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REVIEW PAPER

Some quantum optical states as realizations of Lie groups

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Abstract We start with the Heisenberg–Weyl algebra and after the definitions of the Fock states we give the definition of the coherent state of this group. This is followed by the exposition of the $SU(2)$ and $SU(1,1)$ algebras and their coherent states. From there we go on describing the binomial state and its extensions as realizations of the $SU(2)$ group. This is followed by considering the negative binomial states, and some squeezed states as realizations of the $SU(1,1)$ group. Generation schemes based on physical systems are mentioned for some of these states.

1. Introduction

With the advances in the field of quantum optics which began with the 60s, group theory started to infiltrate in this branch. Groups involving simple Lie algebras such as $SU(2)$ and $SU(1,1)$ and their simple generalizations have been used to study different aspects in quantum optics. However, the use of the theory of groups in quantum mechanics started with the early days of that theory. Weyl's book that was first published in German in 1928 [1] is a standing witness on this. Wider dimensions in various branches of physics benefited greatly from the use of the group theory.

Some states used in the field of quantum optics as realizations of the $SU(2)$ or $SU(1,1)$ groups are reviewed. We start by some preliminaries about the annihilation and creation operators and the number operators which constitute the corner stones of the Hesenbeg-Weyl algebra, then their eigenstates and their coherent states are defined. The familiar algebras of the $SU(2)$ and $SU(1,1)$ are introduced. Then some quantum states which are realizations of the $SU(2)$ are reviewed in Section 3. Section 4 is devoted to states as realizations of $SU(1,1)$ group. Some comments are given about the generations of some of these states through physical processes.

2. Preliminaries

2.1. The harmonic oscillator

In the study of the harmonic oscillator, the following operators are introduced: the annihilation operator $\hat{a}$ the creation operator $\hat{a}^\dagger$ and the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. They satisfy the commutation relations

$$[a, a^\dagger] = I, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a. \tag{1}$$

The eigen-states $|n\rangle$ of the number operator $\hat{n}$ are called Fock states or number states. They satisfy

$$\hat{n}|n\rangle = n|n\rangle. \tag{2}$$

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The non-negative integer n can be looked upon as the number of particles in the state. When $n = 0$ we call $|0\rangle$ the vacuum state with no particles present.

The operations of a and $a^\dagger$ on $|n\rangle$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{3}$$

The states $\{|n\rangle\}$ form a complete set and resolve the unity

$$\sum_n |n\rangle\langle n| = I. \tag{4}$$

2.1.1. Coherent states

The coherent state $|\alpha\rangle$ can be looked upon as an eigenstate of the operator a such that

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{5}$$

Also, it can be produced by applying the Glauber displacement operator which is a unitary operator on the vacuum state $|0\rangle$ [2,3].

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle.$$

This is the coherent state of the Heisenberg–Weyl group [4,5].

This state, which is a superposition of infinite series of the Fock states with their distribution being Poissonian. It is given by its expansion in the number state as

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}. \tag{6}$$

This state describes to a great deal the laser field where the phase is fixed while the number is not. The sates $\{|\alpha\rangle\}$ are overcomplete and they satisfy $\int |\alpha\rangle\langle\alpha|\frac{d^2\alpha}{\pi} = I$.

2.2. The angular momentum (SU(2) group)

The angular momentum defined as $\hat{r} \times \hat{p}$ as well as the spin, are described by the three operators J_x, J_y, and J_z which satisfy the commutation relations (we take $\hbar = 1$)

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y, \tag{7}$$

with

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

which commutes with each component. Raising and lowering operators are introduced through the relations

$$J_\pm = J_x \pm iJ_y.$$

Hence the commutation relations (7) become

$$[J_z, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = 2J_z. \tag{8}$$

The simultaneous eignestates of the operators J_z and J^2 denoted by $|j,m\rangle$ are given from [5,6]

$$J^2|j,m\rangle = j(j+1)|j,m\rangle \quad \text{and} \quad J_z|j,m\rangle = m|j,m\rangle, \tag{9}$$

with $|m| \leqslant j$, j half integers.

The operations of J_+ and J_- on $|j,m\rangle$ are given by

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\ |j,m+1\rangle \\ J_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\ |j,m-1\rangle. \end{aligned} \tag{10}$$

The operators J_α are the generators of the group $SU(2)$. The angular momentum coherent state is defined by the action of the rotation operator

$$\widehat{R}(\theta,\phi) = \exp\left[\frac{1}{2}\theta\left(e^{-i\phi}J_+ - e^{i\phi}J_-\right)\right], \tag{11}$$

on the state $|j,-j\rangle$.

The angular momentum coherent state $|\theta,\phi\rangle$ is given by

$$|\theta,\phi\rangle = \widehat{R}(\theta,\phi)|j,-j\rangle = \left(\cos\frac{1}{2}\theta\right)^{2j}\sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}\left(\tan\frac{1}{2}\theta e^{-i\phi}\right)^{j+m}|j,m\rangle. \tag{12}$$

They resolve the identity operator on the space with total angular momentum j as follows

$$\frac{2j+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi|\theta,\phi\rangle\langle\theta,\phi| = I. \tag{13}$$

2.3. The SU(1,1) group

The notion of coherent states can be extended to any set of operators obeying a Lie algebra. The $SU(1,1)$ is the simplest non-abelian noncompact Lie group with a simple Lie algebra (For a comprehensive review we may refer to [6] and the recent review book [7]).

The $SU(1,1)$ algebra is spanned by the three operators K_1, K_2, K_3 which satisfy the commutation meatiness

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2.$$

By using the operators $K_\pm = K_1 \pm iK_2$, hence

$$[K_3, K_\pm] = \pm K_\pm \quad \text{and} \quad [K_+, K_-] = -2K_3. \tag{14}$$

The Casimir operator $C^2 = K_3^2 - K_1^2 - K_2^2$ has the value $C^2 = k(k-1)I$ for any irreducible representation. Thus, representation is determined by the parameter k which is called the Bargmann number. The corresponding Hilbert space is spanned by the complete orthonormal basis $\{|k,n\rangle\}$ which are the eigenstates of C^2 and K_3, such that

$$\langle k,n|k,m\rangle = \delta_{nm} \quad \text{and} \quad \sum_{n=0}^{\infty}|k,n\rangle\langle k,n| = I.$$

The operations of the operators $K_\pm$ and K_3 on $|k,n\rangle$ are given by [5]

$$\begin{aligned} K_+|k,n\rangle &= \sqrt{(n+1)(2k+n)}\ |k,n+1\rangle \\ K_-|k,n\rangle &= \sqrt{n(2k+n-1)}\ |k,n-1\rangle \\ K_3|k,n\rangle &= (k+n)|k,n\rangle \end{aligned} \tag{15}$$

The ground state $|k,0\rangle$ satisfies $K_-|k,0\rangle = 0$ while

$$|k,m\rangle = \sqrt{\frac{\Gamma(2k)}{n!\Gamma(2k+m)}}K_+^m|k,0\rangle.$$

There are two sets of coherent states related to the $SU(1,1)$ group namely:

(i) The Perelomov coherent states.By applying the unitary operator

$$D_{Per}(\xi) = \exp(\xi K_+ - \xi^* K_-),$$

on the ground state $|k,0\rangle$ to get [2]

$$|\alpha,k\rangle_{Per} = D_{Per}(\xi)|k,0\rangle = (1-|\alpha|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{n!\Gamma(2k)}} \alpha^n |k,n\rangle, \quad (16)$$

with $\xi = |\xi| e^{i\theta}$, $\alpha = (\tan h|\xi|)e^{i\theta}$.(95)

(ii) The Barut-Girardello coherent states It is defined as the eigenstate [7]

$K_-|\alpha,k\rangle_{BG} = \alpha|\alpha,k\rangle_{BG}$,

which can be expressed as

$$|\alpha,k\rangle_{BG} = \sqrt{\frac{\alpha^{2k+1}}{I_{2k-1}(2|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!\Gamma(n+2k)}} |k,n\rangle. \quad (17)$$

$I_\nu(x)$ is the modified Bessel function of the 1st kind.

After this very quick review of these preliminaries we look at some states which are realization of the $SU(2)$ and $SU(1,1)$ groups.

3. $SU(2)$ realizations

We look at some states which can be looked upon as realizations of $SU(2)$ group.

3.1. The single mode Binomial state

These states are of the form [8]

$$|M \cdot \eta\rangle = \sum_{n=0}^{M} \sqrt{\binom{M}{n}} \eta^n (1-|\eta|^2)^{\frac{M-n}{2}} |n\rangle, \quad (18)$$

$M \in \mathbb{Z}^+$, $\eta \in \mathbb{C}$, $|\eta|^2 \leqslant 1$.

They have the photon-number distribution (probability of finding n photons) as

$$P(n) = \binom{M}{n} |\eta|^{2n} (1-|\eta|^2)^{M-n},$$

which is the binomial distribution.

They are the eigen states of the operator [7]

$$B = \eta a^\dagger a + \sqrt{1-|\eta|^2}\sqrt{MI - a^\dagger a}\, a,$$

with the eigen-value ηM, i.e.

$$B|M,\eta\rangle = \eta M|M,\eta\rangle. \quad (19)$$

We may note the vacuum, Fock and coherent state as limiting cases of this state [7,8].

As it is mentioned in Section 2.2 when we looked at the $SU(2)$ representations, the angular momentum operators $\underline{J}$ satisfy the relations (7); and the $SU(2)$ coherent states which is defined as the action of the rotation operator (11) on the ground state. Hence Eq. (12) is the $SU(2)$ coherent state. Thus when we take $\eta = \sin\frac{\theta}{2} e^{-i\phi}$ and take $n = j + m$ and $M = 2j$, the binomial state $|2j, \sin\frac{\theta}{2} e^{-i\phi}\rangle$ is the coherent state of the $SU(2)$ group.

This state can be generated through the following scheme. An atom under a classical magnetic field $\underline{B}$ has the interaction Hamiltonian $H = -\underline{J} \cdot \underline{B}$ with the field along the direction x. Under this Hamiltonian the state $|j,-j\rangle$ evolves to the binomial state. The evolution operator $U(t)$ is given by

$$U = \exp -itH = \exp itB(J_+ + J_-).$$

Then $|\psi(t)\rangle = U|j,-j\rangle$ is the coherent state (12) with $\theta = 2Bt$ and $\phi = -\frac{\pi}{2}$.

3.2. Finite dimensional pair coherent state

It may be termed as the two-mode binomial state $|\xi,q\rangle$. It can be defined as the eigen state of the operators $\left(a^\dagger b + \frac{\xi^{q+1}(ab^\dagger)^q}{(q!)^2}\right)$ and $(a^\dagger a + b^\dagger b)$ where a, b are annihilation operators for the two modes. The states satisfy the eigen value equations [9].

$$\left(a^\dagger b + \frac{\xi^{q+1}(ab^\dagger)^q}{(q!)^2}\right)|\xi,q\rangle = \xi|\xi,q\rangle; \quad (a^\dagger a + b^\dagger b)|\xi,q\rangle = q|\xi,q\rangle, \quad (20)$$

and takes the form

$$|\xi,q\rangle = N_q \sum_{n=0}^{q} \xi^n \sqrt{\frac{(q-n)!}{q!n!}} |q-n,n\rangle, \quad N_q^{-2} = \sum_{n=0}^{q} |\xi|^{2n} \frac{(q-n)!}{q!n!}. \quad (21)$$

This is a type of the entangled states where we find $(q-n)$ particles in 1st mode and (n) particles in the 2nd mode.

When we define

$$J_x = \frac{a^\dagger b + ab^\dagger}{2}, \quad J_y = \frac{a^\dagger b - ab^\dagger}{2i}, \quad J_z = \frac{a^\dagger a - b^\dagger b}{2}, \quad (22)$$

which are the generators of the $SU(2)$ group. Hence the raising and lowering operators are $J_+ = J_x + iJ_y = a^\dagger b$, $J_- = J_x - iJ_y = ab^\dagger$.

Thus is the state (21) are the coherent states of $SU(2)$ when we label $q = 2j$ and identify the states $\{|j,m-j\rangle\}$ as the states $\{|q-n,n\rangle\}$. This state can be generated as demonstrated in Ref. [9,10].

3.3. Nonlinear two-mode binomial state

An extension to the earlier state is performed by introducing the nonlinear finite dimensional pair coherent state as the eigen state satisfying

$$\left[f_1(n_a)a^\dagger b f_2(n_b) + \xi^{q+1} \frac{\left(a\frac{1}{f_1(n_a)} \cdot \frac{1}{f_2(n_b)} b^\dagger\right)^q}{(q!)^2}\right] |\xi,q\rangle_f = \xi|\xi,q\rangle_f, \quad (23)$$

and

$$(n_a + n_b)|\xi,q\rangle = q|\xi,q\rangle,$$

with the usual notation.

It is expanded in the Fock states for the two modes as [10]

$$|\xi,q\rangle_f = N_q \sum_{n=0}^{q} \xi^n \sqrt{\frac{(q-n)!}{n!q!} \frac{f_1(q-n)!}{f_1(q)! f_2(n)!}} |q-n,n\rangle, \quad (24)$$

with $f(n)! = f(0) \cdot f(1) \ldots f(n)$ and $f(0) = 1$.

In order to relate these states to $SU(2)$ group realization, we introduce the operators

$$J_x = \frac{f_1(n_a)a^\dagger f_n(n_b)b + af_1^{-1}(n_a)b^\dagger f_2^{-1}(n_b)}{2},$$
$$J_y = \frac{f_1(n_a)a^\dagger f_n(n_b)b - af_1^{-1}(n_a)b^\dagger f_2^{-1}(n_b)}{2i}, \quad J_z = \frac{\hat{n}_a - \hat{n}_b}{2}, \quad (25)$$

which satisfy the relations $[J_x,J_y] = iJ_z$, $[J_y,J_z] = iJ_x$, $[J_z,J_x] = iJ_y$. Note that neither J_x nor J_y is hermitian, hence we define

$$J_+ = f_1(n_a)a^\dagger f_2(n_b)b, \quad J_- = a\frac{1}{f_1(n_a)} b^\dagger \frac{1}{f_2(n_b)},$$

consequently $[J_z,J_\pm] = \pm J_\pm$, $[J_+,J_-] = 2J_z$.

These operators can be thought of as generators of an extended $SU(2)$ group. When we take $2j = q$ and the states $\{|j,n-j\rangle\}$ to correspond to $\{|q-n,n\rangle\}$, then we get an extended $SU(2)$ coherent state similar to Eq. (24). Some properties of these states may be found in Ref. [10] and also a generation scheme.

4. $SU(1,1)$ Realizations

There are a large number of states that can be termed as realizations of the $SU(1,1)$ group reviewed in Section 2.3. Here we mention some of these states.

4.1. The negative binomial states

This state is defined as the Fock state expansion [11]

$$|M,\xi\rangle_N = \sum_{n=0}^{\infty}\sqrt{\frac{(n+M)!}{n!M!}}\xi^n(1-|\xi|^2)^{\frac{M+1}{2}}|n\rangle. \tag{26}$$

This state follows the negative binomial distribution for the photon number distribution

$$P(n) = \frac{(M+n)!}{M!n!}|\xi|^{2n}(1-|\xi|^2)^{M+1}.$$

The special case of $M = 0$ is the Pascal distribution or the thermal distribution. The state (26) interpolates between the pure thermal state and the coherent state ($\xi \to 0$, $M \to \infty$, $M|\xi|^2 \to |\alpha|^2$), hence it is termed as an intermediate state.

The $SU(1,1)$ realization can be achieved by introducing the operators

$$K_+ = a^\dagger\sqrt{MI+\hat{n}}, \quad K_- = \sqrt{MI+\hat{n}}\,a, \quad K_z = \frac{M}{2}I+\hat{n}, \tag{27}$$

which are the raising, lowering and generators of the $SU(1,1)$ group.

Thus the unitary evolution operator $D(\eta) = \exp(\eta K_+ - \eta^* K_-)$ can be applied on the vacuum state to have the state

$$D(\eta)|0\rangle = \exp(\xi K_+)[1-|\xi|^2]^{\frac{M+1}{2}}\exp(-\xi K_-)|0\rangle$$
$$= [1-|\xi|^2]^{\frac{M+1}{2}}\sum_{n=0}^{\infty}\xi^n\sqrt{\frac{(M+n)!}{M!n!}}|n\rangle = |M,\xi\rangle_N,$$

where $\xi = \frac{\eta}{|\eta|}\tanh|\eta|$.

4.2. The non-linear negative binomial state

The nonlinear extension to the above state has been introduced [12]. It amounts to deform the operator a to $A = af(n)$ where $f(n)$ is an operator valued function. Hence the state is given by

$$|M.\xi\rangle_{Nf} = N_f\sum_{n=0}^{\infty}\xi^n(1-|\xi|^2)^{\frac{M+1}{2}}\sqrt{\frac{(M+n)!}{M!n!}}(f^\dagger(n)!)|n\rangle. \tag{28}$$

The commutation relation

$$[A,A^\dagger] = [af(n),\ f^\dagger(n)a^\dagger] = (n+1)|f(n+1)|^2 - n|f(n)|^2.$$

It becomes $[A,A^\dagger] = 1$ for $f^\dagger(n) = f^{-1}(n)$ i.e. unitary operator.

For the operator $f(n)$ being unitary, the following $SU(1,1)$ generators are defined

$$K_+ = a^\dagger f^\dagger(n)\sqrt{MI+n}, \quad K_+ = \sqrt{MI+n}f(n)a, \quad K_z = \frac{M}{2}I+n. \tag{29}$$

The state (28) is obtained by applying $D(\eta)$ of Section 4.1 but with the operators given by (29), on the vacuum state $|0\rangle$ which is the $SU(1,1)$ realization for the nonlinear negative binomial state (28).

The case of the non-unitary f is also considered in Ref. [12].

4.3. Single mode squeezed vacuum and 1st excited states

The squeezed vacuum state is defined as the eigenstate of the operator $b = \mu a + \nu a^\dagger$ with eigenvalue zero and $|\mu|^2 - |\nu|^2 = 1$ [13,14]. It has the expansion

$$|\xi\rangle_0 = \sum_{n=0}^{\infty}\frac{\sqrt{2n!}}{2^n n!}\xi^n(1-|\xi|^2)^{\frac{1}{4}}|2n\rangle, \tag{30}$$

where $\xi = \tanh r\, e^{i\phi}$, $\mu = \cosh r$, $\nu = \sinh r\, e^{i\phi}$.

While the squeezed 1st excited state is obtained as the eigen state of the operator b^2 with egien value *0*. It has the form

$$|\xi\rangle_1 = \sum_{n=0}^{\infty}\frac{\sqrt{(2n+1)!}}{2^n n!}\xi^n(1-|\xi|^2)^{\frac{3}{4}}|2n+1\rangle. \tag{31}$$

This can be cast as a realization of the $SU(1,1)$ group by taking

$$K_+ = \frac{a^{\dagger 2}}{2}, \quad K_- = \frac{a^2}{2}, \quad K_3 = \frac{1}{2}\left(a^\dagger a+\frac{1}{2}\right). \tag{32}$$

The Casamir operator C_2 in this case

$$C_2 = k(k-1)I = \frac{-3}{16}I.$$

The state space associated with $k = \frac{1}{4}$ is the even Fock subspace with $\{|2n\rangle\}$ and that associated with $k = \frac{3}{4}$ is the odd Fock subspace with $\{|2n+1\rangle\}$. The unitary operator (the squeeze operator) is:

$$S(z) = \exp(zK_+ - z^*K_-) = \exp\left(\frac{1}{2}za^{\dagger 2} - \frac{1}{2}z^*a^2\right).$$

The $SU(1,1)$ coherent states are the single-mode squeezed sates. For $k = \frac{1}{4}$ we have squeezed vacuum

$$\left|\xi,\frac{1}{4}\right\rangle = S(z)|0\rangle = |\xi\rangle_0 \quad \text{of (30)} \quad \xi = \frac{z}{|z|}\tanh|z|,$$

for $k = \frac{3}{4}$ we have the squeezed one photon state

$$\left|\xi,\frac{3}{4}\right\rangle = S(\xi)|1\rangle = |\xi\rangle_1 \quad \text{of} \quad (31).$$

4.4. Nonlinear squeezed states

The use of the following operators

$$K_+ = \frac{1}{2}(f^\dagger(n)a^\dagger)^2, \quad K_- = \frac{1}{2}(af(n))^2. \tag{33}$$

For the unitary operator function $f^\dagger(n) = f^{-1}(n)$, we have

$$K_3 = \frac{1}{2}\left(a^\dagger a+\frac{1}{2}\right).$$

Under these operators, we have the nonlinear squeezing operator

$$S_f(z) = \exp\frac{1}{2}(zA^{\dagger 2} - z^\dagger A^2), \quad \text{where} \quad A = af(n).$$

Consequently

$$|\xi,\frac{1}{4}\rangle_f = (1-|\xi|^2)^{\frac{1}{4}}\sum_{n=1}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}(f(2n)!)\xi^n|2n\rangle$$

$$|\xi,\frac{3}{4}\rangle_f = (1-|\xi|^2)^{\frac{3}{4}}\sum_{n=1}^{\infty}\frac{\sqrt{(2n+1)!}}{2^n n!}(f(2n+1)!)\xi^n|2n+1\rangle. \quad (34)$$

The case of the non-unitary f has been considered and discussed in [15].

4.5. Single mode squeezed coherent state

These states are the solutions of eigenvalue problem [16]

$$b|\beta\rangle = \mu a + v a^\dagger|\beta\rangle = \beta|\beta\rangle,$$

with $\mu = \cosh r$, $v = \sinh r\, e^{i\phi}$.

If we write $\xi = -\frac{v}{\mu} = -e^{i\phi}\tanh\ r$, then the state

$$\begin{aligned}|\beta\rangle &= |\beta,\xi\rangle = |\alpha, r\rangle\\ &= (1-|\xi|^2)^{\frac{1}{4}}\\ &\exp\left[-\frac{1}{2}\left\{|\beta|^2 - \frac{|\xi|}{2}(\beta^2 e^{i\phi} + \beta^{*2}e^{-i\phi})\right\}\right]\times\sum_{m=0}^{\infty}\frac{(-\frac{1}{2}\xi)^{\frac{m}{2}}}{\sqrt{m!}}\\ &H_m\left(\frac{\beta\sqrt{1-|\xi|^2}}{\sqrt{-2\xi}}\right)|m\rangle, \end{aligned} \quad (35)$$

with $\beta = \mu\alpha + v\alpha^*$.

When we use the representation (32) for the operators $K_\pm$, K_3, the state (35) can be cast as the operation of the operator

$$S(\xi) = \exp(\xi K_+ - \xi^* K_-) = \exp\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2),$$

on the state $|\beta\rangle$ of the form (6). Therefore we have

$$|\xi,\beta\rangle = S(\xi)|\beta\rangle = S(\xi)D(\beta)|0\rangle.$$

After using the disentanglement of the squeezing operator and applying it to the state —$\beta\rangle$ we get the expression (35). We may use the relation

$$S(z)D(\beta) = D(\alpha)S(z) \quad \text{with} \quad \alpha = \mu\beta + v\beta^*,$$

Hence we get

$$|\xi,\beta\rangle = S(\xi)D(\beta)|0\rangle = D(\alpha)S(\xi)|0\rangle$$

i.e. we displace the squeezed vacuum, or squeeze the coherent state [6].

4.6. Nonlinear squeezed coherent state

The nonlinear operator A is defined as $A = a f(n)$ and $A^\dagger = f^\dagger(n)a^\dagger$ where the operator valued function $f(n)$ is a unitary operator i.e. $f^\dagger = f^{-1}$. In this case we find $[A,A^\dagger] = I$.

The operators $K_\pm$,K_3 of the $SU(1,1)$ are defined as is (33). the nonlinear realization in this case is given in [17]. With the appearance of the function $f(m)$ denotes the effect of the nonlinearity.

Also, the case for non-unitary nonlinear function has been discussed there.

4.7. Squeezed displaced Fock states

These states are defined as application of the squeezed operators and the displacement operators on the Fock state $|m\rangle$ in the following form [18].

$$|\alpha,\xi,m\rangle = S(\xi)D(\alpha)|m\rangle = D(\alpha_0)S(\xi)|m\rangle,$$

with $\alpha_0 = \alpha\cosh r + \alpha^*\sinh r\, e^{i\phi} = \mu\alpha + v\alpha^*$, $\xi = re^{i\phi}$.

Its expansion in the Fock state space is given there [18].

Here $S(\xi)$ is the unitary operator that could be expressed in terms of the generators of the $SU(1,1)$ group as defined in the Eq. (32) .

4.8. Two-mode squeezed vacuum states

These states are obtained by applying the non-degenerate two-mode operator

$$S_2(\xi) = \exp(-\xi a^\dagger b^\dagger + \xi^* ab),$$

on the vacuum state —0_1,$0_2\rangle$ with $\xi = r\, e^{i\phi}$. These states are expressed in terms of the two-mode Fock states in the form [6,7]

$$|\xi\rangle_2 = S_2(\xi)|0_1,0_2\rangle = \frac{1}{\cosh r}\sum(\tanh re^{i\phi})^n|n,n\rangle. \quad (36)$$

These states are considered as a class of entangled states where the numbers of quanta in both modes are equal in each component.

Such state can be considered as realization of the $SU(1,1)$ as coherent state of this group. This is accomplished by defining the generators as follows

$$K_+ = ab, \quad K_- = a^\dagger b^\dagger, \quad K_3 = \frac{1}{2}(n_a + n_b + 1). \quad (37)$$

The Casimir operator

$$C_2 = K_3^2 - \frac{1}{2}(K_+K_- + K_-K_+) = \frac{1}{4}[(n_a - n_b)^2 - 1].$$

Therefore the irreducible representation with $k = \frac{q+1}{2}$, $q = 0,1,2,\ldots$ is the eigenvalue of $(n_a - n_b)$ hence the state $|m,k\rangle \Rightarrow |n+q,n\rangle$ with $n = 0,1,2,\ldots$

$$\left|\xi, k = \frac{1+q}{2}\right\rangle = \sum_{n=0}^{\infty}(1-|\xi|^2)^{\frac{q+1}{2}}\sqrt{\frac{(n+q)!}{n!q!}}\xi^n|n+q,n\rangle. \quad (38)$$

which reduces to (36) when $q = 0$.

There is another coherent state of $SU(1,1)$ (Barut-Girardillo) which is the eigenfunction of the operator K_- namely:

$$K_-|\alpha,k\rangle_{BG} = \alpha|\alpha,k\rangle_{BG}$$

$$\therefore \quad |\alpha,k\rangle_{BG} = \sqrt{\frac{|\alpha|^{2k+1}}{I_{2k-1}(2|\alpha|)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!\Gamma(n+2k)}}|n,k\rangle.$$

4.9. Nonlinear two mode squeezed vacuum state

As before we have $A = af_1(n_a)$, $B = bf_2(n_b)$ and their hermitian conjugates, consequently

$$[A,A^\dagger] = (n_a+1)f_1(n_a+1)f^\dagger(n_a+1) - n_a f_1^\dagger(n_a)f_1(n_a), \quad (39)$$

with a similar formula for $[B,B^\dagger]$.

For the unitary case $f_i^\dagger = f_i^{-1}, \therefore f_i f_i^\dagger = I$, the nonlinear squeezing operator will be a unitary operator and we have

$$|\xi\rangle_{2f} = S_{2f}(\xi)|0,0\rangle = \exp(-\xi A^\dagger B^\dagger + \xi^\dagger AB)|0,0\rangle. \quad (40)$$

The non-unitary case has been discussed also in [17].

4.10. SU (1,1) Intelligent states

For the two self-adjoint operator A, B, one obtains the uncertainty relation

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geqslant \frac{1}{4}|\langle[A,B]\rangle|^2.$$

A state is called an intelligent state (IS) if it satisfies the strict equality. Such states must satisfy the eigen value equation

$$(A - i\lambda B)|\psi\rangle = \eta|\psi\rangle, \tag{41}$$

λ is a positive real parameter, η a complex number. When $[A,B] = cI$ with constant c, the minimum uncertainty states (MUS) coincide with the IS.

For the $SU(1,1)$ the IS —$\psi\rangle$ are solutions of the eigenvalue problem

$$(K_1 - i\lambda K_2)|\psi\rangle = \eta|\psi\rangle,$$

or

$$(\alpha_1 K_- + \beta_1 K_+)|\psi\rangle = \eta|\psi\rangle, \quad \alpha_1 = 1+\lambda,\ \beta_1 = 1-\lambda. \tag{42}$$

In the basis $|n,k\rangle$ of the $SU(1,1)$ of Eq. (15) then $|\psi\rangle$ is given by

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n,k\rangle.$$

The coefficients c_n can be calculated to be related to the Pollaczek polynomial [19,20]

$$\begin{aligned} c_n &= \left(\frac{\beta_1}{\alpha_1}\right)^n P_n\left(\frac{\eta}{\sqrt{\alpha_1\beta_1}}, k\right) \\ &= i^n\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{n}{2}}\left(\frac{\Gamma(n+2k)}{n!\Gamma(2k)}\right)^{\frac{n}{2}} {}_2F_1\left(-m, k+\frac{i\eta}{\sqrt{\alpha_1\beta_1}}; 2k, 2\right). \end{aligned} \tag{43}$$

Some special cases are discussed in [20]. For example:

(i) The one mode realization include as special cases: the Barut-Girardillo state, the Perelomov C.S., and the nonlinear squeezed coherent states.
(ii) The two-mode realizations include as special cases; the pair coherent state as the correlated $SU(1,1)$ CS, and nonlinear realizations.

5. Conclusion

In this article we have tried to review some quantum states and their relations to some algebraic groups. Some of the Lie algebras and some of the relations of their operators and representations especially some of their coherent states are mentioned. As realizations of these groups the discussion included the single mode binomial states, the finite dimensional pair coherent states, and their nonlinear variants. Then came the negative binomial states, single mode squeezed vacuum, squeezed coherent, squeezed displaced Fock states and their nonlinear variants. The two mode squeezed vacuum states and their nonlinear counterparts are discussed. Finally the intelligent states are mentioned. An extended version of this article with appear elsewhere.

6. A tribute

This is a small tribute to the late Prof. Gamal M. Abd AlKader [1963–2009] with whom I have had a very fruitful and most interesting collaboration for almost two decades. Whose friendship and amicable personality, I as well as many of his colleagues and students really miss. May ALLAH accept him in His Mercy.

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