$

of finite perfect subgroups of $G$ such that

$$G = \bigcup_{n \in \mathbb{N}} G_n,$$

the factor group $G_n/Z(G_n)$ is simple and $G_n \cap Z(G_{n+1}) = 1$ for every $n \geq 1$. Application of Lemma 2 of [10] yields now that either $G$ is linear or it contains an element $g \neq 1$ such that $C_G(g)$ has a section which is simple and non-linear. On the other hand, all proper simple sections of $G$ are finite, so that $G$ must be linear, and hence it is finite by Lemma 2.5. This contradiction completes the proof of the lemma.

**Theorem 2.7.** Let $G$ be a locally graded group satisfying the minimal condition on non-FC subgroups. Then $G$ has an ascending normal series whose factors either are abelian or direct products of finite simple non-abelian groups.

**Proof.** If $G$ is not periodic, then it is an FC-group by Theorem 2.4, so that its chief factors are finite (see [18, Theorem 1.13]), and in this case the statement is obvious. Suppose now that $G$ is periodic, and hence locally finite by Lemma 2.3. Since the hypotheses are inherited by homomorphic images, it is enough to prove that $G$ contains a non-trivial normal subgroup which is either abelian or a direct product of finite non-abelian simple groups. Suppose first that $G$ has a non-trivial normal FC-subgroup $N$. Since every minimal normal subgroup of $N$ is finite, the socle $S$ of $N$ has a direct decomposition $S = S_1 \times S_2$, where $S_1$ is abelian and $S_2$ is a direct product of finite simple non-abelian groups. Moreover, $S_1$ and $S_2$ are characteristic subgroups of $N$, and hence they are normal in $G$. Assume now that $G$ has no non-trivial normal FC-subgroups, so that the set of non-trivial normal subgroups of $G$ satisfies the minimal condition and $G$ contains a minimal normal subgroup $M$. It follows from Lemma 2.6 that the group $M$ has a proper non-trivial normal subgroup $K$. Assume that $K$ is not an FC-group. Then $M/K$ satisfies the minimal condition on subgroups, and hence it is a Cernikov group (see [17]). Since $M$ is perfect, it follows that $M$ has a proper subgroup of finite index, and hence it is residually finite. Then $M$ is an FC-group of Theorem 2.2, and this contradiction shows that $K$ must be an FC-group, so that it contains a minimal subnormal subgroup $E$. If $E$ is abelian, then $M = E^G$ is locally nilpotent and so even abelian (see [9, p. 11]), a contradiction. Therefore $E$
is a finite non-abelian simple subnormal subgroup of $G$, and hence $M = E^G$ is a direct product of finite non-abelian simple groups (see [14, Part 1, Lemma 5.44]). The theorem is proved.

3. CONJUGACY CLASSES OF NON-FC SUBGROUPS

Another natural interpretation of the requirement that the group $G$ has only few subgroups which do not have a certain property $\chi$ is the condition that $G$ has only finitely many conjugacy classes of non-$\chi$ subgroups. This point of view was, for instance, adopted in [15, 16], where groups with finitely many conjugacy classes of non-soluble subgroups and those with the same property for non-nilpotent subgroups were considered.

**Lemma 3.1.** Let $G$ be a group having finitely many conjugacy classes of non-FC subgroups. Then:

(a) Every homomorphic image of $G$ has finitely many conjugacy classes of non-FC subgroups.

(b) Every subgroup of finite index of $G$ has finitely many conjugacy classes of non-FC subgroups.

**Proof.** The first part of the statement is obvious. To prove (b), it is enough to observe that, if $K$ is a subgroup of finite index of $G$ and $H$ is any subgroup of $K$, then the conjugacy class of $H$ in $G$ contains only finitely many conjugacy classes of subgroups under the action of $K$.

**Lemma 3.2.** Let $G$ be an infinite finitely generated soluble-by-finite residually finite minimax group. Then $G$ contains a torsion-free subgroup $H$ of finite index such that $H/H'$ is infinite.

**Proof.** It is well known that the Fitting subgroup $F$ of $G$ is nilpotent and the factor group $G/F$ is abelian-by-finite (see [14, Part 2, Theorem 10.33]). Let $N/F$ be a torsion-free abelian normal subgroup of $G/F$ such that $G/N$ is finite. Since the subgroup $T$ consisting of all elements of finite order of $F$ is finite, there exists a normal subgroup of finite index $K$ of $N$ such that $K \cap T = 1$. Then $K$ is a torsion-free soluble subgroup of finite index of $G$. Let $i$ be the largest positive integer such that the $i$th term $H = K^{(i)}$ of the derived series of $K$ has finite index in $K$. Then $H$ is a torsion-free subgroup of finite index of $G$ and $H/H'$ is infinite.

It was proved in [16] that every locally graded group with finitely many conjugacy classes of non-nilpotent subgroups is locally (soluble-by-finite). This is a consequence of our next result. Recall here that, if $G$ is a soluble-by-finite minimax group, the set of all prime numbers $p$ such that $G$ has a section of type $p^*$ is an invariant of $G$, called the spectrum of $G$. 
Proposition 3.3. Let $\mathcal{X}$ be a subgroup closed class of groups, and let $G$ be a locally graded group having finitely many conjugacy classes of non-$\mathcal{X}$ subgroups. Then $G$ is locally $(\mathcal{X}$-by-finite).

Proof. Assume by contradiction that $G$ contains a finitely generated subgroup $E$ which is not $\mathcal{X}$-by-finite, and let $J$ be the finite residual of $E$. Then $E/J$ is an infinite residually finite group. Since $G$ has finitely many conjugacy classes of non-$\mathcal{X}$ subgroups, there exists a positive integer $r$ such that every finitely generated subgroup of $G$ which is not in $\mathcal{X}$ can be generated by at most $r$ elements. In particular, each subgroup of finite index of $E$ can be generated by at most $r$ elements, and so $E/J$ is a soluble-by-finite minimax group (see [11, Theorem A]). It follows from Lemma 3.2 that $E/J$ contains a torsion-free subgroup of finite index $H/J$ such that $H/H'$ is infinite. Clearly the group $H/H'$ is residually finite, so that $J$ is contained in $H'$. Thus, if $p$ is any prime number, we have $JH^p < JH'^p$ if $m < n$. Let $\pi$ be the spectrum of the minimax group $H/J$.

For every prime $p$ which is not in $\pi$ and for every positive integer $n$, the subgroup $JH^p$ has finite index in $E$ and hence it is not an $\mathcal{X}$-group. Therefore there exist positive integers $m$ and $n$ such that $m < n$ and $(JH^p)^g = H'^p$ for some $g \in G$. Clearly $J$ is the finite residual of $JH^p$, so that $J^g$ is the finite residual of $JH'^p$, and hence $J^g = J$. Moreover $X = JH^p$ is a characteristic subgroup of $X^g$, and $X^g/X$ is a finite $p$-group. Let $X_p$ be the union of the ascending chain.

$$X < X^g < \cdots < X^{g^n} < \cdots.$$ 

Then every $X^{g^n}$ is a normal subgroup of $X_p$, and $X_p/X$ is an infinite Chernikov $p$-group. It follows that $X_p/J$ is a torsion-free soluble-by-finite minimax group with spectrum $\pi \cup \{p\}$, and $J$ is the finite residual of $X_p$.

By hypothesis there exist two different primes $p$ and $q$ which do not belong to $\pi$ such that $(X_p)^h = X_q$ for some element $h$ of $G$. Then $J^h = J$, and hence the groups $X_p/J$ and $X_q/J$ are isomorphic. This contradiction completes the proof of the proposition. 

Groups satisfying the minimal condition on non-$FC$ subgroups are naturally involved in the investigations concerning groups with a finite number of conjugacy classes of non-$FC$ subgroups. This is a consequence of the following result of Zaicev (see [1, Lemma 4.6.3]) and a related corollary.

Lemma 3.4. Let $G$ be a group locally satisfying the maximal condition on subgroups. If $H$ is a subgroup of $G$ such that $H^x \leq H$ for some element $x$ of $G$, then $H^x = H$.

Corollary 3.5. Let $G$ be a locally graded group having finitely many conjugacy classes of non-$FC$ subgroups. Then $G$ satisfies both the minimal and
the maximal condition on non-FC subgroups. In particular, if \( G \) is not periodic, then it is an FC-group.

**Proof.** Since every finitely generated FC-group is polycyclic-by-finite, it follows from Proposition 3.3 that the group \( G \) is locally polycyclic-by-finite. Therefore \( G \) satisfies both the minimal and the maximal condition on non-FC subgroups by Lemma 3.4. Finally, if \( G \) is not periodic, it follows from Theorem 2.4 that \( G \) is an FC-group.

The main result of this section gives a characterization of locally graded groups with finitely many conjugacy classes of non-FC subgroups. To prove this, we need a series of lemmas, most of which are designed to produce infinitely many conjugacy classes of complements of particular subgroups.

**Lemma 3.6.** Let \( G \) be an FC-group, and let \( E \) be a finite subgroup of \( G \). Then \( E \) is contained in a characteristic subgroup \( K \) of \( G \) with finite exponent.

**Proof.** The normal closure \( E^G \) of \( E \) is finite by Dic'man's lemma, so that without loss of generality it can be assumed that \( E \) is a finite normal subgroup of \( G \). Then \( E \) contains a minimal normal subgroup of \( G \), and hence \( E \cap S \neq 1 \), where \( S \) is the socle of \( G \). Since \( S \) is a direct product of finite simple groups, the subgroup \( E \cap S \) is a direct factor of \( S \), and it is also a direct product of finite simple groups (see [14, Part 1, Theorem 5.45]). It follows that \( E \cap S \) is contained in a characteristic subgroup of finite exponent \( L \) of \( S \). By induction on the order of \( E \) we may suppose that \( EL/L \) is contained in a characteristic subgroup of finite exponent \( K/L \) of \( G/L \), and hence \( K \) is a characteristic subgroup of finite exponent of \( G \) containing \( E \).

Recall here that if \( G \) is a locally finite group and \( p \) is a prime number, then a Sylow \( p \)-subgroup of \( G \) is a maximal element of the set of all \( p \)-subgroups of \( G \). It is not difficult to prove that the following holds.

**Lemma 3.7.** Let the group \( G = \bigoplus_{n \in \mathbb{N}} G_n \) be the direct product of infinitely many non-abelian groups.

(a) If every factor \( G_n \) is a semidirect non-direct product \( G_n = B_n \ltimes A_n \), then the complements of \( A = \bigoplus_{n \in \mathbb{N}} A_n \) in \( G \) fall into infinitely many conjugacy classes.

(b) If every factor \( G_n \) is a finite simple group whose order is divisible by a prime \( p \), then the Sylow \( p \)-subgroups of \( G \) fall into infinitely many conjugacy classes.

**Lemma 3.8.** Let \( \pi \) be a set of primes, and let the periodic group \( G = H \ltimes K \) be the semidirect product of a \( \pi \)-subgroup \( H \) and a normal \( \pi' \)-subgroup \( K \) such that every proper subgroup of \( G \) containing \( K \) is an
FC-group and $H$ has no maximal subgroups. If $L$ is a subgroup of $G$ containing $K$ and $X$ is a complement of $K$ in $L$, then there exists a complement of $K$ in $G$ containing $X$.

Proof. Clearly it can be assumed that $L$ is a proper subgroup of $G$, so that it is an FC-group. Since $L = K(H \cap L) = KX$, there exists a locally inner automorphism $\alpha$ of $L$ such that $(H \cap L)\alpha = X$ (see [18, Theorem 5.25]). Moreover, it follows from the hypotheses that $H$ is the union of an ascending chain $(H_i)_{i \in I}$ of proper subgroups containing $H \cap L$. Then each product $KH_i$ is an FC-group and contains $L$, so that for every $i \in I$ there exists a locally inner automorphism $\beta_i$ of $KH_i$ such that $\beta_i(x) = \alpha(x)$ for all $x \in L$ and $\beta_i(g) = \beta_i(g)$ if $g \in KH_j$ and $H_j \leq H_i$ (see [18, Theorem 4.18(i)]). Since

$$G = \bigcup_{i \in I} KH_i,$$

for each element $g$ of $G$ there exists $i \in I$ such that $g \in KH_i$, and the position $\beta(g) = \beta(g)$ defines an automorphism $\beta$ of $G$ extending $\alpha$. Therefore $H^\beta$ is a complement of $K^\beta = K$ in $G$ containing $X$. $lacksquare$

The following elementary result on direct products of simple non-abelian groups is probably well known.

**Lemma 3.9.** Let $G$ be a group, and let $K$ be a normal subgroup of $G$ which is a direct product of simple non-abelian groups. Then $K = C_K(H \times K)$ for every subgroup $H$ of $G$.

**Lemma 3.10.** Let the periodic FC-group $G = H \times K$ be the semidirect product of an elementary abelian $p$-subgroup $H$ and a normal $p'$-subgroup $K$ which either is abelian or a direct product of finite simple non-abelian groups. If $H/C_H(K)$ is infinite, then the complements of $K$ in $G$ fall into infinitely many conjugacy classes.

Proof. The core $H_C$ of $H$ in $G$ is obviously contained in every complement of $K$ in $G$, so that without loss of generality it can be assumed that $H_C = 1$, and in particular $C_H(K) = 1$. Consider an element $y_1 \neq 1$ of $H$, and put $L_1 = \langle y_1 \rangle$. Assume now that for some integer $n > 1$ we have defined a subgroup

$$L_{n-1} = \langle y_1 \rangle \times \cdots \times \langle y_{n-1} \rangle$$

of $H$ of order $p^{n-1}$, such that

$$[K, L_{n-1}] = [K, \langle y_1 \rangle] \times \cdots \times [K, \langle y_{n-1} \rangle].$$
As $G$ is a periodic $FC$-group, $[K, L_{n-1}]$ is a finite normal subgroup of $G$, so that $H/C_H([K, L_{n-1}])$ is finite, and $C_H([K, L_{n-1}]) \neq 1$. Let $y_n$ be a non-trivial element of $C_H([K, L_{n-1}])$, and let $L_n = L_{n-1} \times \langle y_n \rangle$. Suppose first that $K$ is abelian. Then

$$K = C_K(L_{n-1}) \times [K, L_{n-1}]$$

(see [7, Theorem 5.2.3]), so that

$$[K, \langle y_n \rangle] = [C_K(L_{n-1}), \langle y_n \rangle] \leq C_K(L_{n-1}),$$

and hence

$$[K, L_n] = [K, L_{n-1}] \times [K, \langle y_n \rangle] = [K, \langle y_1 \rangle] \times \cdots \times [K, \langle y_n \rangle].$$

Suppose now that

$$K = \bigoplus_{i \in I} K_i$$

is a direct product of finite simple non-abelian groups. Then there exists a subset $J$ of $I$ such that

$$[K, L_{n-1}] = \bigoplus_{j \in J} K_j,$$

and hence $K = [K, L_{n-1}] \times N$, where

$$N = \bigoplus_{i \in I \setminus J} K_i$$

is a normal subgroup of $G$ (see [14, Part 1, Lemma 5.44]). Therefore

$$[K, \langle y_n \rangle] = [N, \langle y_n \rangle] \leq N,$$

and hence

$$[K, L_{n-1}] \cap [K, \langle y_n \rangle] = 1.$$
and hence

\[ [K, L]L = \bigcap_{n \in \mathbb{N}} [K, \langle y_n \rangle] \langle y_n \rangle. \]

As \( H \) does not contain non-trivial normal subgroups of \( G \), each subgroup \( \langle y_n \rangle^G = [K, \langle y_n \rangle] \langle y_n \rangle \) is not abelian, and it follows from Lemma 3.7 that there exists an infinite sequence \((X_n)_{n \in \mathbb{N}}\) of pairwise non-conjugate complements of \([K, L]L\) in \([K, L]L\). In particular, every \( X_n \) is also a complement of \( K \) in \( KL \). Let \( n \) be any positive integer, and let \( P_n \) be a Sylow \( p \)-subgroup of \( G \) containing \( X_n \). Since the Sylow \( p \)-subgroups of \( G \) are locally conjugate (see [18, Theorem 5.2]) and \( G = HK \), we obtain that \( P_n \) is a complement of \( K \) in \( G \). Assume that \( P_n^g = P_m \) for some element \( g \) of \( K \). The subgroup \([K, L]L\) is normal in \( G \), and

\[ P_i \cap [K, L]L = X_i \]

for all \( i \), so that \( X_i^g = X_n \). It follows from Lemma 5.2.3 of [7] or from Lemma 3.9 that \( g = uv \), where \( u \in C_K(X_n) \) and \( v \in [K, X_n] \). Thus

\[ X_m = X_n^g = X_n^{uv} = X_n^v, \]

and hence the subgroups \( X_n \) and \( X_m \) are conjugate in \([K, L]L\). This contradiction shows that the subgroups \( P_n \) (\( n \in \mathbb{N} \)) are pairwise non-conjugate, and completes the proof of the lemma. \[ \blacksquare \]

**Lemma 3.11.** Let the locally finite group \( G = HK \) be the product of a countable \( p \)-subgroup \( H \) and a normal subgroup \( K \) such that every proper subgroup of \( G \) containing \( K \) is an FC-group. If \( P \) is any Sylow \( p \)-subgroup of \( K \), there exists a Sylow \( p \)-subgroup \( P^* \) of \( G \) such that \( P^* \cap K = P \) and \( G = P^*K \).

**Proof.** Since the factor group \( G/K \) is countable, there exists an ascending chain

\[ G_1 < G_2 < \cdots < G_n < \cdots \]

of proper subgroups of \( G \) containing \( K \) such that

\[ G = \bigcup_{n \in \mathbb{N}} G_n. \]

For each positive integer \( n \) let \( P_n \) be a Sylow \( p \)-subgroup of \( G_n \), such that

\[ P \leq P_n \leq P_{n+1}. \]
Then the set
\[ P^* = \bigcup_{n \in \mathbb{N}} P_n \]
is a Sylow $p$-subgroup of $G$ such that $P^* \cap K = P$. For every positive integer $n$, the group $G_n$ is FC, and so its Sylow $p$-subgroups are locally conjugate (see [18, Theorem 5.2]), and so there exists a locally inner automorphism $\varphi$ of $G_n$ such that $H \cap G_n$ is contained in $P_n^\varphi$. Therefore
\[ G_n = (H \cap G_n)K = P_n^\varphi K, \]
so that $G_n = P_nK$ and hence $G = P^*K$.

Our next lemma produces an infinite number of conjugacy classes of $p$-subgroups. This will be used in the study of groups with finitely many conjugacy classes of non-FC subgroups.

**Lemma 3.12.** Let $G$ be a perfect locally finite group satisfying both the minimal and the maximal condition on non-FC subgroups and having no proper subgroups of finite index, and let $H$ be a minimal-non-FC subgroup of $G$. Then $H$ is a perfect $p$-group for some prime $p$ and $G = HK$, where $K$ is a normal FC-subgroup of $G$ such that every proper subgroup of $G$ containing $K$ is an FC-group. Moreover, $K$ can be chosen minimal with respect to this condition, and if $K$ is a minimal normal subgroup of $G$, then there exist infinitely many conjugacy classes of Sylow $p$-subgroups of $G$ supplementing $K$.

**Proof.** Let $N$ be a normal subgroup of $G$ which is not an FC-group. Then the factor group $G/N$ satisfies both the minimal and the maximal condition on subgroups, and hence it is finite. Therefore $N = G$ and every proper normal subgroup of $G$ is an FC-group. By Lemma 2.6 the group $G$ has no maximal normal subgroups, and so it is the union of its proper normal subgroups. Since $H$ is not an FC-group, there exists a finite chain
\[ H = G_0 < G_1 < \cdots < G_t = G \]
such that $G_i$ is a maximal subgroup of $G_{i+1}$ for each $i < t$, and hence $G = \langle H, x_1, \ldots, x_t \rangle$ for suitable elements $x_1, \ldots, x_t$ of $G$. The subgroup $\langle x_1, \ldots, x_t \rangle$ is contained in a proper normal subgroup of $G$, so that by Lemma 3.6 there exists a proper normal FC-subgroup $K$ of $G$ with finite exponent containing $x_1, \ldots, x_t$, and so $G = HK$. If $X$ is any proper subgroup of $G$ containing $K$, the intersection $H \cap X$ is an FC-group and $X = (H \cap X)K$ has a descending series whose factors either are finite or abelian. Application of Theorem 2.2 yields that $X$ is either an FC-group or a Černikov group. In the latter case the subgroup $K$ must be finite, and so $X$ is an FC-group. Therefore every proper subgroup of $G$ containing $K$
is an $FC$-group. Since $G$ is a perfect group having no proper subgroups of finite index, the subgroup $H$ cannot be a Černikov group, and it follows from a result of Belyaev (see [4, 10]) that $H$ is a perfect $p$-group for some prime $p$. In order to prove that $K$ can be chosen minimal with respect to the above condition, it is clearly enough to show that for every descending chain

$$K_1 > K_2 > \cdots > K_n > \cdots$$

of $G$-invariant subgroups of $K$ such that $G = HK_n$ for all $n$, we also have $G = HK_0$, where

$$K_0 = \bigcap_{n \in \mathbb{N}} K_n.$$ 

In fact, since $K = (H \cap K)K_n$, the factor group $K/K_n$ is a $p$-group for every $n$, and so also $K/K_0$ is a $p$-group. Then $G/K_0$ is a $p$-group, so that $G/K_0$ has finite index in $G/K_n$ for all $i < t$, and hence $HK_0$ has finite index in $G$. Therefore $G = HK_0$.

Assume now that $K$ is a minimal normal subgroup of $G$, so that it follows from Theorem 2.7 that $K$ is either an infinite abelian group of prime exponent $q$ or a direct product of infinitely many finite simple nonabelian groups. Suppose first that $K$ has an element of order $p$. As $K$ is infinite, $G$ is not a $p$-group, so that in this case $K$ cannot be abelian, and hence by Lemma 3.7 the Sylow $p$-subgroups of $K$ fall into infinitely many conjugacy classes under the action of $K$. Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of pairwise non-conjugate Sylow $p$-subgroups of $K$. Since the factor group $G/K$ is countable (see [18, Lemma 8.14]), application of Lemma 3.11 yields that for each positive integer $n$ there exists a Sylow $p$-subgroup $P_n$ of $G$ such that $P_n \cap K = L_n$ and $G = P_nK$. Assume that $P_n \supset P_m$ for some element $g$ of $G$, and write $g = ux$ where $u \in P_n$ and $x \in K$. Then $P_n = P_m$, and hence

$$L_n^x = (K \cap P_n)^x = K \cap P_n^x = K \cap P_m = L_m,$$

so that $m = n$ and the subgroups $P_n$ ($n \in \mathbb{N}$) are pairwise non-conjugate in $G$. Suppose finally that $K$ has no elements of order $p$, so that in particular $H \cap K = 1$. Since the core $H_G$ of $H$ is contained in every Sylow $p$-subgroup of $G$, replacing $G$ by $G/H_G$ it can be assumed that $H$ does not contain non-trivial normal subgroups of $G$. It follows from Theorem 2.7 that the $p$-group $H$ is hyperabelian, so that it contains a non-trivial abelian normal subgroup $A$ of exponent $p$. Suppose that $A$ is finite. Then $A$ is contained in $Z(H)$ since $H$ is perfect. Moreover, $AK$ is an $FC$-group, and so $[K, A]$ must be a finite normal subgroup of $G$. In particular, $[K, A]$ is properly contained in $K$, so that $[K, A] = 1$ and $A$ is normal in $G$. This
contradiction shows that $A$ is infinite. Clearly $C_A(K)$ is normal in $G$, so that $C_A(K) = 1$, and application of Lemma 3.10 yields that the complements of $K$ in the FC-group $AK$ fall into infinitely many conjugacy classes. Let $(X^\gamma_n)_{n \in \mathbb{N}}$ be a sequence of pairwise non-conjugate complements of $K$ in $AK$. It follows from Lemma 3.8 that for each positive integer $n$ there exists a complement $H_n$ of $K$ in $G$ such that $X_1^\gamma = H_n$, and clearly every $H_n$ is a Sylow $p$-subgroup of $G$. Assume that $H_n^g = H_m$ for some element $g$ of $G$, and let $g = hx$, where $h \in H_n$ and $x \in K$. Then $H_n^x = H_m$, and hence

$$X_n^\gamma = (H_n \cap AK)^x = H_n^x \cap AK = H_m \cap AK = X_m.$$

Therefore $m = n$, and the subgroups $H_n$ ($n \in \mathbb{N}$) are pairwise non-conjugate in $G$. The lemma is proved.

It is now possible to prove the main result of this section.

**Theorem 3.13.** Let $G$ be a locally graded group. Then $G$ has finitely many conjugacy classes of non-FC subgroups if and only if it satisfies one of the following conditions:

(a) $G$ is an FC-group.

(b) $G$ is a Černikov group whose finite residual $J$ has no infinite proper $\langle x \rangle$-invariant subgroups for every element $x$ of $G \setminus C_G(J)$.

(c) $G$ contains a perfect normal $p$-subgroup $H$ which is minimal-non-FC, the factor group $G/H$ is finite, and every subgroup of $G$ which does not contain $H$ is an FC-group.

**Proof.** Suppose first that $G$ has finitely many conjugacy classes of non-FC subgroups. If $G$ is not an FC-group, it follows from Corollary 3.5 and Lemma 2.3 that $G$ is a locally finite group satisfying both the minimal and the maximal condition on non-FC subgroups. Clearly $G$ contains a normal subgroup $M$ which is not an FC-group and is minimal with respect to this condition. Then the factor group $G/M$ satisfies both the minimal and the maximal condition on subgroups, and hence it is finite. Assume that $M$ has a proper homomorphic image which is either abelian or finite. Then $M$ contains a proper characteristic subgroup $M_0$ such that $M/M_0$ is either abelian or residually finite. By the minimal choice of $M$ the normal subgroup $M_0$ of $G$ is an FC-group, and hence $M$ has a descending series whose factors either are finite or abelian. Then $M$ is a Černikov group by Theorem 2.2, so that $G$ itself is a Černikov group. Let $J$ be the finite residual of $G$, and let $x$ be any element of $G \setminus C_G(J)$. Then the subgroup $\langle x, J \rangle$ is not an FC-group, and hence it contains a subgroup $L$ which is minimal-non-FC. For each positive integer $n$ let $J_n$ be the $n$th term of the socle series of $J$. If $J$ is not contained in $L$, then the subgroups $LJ_n$
determine infinitely many isomorphism classes of non-FC subgroups, a contradiction. Therefore $J$ is the finite residual of $L$, so that $J$ does not contain infinite proper $L$-invariant subgroups (see [5]), and in particular $J$ has no infinite proper $\langle x \rangle$-invariant subgroups. Suppose now that $G$ is not a Černikov group, so that $M$ is perfect and has no proper subgroups of finite index. Let $H$ be a minimal-non-FC subgroup of $M$. Application of Lemma 3.12 yields that $H$ is a perfect $p$-group for some prime $p$ and $M = HK$ for some normal FC-subgroup $K$ of $M$ which can be chosen minimal with respect to this condition. Assume that $K \neq 1$. Since $G$ satisfies the maximal condition on non-FC subgroups, there exists a normal subgroup $N$ of $M$ properly contained in $K$ such that $HN$ is maximal, and of course $N$ can also be chosen maximal with respect to this condition. Then $K/N$ is a minimal normal subgroup of $M/N$. Obviously $HN/N$ is also minimal-non-FC, so that it follows from Lemma 3.12 that there exist infinitely many conjugacy classes of Sylow $p$-subgroups of $M/N$ supplementing $K$. In particular there exist infinitely many conjugacy classes of subgroups of $M$ supplementing $K$, and all such subgroups are not FC-groups, a contradiction, as $M$ has finitely many conjugacy classes of non-FC subgroups by Lemma 3.1. Therefore $K = 1$ and $H = M$ is a normal subgroup of finite index of $G$. Assume now that there exists a subgroup $X$ of $G$ which is not an FC-group and does not contain $H$, and choose $X$ maximal with respect to these conditions. Since $H \cap X$ is an FC-group and $X/H \cap X$ is finite, the group $X$ has a descending series whose factors are either finite or abelian, and it follows from Theorem 2.2 that $X$ must be a Černikov group. Since $H \cap X$ is properly contained in a proper subgroup of $H$, there exists a subgroup $V$ of $H$ such that $H \cap X \subset V$ and the index $[V: H \cap X]$ is finite. Moreover, it is clear that $V$ can be chosen to be $X$-invariant. Then $VX$ is a subgroup of $G$ which properly contains $X$ and does not contain $H$. This last contradiction proves that every subgroup of $G$ which does not contain $H$ is an FC-group.

Conversely, it is clearly enough to show that in case (b) every subgroup $X$ of $G$ which does not contain $J$ is an FC-group. If $X$ is contained in $C_G(J)$, the subgroup $X$ is central-by-finite and so FC. Suppose that $X$ is not contained in $C_G(J)$. Then $J$ does not contain infinite proper $X$-invariant subgroups, so that $X \cap J$ must be finite, and the subgroup $X$ itself is finite.

Corollary 3.14. Let $G$ be a locally graded group having finitely many conjugacy classes of non-FC subgroups. Then $G$ has only finitely many non-FC subgroups.

Clearly conditions similar to those considered in this article can be investigated for other classes of groups defined by restrictions on conjugacy classes. Recall here that a group $G$ is called a CC-group if
$G/C_G((x)^G)$ is a Černikov group for each element $x$ of $G$. The structure of minimal-non-CC groups has recently been investigated in [3, 12, 8]. Here we note the following.

**Theorem 3.15.** Let $G$ be a group having a descending series whose factors either are finite or abelian. The following statements are equivalent:

(a) $G$ is a CC-group.
(b) $G$ satisfies the minimal condition on non-CC subgroups.
(c) $G$ has finitely many conjugacy classes of non-CC subgroups.

**Proof.** Suppose that the group $G$ has finitely many conjugacy classes of non-CC subgroups. Then by Proposition 3.3 every finitely generated subgroup of $G$ is a finite extension of a CC-group, and hence it is polycyclic-by-finite. The proof of Corollary 3.5 yields now that $G$ satisfies the minimal condition on non-CC subgroups. Assume that $G$ is not a CC-group, so that it contains a subgroup $H$ which is not a CC-group, while all proper subgroups of $H$ are CC-groups, contradicting the main theorem of [12]. Therefore $G$ is a CC-group. $lacksquare$

**References**