Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 428 (2008) 44–136
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Observer theory

P.A. Fuhrmann [1]

Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, Israel

Received 17 August 2006; accepted 24 August 2007
Available online 30 October 2007
Submitted by J. Rosenthal

Abstract

The paper is devoted to a comprehensive exposition of the theory of partial state observers in the state space context and the elucidation of the connection between this theory and the theory of observers in the behavioral context, as developed in [M.E. Valcher, J.C. Willems, Observer synthesis in the behavioral framework, IEEE Trans. Aut. Contr. 44 (1999) 2297–2307]. For this we use several techniques, including geometric control theory, polynomial and rational models, shift realizations, coprime factorizations, partial realizations and the basic results on behaviors and behavior homomorphisms. A connection between observers and the construction of state maps is made.

Keywords: Partial state observers; Geometric control; Partial realizations; Internal model principle; Behaviors; Behavior homomorphisms; Behavioral observers; State maps

1. Introduction

To a certain extent, the description of a mathematical theory is analogous to the description of a forest. Do we describe a forest by delineating its boundaries, by specifying the most impressive trees in it, or by describing the main roads crossing it? We take the position that to know a forest well, one has to take all the paths in it, have an access to all main views and see how to go from one to the other. This is the point of view taken in this paper, devoted to the description of observer theory. Thus we are not only concerned with specific results, but our aim is to show how various results relate to each other, in other words we aim not only to knowledge but, more importantly,

E-mail address: fuhrmannbgu@gmail.com
[1] Partially supported by ISF under Grant No. 1282/05.

doi:10.1016/j.laa.2007.08.042

to understanding. Once getting deeper into the subject, one is overwhelmed by its intricacy and the variety of problems that crop up. In this respect, observer theory is a reflection of the general theory of linear systems.

The underlying idea of control theory is that of feedback. Any real life system is operating in a noisy environment and a rational decision making has to take this into account. Thus, open loop control is unrealistic and the use of feedback is an indispensable necessity. For the running of a given process, knowledge of the state of the system at each point in time is sufficient, but in many cases it is more than necessary. So, generally, one would be satisfied with partial knowledge of the state, so long that it is the part that is needed for controlling the system in order to attain a particular objective. In a complex system, even partial state information may not be measurable. So we are faced with the necessity of estimating partial states, or functions of the state, from given observations.

It is folk wisdom that control and observation are dual concepts. This is true to a certain extent, certainly on the level of state space descriptions of linear systems. However, even in the classical algebraic approach to linear systems, as promoted by Kalman, see Kalman et al. [19], and even more so in the behavioral setting, the situation is not symmetric with respect to inputs and outputs. In fact, in the discrete time setting, the input function space is identifiable with vector polynomials, whereas the output function space is identifiable with the space of strictly proper rational vector functions or even of formal power series. This is even more pronounced in the behavioral approach to linear system theory where controllability is an intrinsic property of the system, whereas observability is a property of a particular representation. Thus it seems reasonable to study the problem of observation ab initio and not via dualization of specific control problems. Moreover, to a large extent the problem of observation is more fundamental than the problem of control. If one looks at biological systems for guidance, this becomes immediately obvious. Whether in humans or animals, the main activity of the brain is devoted to visual information processing. Thus it is not surprising that the original work on observers, see [4] or [21], dates back to the early stages of modern control theory. What is surprising is that over the years, the analysis and synthesis of observers, i.e. observer theory, did not get the appropriate attention from the control community that it so rightly deserves. Moreover, in the system literature, there are many gaps, faulty proofs and lack of insights that only now are beginning to be filled in. In this connection we point out [14] which contains a fairly complete account of asymptotic observers, and the thesis of Trumpf [27] which focuses on certain geometric properties that relate to observer theory.

In recent years, a new approach to the modelling of linear systems, namely the behavioral approach, one that avoids the input/output point of view, has been initiated and developed by Willems [30–32] and coworkers. Thus the study of observer design in the behavioral context was called for, and indeed this was worked out in the important paper [28]. In the ongoing debate on the potential usefulness of the behavioral approach for the study of linear systems that paper, in this author's point of view, is a major contribution in favor of behaviors. The contribution is twofold. On the one hand the concepts of observability and detectability are introduced and studied on a conceptual and representation free level, that of system trajectories. On the other hand, polynomial methods are applied to observer synthesis. Although that paper pointed out how some conventional state observers fit into the behavioral approach, a full study of the connections between conventional and behavioral observer theories has not been undertaken.

The object of the present paper is to remedy that and undertake an in depth study of some of the various facets of observer theory. In order to minimize duplication, in Section 3 we avoid the discussion of singular observers and their characterization in terms of almost observability subspaces. Also, we mention dead-beat observers only in passing. For a full discussion of singular observers, Trumpf [27] is the best reference. Dead beat observers are related by duality to singular observers. For a fairly thorough discussion of this see Fuhrmann and Trumpf [16], where also a deeper study of observability subspaces is undertaken.

The paper is structured as follows. In Section 2 we collect some basic material with special emphasis on subbehaviors and factorizations. This is based on the analysis of behavior homomorphisms introduced in Fuhrmann [10,11]. Section 3 is a summary of the state space approach to observer theory and is an extension of Fuhrmann and Helmke [14]. An underlying theme is the relation between the properties of the observation process (observability, detectability), and the properties of the corresponding observer. This multitiered approach to observers can be found in the previously mentioned thesis of Trumpf as well as, in the context of behaviors, in the Valcher and Willems paper. In the present paper, this is best represented in Theorem 3.3. In Section 4, we outline the behavioral approach to observers. Finally, in Section 5, we analyse the connections to conventional observer theory. A key tool in this is the characterization and the analysis of behavior homomorphisms introduced in [11]. This is also related to a version of the internal model principle. We analyze briefly the connection between state observers and the construction of state maps. Finally, in Section 6, we analyse some simple examples to illustrate the theory.

The author is greatly indebted to Jochen Trumpf for many enlightening conversations that took place over the years that this paper was in writing.

2. Preliminaries

Our interest in this paper is in discrete time systems, therefore we find it unnecessary to restrict ourselves to the real or complex field and we will work with linear spaces over an arbitrary field $\mathbb{F}$.

An infinite sequence $\{x_t\}_{t=1}^{\infty}$, $t \in \mathbb{Z}_+$, $x_t \in \mathbb{F}^n$ is a time trajectory. We associate with it the formal power series $x = \sum_{t=1}^{\infty} x_t z^{-t}$. The space of all such formal power series is $z^{-1}\mathbb{F}[[z^{-1}]]^n$.

Let $\mathbb{F}((z^{-1}))^m$ be the space of vectorial truncated Laurent series. We have, as $\mathbb{F}$-linear spaces, the direct sum representation

$$\mathbb{F}((z^{-1}))^m = \mathbb{F}[z]^m \oplus z^{-1}\mathbb{F}[[z^{-1}]]^m. \tag{1}$$

We denote by π_+ and π_- the projections of $\mathbb{F}((z^{-1}))^m$ on $\mathbb{F}[z]^m$ and $z^{-1}\mathbb{F}[[z^{-1}]]^m$ respectively. Clearly, π_+ and π_- are complementary projections. At some point we find it convenient to use row space version of the above spaces. In particular, $\mathbb{F}_r[z]^m$ is the space of m-row vectors with entries in $\mathbb{F}[z]$.

The space $\mathbb{F}((z^{-1}))^m$ is endowed with a natural $\mathbb{F}[z]$-module structure, given by multiplication with $\mathbb{F}[z]^m$ as a submodule. In particular, $S : \mathbb{F}((z^{-1}))^m \longrightarrow \mathbb{F}((z^{-1}))^m$ is defined by

$$Sf(z) = zf(z). \tag{2}$$

As $\mathbb{F}[z]^m$ is an $\mathbb{F}[z]$-submodule, we can induce a module structure on it by restricting the module structure on $\mathbb{F}((z^{-1}))^m$. In particular, we define $S_+ : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m$ by $S_+ = S|\mathbb{F}[z]^m$. We can induce in the space $z^{-1}\mathbb{F}[[z^{-1}]]^m$ an $\mathbb{F}[z]$-module structure via the isomorphism $z^{-1}\mathbb{F}[[z^{-1}]]^m \simeq \mathbb{F}((z^{-1}))^m/\mathbb{F}[z]^m$. This $\mathbb{F}[z]$-module structure is equal to the one induced by the **left** or **backward shift operator** S_- or, for reasons of compatibility with behavioral theory usage, σ defined by

$$S_-h = \sigma h = \pi_- zh, \quad h \in z^{-1}\mathbb{F}[[z^{-1}]]^m. \tag{3}$$

More generally, given $R \in \mathbb{F}[z]^{p\times m}$, we define the **Toeplitz operator** $R(\sigma) : z^{-1}\mathbb{F}[[z^{-1}]]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^p$ by

$$R(\sigma)w = \pi_- Rw, \quad w \in z^{-1}\mathbb{F}[[z^{-1}]]^m. \tag{4}$$

Similarly, given a $p \times m$ rational matrix function G, we define the **Hankel operator** $H_G : \mathbb{F}[z]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^p$ by

$$H_G u = \pi_- Gu, \quad u \in \mathbb{F}[z]^m. \tag{5}$$

We proceed by giving a concise introduction to polynomial and rational models, first introduced in [6].

Given a nonsingular polynomial matrix D in $F[z]^{m\times m}$ we define two projections π_D in $F[z]^m$ and π^D in $z^{-1}F[[z^{-1}]]^m$ by

$$\pi_D f = D\pi_- D^{-1} f \quad \text{for } f \in F^m[z], \tag{6}$$

$$\pi^D h = \pi_- D^{-1}\pi_+ Dh \quad \text{for } h \in z^{-1}F[[z^{-1}]]^m \tag{7}$$

and define two linear subspaces of $\mathbb{F}[z]^m$ and $z^{-1}\mathbb{F}[[z^{-1}]]^m$ by

$$X_D = \operatorname{Im} \pi_D, \tag{8}$$

and

$$X^D = \operatorname{Im} \pi^D. \tag{9}$$

We refer to X_D as a **polynomial model** whereas to X^D as a **rational model**.

An element f of $\mathbb{F}[z]^m$ belongs to X_D if and only if $\pi_+ D^{-1} f = 0$, i.e. if and only if $D^{-1}f$ is a strictly proper rational vector function. Similarly, an element $h \in z^{-1}\mathbb{F}[[z^{-1}]]^m$ belongs to X^D if and only if $D(\sigma)h = \pi_- Dh = 0$, i.e. Dh is a polynomial vector. Thus, for a rational model, we have the kernel representation

$$X^D = \operatorname{Ker} D(\sigma). \tag{10}$$

We turn X_D into an $\mathbb{F}[z]$-module by defining

$$p \cdot f = \pi_D pf \quad \text{for } p \in \mathbb{F}[z],\ f \in X_D. \tag{11}$$

Since $\operatorname{Ker} \pi_D = D\mathbb{F}^m[z]$ it follows that X_D is isomorphic to the quotient module $\mathbb{F}[z]^m/D\mathbb{F}[z]^m$. Similarly, we introduce in X^D a module structure by

$$p \cdot h = \pi_- ph \quad \text{for } p \in \mathbb{F}[z], h \in X^D. \tag{12}$$

As a consequence, we have also the following description of the polynomial model X_D, namely

$$X_D = \{f \in \mathbb{F}[z]^m | f = Dh, h \in z^{-1}\mathbb{F}[[z^{-1}]]^m\}. \tag{13}$$

The advantage of this characterization is that it makes sense for an arbitrary, rectangular, $p \times m$ polynomial matrix R. Thus, following Emre and Hautus [5], we define

$$X_R = \{f \in \mathbb{F}[z]^p | f = Rh, h \in z^{-1}\mathbb{F}[[z^{-1}]]^m\}. \tag{14}$$

This definition extends the concept of a polynomial model. In a similar fashion, we can extend, using (10), the definition of rational models to the case of rectangular symbols by defining

$$X^R = \operatorname{Ker} R(\sigma). \tag{15}$$

This class of subspaces turns out to be the same as behaviors, to be discussed in Section 2.2.

In the polynomial model X_D we will focus on a special map S_D, a generalization of the classical companion matrix, which corresponds to the action of the identity polynomial z, i.e.,

$$S_D f = \pi_D z f \quad \text{for } f \in X_D. \tag{16}$$

It is easily checked that

$$(S_D f)(z) = zf(z) - D(z)\xi_f, \tag{17}$$

where the constant vector ξ_f depends linearly on f. In fact we have $\xi_f = \pi_+ zD(z)^{-1} f$. It follows from (16) that the module structure in X_D is identical to the module structure induced by S_D through $p \cdot f = p(S_D) f$. With this definition the study of S_D is identical to the study of the module structure of X_D. In particular the invariant subspaces of S_D are just the submodules of X_D and they are related to factorizations of the polynomial matrix D.

Similarly, we introduce in X^D a module structure, given by

$$S_D h = \pi_- z h \quad h \in X^D, \tag{18}$$

i.e. S^D is the restriction of the backward shift operator to the backward shift invariant subspace X^D.

Polynomial and rational models are similar, the similarity given by the invertible map $\rho_D : X^D \longrightarrow X_D$ defined by $\rho_D f = D^{-1} f$. Moreover we have $S_D \rho_D = \rho_D S^D$, i.e. ρ_D is an $F[z]$-isomorphism.

2.1. The shift realization

The following is a version of the shift realization as proved in [6,7].

Theorem 2.1. *Let $G = VT^{-1}U + W$ be a representation of a proper, $p \times m$ rational function. With this representation we associate, see [25], the* **polynomial system matrix** $\begin{pmatrix} T & -U \\ V & W \end{pmatrix}$. *In the state space X_T we define a system by*

$$\begin{cases} Af = S_T f & f \in X_T, \\ B\xi = \pi_T U \xi, & \xi \in \mathbb{F}^m, \\ Cf = (VT^{-1} f)_{-1} & f \in X_T, \\ D = G(\infty). \end{cases} \tag{19}$$

Then this is a realization of G. This realization is observable if and only if V and T are right coprime and it is reachable if and only if T and U are left coprime. We will call (19) the **shift realization** *and denote it by $\Sigma(VT^{-1}U + W)$.*

We note that no coprimeness assumptions are made as far as the realization itself is concerned. The coprimeness assumptions relate to reachability and observability. In particular, this allows us to realize systems with no inputs or outputs. This of course turns out to be very useful when dealing with behaviors. An extreme case would be that of an autonomous behavior $X^T = \operatorname{Ker} T(\sigma)$.

A special case of importance for us is the case of a nonsingular polynomial matrix $T(z)$ considered as a left denominator of a matrix fraction. We define the pair (C_T, A_T), acting in the state space X_T, by

$$\begin{cases} A_T f = S_T f & f \in X_T, \\ C_T f = (T^{-1} f)_{-1} & f \in X_T. \end{cases} \tag{20}$$

Note that in the realization (20) the pair (C_T, A_T) depends only on T. An isomorphic pair is obtained by taking the state space to be the rational model X^T with (C^T, A^T) defined by

$$\begin{cases} A^T f = S^T f & f \in X_T, \\ C^T f = (f)_{-1} & f \in X^T. \end{cases} \tag{21}$$

2.2. Behaviors and B-homomorphisms

Behaviors are a relatively recent addition to the systems and control literature. We find that its importance lies not only in the new paradigm for linear systems, dispensing with input/output thinking, but also in its natural appearance in the analysis of various problems in the classical formulation, even within the state space approach. In particular, in connection with geometric control theory. In this subsection we review briefly the results needed in the sequel.

In $z^{-1}\mathbb{F}[[z^{-1}]]^m$ we define the projections P_n, $n \in \mathbf{Z}_+$ by

$$P_n \sum_{i=1}^{\infty} \frac{h_i}{z^i} = \sum_{i=1}^{n} \frac{h_i}{z^i}. \tag{22}$$

We say that a subset $\mathscr{B} \subset z^{-1}\mathbb{F}[[z^{-1}]]^m$ is **complete** if for any $w = \sum_{i=1}^{\infty} w_i z^{-i} \in z^{-1}\mathbb{F}^m[[z^{-1}]]$ and for each positive integer N, $P_N w \in P_N(\mathscr{B})$ implies $w \in \mathscr{B}$.

A **behavior** in our context is defined as a linear, shift invariant and complete subspace of $z^{-1}\mathbb{F}[[z^{-1}]]^m$. Behaviors can be algebraically characterized. A basic result of behavioral theory, see Willems [29, Theorem 5] or [11], is that a subspace $\mathscr{B} \subset z^{-1}\mathbb{F}[[z^{-1}]]^m$ is a behavior if and only if it admits a kernel representation of the form $\mathscr{B} = \operatorname{Ker} R(\sigma)$, where $R(z)$ is a polynomial matrix. A behavior $\mathscr{B}$ is **autonomous** if it is finite dimensional. A behavior $\mathscr{B} \subset z^{-1}\mathbb{F}[[z^{-1}]]^m$ is autonomous if and only if it is a rational model. Using (10), and to empasize the connection of behaviors to rational models, we will use also the notation $X^R = \operatorname{Ker} R(\sigma)$.

A central tool in behavior theory, introduced in [11] is that of a behavior homomorphism. Given two behaviors $\mathscr{B}_1, \mathscr{B}_2$, we define for the backward shift operator σ its restriction to the behaviors by $\sigma^{\mathscr{B}_i} = \sigma|\mathscr{B}_i$. If the behaviors are given in kernel representations $\mathscr{B}_i = \operatorname{Ker} P_i(\sigma)$, we will write also σ^{P_i} for $\sigma^{\mathscr{B}_i}$. A behavior homomorphism Z is an $\mathbb{F}[z]$- homomorphism with respect to the natural $\mathbb{F}[z]$-module structure in the behaviors, i.e. it satisfies $Z\sigma^{P_1} = \sigma^{P_2}Z$. Our interest is in the characterization of behavior homomorphisms. It turns out that no general characterization of behavior homomorphisms is available. However, adding some continuity constraints makes the problem tractable by duality theory. The appropriate continuity is with respect to the w^* topologies on the two behaviors. For a full discussion of this see Fuhrmann [12]. Thus we can state.

Theorem 2.2. *Let $M \in \mathbb{F}[z]^{p\times m}$ and $\overline{M} \in \mathbb{F}[z]^{\bar{p}\times\bar{m}}$ be of full row rank. Then* $\operatorname{Ker} M(\sigma)$ *is an $\mathbb{F}[z]$-submodule of $z^{-1}\mathbb{F}[[z^{-1}]]^m$ and* $\operatorname{Ker} \overline{M}(\sigma)$ *is an $\mathbb{F}[z]$-submodule of $z^{-1}\mathbb{F}[[z^{-1}]]^{\bar{m}}$. Moreover $Z : \operatorname{Ker} M(\sigma) \longrightarrow \operatorname{Ker} \overline{M}(\sigma)$ is a continuous behavior homomorphism, if and only if there exist $\overline{U} \in \mathbb{F}[z]^{\bar{p}\times p}$ and U in $\mathbb{F}[z]^{\bar{m}\times m}$ such that*

$$\overline{U}(z)M(z) = \overline{M}(z)U(z) \tag{23}$$

and

$$Zh = U(\sigma)h \quad h \in \operatorname{Ker} M(\sigma). \tag{24}$$

The next theorem, see [11], summarizes the invertibility properties of continuous behavior homomorphisms.

Theorem 2.3. *Given two full row rank polynomial matrices $M \in \mathbb{F}[z]^{p\times m}$, $\overline{M} \in \mathbb{F}[z]^{\bar{p}\times\bar{m}}$ describing the behaviors $\mathscr{B} = \operatorname{Ker} M(\sigma)$ and $\overline{\mathscr{B}} = \operatorname{Ker} \overline{M}(\sigma)$ respectively. Let $\overline{U}, U$ be appropriately sized polynomial matrices satisfying*

$$\overline{U}(z)M(z) = \overline{M}(z)U(z), \tag{25}$$

and let $Z : \mathrm{Ker}\, M(\sigma) \longrightarrow \mathrm{Ker}\, \overline{M}(\sigma)$ *be the continuous behavior homomorphism defined by*

$$Zh = U(\sigma)h = \pi_- Uh \quad h \in \mathrm{Ker}\, M(\sigma). \tag{26}$$

Then

1. *Z is injective if and only if M, U are right coprime.*
2. *Z is surjective if and only if $\overline{U}, \overline{M}$ are left coprime and*
$$\mathrm{Ker}\begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix} = \mathrm{Im}\begin{pmatrix} M(z) \\ U(z) \end{pmatrix}. \tag{27}$$
3. *Z as defined above is the zero map if and only if, for some appropriately sized polynomial matrix $L(z)$, we have*
$$U(z) = L(z)M(z). \tag{28}$$
4. *Z defined in (26) is invertible if and only if there exists a doubly unimodular embedding*
$$\begin{pmatrix} \overline{X} & -\overline{Y} \\ -\overline{U} & \overline{M} \end{pmatrix}\begin{pmatrix} M & Y \\ U & X \end{pmatrix} = \begin{pmatrix} M & Y \\ U & X \end{pmatrix}\begin{pmatrix} \overline{X} & -\overline{Y} \\ -\overline{U} & \overline{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{29}$$
of $\begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix}$ *and* $\begin{pmatrix} M(z) \\ U(z) \end{pmatrix}$.
5. *If Z is invertible, then in terms of the doubly unimodular embedding (29), its inverse Z^{-1} : $\mathrm{Ker}\, \overline{M}(\sigma) \longrightarrow \mathrm{Ker}\, M(\sigma)$ is given by*
$$Z^{-1} = -\overline{Y}(\sigma). \tag{30}$$

Note that a special case of Theorem 2.3 is the following, well known, relation between factorizations of polynomial matrices and behavior inclusions. The inclusion $\mathrm{Ker}\, M_2(\sigma) \subset \mathrm{Ker}\, M(\sigma)$ is equivalent to a factorization $M(z) = M_1(z)M_2(z)$.

For the purpose of characterizing continuous behavior homomorphisms when the (controllable) behaviors are given in terms of image representations as well as for the proof of the elimination theorem, we present a short discussion of annihilators. Any $\mathbb{F}[z]$-submodule $\mathscr{M} \subset \mathbb{F}[z]^m$ has a representation $\mathscr{M} = M(z)\mathbb{F}[z]^k$ for some $m \times k$ polynomial matrix. If we require M to have full column rank, then it has a factorization of the form $M(z) = \overline{M}(z)E(z)$ with E nonsingular and $\overline{M}$ right prime. We call such a factorization an **external/internal factorization**. $\overline{M}(z)$ is uniquely determined up to a right unimodular factor. Similarly we can define internal/external factorizations. Given a $p \times m$ polynomial matrix $R(z)$, the set $\mathscr{M} = \{f \in \mathbb{F}[z]^m | R(z)f(z) = 0\}$ is a submodule, hence has a representation $\mathscr{M} = M(z)\mathbb{F}[z]^k$, with $M(z)$ not only of full column rank but necessarily left prime. We call M a **minimal right annihilator** or **MRA** for short. Similarly, given a $p \times m$ polynomial matrix $R(z)$, we say M is a **minimal left annihilator**, or **MLA** for short, if $\tilde{M}$ is a MRA of $\tilde{R}$. Here $\tilde{R}$ denotes the transpose of the polynomial matrix R. Note that a MLA is always left prime. The concepts of minimal left/right annihilators was introduced, in the context of behavior theory, in [24].

Computation of minimal left annihilators is closely related to coprime factorizations. In fact, we have the following simple lemma.

Proposition 2.1. *Let*

$$\overline{D}^{-1}\overline{N} = ND^{-1} \tag{31}$$

be coprime factorizations. Then

1. $\begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix}$ *is a MLA of* $\begin{pmatrix} D \\ N \end{pmatrix}$.
2. $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ *is a rational left annihilator of* $\begin{pmatrix} D \\ N \end{pmatrix}$ *if and only if*
$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = W \begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix}, \tag{32}$$
where W is an arbitrary, appropriately sized, rational function.

Proof

1. Eq. (31) can be rewritten as $\begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix} \begin{pmatrix} D \\ N \end{pmatrix} \begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix} = 0$, which shows that $\begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix}$ is a left annihilator.
Let now $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ be any polynomial left annihilator of $\begin{pmatrix} D \\ N \end{pmatrix}$. Extend (31) to a doubly coprime factorization
$$\begin{pmatrix} -\overline{N} & \overline{D} \\ X & -Y \end{pmatrix} \begin{pmatrix} \overline{Y} & D \\ \overline{X} & N \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{33}$$
We have a unique representation
$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} = Q \begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix} + R \begin{pmatrix} X & -Y \end{pmatrix}$$
Since $(P_1 \quad P_2)$ is a left annihilator, we have
$$\begin{aligned} 0 &= \begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} D \\ N \end{pmatrix} \\ &= \left[Q \begin{pmatrix} -\overline{N} & \overline{D} \end{pmatrix} + R \begin{pmatrix} X & -Y \end{pmatrix} \right] \begin{pmatrix} D \\ N \end{pmatrix} = R, \end{aligned}$$
so $(P_1 \quad P_2) = Q(-\overline{N} \quad \overline{D})$, which shows that $(-\overline{N} \quad \overline{D})$ is a MLA.
2. If $(Z_1 \quad Z_2)$ has the representation (32), then it clearly is a left annihilator.
Conversely, assume $(Z_1 \quad Z_2)$ is a rational left annihilator. Let $E(z)$ be any nonsingular polynomial matrix for which $E(Z_1 \quad Z_2) = (P_1 \quad P_2)$ is polynomial. Since $(P_1 \quad P_2)$ is a left annihilator, we have $(P_1 \quad P_2) = Q(-\overline{N} \quad \overline{D})$. In turn, this implies (32) with $W = E^{-1}Q$. □

In the next theorem we characterize continuous behavior homomorphisms when the behaviors are given in terms of image representations. Of course, this means that we are dealing with the special situation of controllable behaviors. It is well known that if a behavior has an image representation $\mathscr{B} = \operatorname{Im} N(\sigma)$, it has also one with the polynomial matrix being right prime, see [20]. In this case, there exists a doubly unimodular embedding of the form

$$\begin{pmatrix} \overline{M} \\ M \end{pmatrix} \begin{pmatrix} N & \overline{N} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{34}$$

For more on doubly unimodular embeddings, see [11]. If $M(z)$ is a MLA of $N(z)$, then we have $\operatorname{Im} N(\sigma) = \operatorname{Ker} M(\sigma)$. We use this passage from image to kernel representations to prove the following theorem.

Theorem 2.4. *Given two behaviors in image* (*MA*) *representations*

$$\mathscr{B}_i = \operatorname{Im} N_i(\sigma), \quad i = 1, 2, \tag{35}$$

where we assume the N_i are right prime polynomial matrices. Then a map $Z : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$ is a continuous behavior homomorphism if and only if there exist appropriately sized polynomial matrices U_2, V_1 such that

$$U_2(z)N_1(z) = N_2(z)V_1(z) \tag{36}$$

and

$$Zh = U_2(\sigma)h \quad h \in \mathscr{B}_1. \tag{37}$$

Proof. Let M_1, M_2 be the minimal left annihilators of N_1, N_2 respectively. Assume $Z : \operatorname{Im} N_1(\sigma) \longrightarrow \operatorname{Im} N_2(\sigma)$ is given by (37) with (36) holding. Applying $M_2(z)$ to both sides of the equality $U_2(z)N_1(z) = N_2(z)V_1(z)$, and using $M_2(z)N_2(z) = 0$, we obtain $0 = M_2(z)(U_2(z)N_1(z)) = (M_2(z)U_2(z))N_1(z)$. Since M_1 is a MLA of N_1, there exists a polynomial matrix W_2 satisfying

$$W_2(z)M_1(z) = M_2(z)U_2(z). \tag{38}$$

By Theorem 2.2, the map $Z = U_2(\sigma)$ is continuous.

Conversely, assume $Z : \operatorname{Im} N_1(\sigma) \longrightarrow \operatorname{Im} N_2(\sigma)$ is a continuous behavior homomorphism. Since $\operatorname{Im} N_i(\sigma) = \operatorname{Ker} M_i(\sigma)$, $i = 1, 2$, we can apply Theorem 2.2 to conclude the existence of polynomial matrices W_2, U_2 for which (38) holds and in terms of which $Z = U_2(\sigma)$. □

An indispensible tool for the analysis of behaviors is the elimination theorem, see [[30], Prop. 4.1.c], [20] and Polderman [22], which gives a procedure for the elimination of latent variables and thus allows passing to a kernel representation. For a new proof, using the analysis of invertibility of B-homomorphisms, see [15].

Theorem 2.5. *Let a behavior $\mathscr{B}$ be given by the latent variable representation*

$$Q(\sigma)w = P(\sigma)\xi, \tag{39}$$

i.e. $\mathscr{B} = \{w \mid \exists \xi;\ Q(\sigma)w = P(\sigma)\xi\}$. Let $N(z)$ be a MLA of $P(z)$ and define $R(z) = N(z)Q(z)$. Then

1. *We have the equality*

   $$N(z)\begin{pmatrix} Q(z) & -P(z) \end{pmatrix} = R(z)\begin{pmatrix} I & 0 \end{pmatrix}, \tag{40}$$

   with

   $$\operatorname{Ker}\begin{pmatrix} -N(z) & R(z) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} Q(z) & -P(z) \\ I & 0 \end{pmatrix} \tag{41}$$

   holding.
2. *The projection $\pi_w : \operatorname{Ker}(Q(\sigma) \quad -P(\sigma)) \longrightarrow \operatorname{Ker} R(\sigma)$, defined by $\pi_w\binom{w}{x} = (I \quad 0)\binom{w}{x} = w$ is a surjective B-homomorphism.*
3. *A kernel representation of $\mathscr{B}$ is given by*

   $$\mathscr{B} = \operatorname{Ker} R(\sigma). \tag{42}$$

We know that, given a module X, a subset $M \subset X$ is a submodule if and only if M is the kernel of a module homomorphism. We would like to state the analog of this result in the behavioral setting. There is however a difficulty, as noted in the analysis of subbehaviors in [11], stemming from the fact that not every shift invariant subspace of a behavior is itself a behavior. It is quite easy to resolve this difficulty by restricting ourselves to continuous behavior homomorphisms, and we state the following.

Proposition 2.2. *Given a behavior* $\mathscr{B} = \operatorname{Ker} M(\sigma)$. *Then*

1. (a) *If* $M = M_2M_1$ *and* $\mathscr{B}_1 = \operatorname{Ker}[M_1(\sigma)|\operatorname{Ker} M(\sigma)]$, *then* $Z : \mathscr{B} \longrightarrow \mathscr{B}_2 = \operatorname{Ker} M_2(\sigma)$ *defined by*
$$Zh = M_1(\sigma)h, \quad h \in \operatorname{Ker} M(\sigma) \tag{43}$$
is a continuous behavior homomorphism with $\operatorname{Ker} Z = \mathscr{B}_1$.
   (b) *Under the previous assumptions, assuming additionally that in the factorization* $M = M_2M_1$, M_1 *has full row rank and* M_2 *has full column rank, we have the module isomorphism*
$$\operatorname{Ker} M_2(\sigma) \simeq \operatorname{Ker} M(\sigma)/\operatorname{Ker} M_1(\sigma). \tag{44}$$
   (c) *A subset* $\mathscr{B}_0 \subset \mathscr{B}$ *is a subbehavior if and only if* $\mathscr{B}_0$ *is the kernel of a continuous behavior homomorphism.*
2. (a) *If* $\mathscr{B} = \operatorname{Ker} M(\sigma)$ *and* $\mathscr{B}_1 = \operatorname{Ker} M_1(\sigma)$ *for a factorization* $M = M_2M_1$ *then* $\mathscr{B}_1 = \operatorname{Im} Z$ *where* $Z : \mathscr{B}_1 \longrightarrow \mathscr{B}$ *is the continuous behavior homomorphism defined by*
$$Zh = h, \quad h \in \mathscr{B}_1. \tag{45}$$
   (b) *A subset* $\mathscr{B}_1 \subset \mathscr{B}$ *is a subbehavior if and only if* $\mathscr{B}_1$ *is the image of a continuous behavior homomorphism.*

Proof

1. (a) Note that $\operatorname{Ker} M_1(\sigma) \subset \operatorname{Ker} M(\sigma)$ and hence $\operatorname{Ker} M_1(\sigma)|\operatorname{Ker} M(\sigma) = \operatorname{Ker} M_1(\sigma)$. Writing $I(M_2M_1) = M_2 \cdot M_1$, it follows from Theorem 2.2 that Z, defined by (43), is a continuous behavior homomorphism with $\operatorname{Ker} Z = \operatorname{Ker} M_1(\sigma)$. Clearly we have $\operatorname{Ker}(I \quad -M_2(z)) = \operatorname{Im}\begin{pmatrix} M_2(z) \\ I \end{pmatrix}$. If $M_1(z) \in \mathbb{F}[z]^{k\times m}$ is left prime, then $\operatorname{Im} M_1(\sigma) = z^{-1}\mathbb{F}[[z^{-1}]]^k$.
   (b) The extra assumption guarantees that the homomorphism Z, defined by (43), is surjective, hence the isomorphism (44).
   (c) Assume that $\mathscr{B}_1 \subset \mathscr{B} = \operatorname{Ker} M(\sigma)$ is a subbehavior. Thus there exists a factorization $M = M_2M_1$ for which $\mathscr{B}_1 = \operatorname{Ker} M_1(\sigma)$. Let $Z : \mathscr{B} \longrightarrow \mathscr{B}_2 = \operatorname{Ker} M_2(\sigma)$ be defined by (43), then Z is a continuous behavior homomorphism with $\operatorname{Ker} Z = \mathscr{B}_1$.
   Conversely, assume $\mathscr{B}_1 \subset \mathscr{B}$ is the kernel of a continuous behavior homomorphism $Z : \mathscr{B} = \operatorname{Ker} M(\sigma) \longrightarrow \overline{\mathscr{B}} = \operatorname{Ker} \overline{M}(\sigma)$ with $\mathscr{B}_1 = \operatorname{Ker} Z$. By Theorem 2.2, there exist polynomial matrices $\overline{U}, U$ satisfying $\overline{U}M = \overline{M}U$ and for which $Z = U(\sigma)$. Thus
$$\begin{aligned} \operatorname{Ker} Z &= \{h \in \operatorname{Ker} M(\sigma) | U(\sigma)h = 0\} \\ &= \operatorname{Ker} M(\sigma) \cap \operatorname{Ker} U(\sigma) = \operatorname{Ker} \begin{pmatrix} M(\sigma) \\ U(\sigma) \end{pmatrix}. \end{aligned}$$
   This shows that $\operatorname{Ker} Z$ is a behavior. That $\operatorname{Ker} Z \subset \mathscr{B}$ is trivial, and it can also be seen from the factorization $M(z) = (I \quad 0)\begin{pmatrix} M(z) \\ U(z) \end{pmatrix}$.

2. (a) Clearly, $M_2M_1 = MI$, it follows from Theorem 2.2 that the map Z defined by (45) is a continuous behavior homomorphism. Moreover, we have $\operatorname{Im} Z = \operatorname{Ker} M_1(\sigma)$.
   (b) That a subbehavior is the image of a continuous behavior homomorphism follows from the previous part.
   To prove the converse, let $Z : \operatorname{Ker} \overline{M}(\sigma) \longrightarrow M(\sigma)$ be a continuous behavior homomorphism. Thus there exist polynomial matrices $\overline{U}, U$ satisfying $\overline{U}M = \overline{M}U$ and for which $Z = U(\sigma)|\operatorname{Ker} M(\sigma)$. Thus

$$\begin{aligned}\operatorname{Im} Z &= U(\sigma)\operatorname{Ker} M(\sigma)\\ &= \left\{w \mid \begin{pmatrix}0\\ I\end{pmatrix} w = \begin{pmatrix}M(\sigma)\\ U(\sigma)\end{pmatrix}\xi\right\}\end{aligned}$$

   Thus $\operatorname{Im} Z$ is clearly a behavior as it has an ARMA representation. □

Corollary 2.1. *Let $P(s) \in \mathbb{F}[z]^{p\times m}$ and $R(s) \in \mathbb{F}[z]^{r\times m}$. Then*

$$\operatorname{Ker}\begin{pmatrix}P(\sigma)\\ R(\sigma)\end{pmatrix} \subset \operatorname{Ker} P(\sigma). \tag{46}$$

If $P_1 = g.c.r.d(P, R)$ and $P = P_2P_1$, then we have the isomorphism

$$\operatorname{Ker} P(\sigma)/\operatorname{Ker}\begin{pmatrix}P(\sigma)\\ R(\sigma)\end{pmatrix} \simeq \operatorname{Ker} P_2(\sigma). \tag{47}$$

Proof. We have $\operatorname{Ker}\begin{pmatrix}P(\sigma)\\ R(\sigma)\end{pmatrix} = \operatorname{Ker} P(\sigma) \cap \operatorname{Ker} R(\sigma) = \operatorname{Ker} P_1(\sigma)$. By Proposition 2.2, we have (47). □

Before proceeding, we state and prove a technical lemma, a slight refinement of Lemma 6.2 in [11].

Lemma 2.1. *Let $Q \in \mathbb{F}[z]^{p\times m}$ be of full row rank and have degree l, i.e. $Q(z) = Q_0 + Q_1z + \cdots + Q_lz^l$. Then $h = \sum_{j=1}^{\infty} \frac{h_j}{z^j} \in \operatorname{Ker} Q(\sigma)$ and $h_1 = \cdots = h_l = 0$ imply $h = 0$ if and only if Q is a square, nonsingular polynomial matrix.*

Proof. Assume Q is a square, nonsingular polynomial matrix. We have $\operatorname{Ker} Q(\sigma) = X^Q = \operatorname{Im} \pi^Q$. So $h \in X^Q$ if and only if $h = \pi^Q h$. However, under our assumptions $\pi^Q h = \pi_- Q^{-1}\pi_+ Qh = \pi_- Q^{-1}\pi_+(Qz^{-l})(z^l h)$. Clearly Qz^{-l} is proper whereas $z^l h$ is strictly proper, so the product is strictly proper and $\pi_+(Qz^{-l})(z^l h) = 0$ which implies $h = 0$.

Conversely, assume $h \in \operatorname{Ker} Q(\sigma)$ and $h_1 = \cdots = h_l = 0$ implies $h = 0$. As Q is assumed to be of full row rank, we may, by permuting variables, assume without loss of generality that $Q = (D \quad -N)$ with D nonsingular and $D^{-1}N$ proper. Writing $h = \begin{pmatrix}h'\\ h''\end{pmatrix}$ in agreement with the previous representation, it is clear that h'' can be chosen arbitrarily. Thus under our assumption we must have $Q = D$, i.e. Q is square, nonsingular. □

2.3. Behaviors and geometric control

Geometric control has been introduced by Basile and Marro [2], as well as by Wonham and Morse [34], as a tool for the study of linear systems. The introduction of behaviors in [30,31] was not only yet another tool but represented a paradigm shift from input/output descriptions to a somewhat more general conception of what a linear system is, as distinct from its many possible representations. For our purpose, namely the study of observers, we would like to understand how the basic concepts of geometric control, those that play a significant role in observer theory, appear in the behavioral context.

Since behaviors are actually spaces of trajectories of system variables, and as geometric control deals with trajectories belonging to subspaces, it is only to be expected that the behavioral language should be amenable to the description of geometric objects.

To begin, assume we look at an autonomous system, given by the dynamic equation $x_{k+1} = Ax_k$ or, in behavioral notation by the dynamic equation $\sigma x = Ax$. Clearly, the corresponding behavior is given by $\mathscr{B} = \operatorname{Ker}(\sigma I - A)$. We note that

$$\operatorname{Ker}(\sigma I - A) = \{(zI - A)^{-1}x | x \in \mathbb{C}^n\} = \left\{\sum_{i=0}^{\infty} \frac{A^i x}{z^{i+1}} | x \in \mathbb{C}^n\right\}. \tag{48}$$

If $d(z) = \det(zI - A)$ and $d = d_2 d_1$ is a factorization into coprime factors, then it induces, essentially unique, factorizations

$$zI - A = \overline{S}_2(z)S_1(z) = \overline{S}_1(z)S_2(z) \tag{49}$$

with $d_i = \det \overline{S}_i = \det S_i$. These factorizations imply the direct sum decompositions

$$\operatorname{Ker}(\sigma I - A) = \operatorname{Ker} S_1(\sigma) \oplus \operatorname{Ker} S_2(\sigma). \tag{50}$$

We refer to (50) as a **spectral decomposition**. Of course, this is just the behavioral representation of the spectral decomposition

$$\mathbb{C}^n = X_1(A) \oplus X_2(A), \tag{51}$$

where $X_i(A) = \operatorname{Ker} d_i(A)$, $i = 1, 2$. As a matter of fact, we note that the map $\phi : \mathbb{C}^n \longrightarrow \operatorname{Ker}(\sigma I - A)$ defined by

$$\phi(x) = (zI - A)^{-1}x \tag{52}$$

is an isomorphism satisfying

$$\phi(Ax) = \sigma\phi(x). \tag{53}$$

Thus we have

$$\phi(X_i(A)) = \operatorname{Ker} S_i(\sigma), \quad i = 1, 2. \tag{54}$$

The following simple, but very useful, lemma is a slight extension of Lemma 5.1 in [14].

Lemma 2.2. *Let A be a linear transformation in a linear space $\mathscr{X}$, W and Z linear transformations from $\mathscr{X}$ to $\mathscr{X}'$. Then the following statements are equivalent.*

1. *We have*

$$A \operatorname{Ker} W \subset \operatorname{Ker} Z. \tag{55}$$

2. *We have*

$$\operatorname{Ker} W \subset \operatorname{Ker} ZA. \tag{56}$$

3. *There exists a linear transformation F in $\mathscr{X}'$ such that*

$$ZA = FW. \tag{57}$$

Let $\mathscr{X}$, $\mathscr{Y}$ be $\mathbb{F}$-linear spaces and let $A : \mathscr{X} \longrightarrow \mathscr{X}$ and $C : \mathscr{X} \longrightarrow \mathscr{Y}$ be linear maps. The pair (C, A) represents the state/output linear system

$$\begin{cases} \sigma x = Ax, \\ y = Cx. \end{cases} \tag{58}$$

We say that a subspace $\mathscr{V} \subset \mathscr{X}$ is **conditioned invariant** for the pair (C, A) if there exists an **output injection map** $L : \mathscr{Y} \longrightarrow \mathscr{X}$ such that $(A - LC)\mathscr{V} \subset \mathscr{V}$. Such a map L is called a **friend** of $\mathscr{V}$. Note that, by Lemma 2.2, as $\operatorname{Ker}\begin{pmatrix} Z \\ C \end{pmatrix} = \operatorname{Ker} Z \cap \operatorname{Ker} C$, the subspace $\mathscr{V} = \operatorname{Ker} Z$ is conditioned invariant if and only if $A \operatorname{Ker}\begin{pmatrix} Z \\ C \end{pmatrix} \subset \operatorname{Ker} Z$. A subspace $\mathscr{V}$ is **outer detectable** if there exists an output injection map L such that $\mathscr{V}$ is $(A - LC)$-invariant and the induced map in the quotient space, $(A - LC)|_{\mathscr{X}/\mathscr{V}}$, is stable. Similarly, a subspace $\mathscr{V}$ is **outer reconstructible** if there exists an output injection map L such that $\mathscr{V}$ is $(A - LC)$-invariant and the induced map in the quotient space, $(A - LC)|_{\mathscr{X}/\mathscr{V}}$, is monomic. Finally if for any polynomial d satisfying $\deg d = \operatorname{codim} \mathscr{V}$, there exists a friend L of $\mathscr{V}$ for which d is the characteristic polynomial of $(A - LC)|_{\mathscr{X}/\mathscr{V}}$ then $\mathscr{V}$ will be called an **observability subspace**. Actually an **outer observability subspace** would be a better description.

The sets of conditioned invariant, outer detectability, outer reconstructibility and outer observability subspaces are all closed under intersections, thus in each class there exists a smallest element. The smallest conditioned invariant subspace is clearly $\{0\}$. The other subspaces may be nonzero and will be denoted by $\mathscr{D}_*$, $\mathscr{W}_*$, $\mathscr{O}_*$ respectively. The smallest outer observability subspace is given by $\cap_{i=0}^{\infty} \operatorname{Ker} CA^i$. The smallest outer detectability subspace is given by

$$\mathscr{D}_* = X_+(A) \cap \operatorname{Ker} C = X_+(A) \cap (\cap_{i=0}^{\infty} \operatorname{Ker} CA^i) = X_+(A) \cap \mathscr{O}_*, \tag{59}$$

where $X_+(A)$ is the A-invariant subspace of nonstable vectors. Clearly, the pair (C, A) is observable if and only if $\{0\}$ is the smallest outer observability subspace. The concepts of controlled and conditioned invariant subspaces were first introduced by [2].

Define the **nonobservable subspace** of the pair (C, A) by

$$\mathscr{O}_* = \cap_{i=0}^{\infty} \operatorname{Ker} CA^i. \tag{60}$$

Clearly, $\mathscr{O}_*$ is an A-invariant subspace contained in $\operatorname{Ker} C$, in fact the largest such subspace. At the same time $\mathscr{O}_*$ is the smallest outer observability subspace in $\mathbb{F}^n$. To see this, note that $\mathscr{O}_*$ is A-invariant and, taking an arbitrary complementary subspace $\mathscr{W}$ of $\mathscr{O}_*$, we have with respect to the direct sum representation

$$\mathbb{F}^n = \mathscr{W} \oplus \mathscr{O}_* \tag{61}$$

the representation

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & 0 \end{pmatrix} \end{aligned} \tag{62}$$

with $\cap_{i=0}^{\infty} \operatorname{Ker} C_1 A_{11}^i = \{0\}$, i.e. with (C_1, A_{11}) an observable pair. Thus, the representation (62) shows that $\mathscr{O}_*$ is indeed the smallest outer observability subspace in $\mathbb{C}^n$.

We note that, as a consequence of (48), we have

$$\operatorname{Ker}\begin{pmatrix} \sigma I - A \\ C \end{pmatrix} = \operatorname{Ker}(\sigma I - A) \cap \operatorname{Ker} C$$

$$= \left\{ \sum_{i=0}^{\infty} \frac{A^i x}{z^{i+1}} \Big| \sum_{i=0}^{\infty} \frac{CA^i x}{z^{i+1}} = 0, x \in \mathbb{C}^n \right\}$$
$$= \left\{ \sum_{i=0}^{\infty} \frac{A^i x}{z^{i+1}} \Big| x \in \cap_{i=0}^{\infty} \mathrm{Ker}\, CA^i \right\}$$
$$= \{(zI - A)^{-1} x | x \in \mathcal{O}_*\}.$$

Clearly, under the isomorphism ϕ, defined in (52), we have

$$\phi(\mathcal{O}_*) = \mathrm{Ker} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix}. \tag{63}$$

Letting

$$\begin{pmatrix} zI - A \\ C \end{pmatrix} = \begin{pmatrix} L_1(z) \\ L_2(z) \end{pmatrix} E(z) \tag{64}$$

be an external/internal factorization, i.e. E is a greatest common right divisor of $C, zI - A$, we have

$$\mathrm{Ker} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} = \mathrm{Ker}\, E(\sigma). \tag{65}$$

Via the isomorphism ϕ of (52), Ker $E(\sigma)$ corresponds to the largest A-invariant subspace of $\mathbb{C}^n$ contained in Ker C, i.e. to $\mathcal{O}_*$. We note that, using the representation (62), we can get a concrete representation of the factorization (64). In fact, we have the factorization

$$\begin{pmatrix} zI - A \\ C \end{pmatrix} = \begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & zI - A_{22} \\ C_1 & 0 \end{pmatrix} = \begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & I \\ C_1 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & zI - A_{22} \end{pmatrix} \tag{66}$$

where clearly $\begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & I \\ C_1 & 0 \end{pmatrix}$ is right prime. This follows from the observability of the pair (C_1, A_{11}).

We proceed next to look at outer detectability suspaces. Assume that the underlying field is the complex field $\mathbb{C}$. The choice of $\mathbb{R}$ does not require major changes. For continuous time systems, a polynomial will be called **stable** if all its zeros lie in the open left half plane, whereas for discrete time systems if all its zeros lie in the open unit disk. Clearly any real or complex polynomial p has a factorization of the form $p = p_- p_+$ with p_- stable and p_+ coprime with all stable polynomials. Given a linear transformation $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, let $d(z) = \det(zI - A)$ be its characteristic polynomial and let $d = d_- d_+$ be its factorization into a stable and antistable factors. We saw already that such a factorization induces essentially unique factorizations

$$zI - A = \overline{S}_+(z) S_-(z) = \overline{S}_-(z) S_+(z), \tag{67}$$

with $\overline{S}_-(z)$, $S_-(z)$ stable and $\overline{S}_+(z)$, $S_+(z)$ antistable. This leads to the spectral decomposition

$$\mathbb{C}^n = X_{zI-A} = \overline{S}_+(z) X_{S_-} \oplus \overline{S}_-(z) X_{S_+} = X_-(A) \oplus X_+(A), \tag{68}$$

where the subspaces $X_-(A)$, $X_+(A)$ are the generalized eigenspaces associated with the sets of stable and antistable eigenvalues respectively.

From the state space point of view, we note that from (61) and (62) we have $\det(zI - A_{22})$ is a left factor of $d(z) = \det(zI - A)$. Thus it also has a factorization $\det(zI - A_{22}) = e_- e_+$. As a consequence, the A-invariant subspace $\mathcal{O}_*$ has a direct sum representation

$$\mathscr{O}_* = \mathscr{O}_*^+ \oplus \mathscr{O}_*^- \tag{69}$$

with $\mathscr{O}_*^\pm$ the spectral subspaces of stable and nonstable vectors. Both $\mathscr{O}_*^\pm$ are of course A-invariant.

The representation (62) has therefore a finer representation given by

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21}^- & A_{22}^- & 0 \\ A_{21}^+ & 0 & A_{22}^+ \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{70}$$

This shows that $\mathscr{O}_*^- \subset \mathscr{O}_*$ is the smallest outer detectability subspace for the pair (C, A). Now it is easy to establish how $\mathscr{O}_*^-$ is represented under the isomorphism ϕ of (52). We use the factorization (67) to get

$$\begin{aligned} \begin{pmatrix} zI - A \\ C \end{pmatrix} &= \begin{pmatrix} \overline{S}_+(z)S_-(z) \\ C \end{pmatrix} = \begin{pmatrix} \overline{S}_-(z)S_+(z) \\ C \end{pmatrix} \\ &= \begin{pmatrix} \overline{S}_+(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_-(z) \\ C \end{pmatrix} = \begin{pmatrix} \overline{S}_-(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_+(z) \\ C \end{pmatrix}. \end{aligned} \tag{71}$$

Since $\begin{pmatrix} \overline{S}_+(z) & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} \overline{S}_-(z) & 0 \\ 0 & I \end{pmatrix}$ are left coprime, applying Theorem 3.4 in [11], we have

$$\operatorname{Ker} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} S_-(\sigma) \\ C \end{pmatrix} \oplus \operatorname{Ker} \begin{pmatrix} S_+(\sigma) \\ C \end{pmatrix}.$$

Let

$$\begin{aligned} \begin{pmatrix} S_+(z) \\ C \end{pmatrix} &= \begin{pmatrix} H_1^+(z) \\ H_2^+(z) \end{pmatrix} D_+(z), \\ \begin{pmatrix} S_-(z) \\ C \end{pmatrix} &= \begin{pmatrix} H_1^-(z) \\ H_2^-(z) \end{pmatrix} D_-(z) \end{aligned}$$

be external/internal factorizations, then it follows that

$$\operatorname{Ker} \begin{pmatrix} S_\pm(\sigma) \\ C \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} H_1^\pm(\sigma) \\ H_2^\pm(\sigma) \end{pmatrix} D_\pm(\sigma) = \operatorname{Ker} D_\pm(\sigma).$$

Thus we have

$$\phi(\mathscr{O}_*^\pm) = \operatorname{Ker} \begin{pmatrix} H_1^\pm(\sigma) \\ H_2^\pm(\sigma) \end{pmatrix} D_\pm(\sigma) = \operatorname{Ker} D_\pm(\sigma) = \{(zI - A)^{-1}x | x \in \mathscr{O}_*^\pm\}.$$

Note that from (71) it follows that

$$\begin{aligned} \begin{pmatrix} zI - A \\ C \end{pmatrix} &= \begin{pmatrix} \overline{S}_+(z)S_-(z) \\ C \end{pmatrix} = \begin{pmatrix} \overline{S}_+(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_-(z) \\ C \end{pmatrix} \\ &= \begin{pmatrix} \overline{S}_+(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} H_1^-(\sigma) \\ H_2^-(\sigma) \end{pmatrix} D_-(\sigma) \end{aligned}$$

and in particular $(zI - A) = \overline{S}_+(z)H_1^-(z)D_-(z)$, i.e. D_- is a nonsingular right factor of $(zI - A)$, hence $\operatorname{Ker} D_-(\sigma)$ is a subbehavior of $\operatorname{Ker}(\sigma I - A)$. By the isomorphism ϕ and (53), it follows that $\mathscr{O}_*^+ = \phi^{-1}(\operatorname{Ker} D_+(\sigma))$ is an A-invariant subspace of $\mathbb{C}^n$. This can easily be identified as

$$\phi^{-1}(\operatorname{Ker} D_+(\sigma)) = \{x \in \mathbb{C}^n | x \in \cap_{i=0}^{\infty} \operatorname{Ker} CA^i \cap X_+(A)\}.$$

One other thing to note is that, given the pair (C, A), $\mathcal{O}_*$ is invariant under output injection. This is clear from the factorization

$$\begin{pmatrix} zI - A + JC \\ C \end{pmatrix} = \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \begin{pmatrix} zI - A \\ C \end{pmatrix},$$

which, using the unimodularity of $\begin{pmatrix} I & J \\ 0 & I \end{pmatrix}$, shows that

$$\operatorname{Ker} \begin{pmatrix} \sigma I - A + JC \\ C \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix}.$$

The same holds of course for the smallest outer detectability subspace. Indeed

$$\begin{pmatrix} zI - A \\ C \end{pmatrix} = \begin{pmatrix} \overline{S}_+(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_-(z) \\ C \end{pmatrix}$$

implies

$$\begin{aligned} \begin{pmatrix} zI - A + JC \\ C \end{pmatrix} &= \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{S}_+(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S_-(z) \\ C \end{pmatrix} \\ &= \begin{pmatrix} \overline{S}_+(z) & J \\ 0 & I \end{pmatrix} \begin{pmatrix} S_-(z) \\ C \end{pmatrix} = \begin{pmatrix} \overline{S}_-(z) & J \\ 0 & I \end{pmatrix} \begin{pmatrix} S_+(z) \\ C \end{pmatrix} \end{aligned}$$

Now $\begin{pmatrix} \overline{S}_+(z) & J \\ 0 & I \end{pmatrix}$, $\begin{pmatrix} \overline{S}_-(z) & J \\ 0 & I \end{pmatrix}$ are clearly left coprime (as $\overline{S}_+(z)$, $\overline{S}_-(z)$ are) and hence we have the direct sum representation

$$\operatorname{Ker} \begin{pmatrix} \sigma I - A + JC \\ C \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} S_-(\sigma) \\ C \end{pmatrix} \oplus \operatorname{Ker} \begin{pmatrix} S_+(\sigma) \\ C \end{pmatrix}. \tag{72}$$

We can summarize the preceding discussion in the following.

Theorem 2.6. *Let (C, A) be a, not necessarily observable, output pair acting in $\mathbb{C}^n$. Let d be the characteristic polynomial of A, factored as $d = d_+d_-$ with the zeros of d_- stable and those of d_+ nonstable. Let $\mathcal{O}_*$ be the smallest outer observability subspace in $\mathbb{C}^n$, $\mathcal{O}_*^-$, $\mathcal{O}_*^+$ be the smallest outer detectability and smallest outer nondetectability subspaces respectively.*

1. *Denote the spectral subspaces by*

$$\begin{aligned} X_-(A) &= \operatorname{Ker} d_-(A), \\ X_+(A) &= \operatorname{Ker} d_+(A), \end{aligned} \tag{73}$$

then

$$\mathbb{C}^n = X_-(A) \oplus X_+(A). \tag{74}$$

2. *The factorization $d = d_+d_-$ induces an essentially unique factorization*

$$(zI - A) = \overline{S}_+(z)S_-(z) = \overline{S}_-(z)S_+(z) \tag{75}$$

with

$$d_\pm(z) = \det \overline{S}_\pm(z) = \det S_\pm(z). \tag{76}$$

3. *Defining the map* $\phi : \mathbb{C}^n \longrightarrow \mathrm{Ker}(\sigma I - A)$ *by* (52). *Then*
   (a) ϕ *is an isomorphism satisfying* (53).
   (b) *We have*

$$\phi(X_{\pm}(A)) = \mathrm{Ker}\, S_{\pm}(\sigma). \tag{77}$$

4. *We have*

$$\text{(a) } \mathrm{Ker}(\sigma I - A) = \{(zI - A)^{-1}x | x \in \mathbb{C}^n\} = \left\{ \sum_{i=0}^{\infty} \frac{A^i x}{z^{i+1}} | x \in \mathbb{C}^n \right\} \tag{78}$$

   (b) $\mathcal{O}_*$, *the smallest outer observability subspace for the pair* (C, A), *has the representation*

$$\mathcal{O}_* = \cap_{i=0}^{\infty} \mathrm{Ker}\, CA^i. \tag{79}$$

   (c) *We have*

$$\mathrm{Ker} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} = \{(zI - A)^{-1}x | x \in \mathcal{O}_*\} \tag{80}$$

$$\text{(d) } \phi(\mathcal{O}_*) = \mathrm{Ker} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} = \mathrm{Ker}\, E(\sigma), \tag{81}$$

   where E *is the g.c.r.d. of* $zI - A$ *and* C.
   (e) *We have*

$$\phi(\mathcal{O}_*^-) = \mathrm{Ker} \begin{pmatrix} S_+(\sigma) \\ C \end{pmatrix} = \mathrm{Ker}\, D_+(\sigma),$$

$$\phi(\mathcal{O}_*^+) = \mathrm{Ker} \begin{pmatrix} S_-(\sigma) \\ C \end{pmatrix} = \mathrm{Ker}\, D_-(\sigma), \tag{82}$$

   where D_+ *is the g.c.r.d. of* S_+ *and* C *and* D_- *is the g.c.r.d. of* S_- *and* C.
5. *The spaces* $\mathcal{O}_*, \mathcal{O}_*^-, \mathcal{O}_*^+$ *are all output injection invariant, i.e. we have* $\mathcal{O}_*(C, A) = \mathcal{O}_*(C, A - JC)$ *etc.*

Remarks

1. The isomorphism ϕ, defined in (52) is related to an all important result of Rota [26] on the universality of the backward shift operator. In this connection, see also [9].
2. A similar analysis of outer reconstructibility subspaces, based on factorizations of a polynomial $d = d_0 d_{nm}$ into a maximal monomic factor d_0 and a factor coprime to it. We omit the details.

3. Observers

3.1. Introduction

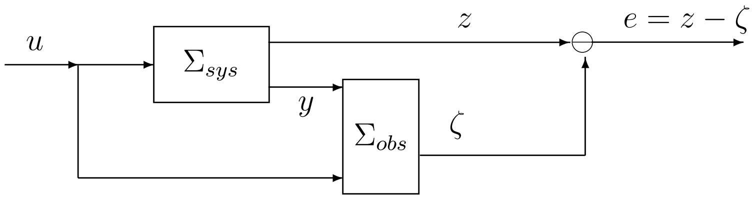
In the later part of this paper we will be interested in the construction of observers for linear dynamical systems given in the behavioral framework. Considering the long history of observer theory within the conventional, input/output framework, a history laden with vagueness, imprecision, incomplete or even false proofs, it seems that in this particular part of system theory the behavioral approach, as developed and presented in the important paper Valcher and Willems

[28], has distinct advantages, both in terms of clarity of concepts as well as in the area of observer design. It is therefore of the utmost importance to fully understand the relation of observer theory in the behavioral setting to the conventional one, and this will be the central theme of this paper. We will draw heavily on the following sources for the conventional theory [1,14,27].

We begin our discussion of observers by outlining the principal results in the state space context. This also allows us to check that the definitions and results in the behavioral framework are consistent with those in the state space.

The problem of observation arises from the fact that, in a given system Σ, driven by the inputs u, the observed variables y are not necessarily the variables z which are unobserved and which we need to estimate. Thus we are after a mechanism that allows us to use observed variables to estimate the variables of interest. Loosely speaking, an observer for the system is itself a linear system Σ_{est}, driven by the variables u, y, and whose output is the desired estimate ζ of z. Thus we have the following scheme.

Diagram 3.1



Of course, the theory of observers depends strongly on the interpretation of what a linear system is and how it is represented. We choose to take the state space representation of a finite dimensional, time invariant linear system as our starting point. Our next choice is to work with discrete time systems as this simplifies matters when moving into the behavioral domain. However, as most of the statements remain true for continuous time systems, we shall adopt the usage of the operator σ which for discrete time is interpreted as the backward shift whereas for continuous time as the differentiation operator. Thus $\sigma x = Ax + Bu$ is interpreted in discrete time as $x_{t+1} = Ax_t + Bu_t$ and in continuous time as $\dot{x} = Ax + Bu$.

3.2. Observation properties

One of Kalman's major achievements is the introduction of the concepts of controllability and observability as distinct from compensator, or observer, design. This separation is reflected in [28] where the observability or detectability of one set of system variables from another is studied before observer design is attempted. We adopt this philosophy in this section.

Clearly, observers depend on observability properties of a system and we shall outline a few gradations of observability. Naturally, one expects that the stronger the observation properties of a system are, the better behaved should be the corresponding observers. How the observation properties of the system are reflected in the corresponding observers will be studied in Section 3.3.

We define now some of the possible observation properties, i.e. to what extent do the observed variables y determine the relevant variables z.

Definition 3.1. Given a linear system Σ in the state space representation

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx, \end{cases} \tag{83}$$

x, y, u, z taking values in $\mathbb{F}^n, \mathbb{F}^p, \mathbb{F}^m, \mathbb{F}^k$ respectively. We say that

1. z is **trackable** from $\binom{y}{u}$ if there exists a positive integer T such that given any two solutions $\begin{pmatrix} x \\ y \\ u \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ u \\ \bar{z} \end{pmatrix}$ of (83), then $\bar{z}_k = z_k$ for $1 \leqslant k \leqslant T$ implies $\bar{z} = z$.
2. z is **detectable** from $\binom{y}{u}$ if given any two solutions $\begin{pmatrix} x \\ y \\ u \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ u \\ \bar{z} \end{pmatrix}$ of (83), then $\lim_{k\to\infty} z_k - \bar{z}_k = 0$.
3. z is **reconstructible** from $\binom{y}{u}$ if given any two solutions $\begin{pmatrix} x \\ y \\ u \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ u \\ \bar{z} \end{pmatrix}$ of (83), then there exists a positive integer T such that $(z_k - \bar{z}_k) = 0$ for $k > T$.
4. z is **observable** from $\binom{y}{u}$ if given any two solutions $\begin{pmatrix} x \\ y \\ u \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ u \\ \bar{z} \end{pmatrix}$ of (83), then $\bar{z} = z$.

We will say that a system Σ given by (83) is **trackable** if z is **trackable** from y. Similarly, we define **detectability**, **reconstructibility** and **observability**.

Paraphrasing Valcher and Willems [28], intuitively it is clear that, since we assume perfect knowledge of the system, the effect of the input, or control, variable on the estimate can be removed without affecting the observation properties. We state, without proof, the following simple proposition.

Proposition 3.1. *Given the linear system Σ in the state space representation* (83). *Associate with it the system Σ' given by*

$$\Sigma' := \begin{cases} \sigma x = Ax, \\ y = Cx, \\ z = Kx. \end{cases} \tag{84}$$

Then

1. *z is trackable from $\binom{y}{u}$ with respect to Σ if and only if z is trackable from y with respect to Σ'.*
2. *z is detectable from $\binom{y}{u}$ with respect to Σ if and only if z is detectable from y with respect to Σ'.*
3. *z is reconstructible from $\binom{y}{u}$ with respect to Σ if and only if z is reconstructible from y with respect to Σ'.*
4. *z is observable from $\binom{y}{u}$ with respect to Σ if and only if z is observable from y with respect to Σ'.*

What is of principal interest, is to find characterizations of the properties introduced in Definition 3.1. This depends very much on the functional relation between the observed variables y and the to be estimated variables z. Of course, in the state space representation (84) of the system Σ', this relation is indirect. To get a direct relation, we need to eliminate the state variable x from (84). As the system can be written in the latent variable form

$$\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \sigma I - A \\ C \\ K \end{pmatrix} x, \tag{85}$$

the elimination theorem requires computing a MLA of the polynomial matrix $\begin{pmatrix} zI - A \\ C \\ K \end{pmatrix}$ and this is best done via a coprime factorization of $\begin{pmatrix} C \\ K \end{pmatrix}(zI - A)^{-1}$. Apriori, there is no reason to assume that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is observable, but the following proposition shows that this entails no loss of generality.

Proposition 3.2

1. *Given the linear system* (83), *we can assume without loss of generality that the pair* $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *is observable*.
2. *If the pair* $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *is observable but* (C, A) *is not*, *then the system* Σ' *has a representation of the form*
$$\begin{aligned} A &= \begin{pmatrix} A_{11} & 0, \\ A_{21} & A_{22} \end{pmatrix} \\ C &= \begin{pmatrix} C_1 & 0 \end{pmatrix}, \\ K &= \begin{pmatrix} K_1 & K_2 \end{pmatrix}, \end{aligned} \tag{86}$$
with both pair (C_1, A_{11}) *and* (K_2, A_{22}) *observable*.

Proof

1. If the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is not observable, we can reduce the system to an observable one. Specifically, with respect to a direct sum decomposition of the state space into $\mathscr{X} = \mathscr{W} \oplus \mathscr{V}$, where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 \in \mathscr{W}$ and $x_2 \in \mathscr{V}$, $\mathscr{V} = \cap_{j \geqslant 0} \mathrm{Ker} \begin{pmatrix} C \\ K \end{pmatrix} A^j$ is the nonobservable subspace for the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ and $\mathscr{W}$ an arbitrary complementary subspace. Therefore, we have the block matrix representations
$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad \begin{pmatrix} C \\ K \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ K_1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$
Necessarily, $\left(\begin{pmatrix} C_1 \\ K_1 \end{pmatrix}, A_{11}\right)$ is an observable pair and we can replace (121) by
$$\begin{cases} \sigma x = A_{11} x + B_1 u, \\ y = C_1 x, \\ z = K_1 x \end{cases}$$
for C_2, K_2 are both zero and hence x_2 plays no role.

2. If (C, A) is not an observable pair, let $\mathscr{O}_* \subset \mathbb{F}^n$ be the corresponding unobservable subspace $\mathscr{O}_*$. Let $\mathscr{W}$ be any complementary subspace. Thus, with respect to the direct sum decomposition (61), we have the block matrix representation (86). By construction, the pair (C_1, A_{11}) is observable. Also A_{22} is similar to $A|\mathscr{O}_*$. Our assumption that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is observable implies that also the pair (K_2, A_{22}) is observable. □

Coprime factorizations were an important tool in bridging the gap between frequency domain and state space methods. This was done via realization theory, see [6]. As a consequence of Theorem 2.5, coprime factorizations turn out to be also an indispensable tool for the passage from a state space representation of a dynamical system to a functional representation, which in the context of this paper means a behavioral representation. The clarification of the connections between the two types of representations will play a major role in the characterization of various classes of observers. This is important to understand as, in the past, characterizations of observers were given, among others, by geometric control objects. The following theorem will study the corresponding functional characterizations.

Theorem 3.1. *Given the state space system*

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx. \end{cases} \tag{87}$$

We assume that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *is observable.*

Define

$$\begin{aligned} Z_K(z) &= K(zI - A)^{-1}, \\ Z_C(z) &= C(zI - A)^{-1}. \end{aligned} \tag{88}$$

Then,

1. *There exists a left coprime factorization of the form*

$$\begin{pmatrix} C \\ K \end{pmatrix} (zI - A)^{-1} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}. \tag{89}$$

More specifically, there exists a left coprime factorization of the form

$$\begin{pmatrix} C_1 & 0 \\ K_1 & K_2 \end{pmatrix} \begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & zI - A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Theta_{11} & 0 \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \tag{90}$$

for which

(a) $D_{11}(z)^{-1}\Theta_{11}(z)$ *is a left coprime factorization of* $C(zI - A)^{-1}$ *and* $D_{22}(z)^{-1}\Theta_{22}(z)$ *is a left coprime factorization of* $K_2(zI - A_{22})^{-1}$.

(b) D_{11} *and* D_{22} *are row proper.*

(c) $D_{21}D_{11}^{-1}$ *and* $D_{22}^{-1}D_{21}D_{11}^{-1}$ *are strictly proper.*

Moreover, we have

$$\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} = \begin{pmatrix} D_{11}^{-1} & 0 \\ -D_{22}^{-1}D_{21}D_{11}^{-1} & D_{22}^{-1} \end{pmatrix}. \tag{91}$$

2. *We have*
   (a) $n = \deg\det(zI - A) = \deg\det D_{11} + \deg\det D_{22}$.
   (b) *The map* $\psi : \cap_{i=0}^{\infty}\mathrm{Ker}\, CA^i \longrightarrow X_{D_{22}}$ *defined by*

$$\psi x = \Theta_{22}x, \quad \textit{for } x \in \cap_{i=0}^{\infty}\mathrm{Ker}\, CA^i \tag{92}$$

is a bijective map satisfying

$$\psi A_{22} = S_{D_{22}}\psi, \tag{93}$$

which implies the isomorphism

$$S_{D_{22}} \simeq A_{22} = A| \cap_{i=0}^{\infty} \mathrm{Ker}\, CA^i. \tag{94}$$

   (c) i. D_{22} *is a nonsingular polynomial matrix*.
   ii. D_{22} *is a stable matrix if and only if the pair* (C, A) *is detectable*.
   iii. D_{22} *is a monomic matrix if and only if the pair* (C, A) *is reconstructible*.
   iv. D_{22} *is a unimodular matrix if and only if the pair* (C, A) *is observable*.

3. *Let* $\phi : \mathbb{C}^n \longrightarrow \mathrm{Ker}(\sigma I - A)$ *be defined by* (52), *i.e.* $\phi(x) = (zI - A)^{-1}x$. *Then*

$$\phi(\mathcal{O}_*) = \mathrm{Ker}\begin{pmatrix} I & 0 \\ 0 & \sigma I - A_{22}\end{pmatrix} = \{0\} \oplus \mathrm{Ker}(\sigma I - A_{22}). \tag{95}$$

4. (a) *The map* $\psi : \mathcal{O}_* \longrightarrow X_{D_{22}}$ *defined by*

$$\psi(x) = \Theta_{22}x \tag{96}$$

is an isomorphism.
   (b) *The map* $K_2 : \mathrm{Ker}(\sigma I - A_{22}) \longrightarrow \mathrm{Ker}\, D_{22}(\sigma)$ *is a B-isomorphism*.
5. *The map* $\Psi : \mathcal{O}_* \longrightarrow \mathrm{Ker}\, D_{22}(\sigma)$ *defined by* $x \mapsto D_{22}^{-1}\Theta_{22}x$ *is an isomorphism satisfying*

$$\Psi(Ax) = \sigma\Psi(x), \tag{97}$$

i.e.

$$A|\mathcal{O}_* \simeq S_{D_{22}} = \sigma|\mathrm{Ker}\, D_{22}(\sigma). \tag{98}$$

6. *The following diagram is commutative*.

$$\begin{tikzcd}
\mathcal{O}_* \arrow[r, "\psi"] \arrow[d, "\phi|\mathcal{O}_*"'] \arrow[dr, "\Psi"] & X_{D_{22}} \arrow[d, "D_{22}^{-1}"] \\
\mathrm{Ker}\,(\sigma I - A_{22}) \arrow[r, "K_2"] & \mathrm{Ker}\, D_{22}(\sigma)
\end{tikzcd}$$

Proof

1. (a) Let $D_{11}^{-1}\Theta_{11}$ be a left coprime factorization of $C(zI - A)^{-1}$. Applying Proposition 3.2, with respect to the direct sum decomposition (61), we have the block matrix representation

(86) with the pairs (C_1, A_{11}) and (K_2, A_{22}) observable. Let $D_{11}^{-1}\Theta_{11}$ and $D_{22}^{-1}\Theta_{22}$ be left coprime factorizations of $C_1(zI - A_{11})^{-1}$ and $K_2(zI - A_{22})^{-1}$ respectively.
Clearly,

$$\begin{aligned}
&\begin{pmatrix} C \\ K \end{pmatrix}(zI - A)^{-1} \\
&\quad = \begin{pmatrix} C_1 & 0 \\ K_1 & K_2 \end{pmatrix}\begin{pmatrix} zI - A_{11} & 0 \\ -A_{21} & zI - A_{22} \end{pmatrix}^{-1} \\
&\quad = \begin{pmatrix} C_1 & 0 \\ K_1 & K_2 \end{pmatrix}\begin{pmatrix} (zI - A_{11})^{-1} & 0 \\ (zI - A_{22})^{-1}A_{21}(zI - A_{11})^{-1} & (zI - A_{22})^{-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} C_1(zI - A_{11})^{-1} & 0 \\ K_1(zI - A_{11})^{-1} + K_2(zI - A_{22})^{-1}A_{21}(zI - A_{11})^{-1} & K_2(zI - A_{22})^{-1} \end{pmatrix}
\end{aligned}$$

Note that $K_2(zI - A_{22})^{-1}$ is generally different from $K(zI - A)^{-1}$. We claim that there exist polynomial matrices D_{21} and Θ_{21} such that

$$\begin{aligned}
\begin{pmatrix} C \\ K \end{pmatrix}(zI - A)^{-1} &= \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1}\begin{pmatrix} \Theta_{11} & 0 \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \\
&= \begin{pmatrix} D_{11}^{-1} & 0 \\ -D_{22}^{-1}D_{21}D_{11}^{-1} & D_{22}^{-1} \end{pmatrix}\begin{pmatrix} \Theta_{11} & 0 \\ \Theta_{21} & \Theta_{22} \end{pmatrix}
\end{aligned} \tag{99}$$

For equality (99) to hold, we must have

$$\begin{aligned}
&\begin{pmatrix} C_1(zI - A_{11})^{-1} & 0 \\ K_1(zI - A_{11})^{-1} + K_2(zI - A_{22})^{-1}A_{21}(zI - A_{11})^{-1} & K_2(zI - A_{22})^{-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} D_{11}^{-1} & 0 \\ -D_{22}^{-1}D_{21}D_{11}^{-1} & D_{22}^{-1} \end{pmatrix}\begin{pmatrix} \Theta_{11} & 0 \\ \Theta_{21} & \Theta_{22} \end{pmatrix}.
\end{aligned}$$

Thus all that remains to be proved is the existence of polynomial matrices D_{21} and Θ_{21} such that the equality

$$K_1(zI - A_{11})^{-1} + K_2(zI - A_{22})^{-1}A_{21}(zI - A_{11})^{-1} = -D_{22}^{-1}D_{21}D_{11}^{-1}\Theta_{11} + D_{22}^{-1}\Theta_{21}$$

holds. Multiplying by D_{22} on the left and by $(zI - A_{11})$ on the right and noting that $D_{11}^{-1}\Theta_{11}(zI - A_{11}) = C_1$, leads to

$$D_{22}K_1 + \Theta_{22}A_{21} = -D_{21}C_1 + \Theta_{21}(zI - A_{11}). \tag{100}$$

That this equation has a polynomial solution follows from the fact that the pair (C_1, A_{11}) is observable which in turn is equivalent to the right coprimeness of C_1 and $(zI - A_{11})$ and hence to the solvability of the corresponding Bezout equation from which (100) easily follows.

(b) A left coprime factorization is only unique up to a common left unimodular factor. We can use this freedom, employing a block diagonal unimodular matrix, to reduce D_{11} and D_{22} to row proper form.

(c) Let $\pi_+ D_{21} D_{11}^{-1} = Q$, then $(D_{21} - Q D_{11}) D_{11}^{-1}$ is strictly proper. Applying the unimodular matrix $\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}$ to the coprime factors, we can assume, changing notation, that $D_{21} D_{11}^{-1}$ is strictly proper. This does not change the row properness of D_{11} and D_{22}.
That $D_{22}^{-1} D_{21} D_{11}^{-1}$ is strictly proper follows from the strict properness of $D_{21} D_{11}^{-1}$ and the fact that D_{22} as a nonsingular, row proper polynomial matrix, has a proper inverse.

2. (a) From the observability assumption on the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ and the left coprime factorization (90), we conclude that

$$n = \deg\det(zI - A) = \deg\det \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix} = \deg\det D_{11} + \deg\det D_{22}.$$

(b) From the coprime factorization (90), we obtain in particular the coprime factorization

$$D_{22}^{-1} \Theta_{22} = K_2 (zI - A_{22})^{-1}, \tag{101}$$

which is equivalent to the intertwining relation

$$\Theta_{22}(zI - A_{22}) = D_{22} K_2. \tag{102}$$

Applying Theorems 4.5 and 4.7 in [6] proves the intertwining relation (93) as well as the invertibility of ψ.
The isomorphism (94) follows from (93) and the invertibility of ψ.

(c) i. The nonsingularity of D_{22} follows from the nonsingularity of $\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}$.
ii. The pair (C, A) is detectable if and only if $A_{22} \simeq A| \cap_{i=0}^{\infty} \operatorname{Ker} CA^i$ is stable. By the isomorphism (94), this is equivalent to the stability of D_{22}.
iii. The pair (C, A) is reconstructible if and only if $A_{22} \simeq A| \cap_{i=0}^{\infty} \operatorname{Ker} CA^i$ is nilpotent. As before, this is equivalent to D_{22} being monomic.
iv. The pair (C, A) is observable if and only if $n = \deg\det(zI - A) = \deg\det D_{11}$. This is equivalent to $\deg\det D_{22} = 0$, i.e. to D_{22} being unimodular.

3. Applying Theorems 2.2 and 2.3, we conclude that the map $I : \operatorname{Ker}\begin{pmatrix} I & 0 \\ 0 & \sigma I - A_{22} \end{pmatrix} \longrightarrow \operatorname{Ker}\begin{pmatrix} \sigma I - A \\ C \end{pmatrix}$ is an injective B-homomorphism. However, $\operatorname{Ker}(\sigma I - A_{22}) = \{(zI - A)^{-1} x | x \in \mathcal{O}_*\} = \phi(\mathcal{O}_*)$ which proves (95).

4. (a) The coprime factorizations $D_{22}(z)^{-1} \Theta_{22}(z) = K_2 (zI - A_{22})^{-1}$ yields the intertwining relation

$$\Theta_{22}(z)(zI - A_{22}) = D_{22}(z) K_2. \tag{103}$$

Thus the map $\psi : X_{zI - A_{22}} \longrightarrow X_{D_{22}}$ defined by $\psi(x) = \pi_{D_{22}} \Theta_{22} x = \Theta_{22} x$ is an isomorphism. Note that $X_{zI - A_{22}} = \mathcal{O}_*$.
(b) Follows from (103), and the coprimeness conditions, by applying Theorems 2.2 and 2.3.

5. Since autonomous behaviors are equal to rational models, see [11], we have in particular that $X^{D_{22}} = \operatorname{Ker} D_{22}(\sigma)$. Now the multiplication map $D_{22}^{-1} : X_{D_{22}} \longrightarrow X^{D_{22}}$ is an $\mathbb{F}[z]$-module isomorphism, it follows that the composed map $\Psi = D_{22}^{-1} \psi$ is also an $\mathbb{F}[z]$-module isomorphism from $\mathcal{O}_*$ onto $\operatorname{Ker} D_{22}(\sigma)$.

6. For $x \in \mathcal{O}_*$, and using the coprime factorizations (101), we compute

$$K_2 \phi(x) = K_2 (zI - A)^{-1} x = D_{22}^{-1} \Theta_{22} x = \Psi x = D_{22}^{-1} \psi(x). \qquad \square$$

Our next objective is the characterization of trackability, detectability, reconstructibility, and observability. This is done in essentially two ways, geometric and functional. For the analysis of detectability, we will always assume that the underlying space is either $\mathbb{C}^n$ or $\mathbb{R}^n$.

Proposition 3.3. *Given the system Σ' in the state space representation* (84) *with the pair* $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *observable. We assume the coprime factorization* (89). *Then*

1. *The following statements are equivalent.*
   (a) *z is trackable from y.*
   (b) *There exists a rational, strictly proper solution to the equation*

$$\begin{pmatrix} Z_1(z) & Z_2(z) \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = K. \tag{104}$$

   (c) *There exists a conditioned invariant subspace* $\mathscr{V} \subset \operatorname{Ker} K$.
2. *Assume $\mathbb{F} = \mathbb{C}$. The following statements are equivalent.*
   (a) *z is detectable from y.*
   (b) *There exists a rational, strictly proper and stable solution to equation* (*104*).
   (c) *There exists a outer detectability subspace* $\mathscr{D} \subset \operatorname{Ker} K$.
   (d) *We have*

$$\operatorname{Ker} \begin{pmatrix} C \\ S_+(\sigma) \end{pmatrix} \subset \operatorname{Ker} K. \tag{105}$$

   (e) *D_{22} is a stable polynomial matrix.*
3. *The following statements are equivalent.*
   (a) *z is reconstructible from y.*
   (b) *There exists a rational, strictly proper solution with monomic denominator to equation* (*104*).
   (c) *There exists an outer reconstructibility subspace* $\mathscr{V} \subset \operatorname{Ker} K$.
   (d) *We have*

$$\operatorname{Ker} \begin{pmatrix} C \\ S_{nm}(\sigma) \end{pmatrix} \subset \operatorname{Ker} K. \tag{106}$$

   (e) *D_{22} is a monomic polynomial matrix.*
4. *The following statements are equivalent.*
   (a) *z is observable from y.*
   (b) *There exists a polynomial solution* $\begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix}$ *to equation* (*104*).
   (c) *There exists an outer observability subspace* $\mathscr{O} \subset \operatorname{Ker} K$.
   (d) *We have*

$$\operatorname{Ker} \begin{pmatrix} C \\ \sigma I - A \end{pmatrix} \subset \operatorname{Ker} K. \tag{107}$$

   (e) *D_{22} is unimodular.*

5. *The existence of a polynomial solution to* (104) *implies the existence of a strictly proper solution*.

Proof

1. (1b) $\Leftrightarrow$ (1a)
There always exists one obvious rational, strictly proper solution to Eq. (104) and it is given by $(0 \quad K(zI-A)^{-1})$. In particular, this is implied by the trackability of the system Σ.
Conversely, assume there exists a rational, strictly proper solution to Eq. (104). Writing $Z_i = E^{-1}F_i$, we have $(F_1 \quad F_2)\begin{pmatrix} C \\ zI-A \end{pmatrix} = E(z)K$. Since $E(z)$ is a nonsingular polynomial matrix, by Proposition 3.4 in [11], there exists a representation $X^E = \{J(zI-F)^{-1}|\xi \in \mathbb{F}^q\}$ with the pair J, F observable. Choosing a basis matrix $\Phi(z)$ for the polynomial model X_E, we have without loss of generality the equality $J(zI-F)^{-1} = E(z)^{-1}\Phi(z)$. Since $E^{-1}Z_i$ are strictly proper, there exist constant matrices G, Z for which $F_1(z) = \Phi(z)G$ and $F_2(z) = \Phi(z)Z$. It follows from
$$\begin{pmatrix}\Phi_2(\sigma) & \Phi_1(\sigma)\end{pmatrix}\begin{pmatrix}\sigma I - A \\ C\end{pmatrix} = E(\sigma)K \tag{108}$$
that $\xi \in \operatorname{Ker}\begin{pmatrix}\sigma I - A \\ C\end{pmatrix}$ and, applying Lemma 2.1, the trackability of Σ follows.
(1b) $\Leftrightarrow$ (1c)
Assume there exists a conditioned invariant subspace $\mathscr{V} \subset \operatorname{Ker} K$. Let $\mathscr{V} = \operatorname{Ker} Z$ be a kernel representation. The inclusion $\operatorname{Ker} Z \subset \operatorname{Ker} K$ implies the existence of a map J for which $K = JZ$. Since $\operatorname{Ker} Z$ is conditioned invariant, it follows from Lemma 2.2, that there exist maps L and F for which $Z(A-LC) = FZ$. Defining
$$\begin{pmatrix}Z_1 & Z_2\end{pmatrix} = \left(\begin{array}{c|cc} F & ZL & Z \\ \hline J & 0 & 0\end{array}\right),$$
we compute
$$\begin{aligned}\begin{pmatrix}Z_1 & Z_2\end{pmatrix}\begin{pmatrix}C \\ zI-A\end{pmatrix} &= J(zI-F)^{-1}ZLC + J(zI-F)^{-1}Z(zI-A) \\ &= J(zI-F)^{-1}[ZLC + Z(zI-A)] \\ &= J(zI-F)^{-1}[ZLC - ZA + zZ] \\ &= J(zI-F)^{-1}[-FZ + zZ] = J(zI-F)^{-1}(zI-F)Z \\ &= JZ = K.\end{aligned}$$
Conversely, assume
$$\begin{pmatrix}Z_1 & Z_2\end{pmatrix} = \left(\begin{array}{c|cc} F & L & L' \\ \hline J & 0 & 0\end{array}\right)$$
is a minimal realization of a strictly proper rational solution of equation (104). Thus we have $J(zI-F)^{-1}LC + J(zI-F)^{-1}L'(zI-A) = K$, or equivalently $J(zI-F)^{-1}[LC + L'(zI-A)] - K = 0$. As $J(zI-F)^{-1}$ is a right coprime factorization, it follows from Lemma 5.2 in [14] that for some Z we have $LC + L'(zI-A) = (zI-F)Z$. Equating

coefficients, we conclude that $L' = Z$. In turn, this implies the equality $LC - ZA = -FZ$ which shows that Ker Z is conditioned invariant. Substituting the equality $LC - ZA = -FZ$ back, we have the factorization $K = JZ$ which proves the inclusion Ker $Z \subset$ Ker K.

2. (2a) $\Leftrightarrow$ (2b)

Assume z is detectable from y. This means that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ \bar{z} \end{pmatrix} \in \mathscr{B}$ implies $z_k - \bar{z}_k \to 0$. Equivalently, by subtraction, setting $\xi = x - \bar{x}$ and $\zeta = z - \bar{z}$, we have $\begin{pmatrix} \xi \\ 0 \\ \zeta \end{pmatrix} \in \mathscr{B}$ implies $\zeta_k \to 0$. The condition $\begin{pmatrix} \xi \\ 0 \\ \zeta \end{pmatrix} \in \mathscr{B}$ translates to

$$\begin{pmatrix} \sigma I - A \\ C \end{pmatrix} \xi = 0, \qquad \zeta = K\xi, \tag{109}$$

or

$$\begin{pmatrix} \sigma I - A \\ C \\ K \end{pmatrix} \xi = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \zeta$$

which is the latent variable representation of the manifest variable ζ. Using the coprime factorization (89), and the elimination theorem, we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\Theta_1(\sigma) & D_{11}(\sigma) & 0 \\ -\Theta_2(\sigma) & D_{21}(\sigma) & D_{22}(\sigma) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \zeta = \begin{pmatrix} 0 \\ D_{22}(\sigma) \end{pmatrix} \zeta,$$

or $D_{22}(\sigma)\zeta = 0$, with $D_{22}(z)$ necessarily a stable polynomial matrix. The system equations (109) imply now that if $\xi \in \operatorname{Ker}\begin{pmatrix} \sigma I - A \\ C \end{pmatrix}$ then $D_{22}(\sigma)K\xi = 0$, i.e. $\operatorname{Ker}\begin{pmatrix} \sigma I - A \\ C \end{pmatrix} \subset \operatorname{Ker} D_{22}(\sigma)K$. Thus, as behavior inclusion is expressible in terms of factorizations, there exist polynomial matrices P_1, P_2 for which

$$\begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = D_{22}(z)K,$$

or that $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = D_{22}^{-1} \begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix}$ solve Eq. (104). By the stability of D_{22}, it follows that Z_1, Z_2 are stable rational matrices. Clearly (104) shows that the strict properness of Z_1 implies that of Z_2. We will show now that we can modify Z_1, Z_2 so that both are strictly proper. Let $\overline{Z}_1, \overline{Z}_2$ be any other solution of (104). Since the MLA of $\begin{pmatrix} C \\ zI - A \end{pmatrix}$ is $\begin{pmatrix} D_{11} & -\Theta_1 \end{pmatrix}$, the general solution is given by

$$\begin{pmatrix} \overline{Z}_1 & \overline{Z}_2 \end{pmatrix} = \begin{pmatrix} Z_1 & Z_2 \end{pmatrix} - W \begin{pmatrix} D_{11} & -\Theta_1 \end{pmatrix}$$

with W an appropriately sized but otherwise arbitrary rational function. We write $W = XY^{-1}$ and apply the projection π_+ to $\overline{Z}_1 = Z_1 - WD_{11} = Z_1 - XY^{-1}D_{11}$. Assuming $\overline{Z}_1$ to be strictly proper, we have $\pi_+ Z_1 = \pi_+ XY^{-1}D_{11}$. Choosing Y to be nonsingular, stable and such that $Y^{-1}D_{11}$ is biproper, and using the invertibility of the Toeplitz operator $X \mapsto \pi_+ XY^{-1}D_{11}$ as well as the stability of Y, we get the strict properness of $\overline{Z}_1$ as well as its stability. Letting $\overline{Z}_2 = Z_2 + XY^{-1}\Theta_1$, we have obtained a strictly proper and stable solution of (104).

Conversely, assume there exists a rational, strictly proper and stable solution Z_1, Z_2 to Eq. (104). Let $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = E^{-1} \begin{pmatrix} P_1 & P_2 \end{pmatrix}$ be a left coprime factorization with E stable. Since

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = E^{-1} \begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = K,$$

we have

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = E(z)K. \tag{110}$$

We have to show that $\begin{pmatrix} \xi \\ 0 \\ \zeta \end{pmatrix} \in \mathscr{B}$ implies $\zeta_k \to 0$. Using (110) and Eqs. (109), we have

$$E(\sigma)K\xi = E(\sigma)\zeta = 0.$$

Since $E(z)$ is stable, it follows that $\zeta_k \to 0$.
(2b) $\Leftrightarrow$ (2c): Assume there exists an outer detectability subspace $\mathscr{V} = \mathrm{Ker}\, Z \subset \mathrm{Ker}\, K$. Let L be an output injection for which the induces map $(A - LC)|\mathscr{X}/\mathrm{Ker}\, Z$ is stable. Thus, there exists an F for which $ZA - FZ = LC$ and the isomorphism $F \simeq (A - LC)|\mathscr{X}/\mathrm{Ker}\, Z$ implies the stability of F. Now, by applying Part 1, $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \left(\begin{array}{c|cc} F & L & Z \\ \hline J & 0 & 0 \end{array}\right)$ is a rational, strictly proper and stable solution of (104).
Conversely, assume there exists a rational, strictly proper and stable solution $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ to Eq. (104) with a minimal realization $\left(\begin{array}{c|cc} F & L & Z \\ \hline J & 0 & 0 \end{array}\right)$, F being stable. Since this implies, by Part 1, that $Z(A - LC) = FZ$, it follows that $\mathscr{V} = \mathrm{Ker}\, Z$ is conditioned invariant and $(A - LC)|\mathscr{X}/\mathrm{Ker}\, Z$ is stable, i.e. $\mathscr{V} = \mathrm{Ker}\, Z$ is an outer detectability subspace.
(2b) $\Leftrightarrow$ (2e): Assume (104) has a strictly proper and stable solution with a left coprime factorization $\begin{pmatrix} Z_1(z) & Z_2(z) \end{pmatrix} = E^{-1} \begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix}$. Necessarily $E(z)$ is stable. Thus we have $\begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = E(z)K$. This can be rewritten as

$$\begin{pmatrix} P_1 & -E & P_2 \end{pmatrix} \begin{pmatrix} C \\ K \\ zI - A \end{pmatrix} = 0.$$

Using the coprime factorization (89), it follows that the MLA of $\begin{pmatrix} C \\ K \\ zI - A \end{pmatrix}$ is given by $\begin{pmatrix} D_{11} & 0 & -\Theta_1 \\ D_{21} & D_{22} & -\Theta_2 \end{pmatrix}$. Thus, there exist polynomial matrices X, Y for which

$$\begin{pmatrix} P_1 & -E & P_2 \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} D_{11} & 0 & -\Theta_1 \\ D_{21} & D_{22} & -\Theta_2 \end{pmatrix}.$$

In particular we get $-YD_{22} = E$ which, by the stability of E, shows that D_{22} is necessarily stable.
Conversely, assume that in the coprime factorization (89), D_{22} is stable. Using that factorization and (104), we have $K = -D_{22}^{-1}D_{21}C + D_{22}^{-1}\Theta_2(zI - A)$ and $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \begin{pmatrix} -D_{22}^{-1}D_{21} & D_{22}^{-1}\Theta_2 \end{pmatrix}$ is a stable rational function. Applying Toeplitz operators as in a previously used argument, the existence of a strictly proper solution is guaranteed.
(2c) $\Leftrightarrow$ (2d): Suppose there exists an outer detectable subspace $\mathscr{D} \subset \mathrm{Ker}\, K$. Let $\mathscr{D}_*$ be the smallest outer detectability subspace. Clearly, we have $\mathscr{D}_* \subset \mathscr{D} \subset \mathrm{Ker}\, K$. In behavioral terms we can rewrite (59) as

$$\mathrm{Ker}\, D_*(\sigma) = \mathrm{Ker}\, S_+(\sigma) \cap \mathrm{Ker}\, C = \mathrm{Ker}\, S_+(\sigma) \cap \mathrm{Ker} \begin{pmatrix} C \\ \sigma I - A \end{pmatrix}$$

We note that, as $S_+(z)$ is a right factor of $zI - A$, the greatest common right divisor of $S_+, \begin{pmatrix} C \\ zI - A \end{pmatrix}$ is equal to the greatest common right divisor of $\begin{pmatrix} C \\ S_+ \end{pmatrix}$. Now $\operatorname{Ker} S_+(\sigma) \cap \operatorname{Ker}\begin{pmatrix} C \\ \sigma I - A \end{pmatrix} = \operatorname{Ker} D_*(\sigma)$. The inclusion $\mathscr{D} \subset \operatorname{Ker} K$ now implies $\operatorname{Ker}\begin{pmatrix} C \\ S_+(\sigma) \end{pmatrix} = \operatorname{Ker} D_*(\sigma) \subset \operatorname{Ker} K$.

To prove the converse, assume $\operatorname{Ker}\begin{pmatrix} C \\ S_+(\sigma) \end{pmatrix} \subset \operatorname{Ker} K$. Since, by Theorem 2.6, we have $\phi(\mathscr{O}_*^-) = \operatorname{Ker}\begin{pmatrix} S_+(\sigma) \\ C \end{pmatrix} = \operatorname{Ker} D_+(\sigma)$ and $\phi^{-1}(\operatorname{Ker}(\sigma I - A) \cap \operatorname{Ker} K) = \operatorname{Ker} K$, it follows that $\mathscr{O}_*^- \subset \operatorname{Ker} K$, i.e. there exists an outer detectability subspace in $\operatorname{Ker} K$. (Note that $\operatorname{Ker} K$ is used here in two different interpretation, one as a subspace of $\mathbb{C}^n$, the other as a subbehavior of $z^{-1}\mathbb{C}[[z^{-1}]]^n$. The exact meaning follows from the context).

3. The proof follows exactly the line of proof of Part 2, but with a different interpretation of stability.
4. (4a) $\Leftrightarrow$ (4d): Assume z is observable from y. This means that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ \bar{z} \end{pmatrix} \in \mathscr{B}$ implies $z = \bar{z}$.

Since the behavior $\mathscr{B}$ is closed under linear combinations, we have $\begin{pmatrix} x - \bar{x} \\ 0 \\ z - \bar{z} \end{pmatrix} \in \mathscr{B}$. Setting $\xi = x - \bar{x}$ and $\zeta = z - \bar{z}$, we have $\begin{pmatrix} \xi \\ 0 \\ \zeta \end{pmatrix} \in \mathscr{B}$ implies $\zeta = 0$. The condition $\begin{pmatrix} \xi \\ 0 \\ \zeta \end{pmatrix} \in \mathscr{B}$ translates to

$$\begin{pmatrix} \sigma I - A \\ C \end{pmatrix} \xi = 0,$$
$$\zeta = K\xi.$$

This shows the inclusion (107) holds.

Conversely, assume the inclusion (107). The system equations (83) can be rewritten as

$$\begin{pmatrix} \sigma I - A \\ C \end{pmatrix} x = \begin{pmatrix} 0 \\ I \end{pmatrix},$$
$$z = Kx,$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ \bar{z} \end{pmatrix} \in \mathscr{B}$. By subtraction, we get

$$\begin{pmatrix} \sigma I - A \\ C \end{pmatrix} (x - \bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$(z - \bar{z}) = K(x - \bar{x}).$$

The inclusion (107) implies now that $z - \bar{z} = 0$, i.e. z is observable from y.

(4b) $\Leftrightarrow$ (4e): Assume (104) has a polynomial solution $P_1(z), P_2(z)$. Thus we have also

$$\begin{pmatrix} P_1 & -I & P_2 \end{pmatrix} \begin{pmatrix} C \\ K \\ zI - A \end{pmatrix} = 0.$$

By the same argument used in the proof of Part 2, there exist polynomial matrices X, Y for which

$$\begin{pmatrix} P_1 & -I & P_2 \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} D_{11} & 0 & -\Theta_1 \\ D_{21} & D_{22} & -\Theta_2 \end{pmatrix}.$$

In particular we get $-YD_{22} = I$ which shows that D_{22} is unimodular, and without loss of generality we can take $D_{22} = I$.
To prove the converse, observe that from the coprime factorization (89), we have

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}(zI - A) = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} C \\ K \end{pmatrix}.$$

Assuming $D_{22} = I$, it follows that $\Theta_2(zI - A) - D_{21}C = K$, i.e. Eq. (104) has a polynomial solution given by $\begin{pmatrix} -D_{21} & \Theta_2 \end{pmatrix}$.
(4b) ⇔ (4d)
Follows from Theorem 2.3.
(4b) ⇔ (4c)
Assume $P_1(z)$, $P_2(z)$ is a polynomial solution to (104). This factorization implies the behavioral inclusion

$$\operatorname{Ker}\begin{pmatrix} C \\ \sigma I - A \end{pmatrix} \subset \operatorname{Ker} K. \tag{111}$$

Now $\operatorname{Ker}\begin{pmatrix} C \\ \sigma I - A \end{pmatrix} = \operatorname{Ker}(\sigma I - A) \cap \operatorname{Ker} C$. It is easily verified that

$$\operatorname{Ker}(\sigma I - A) = \left\{ \sum_{j=1}^{\infty} \frac{A^{j-1}\xi}{z^j} | \xi \in \mathbb{F}^n \right\}$$

so

$$\begin{aligned} \operatorname{Ker}\begin{pmatrix} C \\ \sigma I - A \end{pmatrix} &= \operatorname{Ker}(\sigma I - A) \cap \operatorname{Ker} C \\ &= \left\{ \sum_{j=1}^{\infty} \frac{A^{j-1}\xi}{z^j} | \xi \in \mathbb{F}^n, CA^{j-1}\xi = 0 \right\} \qquad (112) \\ &= \left\{ \sum_{j=1}^{\infty} \frac{A^{j-1}\xi}{z^j} | \xi \in \mathscr{O}_* = \cap_{j=1}^{\infty} \operatorname{Ker} CA^j \right\}. \end{aligned}$$

Here $\mathscr{O}_*$ is the smallest outer observability subspace for the pair (C, A). This proves the existence of an outer observability subspace included in Ker K.
Conversely, assume there exists an outer observability subspace $\mathscr{O} \subset \operatorname{Ker} K$. Clearly, we have $\mathscr{O}_* \subset \mathscr{O} \subset \operatorname{Ker} K$ which implies the inclusion (111) which in turn implies the factorization $\begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix}\begin{pmatrix} C \\ zI - A \end{pmatrix} = K$ with the P_i polynomial.

5. This follows from the fact that an observability subspace is also conditioned invariant.
It may be instructive to give another, constructive, proof of this statement. Suppose $\begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix}$ is a polynomial solution of equation (104). As $\begin{pmatrix} D_{11} & -\Theta_1 \end{pmatrix}$ is a MLA of $\begin{pmatrix} C \\ zI - A \end{pmatrix}$, the general rational solution is given, with W an appropriately sized but otherwise arbitrary rational function, by

$$\begin{pmatrix} Z_1(z) & Z_2(z) \end{pmatrix} = \begin{pmatrix} P_1(z) & P_2(z) \end{pmatrix} - W\begin{pmatrix} D_{11} & -\Theta_1 \end{pmatrix}, \tag{113}$$

i.e. by

$$\begin{aligned} Z_1(z) &= P_1 - WD_{11}, \\ Z_2(z) &= P_2 + W\Theta_1. \end{aligned} \tag{114}$$

We intend to find Z_i that are strictly proper. Clearly, the strict properness of Z_1 implies that of Z_2. with the coprime factorizations $W = \overline{X}\overline{Y}^{-1} = Y^{-1}X$, we have $Z_1 = P_1 - \overline{X}\overline{Y}^{-1}D_{11}$. If Z_1 is strictly proper, then, by applying the projection π_+, we have $P_1 = \pi_+\overline{X}\overline{Y}^{-1}D_{11}$. Now, for every nonsingular polynomial matrix $\overline{Y}$ for which $\overline{Y}^{-1}D_{11}$ is biproper, the Toeplitz operator defined by $\overline{X} \mapsto \pi_+\overline{X}\overline{Y}^{-1}D_{11}$ is invertible. Hence for any P_1 there exists a strictly proper Z_1 for which $Z_1(z) = P_1 - WD_{11}$. Obviously, the McMillan degree of Z_1 is bounded by $\deg\det D_{11}$. $\square$

Remarks

1. Note that for the system Σ, defined by (83), $z = Kx$ is always trackable. Indeed $(z_1 \quad z_2) = \begin{pmatrix}0 & K(zI-A)^{-1}\end{pmatrix}$ is clearly a rational, strictly proper solution of (104). In this case $\begin{pmatrix}Z_1 & Z_2\end{pmatrix} = \left(\begin{array}{c|cc} A & 0 & I \\ \hline K & 0 & 0\end{array}\right)$. From the geometric viewpoint, $\mathscr{V} = \{0\}$ is conditioned invariant and clearly $\{0\} \subset \operatorname{Ker} K$.
2. It follows from the proof of Statement 5 that the characteristic polynomial of Z_1 can be freely preassigned, however the shift realization of Z_1 is not necessarily minimal. For a detailed discussion of spectral assignability, see [16]. This relates also to the fact that a subspace is an observability subspace for (C, A) if and only if it is conditioned invariant and at the same time an almost observability subspace.

A special case is that of the system Σ' given in (84) where the relevant variable to be estimated is the state x itself, i.e. the case $K = I$. We have the following proposition.

Corollary 3.1. *Given the system Σ' defined by (84) with $K = I$. Then*

1. *The state x is always trackable from y.*
2. *The state x is detectable from y if and only if (C, A) is a detectable pair, i.e. $\begin{pmatrix} zI-A \\ C\end{pmatrix}$ has full column rank for all unstable values of z or, equivalently, $\xi \in \cap_{i=0}^{\infty}\operatorname{Ker} CA^i$ implies $\lim_{i\to\infty} A^i x = 0$.*
3. *The state x is reconstructible from y if and only if (C, A) is a reconstructible pair, i.e. $\begin{pmatrix} zI-A \\ C\end{pmatrix}$ has full column rank for all nonzero values of z.*
4. *The state x is observable from y if and only if (C, A) is an observable pair, i.e. $\begin{pmatrix} zI-A \\ C\end{pmatrix}$ is right prime or, equivalently $\cap_{i=0}^{\infty}\operatorname{Ker} CA^i = \{0\}$.*

Proof

1. Clearly (84) implies that the state is given by $x_j = A^{j-1}x_0$, i.e. the state trajectory is $x(z) = (zI-A)^{-1}x_0$, x_0 being the state initial condition. If n is the dimension of the state space, then by the Cayley–Hamilton theorem, the equality $\bar{x}_k - x_k$ for $k = 0, \ldots, n-1$ implies $\bar{x}_k = x_k$ for all k. Thus x is always trackable, even in the case that $C = 0$, i.e. when there are no observations at all.
2. The state x is detectable from y if and only if $\mathscr{O}_*^+$, the smallest outer nondetectability subspace is zero. This, by Theorem 2.6, is equivalent to $\begin{pmatrix} S_+(z) \\ C\end{pmatrix}$ being right prime. Since

$\begin{pmatrix} C \\ zI - A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \overline{S}_-(z) \end{pmatrix} \begin{pmatrix} C \\ S_+(z) \end{pmatrix}$, this is equivalent to $\begin{pmatrix} C \\ zI - A \end{pmatrix}$ having full column rank for all nonstable values of z.

3. The proof follows the same lines.
4. The state x is observable from y if and only if $\mathcal{O}_*$, the smallest outer observability subspace is zero. By Theorem 2.6, this is equivalent to the right primeness of $\begin{pmatrix} C \\ zI - A \end{pmatrix}$. □

Remarks

1. The statement of Corollary 3.1.1 shows that trackability is a very weak concept, though useful in clarifying observer theory.
2. The detectability, reconstructibility and observability criteria are generally known as the **Hautus test**.
3. If the field is without a topology, then we interpret $\lim_{t \to \infty} z(t) = 0$ as $z(t) = 0$ for t large enough. In this case, detectability and reconstructibility coincide. For fields with a topology, both notions make sense and do not coincide in general.
4. It is clear that observability implies detectability.

Proposition 3.3 shows that a central role in the characterization of various classes of observers is the solvability of the functional equation (104). This equation is the key to the study of the parametrization of all observers belonging to a given class. Since it is a system of linear nonhomogeneous equations, it may have many solutions. In fact, given a particular solution of (104), all other solution are obtained by adding the general solution of the corresponding homogeneous equation. This equation, i.e. $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = 0$, shows that a solution is a, not necessarily polynomial, left annihilator of $\begin{pmatrix} C \\ zI - A \end{pmatrix}$. We proceed to relate the solutions of this equation to the coprime factorization (89).

Proposition 3.4. *Given the state space system*

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx. \end{cases} \tag{115}$$

We assume that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A \right)$ *is observable as well as the coprime factorization* (89).

1. *The rational matrix function* $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$, *having the coprime factorization*

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} P_1 & P_2 \end{pmatrix}, \tag{116}$$

is a solution of

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = K \tag{117}$$

if and only if there exist polynomial matrices X, Y, *with* Y *nonsingular*, *such that*

$$\begin{pmatrix} -P_2 & -P_1 & Q \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} -\Theta_1 & D_{11} & 0 \\ -\Theta_2 & D_{21} & D_{22} \end{pmatrix} \tag{118}$$

or

$$\begin{aligned} P_1 &= -XD_{11} - YD_{21}, \\ P_2 &= X\Theta_1 + Y\Theta_2, \\ Q &= YD_{22}. \end{aligned} \tag{119}$$

We have, with $W = Y^{-1}X$,

$$\begin{aligned} Z_1 &= -D_{22}^{-1}D_{21} - D_{22}^{-1}WD_{11}, \\ Z_2 &= D_{22}^{-1}\Theta_2 + D_{22}^{-1}W\Theta_1. \end{aligned} \tag{120}$$

2. *The stable (monomic) rational functions* $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ *is a solution of* (117) *if and only if* D_{22} *is stable (monomic) and there exist polynomial matrices* X, Y, *with* Y *stable (monomic), such that* (119) *holds.*
3. *The polynomial function* $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ *is a solution of* (117) *if and only if* D_{22} *is unimodular and there exist polynomial matrices* X, Y, *with* Y *unimodular, such that* (119) *holds.*

Proof

1. Assume Z_1, Z_2 are defined by (116) with Q, P_1, P_2 defined by (119). We compute, using the coprime factorization (89),

$$\begin{aligned} &\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}\begin{pmatrix} C \\ zI - A \end{pmatrix} - K \\ &\quad = Q^{-1}\begin{pmatrix} P_1 & P_2 \end{pmatrix}\begin{pmatrix} C \\ zI - A \end{pmatrix} - K \\ &\quad = Q^{-1}\left[\begin{pmatrix} P_1 & P_2 \end{pmatrix}\begin{pmatrix} C \\ zI - A \end{pmatrix} - Q(z)K\right] \\ &\quad = Q^{-1}\left[\begin{pmatrix} -XD_{11} - YD_{21} & X\Theta_1 + Y\Theta_2 \end{pmatrix}\begin{pmatrix} C \\ zI - A \end{pmatrix} - Q(z)K\right] \\ &\quad = Q^{-1}[-(XD_{11} + YD_{21})C + (X\Theta_1 + Y\Theta_2)(zI - A) - Q(z)K] \\ &\quad = Q^{-1}[-X(D_{11}C - \Theta_1(zI - A)) - Y(D_{21}C - \Theta_2(zI - A)) + YD_{22}K] \\ &\quad = -Q^{-1}Y[D_{21}C - \Theta_2(zI - A) + D_{22}K] = 0 \end{aligned}$$

Conversely, assume $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ is a rational solution of equation (117). Let $Q^{-1}\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ be a left coprime factorization of $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$. Thus (117) can be rewritten as

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix}\begin{pmatrix} C \\ zI - A \end{pmatrix} = QK,$$

or

$$\begin{pmatrix} P_1 & -Q & P_2 \end{pmatrix}\begin{pmatrix} C \\ K \\ zI - A \end{pmatrix} = 0,$$

i.e. $\begin{pmatrix} P_1 & -Q & P_2 \end{pmatrix}$ is a left annihilator of $\begin{pmatrix} C \\ K \\ zI - A \end{pmatrix}$. Applying Proposition 2.1 and using the coprime factorization (89), the MLA is given by $\begin{pmatrix} D_{11} & 0 & -\Theta_1 \\ D_{21} & D_{22} & -\Theta_2 \end{pmatrix}$. Thus, we have the existence of polynomial matrices X, Y for which the factorization (118) holds which, in turn, is equivalent to (119). In particular, the equality $Q = YD_{22}$ shows that the polynomial matrix Y is necessarily nonsingular.

2. Assume $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = Q^{-1}\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ is a stable rational solution of (117), i.e. Q is stable. From (119) we have $Q = YD_{22}$ which implies the stability of D_{22}.
   Conversely, if D_{22} is stable then any choice of $W = XY^{-1}$ with Y nonsingular and stable leads to a stable solution of (117) given by (120).
3. Assume $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ is a polynomial solution of (117), thus $Q = I$ and from $I = YD_{22}$ the unimodularity of D_{22} follows.
   Conversely, if D_{22} is unimodular, and without loss of generality $D_{22} = I$, then by choosing $W = 0$ we get the polynomial solution $\begin{pmatrix} P_1 & P_2 \end{pmatrix} = \begin{pmatrix} -D_{21} & \Theta_2 \end{pmatrix}$. □

The factorization (118) has a behavioral interpretation and we shall return to it in Section 5.

3.3. *Observers: Definitions*

As indicated in the introduction to this section, and emphasized by Diagram 3.1, an observer is itself a dynamical system driven by the controls and observations and whose output is an estimate ζ of the relevant variable z. This leads us to the following definition of the most common observers.

Definition 3.2. Given a linear system

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx \end{cases} \tag{121}$$

with A, B, C, K in $\mathbb{F}^{n\times n}, \mathbb{F}^{n\times m}, \mathbb{F}^{p\times n}, \mathbb{F}^{k\times n}$ respectively. Let another system

$$\Sigma_{est} := \begin{cases} \sigma\xi = F\xi + Gy + Hu, \\ \zeta = J\xi \end{cases} \tag{122}$$

be given with F, G, H, J in $\mathbb{F}^{q\times q}, \mathbb{F}^{q\times p}, \mathbb{F}^{q\times m}, \mathbb{F}^{k\times q}$ respectively, with J of full row rank, and driven by the input u and output y of (121). Define the **estimation error** e by

$$e_t = z_t - \zeta_t = Kx_t - J\xi_t. \tag{123}$$

The **error trajectory** is defined by

$$e = e(z) = \sum_{t=1}^{\infty} e_t z^{-t}. \tag{124}$$

The system Σ_{est} defined by (122) will be called

1. A **finitely determined observer** for K if there exists a $T \in \mathbb{Z}_+$ such that $e_t = 0$ for $t < T$ implies $e = 0$.
2. A **tracking observer** for K if for every $x_1 \in \mathbb{F}^n$ there exists a $\xi_1 \in \mathbb{F}^q$ such that, for all input functions u, the solutions x_t and ξ_t of (121) and (122) respectively, satisfy $e_t = z_t - \zeta_t = Kx_t - J\xi_t = 0$ for all $t \geqslant 1$.
3. A **strongly tracking observer** for K if, given $e_1 = z_1 - \zeta_1 = 0$ implies $e_t = 0$ for all input functions u and all $t \geqslant 1$.
4. A tracking observer is called an **asymptotic tracking observer** for K if for all initial conditions of the states x and ξ and all inputs u, $\lim_{t\to\infty} e_t = \lim_{t\to\infty}(z_t - \zeta_t) = 0$.

In all cases q will be called the **order** of the observer.

In order for an observer to do its estimation properly, we will have to have a closer look at how the error evolves in time, i.e. at the error dynamics. This will be done in Section 3.4.

Naturally, there are two fundamental problems that present themselves, namely:

1. Given the system (121), give a characterization of observers of the various types.
2. Given the system (121), show the existence of observers of the various types and provide a computational procedure for observer construction.

Remarks

1. We shall always assume that C, K are both of full row rank and that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is observable.
2. The definition of a tracking observer clearly implies that the set of the trajectories to be estimated is included in the set of outputs of the tracking observer.
3. A strongly tracking observer is at the same time a tracking observer. This follows from our assumption that J has full row rank. Thus $e_1 = Kx_1 - J\xi_1$ can always be made zero by an appropriate choice of ξ_1. We note also that a strongly tracking observer is finitely determined, with $T = 2$.
4. The definition of asymptotic observers uses convergence, so an underlying topology has to be assumed. In case of the real or complex fields, we use the standard metric topology. As we study discrete time systems, the stability of the map F is equivalent to all its eigenvalues to be located in the open unit disk. In the case of other fields, in particular finite fields, we will interpret $\lim_{t\to\infty} z_t = 0$ as $z_t = 0$ for t large enough. In this setting, as we noted before, the notion of a dead beat tracking observer coincides with that of asymptotic tracking observer. This is not the case when the underlying field is real or complex and these notions are distinct. Also, in this case, a proper matrix rational function $G(z)$ is stable if its only singularity is at $z = 0$, i.e. there exists an integer N such that $z^N G(z)$ is a polynomial matrix. The following theorem covers also the case of dead beat observers when the field has no topology, as in this case the notions of dead beat observer coincides with that of an asymptotic observer. Due to space limitations, we will omit a discussion of dead beat observers in the case that the underlying field is real or complex and the two notions are distinct.
5. In view of Proposition 3.2, the assumption of the observability of the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ entails no loss of generality.
6. Note that, generally, we do not know the initial value of the state of the system which is the core of the estimation or observation problem. Thus, even if we have a tracking observer, there will be a nonzero tracking error whenever the initialization of the observer is incorrect. This points out the importance of asymptotic observers as well as, even better, spectrally assignable observers where we have also control of the rate of convergence to zero of the error.

3.4. *Observers: Characterizations*

Having studied the observation properties of linear systems in Section 3.2 and introduced several classes of observers in Section 3.3, it comes as no great surprise that there is a natural correspondence between the two. This correspondence is addressed next. In the present section we give a state space characterization for classes of observers introduced in Definition 3.2. This characterization is given in terms of Sylvester equations. The existence of various types of observers, or families of observers, will depend on the observation properties of the underlying system. A discussion of this will be undertaken in Section 3.5.

Theorem 3.2. *Given the linear system* Σ_{sys}

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx \end{cases} \tag{125}$$

in the state space $\mathbb{F}^n$. *We assume* $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *is observable. Let the observable system*

$$\Sigma_{est} := \begin{cases} \sigma \xi = F\xi + Gy + Hu, \\ \zeta = J\xi \end{cases} \tag{126}$$

be given in the state space $\mathbb{F}^q$. *Then*

1. *The following conditions are equivalent*:
   (a) *The observable system given by* (126) *is a tracking observer for* K.
   (b) *There exists a uniquely determined, linear transformations* $Z \in \mathbb{F}^{q\times n}$ *such that the* **Sylvester equations**

$$\begin{cases} ZA - FZ = GC, \\ H = ZB, \\ K = JZ \end{cases} \tag{127}$$

hold.
Defining

$$\epsilon = Zx - \xi, \tag{128}$$

the observer error dynamics are given by

$$\begin{cases} \sigma\epsilon = F\epsilon, \\ e = J\epsilon, \end{cases} \tag{129}$$

i.e. the error trajectory is the output of an autonomous linear system. The set $\mathscr{B}_{err}$ *of all error trajectories is an autonomous behavior, having the representation*

$$\mathscr{B}_{err} = X^Q = \operatorname{Ker} Q(\sigma), \tag{130}$$

where

$$Q^{-1}\Pi = J(zI - F)^{-1} \tag{131}$$

are coprime factorizations.

2. *The following conditions are equivalent*:
   (a) *The observable system given by* (126) *is an asymptotic tracking observer for* K.
   (b) *There exists a linear transformations* Z *such that* (127) *holds with* F *stable.*

Proof. In view of Proposition 3.1, and in order to keep complexity to a minimum, we will, without loss of generality, treat the case of no inputs. With this completed, all one needs is to define $H = ZB$ to go over to the general case.

1. (a) $\Leftrightarrow$ (b)
   Assume that Σ_{est}, defined by (126), is a tracking observer. The combined system/observer equations are given by
$$\begin{cases} \begin{pmatrix} \sigma I - A & 0 \\ -GC & \sigma I - F \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ e = z - \zeta = \begin{pmatrix} K & -J \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}, \end{cases} \tag{132}$$
   i.e. the error is the output of an autonomous system.
   Given initial conditions x_1 for Σ and ξ_1 for Σ_{est}, the solution of equations (132) is given by
$$\begin{aligned} \begin{pmatrix} x \\ \xi \end{pmatrix} &= \begin{pmatrix} zI - A & 0 \\ -GC & zI - F \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \xi_1 \end{pmatrix} \\ &= \begin{pmatrix} (zI - A)^{-1} & 0 \\ (zI - F)^{-1}GC(zI - A)^{-1} & (zI - F)^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \xi_1 \end{pmatrix} \end{aligned}$$
$$\begin{aligned} e &= \begin{pmatrix} K & -J \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \\ &= K(zI - A)^{-1}x_1 - J(zI - F)^{-1}GC(zI - A)^{-1}x_1 - J(zI - F)^{-1}\xi_1 \\ &= 0 \end{aligned} \tag{133}$$
   Note that the second equation in (133) describes the error trajectory. Evaluating the residue in (133), we have that for every x_1 there exists a ξ_1, such that $e = 0$. In particular, this shows the inclusion $\operatorname{Im} K \subset \operatorname{Im} J$. Hence, there exists a linear map Z such that $K = JZ$ as well as $\xi_1 = Zx_1$. Substituting back in (133), we get
$$\begin{aligned} 0 &= J[Z(zI - A)^{-1} - (zI - F)^{-1}GC(zI - A)^{-1} - (zI - F)^{-1}Z]x_1 \\ &= J(zI - F)^{-1}[(zI - F)Z - GC - Z(zI - A)](zI - A)^{-1}x_1 \\ &= J(zI - F)^{-1}[ZA - GC - FZ](zI - A)^{-1}x_1 \end{aligned}$$
   Now x_1 is arbitrary, $(zI - A)^{-1}$ nonsingular and the pair (J, F) observable, hence we obtain $ZA - FZ - GC = 0$.
   To show uniqueness, assume there exist two maps Z', Z'' satisfying (127). Setting $Z = Z'' - Z'$, we have $ZA = FZ$ and $JZ = 0$. The intertwining relation implies that for all $k \geqslant 0$ we have $ZA^k = F^kZ$. Thus we have
$$JF^kZ = JZA^k = 0,$$
   i.e. $\operatorname{Im} Z \subset \cap_{k\geqslant 0}\operatorname{Ker} JF^k$. The observability of the pair (J, F) implies now $Z = 0$ or $Z'' = Z'$.
   Conversely, assume the Sylvester equations (127) are satisfied. Given an arbitrary initial condition x_1 for Σ', we choose $\xi_1 = Zx_1$. By (132), the error trajectory is given by
$$\begin{aligned} e &= K(zI - A)^{-1}x_1 - J(zI - F)^{-1}GC(zI - A)^{-1}x_1 - J(zI - F)^{-1}\xi_1 \\ &= JZ(zI - A)^{-1}x_1 - J(zI - F)^{-1}GC(zI - A)^{-1}x_1 - J(zI - F)^{-1}Zx_1 \\ &= J(zI - F)^{-1}[(zI - F)Z - GC - Z(zI - A)](zI - A)^{-1}x_1 = 0. \end{aligned}$$

This shows that Σ_{est} is a tracking observer for Σ'.
Eq. (133) shows that the error dynamics are given by the pair $\left(\begin{pmatrix} JZ & -J \end{pmatrix}, \begin{pmatrix} A & 0 \\ GC & F \end{pmatrix}\right)$. However, this pair is not observable. In order to see this, we apply the state space isomorphism $\begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix}$ and compute, noting that $\begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix}$,

$$\begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix}\begin{pmatrix} A & 0 \\ GC & F \end{pmatrix}\begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix} = \begin{pmatrix} A & 0 \\ ZA - FZ - GC & F \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}$$

$$\begin{pmatrix} JZ & -J \end{pmatrix}\begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix} = \begin{pmatrix} 0 & J \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ Z & -I \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} x \\ Zx - \xi \end{pmatrix} = \begin{pmatrix} x \\ \epsilon \end{pmatrix},$$

where $\epsilon = Zx - \xi$. Thus the observer error dynamics are reduced to (129).
Conversely, assume there exists a linear transformation $Z \in \mathbb{F}^{q \times n}$ such that the Sylvester equations (127) hold. To compute the error dynamics, we define

$$\epsilon = Zx - \xi \tag{134}$$

and compute

$$\begin{aligned} \sigma\epsilon &= \sigma(Zx - \xi) = Z\sigma x - \sigma\xi \\ &= Z(Ax + Bu) - (F\xi + Gy + Hu) \\ &= ZAx + ZBu - FZx + FZx - F\xi - GCx - Hu \\ &= (ZA - FZ - GC)x + F(Zx - \xi) + (ZB - H)u = F\epsilon \end{aligned} \tag{135}$$

This implies that for the estimation error $e = z - \zeta = Kx - J\xi = J(Zx - \xi) = J\epsilon$, the error dynamics is given by

$$\begin{cases} \sigma\epsilon = F\epsilon, \\ e = J\epsilon, \end{cases} \tag{136}$$

so the error trajectory is the output of an autonomous linear system. Choosing $\xi_1 = Zx_1$ implies $\epsilon_1 = Zx_1 - \xi_1 = 0$ and hence $\epsilon_t = 0$ as a solution to a homogeneous system with homogeneous initial conditions. In turn, this implies $e_t = 0$ for $t > 1$. This shows that Σ_{est} is indeed a tracking observer.
The space of error trajectories is given by $\mathscr{B}_{err} = \{J(zI - F)^{-1}\xi | \xi \in \mathbb{F}^q\}$. It is well known, see [6], that for the coprime factorizations (131) we have $\{J(zI - F)^{-1}\xi | \xi \in \mathbb{F}^q\} = X^Q$. That rational models are equal to autonomous behaviors has been shown in [11].

2. (a) $\Leftrightarrow$ (b)
Assume that (126) is an asymptotic tracking observer for (125). By Part 1, there exists a uniquely determined linear transformation Z for which the Sylvester equations (127) hold. Since the error dynamics is given by (129), and we assume (J, F) to be an observable pair, then necessarily F is stable.
Conversely, if there exists a map Z solving the Sylvester equations (127), then Σ_{est} is a consistent tracking observer. The assumed stability of F implies, using the error dynamics (129), that it is actually an asymptotic tracking observer. $\square$

We note that the map Z, whose existence is proved in Theorem 3.2, is not necessarily surjective. There is a simple reduction which allows us to reduce the dimension of the observer state space so that the induced map is surjective. Indeed, if Z is not surjective, then in appropriate basis, we have $Z = \begin{pmatrix} \overline{Z} \\ 0 \end{pmatrix}$ with $\overline{Z}$ surjective, i.e. of full row rank. We have the corresponding representations

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & J_2 \end{pmatrix}. \tag{137}$$

Eqs. (127) can now be rewritten as

$$\begin{aligned} &\begin{pmatrix} \overline{Z} \\ 0 \end{pmatrix} A = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} \overline{Z} \\ 0 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} C, \\ &\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} \overline{Z} \\ 0 \end{pmatrix} B, \\ &K = \begin{pmatrix} J_1 & J_2 \end{pmatrix} \begin{pmatrix} \overline{Z} \\ 0 \end{pmatrix}. \end{aligned} \tag{138}$$

In turn, this implies

$$\begin{cases} \overline{Z}A - F_{11}\overline{Z} = G_1 C, \\ H_1 = \overline{Z}B, \\ K = J_1\overline{Z}, \end{cases} \tag{139}$$

as well as $H_2 = 0$ and $F_{21}\overline{Z}\xi + G_1 C y = 0$, for all ξ, y. Thus we must have, by the surjectivity of $\overline{Z}$, that $F_{21} = 0$. The observability of the pair (J, F) implies now the observability of the pair (J_1, F_{11}). Therefore

$$\begin{cases} \sigma\xi = F_{11}\xi + G_1 y + H_1 u, \\ \zeta = J_1\xi \end{cases} \tag{140}$$

is also a tracking observer for (125).

An interesting question is to analyse the extent of our control over the error dynamics. In particular, we would like to clarify the question: under what conditions can we preassign the error dynamics? In one direction this is easily resolved, and it uses observability subspaces. This is studied in [16, Theorem 14]. To make that result more symmetric, we need to extend Definition 3.2 as follows.

Definition 3.3. Given a linear system (121), we say that a family of (observable) observers (122) is **spectrally assignable** if there exists a fixed map Z such that, given an arbitrary polynomial f, with $\deg f = \operatorname{codim} \operatorname{Ker} Z$, there exists an observer in the family for which the characteristic polynomial of F is f and the Sylvester equations (127) are satisfied.

3.5. Observers: Existence

Our next objective is the analysis of the existence of various types of observers. The existence conditions are given in terms of solvability of certain functional equations as well as in geometric terms. They are summed up in the following theorem that clearly links the observation properties of the system with the existence of appropriate observers. There is a slight asymmetry in the statements of the following theorem, due to the fact that the solution Z of the Sylvester equations (127), whose existence is guaranteed by Theorem 3.2, is not necessarily surjective. Since the proofs of parts 2 and 3 of the following theorem follow from Part 1 by minor modifications, we will omit most of their proofs. The only exception relates to spectral assignability which is much more delicate.

Theorem 3.3. *Given the linear system* Σ_{sys}

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx \end{cases} \tag{141}$$

in the state space $\mathbb{F}^n$. *We assume* $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *is observable. Let the observable system*

$$\Sigma_{est} := \begin{cases} \sigma \xi = F\xi + Gy + Hu, \\ \zeta = J\xi \end{cases} \tag{142}$$

be given in the state space $\mathbb{F}^q$. *Let* Z_C, Z_K *be defined by* (88). *Then*

1. (a) *The following conditions are equivalent*:
      i. *There exists an observable tracking observer for* K *of the form* (142).
      ii. Σ *is trackable*.
   (b) *The following conditions are equivalent*:
      i. *There exists a conditioned invariant subspace* $\mathscr{V} \subset \operatorname{Ker} K$, *with* $\operatorname{codim} \mathscr{V} = q$.
      ii. *There exists a rank* q, *surjective solution* Z *of the Sylvester equations* (127).
   (c) *The existence of a rank* q, *surjective solution* Z *of the Sylvester equations* (127) *implies the existence of an order* q *tracking observer*.
   (d) *The existence of a conditioned invariant subspace* $\mathscr{V} \subset \operatorname{Ker} K$ *with* $\operatorname{codim} \mathscr{V} = q$ *implies the existence of strictly proper, rational functions* Z_1, Z_2, *with the McMillan degree of* $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \leqslant q$, *having the minimal realization*

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \left(\begin{array}{c|cc} F & G & Z \\ \hline J & 0 & 0 \end{array}\right), \tag{143}$$

   that solve

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = K. \tag{144}$$

   (e) *There exist strictly proper rational solutions* Z_1, Z_2 *that solve* (144) *if and only if they solve*

$$Z_K = Z_1 Z_C + Z_2. \tag{145}$$

2. (a) *The following conditions are equivalent*:
   i. *There exists an observable, asymptotic tracking observer for* K.
   ii. Σ *is detectable*.
   (b) *The following conditions are equivalent*:
   i. *There exists an outer detectable subspace* $\mathscr{D} \subset \operatorname{Ker} K$, *with* $\operatorname{codim} \mathscr{D} = q$.
   ii. *There exists a rank* q, *surjective solution* Z *of the Sylvester equations* (127) *with* F *stable*.
   (c) *The existence of a rank* q, *surjective solution* Z *of the Sylvester equations* (127), *with* F *stable implies the existence of an order* q *asymptotic tracking observer*.
   (d) *The existence of a detectability subspace* $\mathscr{V} \subset \operatorname{Ker} K$ *with* $\operatorname{codim} \mathscr{V} = q$ *implies the existence of strictly proper, stable rational functions* Z_1, Z_2, *with the McMillan degree of* $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \leqslant q$, *that solve* (144).
   (e) *There exist strictly proper, stable rational functions* Z_1, Z_2 *that solve* (*145*) *if and only if they solve* (*144*).
3. (a) *The following conditions are equivalent*:
   i. *There exists a spectrally assignable family* (142) *of tracking observer for* K.
   ii. Σ *is observable*.
   (b) *The following conditions are equivalent*:
   i. *There exists an outer observability subspace* $\mathscr{O} \subset \operatorname{Ker} K$, *with* $\operatorname{codim} \mathscr{O} = q$.
   ii. *There exists a rank* q, *surjective solution* Z *of the Sylvester equations* (127) *with* F *having an arbitrarily preassignable characteristic polynomial of degree* q.
   (c) *The existence of a rank* q, *surjective solution* Z *of the Sylvester equations* (127), *with* F *having an arbitrarily preassignable characteristic polynomial of degree* q *implies the existence of an order* q *spectrally assignable family of tracking observer*.
   (d) *The existence of a codimension* q *outer observability subspace* $\mathscr{O} \subset \operatorname{Ker} K$ *implies the existence of strictly proper, rational functions* Z_1, Z_2, *with the McMillan degree of* $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \leqslant q$, *having the minimal realization* (143), *that solve* (144) *and the existence of a polynomial solution* P_1, P_2 *of the equation*

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} C \\ zI - A \end{pmatrix} = Z. \tag{146}$$

Proof. As in the proof of Theorem 3.2, it suffices to treat only the case of no inputs.

1. (a) (i) $\Leftrightarrow$ (ii)
   Assume a tracking observer exists and is given by (142). By Theorem 3.2, this implies the existence of a map Z for which the Sylvester equations (127) are satisfied. Also, the error trajectories are in X^Q where $Q^{-1}\Pi = J(zI - F)^{-1}$. Applying Lemma 2.1, we get trackability.
   Conversely, assume Σ is trackable. Let $\mathscr{B}$ be the space of all solutions of the system (141) and let the space $\mathscr{B}$ be defined by

$$\mathscr{B} = \left\{ \hat{z} = z - \bar{z} \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \bar{x} \\ y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys} \right. \right\}. \tag{147}$$

This space is equivalent to the space of solutions of

$$\begin{cases} \sigma\hat{x} = A\hat{x}, \\ 0 = C\hat{x}, \\ \hat{z} = K\hat{x}. \end{cases} \tag{148}$$

By the assumption of trackability, $\mathscr{B} = X^Q$ for some nonsingular polynomial matrix Q. Let E be a g.c.r.d. of $zI - A$ and C. Since E is only defined up to a left unimodular factor, we can assume without loss of generality, E to be row proper. From Eqs. (148) it follows that necessarily $KE(z)^{-1}$ is strictly proper. Let $Q^{-1}P$ be a left coprime factorization of $KE(z)^{-1}$ and let $Q^{-1}P = J(zI - F)^{-1}G$ be a minimal realization. Define

$$\Sigma_{est} := \begin{cases} \sigma\xi = F\xi + Gy, \\ \zeta = J\xi. \end{cases} \tag{149}$$

We show next that Σ_{est} is a tracking observer. Note that $KE^{-1} = Q^{-1}\Pi$ implies the equality $Q(z)K = \Pi(z)E(z)$. This shows that $K : X^E \longrightarrow X^Q$ is an $\mathbb{F}[z]$-homomorphism which is necessarily surjective by the left coprimeness of Q, Π. Thus Σ_{est} is indeed a tracking observer.

(b) (i) $\Leftrightarrow$ (ii)
Assume a tracking observer exists and is given by (142). By the remark following Theorem 3.2, there exist F and Z surjective of rank q for which the Sylvester equations (127) hold. In particular, $ZA = FZ + GC$ shows that $\mathscr{V} = \operatorname{Ker} Z$ is conditioned invariant. Moreover, $K = JZ$ implies the inclusion $\mathscr{V} = \operatorname{Ker} Z \subset \operatorname{Ker} K$. Since Z has rank q, we have $\dim \operatorname{Ker} Z = n - \operatorname{rank} Z$, which implies $\operatorname{codim} \mathscr{V} = q$.
To prove the converse, assume there exists a conditioned invariant subspace $\mathscr{V} \subset \operatorname{Ker} K$. Let $\mathscr{V} = \operatorname{Ker} Z$, with Z surjective of rank q, be a kernel representation of $\mathscr{V}$. $\operatorname{Ker} Z \subset \operatorname{Ker} K$ shows the existence of a map J for which $K = JZ$. As $\mathscr{V}$ is conditioned invariant, there exists an output injection L such that $(A - LC)\mathscr{V} \subset \mathscr{V}$ or $(A - LC)\operatorname{Ker} Z \subset \operatorname{Ker} Z$. By Lemma 2.2, there exists an F for which $Z(A - LC) = FZ$. So, with $G = ZL$ and defining $H = ZB$, Eqs. (127) hold. Applying Theorem 3.2, this shows the existence of an order q tracking observer.

(c) Assume Z is a rank q surjective solution of the Sylvester equations (127). This implies that $\mathscr{V} = \operatorname{Ker} Z$ is a codimension q conditioned invariant subspace. By Part 0b, there exists an order q tracking observer.

(d) Assume there exists codimension q conditioned invariant subspace. By Part 0b, there exists an order q tracking observer (142). We compute, using equations (127),

$$\begin{aligned} Z_1 Z_C + Z_2 &= J(zI - F)^{-1}GC(zI - A)^{-1} + J(zI - F)^{-1}Z \\ &= J(zI - F)^{-1}[GC + Z(zI - A)](zI - A)^{-1} \\ &= J(zI - F)^{-1}(zI - F)Z(zI - A)^{-1} \\ &= JZ(zI - A)^{-1} = K(zI - A)^{-1} = Z_K, \end{aligned}$$

i.e. we obtain a strictly proper solution of (145).

(e) The equality of the sets of solutions of equations (144) and (145) is trivial.

3. (d)

An observability subspace is simultaneously a conditioned invariant subspace as well as an almost observability subspace, see [27] and [16]. Since $\mathscr{O}$ is conditioned invariant, it follows from Part 1 that there exists a strictly proper, rational solution Z_1, Z_2 of (144) having the minimal realization (143). Note that $\mathscr{O} = \operatorname{Ker} Z$. Since $\mathscr{O}$ is also an almost observability subspace, it follows from Theorem 13 in [16] that there exists a polynomial solution of (146). □

Remarks

1. We already noted, in the remarks following Corollary 3.1, that trackability is a weak concept. Therefore one expects that a tracking observer for Σ, given by (141), should always exist. This is indeed the case. One can define
$$\Sigma_{est} := \begin{cases} \sigma\xi = A\xi + Bu, \\ \zeta = K\xi \end{cases} \tag{150}$$
and check that it is a tracking observer. Also, one notes that one strictly proper solution of (144) is given by $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \begin{pmatrix} 0 & K(zI - A)^{-1} \end{pmatrix}$. This also leads to the observer (150). Finally, the zero subspace is a conditioned invariant subspace for (C, A) and is contained in $\operatorname{Ker} K$. This allows us to take $Z = I$ and hence, from the Sylvester equations, that $F = A$ and $J = K$. So, once again, we are back to the observer (150). Such an observer is of course totally useless as it disregards all the observed data y.
2. We quote from [16]: "Note that the existence of fixed order tracking observers with arbitrary spectrum does not necessarily imply the existence of a suitable observability subspace, not even in the minimal order case. For every given characteristic polynomial there exists a conditioned invariant subspace with the respective outer spectrum but they could all be different". Clearly, Definition 3.3 was introduced to overcome this problem.
3. Note that the existence of a polynomial solution of (146) implies, multiplying through by J, the existence of a polynomial solution of (144) but the converse does not necessarily hold. A sufficient condition for that is that $\operatorname{Ker} K$ is an almost observability subspace.

Corollary 3.2. *Given the system* (125). *The observer* (126) *is finitely determined if and only if it is tracking.*

Proof. Follows from the error dynamics given by (130) and Lemma 2.1. □

It is also of interest to relate the error trajectories to the strictly proper solutions of the equation $Z_K = Z_1 Z_C + Z_2$. Since Z_1 is the transfer function of the observer, one expects that Z_2 will be related to the error estimate. This is indeed the case and we can state.

Proposition 3.5. *Given the system* (125) *and the tracking observer* (126). *Let the initial conditions for the system and the observer be x_0 and ξ_0 respectively. Then, defining*
$$Z_J(z) = J(zI - F)^{-1}, \tag{151}$$
we have $Z_2 = Z_J Z$ and the error trajectory is given by
$$e = Z_2 x_0 - Z_J \xi_0 = Z_J(Z x_0 - \xi_0). \tag{152}$$

Proof. The solution trajectory of the to be estimated variables is given by $Z_K x_0$. On the other hand, the solution of the observer equations, being a nonhomogeneous difference equation, is naturally split into the sum of two terms. The first is the solution of the nonhomogeneous equation with homogeneous initial conditions which is given by $Z_1 Z_C x_0$ and the second is the solution to the homogeneous equation with the nonhomogeneous initial condition ξ_0, and it is given by $Z_J\xi_0$. Therefore the error trajectory is given by

$$e = Z_K x_0 - Z_1 Z_C x_0 - Z_J \xi_0. \tag{153}$$

We note that the equation $Z_K = Z_1 Z_C + Z_2$ implies $Z_K x_0 - Z_1 Z_C x_0 = Z_2 x_0 = Z_J Z x_0$. substituting back into (153), we obtain (152). Eq. (152) gives not only the error dynamics but also the actual error, completely determined by the choice of ξ_0. □

It may be of interest to understand under what conditions the observer equations (126) can be simplified to $\sigma\zeta = F\zeta + Gy + Hu$. The following proposition addresses this question and gives a geometric characterization for the existence of strongly tracking observers.

Proposition 3.6. *Given the system*

$$\begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx. \end{cases} \tag{154}$$

We assume that C, K both have full row rank and that $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is an observable pair. Then

1. *A tracking observer*
$$\Sigma_{est} := \begin{cases} \sigma\xi = F\xi + Gy + Hu, \\ \zeta = J\xi \end{cases} \tag{155}$$
with (J, F) observable, is a strongly tracking observer if and only if J is nonsingular. In that case, we may assume without loss of generality that the observer is given by
$$\sigma\zeta = F\zeta + Gy + Hu. \tag{156}$$
2. *A strongly tracking observer of the form*
$$\sigma\zeta = F\zeta + Gy + Hu \tag{157}$$
exists if and only if Ker K *is a conditioned invariant subspace. In this case, the error dynamics are given by*
$$\sigma e = Fe. \tag{158}$$

Proof

1. Assume J in (155) is nonsingular. The error dynamics are given by (129), and hence
$$\sigma e = J\sigma\epsilon = JF\epsilon = JFJ^{-1}J\epsilon = JFJ^{-1}e.$$
This shows that $e_n = (JFJ^{-1})^{n-1}e_1$ and hence $e_1 = 0$ implies $e_n = 0$, i.e. Σ_{est} is a strongly tracking observer.
Conversely, assume that Σ_{est}, defined by (155), is a strongly tracking observer. The error dynamics are given by (129) and hence $e_n = JF^{n-1}\epsilon_1$. By the property of strong tracking, $e_1 = 0$ implies $e_n = 0$ for all $n \geqslant 1$, i.e. $\epsilon_1 \in \cap \mathrm{Ker}\, JF^{n-1}$. By observability of the pair (J, F), we conclude that $e_1 = 0$ implies $\epsilon(1) = 0$. This shows that J is injective and hence, since we assumed that J has full row rank, actually invertible. Substituting $\xi = J^{-1}\zeta$ in the observer equation and multiplying through by J, we obtain $\sigma\zeta = (JFJ^{-1})\zeta + (JG)y + (JH)u$. Modifying appropriately the definitions of F, G, H, Eq. (156) follows.

2. Assume (157) is a strongly tracking observer. Thus, by Theorem 3.3, there exist a map Z satisfying the Sylvester equations
$$\begin{cases} ZA = FZ + GC, \\ H = ZB, \\ K = Z. \end{cases} \tag{159}$$
For $x \in \operatorname{Ker} K \cap \operatorname{Ker} C$, we have $K(Ax) = 0$, i.e. $Ax \in \operatorname{Ker} K$, so $\operatorname{Ker} K$ is a conditioned invariant subspace.
Conversely, assume $\operatorname{Ker} K$ is a conditioned invariant subspace. Let $Z = K$. There exists a map L such that $(A - LC)\operatorname{Ker} K \subset \operatorname{Ker} K$ and using once again Lemma 2.2, we infer that $K(A - LC) = FK$ for some L. Thus $KA - FK = GC$ with $G = KL$. We set $J = I$ and, defining $H = KB$, we are done.
That the error dynamics are given by (158) follows from (129) and the fact that $J = I$. □

Remarks

1. Clearly, the existence of a spectrally assignable family of observers implies the existence of an asymptotic observer. Thus in particular Theorem 3.3.3.c should imply 2.c and, in the same way, Theorem 3.3.3.d should imply 2.d. To see this directly will become clear from the use of partial realizations. We will return to this subject in Section 5.
2. In view of the geometric characterizations given in Theorem 3.3, in order to find minimal order observers for the system (127), we have to find all maximal dimensional conditioned invariant subspaces $\mathscr{V}$ that satisfy $\mathscr{V} \subset \operatorname{Ker} K$. Since the set of all conditioned invariant subspaces is closed under intersections but not under additions, in general there does not exist a unique, maximal dimensional conditioned invariant subspace $\mathscr{V} \subset \operatorname{Ker} K$. This is at the root of the nonuniqueness of minimal observer construction. For more on this, see [13] and [16].

3.6. Partial realizations and parametrizations

In general, given a system Σ, as defined by (125), there exist many tracking observers for it. Even if we require the observer to have a minimal McMillan degree, it is, in general, not uniquely determined. One way to see this is via the geometric characterization of observers, i.e. the existence of a maximal dimensional, conditioned invariant subspace included in $\operatorname{Ker} K$. Conditioned invariant subspaces are closed under intersections but not under sums, so there may be many conditioned invariant subspaces of maximal dimension. Of course, if $\operatorname{Ker} K$ is itself conditioned invariant, it is automatically maximal. However, even in this case, if $\operatorname{Ker} K$ is not tight, see [14] or [16], there may be many observers of minimal McMillan degree.

Thus, it is of interest to parametrize the set of all conditioned invariant subspaces contained in a given subspace. This problem arose in connection with spectral assignability for observers and was treated in detail in [16] using polynomial matrix completions.

In the present paper, we use partial realizations as a tool to parametrize the set of all observers. This idea has its roots in [1] and will be used extensively.

What have partial realizations to do with observer theory in general and, more specifically, with observer construction? Since this question is central in what follows, we will try to answer it as best we can. One of the characterizations for the existence of a tracking observer for the system (125) is given in geometric terms, i.e. the existence of a conditioned invariant $\mathscr{V} \subset \operatorname{Ker} K$. Now polynomial and rational models provide the ideal setting for constructing realizations. Assuming (C, A) to be observable, this leads to a functional representation. It is simple to check that the existence of an observer for K from the observations $y = Cx$ is invariant under the output injection group. The following observation is elementary, yet has important consequences. It enables us to reduce the system, in the observable case, to Brunovsky form.

Lemma 3.1. *If Σ_{est}, given by (126), is a tracking observer of the system Σ, defined by (125), then*

$$\overline{\Sigma}_{est} := \begin{cases} \sigma\xi = F\xi + (G - ZL)y, \\ \zeta = J\xi \end{cases} \tag{160}$$

is a tracking observer for the system

$$\overline{\Sigma} := \begin{cases} \sigma x = R^{-1}(A - LC)Rx, \\ y = CRx, \\ z = KRx \end{cases} \tag{161}$$

for every nonsingular R.

Proof. Assuming R to be nonsingular, we define

$$\begin{cases} \overline{A} = R^{-1}(A - LC)R, \\ \overline{C} = CR, \\ \overline{K} = KR. \end{cases} \tag{162}$$

By the definition of $\overline{A}$, we have $R\overline{A} = (A - LC)R$ and hence, using the Sylvester equations (127), also

$$\begin{aligned} (ZR)\overline{A} &= ZAR - ZLCR = (FZ - GC)R - ZLCR \\ &= F(ZR) + (G - ZL)(CR) \end{aligned}$$

Defining $\overline{Z} = ZR$ and $\overline{G} = G - ZL$, we have obtained the Sylvester equations

$$\begin{cases} \overline{Z}\overline{A} = F\overline{Z} + \overline{G}\overline{C}, \\ \overline{K} = J\overline{Z}. \end{cases} \tag{163}$$

This shows that $\overline{\Sigma}_{est}$ defined by (160) is indeed a tracking observer for $\overline{\Sigma}$. □

Assume the system given by (83) is observable, then, by the previous lemma, we can assume without loss of generality that our system is given in dual Brunovsky form with observability indices $\mu_1 \geqslant \cdots \geqslant \mu_p \geqslant 0$. That all observability indices are positive is equivalent to the linear independence of the rows of C, i.e. to the assumption that C has full row rank. In particular, we have the coprime factorizations

$$C(zI - A)^{-1} = \Delta(z)^{-1}\Theta(z),$$

with

$$\Delta(z) = \mathrm{diag}(z^{\mu_1}, \ldots, z^{\mu_p}), \tag{164}$$

with $\mu_1 \geqslant \cdots \geqslant \mu_p \geqslant 0$, and

$$\Theta(z)=\left(\begin{array}{ccccc|ccc|cccccc} 1 & z & . & . & z^{\mu_1-1} & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & z & . & . & & z^{\mu_p-1} \end{array}\right). \tag{165}$$

This implies $X_\Delta = X_{z^{\mu_1}} \oplus \cdots \oplus X_{z^{\mu_p}}$ with $\dim X_\Delta = \deg\det\Delta = \sum_{i=1}^p \mu_i = n$. We refer to the pair $\Delta(z)$, $\Theta(z)$ as the **polynomial dual Brunovsky form**. Indeed, applying the shift realization, see [6,11], taking the polynomial model X_Δ as the state space and defining

$$\begin{aligned} &A_\Delta f = \pi_\Delta z f \quad \text{for } f \in X_\Delta, \\ &C_\Delta f = (\Delta^{-1} f)_{-1} \end{aligned} \tag{166}$$

we obtain the standard dual Brunovsky form.

Since we chose a functional representation for the state space, we expect that the map K should have a corresponding one. This is worked out next.

Let $e_1, \ldots, e_p$ be the standard basis in $\mathbb{F}^p$ and $\epsilon_1, \ldots, \epsilon_k$ be the standard basis in $\mathbb{F}^k$. An element $f \in X_\Delta$ can be written uniquely as $f(z) = \sum_{j=1}^p f_j(z) e_j$ with $f_j(z) = \sum_{\nu=1}^{\mu_j} f_{j\nu} z^{\nu-1} \in X_{z^{\mu_j}}$ polynomials of degree $\leqslant \mu_j - 1$.

A map $K : X_\Delta = X_{z^{\mu_1}} \oplus \cdots \oplus X_{z^{\mu_p}} \longrightarrow \mathbb{F}^k$ can be written, with $f(z) = \sum_{j=1}^p f_j(z) e_j$, as $K = \sum_{j=1}^p K_j f_j$, with $K_j : X_{z^{\mu_j}} \longrightarrow \mathbb{F}^k$. Writing $K_j f_j = \sum_{i=1}^k K_{ij} f_j \epsilon_i$ with $K_{ij} : X_{z^{\mu_j}} \longrightarrow \mathbb{F}$ a linear functional on $X_{z^{\mu_j}}$. Thus, there exist uniquely determined numbers $K_{ij}^{(\nu)}$ such that, with $f_j(z) = \sum_{\nu=1}^{\mu_j} f_{j\nu} z^{\nu-1} \in X_{z^{\mu_j}}$, we have

$$K_{ij} f_j = \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} f_{j\nu}. \tag{167}$$

In these terms, we infer that $K_j : X_{z^{\mu_j}} \longrightarrow \mathbb{F}^k$ is given by

$$K_j f_j = \sum_{i=1}^k K_{ij} f_j \epsilon_i = \sum_{i=1}^k \left\{ \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} f_{j\nu} \right\} \epsilon_i \tag{168}$$

and hence

$$\begin{aligned} Kf &= \sum_{j=1}^p K_j f_j = \sum_{j=1}^p \sum_{i=1}^k K_{ij} f_j \epsilon_i = \sum_{i=1}^k \left\{ \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} f_{j\nu} \right\} \epsilon_i \\ &= \sum_{i=1}^k \left\{ \sum_{j=1}^p \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} f_{j\nu} \right\} \epsilon_i \\ &= \sum_{i=1}^k \left\{ \sum_{j=1}^p K_{ij} f_j \right\} \epsilon_i \end{aligned} \tag{169}$$

We proceed next to compute the adjoints of the maps K, K_j, K_{ij}. We use the fact that the dual space of the polynomial model $X_{z^{\mu_j}}$ can be identified with the rational model $X^{z^{\mu_j}}$. Hence, $K_{ij}^* : \mathbb{F} \longrightarrow X^{z^{\mu_j}}$ has the representation

$$K_{ij}^* = \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} z^{-\nu}. \tag{170}$$

Next we compute, with $\xi = \sum_{i=1}^k \xi_i \epsilon_i \in \mathbb{F}^k$,

$$\begin{aligned}\langle f, K^*\xi\rangle &= [Kf, \xi] = \sum_{j=1}^p [K_j f_j, \xi] = \sum_{j=1}^p \left[\sum_{i=1}^k K_{ij} f_j \epsilon_i, \xi\right] \\ &= \sum_{j=1}^p \sum_{i=1}^k \langle f_j, K_{ij}^* \xi_i\rangle = \sum_{i=1}^k \sum_{j=1}^p \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} f_{j\nu} \xi_i \\ &= \sum_{j=1}^p \left\langle f_j, \sum_{i=1}^k \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} z^{-\nu} \xi_i \right\rangle.\end{aligned} \tag{171}$$

This implies that $K_j^* : \mathbb{F}^k \longrightarrow X_{z^{\mu_j}}$ is given by

$$K_j^* \xi = \sum_{i=1}^k \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} z^{-\nu} \xi_i = \sum_{i=1}^k K_{ij}^* \xi_i, \tag{172}$$

where K_{ij}^* is given by (170).

As a result $K^* : \mathbb{F}^k \longrightarrow (X_{z^{\mu_1}} \oplus \cdots \oplus X_{z^{\mu_p}})^* = X^{z^{\mu_1}} \oplus \cdots \oplus X^{z^{\mu_p}}$ is given by

$$K^*\xi = \sum_{j=1}^p (K_j^*\xi) e_j = \sum_{j=1}^p K_j^* \left\{\sum_{i=1}^k \xi_i \epsilon_i\right\} e_j = \sum_{j=1}^p \left\{\sum_{i=1}^k K_{ij}^* \xi_i\right\} e_j, \tag{173}$$

and, using (170),

$$K^*\xi = \sum_{j=1}^p \left\{\sum_{i=1}^k \sum_{\nu=1}^{\mu_j} K_{ij}^{(\nu)} z^{-\nu} \xi_i\right\} e_j \tag{174}$$

Next we introduce the **dual indices**, $\lambda_1, \ldots, \lambda_{\mu_1}$ to the indices μ_i. They are defined by

$$\lambda_k = \sharp\{\mu_j | \mu_j \geqslant k\}, \quad k = 1, \ldots, \mu_1. \tag{175}$$

Obviously, we have

$$\lambda_1 \geqslant \cdots \geqslant \lambda_{\mu_1} > 0 \tag{176}$$

and $\sum_{k=1}^{\mu_1} \lambda_k = \sum_{j=1}^p \mu_j = n$.

For fixed indices $1 \leqslant \nu \leqslant \mu_1$ and $1 \leqslant i \leqslant k$, we have $1 \leqslant j \leqslant \lambda_i$. We use the dual indices to change the order of summation in Eq. (174), which can be rewritten as

$$\langle f, K^*\xi\rangle = \sum_{\nu=1}^{\mu_1} \sum_{i=1}^k \sum_{j=1}^{\lambda_\nu} K_{ij}^{(\nu)} f_{j\nu} \xi_i. \tag{177}$$

So an index $1 \leqslant \nu \leqslant \mu_1$ defines an $k \times \lambda_\nu$ matrix by

$$(K^{(\nu)})_{ij} = (K_{ij}^{(\nu)}). \tag{178}$$

Define a $k \times p$ strictly proper matrix by

$$K(z) = \begin{pmatrix} \sum_{\nu=1}^{\mu_1} K_{11}^{(\nu)} z^{-\nu} & . & . & . & \sum_{\nu=1}^{\mu_p} K_{1p}^{(\nu)} z^{-\nu} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \sum_{\nu=1}^{\mu_1} K_{k1}^{(\nu)} z^{-\nu} & . & . & . & \sum_{\nu=1}^{\mu_p} K_{kp}^{(\nu)} z^{-\nu} \end{pmatrix}$$

$$= \sum_{\nu=1}^{\mu_1} \begin{pmatrix} K_{11}^{(\nu)} & . & . & . & K_{1p}^{(\nu)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ K_{k1}^{(\nu)} & . & . & . & K_{kp}^{(\nu)} \end{pmatrix} z^{-\nu} = \sum_{\nu=1}^{\mu_1} K^{(\nu)} z^{-\nu} \tag{179}$$

Apriori, the coefficient matrices $K^{(\nu)}$ are defined to be of size $k \times \lambda_\nu$, but for summation we complete the to size $k \times p$ by adding zero columns.

Since $K^* : \mathbb{F}^k \longrightarrow (X_{z^{\mu_1}} \oplus \cdots \oplus X_{z^{\mu_p}})^* = X^{z^{\mu_1}} \oplus \cdots \oplus X^{z^{\mu_p}}$, we have

$$(K^*\xi)(z) = \tilde{K}(z)\xi \tag{180}$$

where, using (174), we have

$$\tilde{K}(z) = \begin{pmatrix} \sum_{\nu=1}^{\mu_1} K_{11}^{(\nu)} z^{-\nu} & . & . & . & \sum_{\nu=1}^{\mu_1} K_{k1}^{(\nu)} z^{-\nu} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \sum_{\nu=1}^{\mu_p} K_{1p}^{(\nu)} z^{-\nu} & . & . & . & \sum_{\nu=1}^{\mu_p} K_{kp}^{(\nu)} z^{-\nu} \end{pmatrix}$$

$$= \sum_{\nu=1}^{\mu_1} \begin{pmatrix} K_{11}^{(\nu)} & . & . & . & K_{k1}^{(\nu)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ K_{1p}^{(\nu)} & . & . & . & K_{kp}^{(\nu)} \end{pmatrix} z^{-\nu} \tag{181}$$

Depending on the choice of basis in $X_\Delta = X_{z^{\mu_1}} \oplus \cdots \oplus X_{z^{\mu_p}}$, we have a corresponding matrix representation for the map K. With respect to the standard bases in $\mathbb{F}^k$ and that of X_Δ, the matrix representation of K is given by

$$[K]_{st}^{st} = \left(\begin{array}{ccccc|ccccc|ccccc} K_{11}^{(1)} & K_{11}^{(2)} & . & . & K_{11}^{(\mu_1)} & . & . & . & . & . & K_{1p}^{(1)} & K_{1p}^{(2)} & . & . & K_{1p}^{(\mu_p)} \\ \hline . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ \hline K_{k1}^{(1)} & K_{k1}^{(2)} & . & . & K_{k1}^{(\mu_1)} & . & . & . & . & . & K_{kp}^{(1)} & K_{kp}^{(2)} & . & . & K_{kp}^{(\mu_p)} \end{array}\right), \tag{182}$$

whereas with respect to the permuted basis, represented by the basis matrix

$$\left(\begin{array}{cccccc|cccc|cccc|ccc} 1 &&&&&& z &&&& &&&& z^{\mu_1-1} && \\ & 1 &&&&&& \cdot &&& &&&& & \cdot & \\ && \cdot &&&&&& \cdot && \cdot & \cdot & \cdot & \cdot & && z^{\mu_1-1} \\ &&& \cdot &&&&&& z & &&&& && \\ &&&& \cdot &&&&&& &&&& && \\ &&&&& 1 &&&&& &&&& && \end{array}\right), \tag{183}$$

with the blocks of size $k \times \lambda_i$, the matrix representation is given by

$$[K]^{st}_{per} = \left(\begin{array}{ccccc|cccc|ccccc} K^{(1)}_{11} & K^{(1)}_{12} & \cdot & \cdot & K^{(1)}_{1\lambda_1} & \cdot & \cdot & \cdot & \cdot & K^{(\mu_1)}_{1\lambda_1} & K^{(\mu_1)}_{1\lambda_2} & \cdot & \cdot & K^{(\mu_1)}_{1\lambda_{\mu_1}} \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline K^{(1)}_{k1} & K^{(1)}_{k2} & \cdot & \cdot & K^{(1)}_{k\lambda_1} & \cdot & \cdot & \cdot & \cdot & K^{(\mu_1)}_{k\lambda_1} & K^{(\mu_1)}_{k\lambda_2} & \cdot & \cdot & K^{(\mu_1)}_{k\lambda_{\mu_1}} \end{array}\right). \tag{184}$$

The standard basis in $X^{\tilde{\Delta}} = X^{z^{\mu_1}} \oplus \cdots \oplus X^{z^{\mu_p}}$ is the dual to the standard basis in X_Δ, given by (165), and is given by the basis matrix

$$\left(\begin{array}{ccccc|ccc|ccccc} \frac{1}{z} & \cdot & \cdot & \cdot & \frac{1}{z^{\mu_1}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{z} & \cdot & \cdot & \cdot & \frac{1}{z^{\mu_p}} \end{array}\right). \tag{185}$$

With respect to this basis we obviously have

$$[K^*]^{st}_{st} = \left(\begin{array}{ccccc} K^{(1)}_{11} & \cdot & \cdot & \cdot & K^{(1)}_{k1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K^{(\mu_1)}_{11} & \cdot & \cdot & \cdot & K^{(\mu_1)}_{k1} \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline K^{(1)}_{1p} & \cdot & \cdot & \cdot & K^{(1)}_{kp} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K^{(\mu_p)}_{1p} & \cdot & \cdot & \cdot & K^{(\mu_p)}_{kp} \end{array}\right). \tag{186}$$

Definition 3.4

1. We will say that a sequence of $k \times \lambda_\nu$ matrices $\{K^{(\nu)}\}_{\nu=1}^{\mu_1}$ is a **nice sequence** if $\lambda_1 \geqslant \cdots \geqslant \lambda_{\mu_1}$. We will say that a system $\left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$, in the state space $\mathbb{F}^q$, is an **order q partial realization** of the sequence $\{K^{(\nu)}\}$ if

$$K_{ij}^{(\nu)} = \overline{C}_i \overline{A}^{\nu-1} \overline{B}_j, \quad i = 1, \ldots, k; \quad j = 1, \ldots, \lambda_\nu. \tag{187}$$

It is a minimal partial realization if there exists no partial realization of smaller order. In that case the order will be called the **McMillan degree** of the nice sequence. The **characteristic polynomial of a partial realization** is defined to be the characteristic polynomial of the generating map $\overline{A}$. We will say that a partial realization is stable or monomic or if the characteristic polynomial of the partial realization is stable or monomic respectively. Similarly, we will say that a family of partial realizations is spectrally assignable if the characteristic polynomial of the partial realizations can be arbitrarily preassigned, subject only to degree constraints.

2. Given a $k \times p$, strictly proper rational function $K(z)$, we say a rational function $W(z) = \left(\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & 0 \end{array}\right)$ solves a $K(z)$**-induced** $(\mu_1, \ldots, \mu_p)$ **partial realization problem** if $(K(z) - W(z))$ $\Delta(z)^{-1}$, with $\Delta(z)$ given by (164), is strictly proper. It is a minimal partial realization if there exists no partial realization of smaller order. In that case the order will be called the **McMillan degree**.

Note that, writing $K(z) = \sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{z^\nu}$, the partial realization problem in Definition 3.4.2 is equivalent to (187) with $\lambda_1, \ldots, \lambda_{\mu_1}$ the set of indices dual to $(\mu_1, \ldots, \mu_p)$.

Problem 3.1. Given a nice sequence of $k \times \lambda_\nu$ matrices $\{K^{(\nu)}\}_{\nu=1}^{\mu_1}$, find a system $\left(\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & 0 \end{array}\right)$ in the state space $\mathbb{F}^q$ satisfying (187). We will say that the system is a **partial realization** of the sequence $\{K^{(\nu)}\}$.

Following Antoulas [1], given a nice sequence of $k \times \lambda_\nu$ matrices $\{K^{(\nu)}\}_{\nu=1}^{\mu_1}$, we will denote by Γ_K the set of all strictly proper rational functions $\sum_{\nu=1}^{\infty} L^{(\nu)} z^{-\nu}$ satisfying

$$L_{ij}^{(\nu)} = K_{ij}^{(\nu)}, \quad i = 1, \ldots, k; \ j = 1, \ \ldots, \lambda_\nu.$$

Note that (176) implies that the sequence $K^{(1)}, \ldots, K^{(\mu_1)}$, defined by (178), is a nice sequence. Moreover, by reordering the columns CA^iB_j in the following way

$$\begin{aligned} K &= (K_{\cdot 1}^{(1)}, \ldots, K_{\cdot \mu_1}^{(1)}, \ldots, K_{\cdot 1}^{(p)}, \ldots, K_{\cdot \mu_p}^{(p)}) \\ &= (CB_1, \ldots, CA^{\mu_1 - 1}B_1, \ldots, CB_p, \ldots, CA^{\mu_p - 1}B_p), \end{aligned}$$

the partial realization condition can be written in the form

$$K = C\mathscr{R}_\mu(A, B) = C(B_1, \ldots, A^{\mu_1 - 1}B_1, \ldots, B_p, \ldots, A^{\mu_p - 1}B_p). \tag{188}$$

The definition of the μ-**partial reachability matrix** $\mathscr{R}_\mu(A, B)$ is self evident or, alternatively, can be found in [14].

The following theorem connects partial realizations with classes of observers. The asymmetry in the statements follows from the corresponding asymmetries in Theorem 3.3. We prove only the first part of the theorem. The other parts follow from that by minor modifications.

Theorem 3.4. *Given the observable linear system*

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx, \end{cases} \tag{189}$$

with C, K *of full row rank and which we assume to be in dual Brunovsky form. Then*

1. *If the nice sequence of matrices $K^{(1)}, \ldots, K^{(\mu_1)}$ defined in (178) has an order q minimal partial realization, then the system given by (189) has an order q tracking observer for K.*
   Conversely, if the system given by (189) has an order q tracking observer for K then the nice sequence of matrices $K^{(1)}, \ldots, K^{(\mu_1)}$ defined in (178) has an order $\leqslant q$ minimal partial realization.
2. *If the nice sequence of matrices $K^{(1)}, \ldots, K^{(\mu_1)}$ defined in (178) has an order q minimal, stable partial realization then the system given by (189) has an order q asymptotic tracking observer for K.*
   Conversely, if the system given by (189) has an order q asymptotic tracking observer for K, then the nice sequence of matrices $K^{(1)}, \ldots, K^{(\mu_1)}$ defined in (178) has an order $\leqslant q$ minimal, stable partial realization.
3. *If the nice sequence of matrices $K^{(1)}, \ldots, K^{(\mu_1)}$ defined in (178) has an order q minimal, partial realization with a freely preassigned characteristic polynomial of degree q, then the system given by (189) has an order q spectrally assignable family of tracking observers for K.*
   Conversely, if the system given by (189) has an order q spectrally assignable family of tracking observers for K, then the nice sequence of matrices $K^{(1)}, \ldots, K^{(\mu_1)}$ defined in (178) has an order q minimal, partial realization with a freely preassigned characteristic polynomial of degree $\leqslant q$.

Proof

1. Assume first that the nice sequence $K^{(\nu)}$, $\nu = 1, \ldots, \mu_1$, of $k \times \lambda_\nu$ matrices, defined in (178), has a McMillan degree q, minimal partial realization given by $G = \left(\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & 0 \end{array}\right)$, i.e. (187) holds.
   Now $K : X_\Delta \longrightarrow \mathbb{F}^k$ is, by assumption, surjective. Hence $K^* : \mathbb{F}^k \longrightarrow X^{\tilde{\Delta}}$ is injective. Using the transposition $\tilde{\overline{B}}(zI - \tilde{\overline{A}})^{-1}\tilde{\overline{C}} = \tilde{E}^{-1}\tilde{H} = \tilde{G}$, we have $K^*\xi = \pi^{\tilde{\Delta}}\tilde{G}\xi$. Using (181), it is easy to compute that
   $$(K^*\xi)(z) = \pi^{\tilde{\Delta}}\tilde{G} = \tilde{K}(z)\xi. \tag{190}$$
   From (179), we infer that, with Δ defined by (164), we have $\pi_- K\Delta = 0$, i.e. $K(z)\Delta(z)$ is a polynomial matrix. That G is a partial realization translates into
   $$\pi_+(K - G)\Delta = 0, \tag{191}$$
   i.e. that $(K - G)\Delta$ is strictly proper.
   From (191), it follows that
   $$\pi_+\tilde{\Delta}(\tilde{K} - \tilde{G}) = 0. \tag{192}$$
   However, we have
   $$\pi^{\tilde{\Delta}}\tilde{G} = \pi_-\tilde{\Delta}^{-1}\pi_+\tilde{\Delta}\tilde{G} = \pi_-\tilde{\Delta}^{-1}\pi_+\tilde{\Delta}\tilde{K} = \pi_-\tilde{\Delta}^{-1}\tilde{\Delta}\tilde{K} = \pi_-\tilde{K} = \tilde{K}. \tag{193}$$
   Let now $G = HE^{-1}$ be a right coprime factorization. Clearly, $\deg\det E = q$. The rational model $X^{\tilde{E}} = \operatorname{Im} H_{\tilde{G}}$ is the smallest shift invariant subspace containing $\{\tilde{G}\xi | \xi \in \mathbb{F}^k\}$. By assumption, the map $K : X_\Delta \longrightarrow \mathbb{F}^k$ is surjective. Hence $K^* : \mathbb{F}^k \longrightarrow X^{\tilde{\Delta}}$ is injective. The injectivity of K^* implies that $\pi^{\tilde{\Delta}}|X^{\tilde{E}}$ is injective. By Theorem 3.3 in [14], all the left Wiener-Hopf factorization indices of $\Delta^{-1}E$ are nonpositive. By the assumed realization, we have, for each $\xi \in \mathbb{F}^k$, that $K^*\xi \in \mathscr{W} = \pi^{\tilde{\Delta}}X^{\tilde{E}}$, or $\operatorname{Im} K^* \subset \mathscr{W}$. This clearly implies that the subspace

$\mathscr{W}$ is, by Theorem 4.6 in [17], a controlled invariant subspace of $X^{\tilde{\Delta}}$. Moreover, by the injectivity of $\pi^{\tilde{\Delta}}|X^{\tilde{E}}$, we have $\dim \mathscr{W} = \dim X^{\tilde{E}} = \deg\det \tilde{E} = \deg\det E = q$. The preannihilator of this subspace in X_Δ is clearly $\mathscr{V} = {}^\perp\mathscr{W} = X_\Delta \cap E\mathbb{F}[z]^p$, which is a conditioned invariant subspace. Moreover, see Proposition 3.4 in [14], we have $\operatorname{codim} \mathscr{V} = q$. By Theorem 3.3 an order q tracking observer for K exists.
Conversely, assume there exists an order q tracking observer. By Theorem 3.3, there exists a conditioned invariant subspace $\mathscr{V} \subset \operatorname{Ker} K$ satisfying $\operatorname{codim} \mathscr{V} \leqslant q$. By the characterization of conditioned invariant subspaces with respect to the shift realization, $\mathscr{V} \subset X_\Delta$ has the representation $\mathscr{V} = X_\Delta \cap E\mathbb{F}[z]^p$, with E a polynomial matrix for which all the left Wiener-Hopf factorization indices of $\Delta^{-1}E$ are nonpositive. Using duality theory, as developed in [8,13], we have $\mathscr{W} = \mathscr{V}^\perp = \pi^{\tilde{\Delta}} X^{\tilde{E}} \supset \operatorname{Im} K^*$. Since, by Theorem 3.3 in [14], the map $\pi^{\tilde{\Delta}}|X^{\tilde{E}}$ is injective, it follows that $\dim \mathscr{W} = \dim X^{\tilde{E}} = \deg\det \tilde{E} = \deg\det E = q$.
Now, for each vector $\xi \in \mathbb{F}^k$, we have

$$(K^*\xi)(z) = \tilde{K}(z)\xi \in \pi^{\tilde{\Delta}} X^{\tilde{E}}. \tag{194}$$

Choosing successively $\xi = e_i$ where e_i is the ith unit vector in $\mathbb{F}^k$, we conclude the existence of elements $\tilde{G}_i \in X^{\tilde{E}}$ for which $K^* e_i = \pi^{\tilde{\Delta}} \tilde{G}_i$. Thus there exist polynomial vectors $\tilde{H}_i \in X_{\tilde{\Delta}}$ for which $\tilde{G}_i = \tilde{E}^{-1}\tilde{H}_i$. Let $\tilde{H}(z)$ be the $p \times k$ polynomial matrix whose ith column is $\tilde{H}_i(z)$ and $\tilde{G}(z)$ the $p \times k$ strictly proper rational matrix whose columns are $\tilde{G}_i$. Thus we have the, not necessarily left coprime, matrix fraction representation

$$\tilde{G}(z) = \tilde{E}(z)^{-1}\tilde{H}(z). \tag{195}$$

Furthermore, we have $\pi_+\tilde{\Delta}(\tilde{K} - \tilde{G}) = 0$ and hence $\pi_+(K - G)\Delta = 0$. Thus G is a partial realization of the nice sequence. Using the shift realization, with X^E as the state space, it is clear that the McMillan degree of G is at most q. Denoting this realization by $\left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$, it is clear that

$$K_{ij}^{(\nu)} = \overline{C}_i \overline{A}^{\nu-1} \overline{B}_j, \quad 1 \leqslant i \leqslant k, \quad 1 \leqslant j \leqslant \lambda_\nu, \quad 1 \leqslant \nu \leqslant \mu_1,$$

i.e. the sequence $K^{(1)}, \ldots, K^{(\mu_1)}$ has an order $\leqslant q$ minimal partial realization. □

One of the characterizations, given in Theorem 3.3, for the existence of a tracking observer is the existence of a strictly proper solution Z_1, Z_2 of Eq. (145). If the system Σ is given in terms of state space equations, (125), then with it is associated a transfer function which has a left coprime factorization. This left coprime factorization leads to a rational solution of (145) which is polynomial in the case that the system is observable and generally is not proper. However, since Eq. (145) is a linear nonhomogeneous system, over the field of rational functions, the general solution is given in terms of that particular solution and the general solution of the homogeneous system. This was analyzed in Theorem 3.1. Thus we have the key to the parametrization of the set of all rational solutions. We can proceed to identify the subset of strictly proper solutions which turn out to be related to partial realizations. The following theorem summarizes these results.

Theorem 3.5. *Given the state space system*

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx. \end{cases} \tag{196}$$

We assume that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ *is observable. Let* (90) *be the coprime factorization satisfying conditions* 1a-1c *of Theorem* 3.1.

1. *Assuming* (C, A) *observable and in dual Brunovsky form, then we have the left coprime factorization* $C(zI - A)^{-1} = D_{11}(z)^{-1}\Theta_{11}(z)$ *where*

$$D_{11}(z) = \begin{pmatrix} z^{\mu_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ & & & & z^{\mu_p} \end{pmatrix} \tag{197}$$

and

$$\Theta_{11}(z) = \left(\begin{array}{ccccc|ccc|ccccc} 1 & z & \cdot & \cdot & z^{\mu_1 - 1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & z & \cdot & \cdot & z^{\mu_p - 1} \end{array}\right), \tag{198}$$

is the basis matrix for the polynomial model space $X_{D_{11}}$.
Defining

$$\mathbf{B} = \left(\begin{array}{c|c|c} 1 & & \\ \vdots & & \\ 0 & & \\ \hline & \ddots & \\ \hline & & 1 \\ & & \vdots \\ & & 0 \end{array}\right), \tag{199}$$

the number of rows in the p vertical blocks equal to $\mu_1, \ldots, \mu_p$ *respectively, we have*

$$\Theta_{11}(z)\mathbf{B} = I, \tag{200}$$

$$\Theta_{21}(z)\mathbf{B} = 0, \tag{201}$$

$$Z_K\mathbf{B} = K(zI - A)^{-1}\mathbf{B} = \sum_{\nu=1}^{\mu_1} \frac{K^{(\nu)}}{z^\nu} = -D_{21}D_{11}^{-1}. \tag{202}$$

$$Z_C\mathbf{B} = D_{11}^{-1}. \tag{203}$$

2. *Given the system* Σ, *defined by* (262), *assuming that* (C, A) *is an observable pair. Let* Γ_K *be the set of strictly proper, rational solutions to the nice partial realization problems associated with the system* Σ *via equations* (170)–(178). *Then* Z_1, Z_2 *is a strictly proper rational solution of equation* (145) *if and only if*

$$\begin{cases} Z_1 = -D_{21} - WD_{11}, \\ Z_2 = \Theta_{21} + W\Theta_{11} \end{cases} \tag{204}$$

with $W \in \Gamma_K$. We will refer to equation (204) as the **Antoulas parametrization**.
Let

$$W = Y^{-1}X = \overline{XY}^{-1} \tag{205}$$

be coprime factorizations. Then we have the representations

$$\begin{cases} Z_1 = -Y^{-1}(YD_{21} + XD_{11}), \\ Z_2 = Y^{-1}(Y\Theta_{21} + X\Theta_{11}). \end{cases} \tag{206}$$

3. *Given the system Σ, defined by (262), without assuming that (C, A) is an observable pair. We assume the coprime factorization (89). Let $K(z) = -D_{22}^{-1}D_{21}D_{11}^{-1}$ and Γ_K be the set of strictly proper, rational solutions to the $K(z)$-induced $(\mu_1, \ldots, \mu_p)$ nice partial realization problem. Then Z_1, Z_2 is a strictly proper rational solution of equation (145) if and only if*

$$\begin{cases} Z_1 = -D_{22}^{-1}D_{21} - WD_{11}, \\ Z_2 = D_{22}^{-1}\Theta_{21} + W\Theta_{11} \end{cases} \tag{207}$$

with $W \in \Gamma_K$. We will refer also to equation (207) as the **Antoulas parametrization**.
Let

$$W = Y^{-1}X = \overline{XY}^{-1} \tag{208}$$

be coprime factorizations. Then we have the representations

$$\begin{cases} Z_1 = -D_{22}^{-1}Y^{-1}(YD_{21} + XD_{11}), \\ Z_2 = D_{22}^{-1}Y^{-1}(Y\Theta_{21} + X\Theta_{11}). \end{cases} \tag{209}$$

Proof

1. We saw, as a consequence of Lemma 3.1, that without loss of generality D_{11} has the representation (197). The polynomial matrix Θ_{11} defined by (198) is clearly a basis matrix for $X_{D_{11}}$ and it is easily checked that the shift realization applied to $D_{11}^{-1}\Theta_{11}$ leads to (C, A) in dual Brunovsky form. Note that our assumption that the pair (C, A) is observable implies $D_{22} = I$. With $\mathbf{B}$ defined by (199), it is obvious that (200) holds.
   Next, we compute

   $$Z_C\mathbf{B} = C(zI - A)^{-1}\mathbf{B} = D_{11}^{-1}\Theta_{11}\mathbf{B} = D_{11}^{-1},$$

   i.e. we obtain (203). Using (204) with $W = 0$, we get $Z_K = -D_{21}Z_C + \Theta_{21}$, and we compute further

   $$Z_K\mathbf{B} = -D_{21}Z_C\mathbf{B} + \Theta_{21}\mathbf{B} = -D_{21}D_{11}^{-1} + \Theta_{21}\mathbf{B}.$$

   As $Z_K\mathbf{B}$ and $-D_{21}D_{11}^{-1}$ are both strictly proper whereas $\Theta_{21}\mathbf{B}$ is polynomial, we conclude that necessarily $Z_K\mathbf{B} = -D_{21}D_{11}^{-1}$ and $\Theta_{21}\mathbf{B} = 0$, i.e. (201) holds.

Now A is in Brunovsky form, so we have

$$
(zI - A)^{-1} = \left(\begin{array}{ccccc|cccc|cccc}
z^{-1} & . & . & . & . & . & . & . & . & . & . & . & . \\
z^{-2} & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
z^{-\mu_1} & . & . & z^{-2} & z^{-1} & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & z^{-1} & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & z^{-\mu_p} & . & . & z^{-1}
\end{array}\right) \tag{210}
$$

and hence

$$
(zI - A)^{-1}\mathbf{B} = \left(\begin{array}{c|ccc|c}
z^{-1} & . & . & . & \\
z^{-2} & . & . & . & \\
. & . & . & . & \\
. & . & . & . & \\
z^{-\mu_1} & . & . & & \\
\hline
. & . & . & . & \\
. & . & . & . & \\
. & . & . & . & \\
\hline
. & . & . & . & z^{-1} \\
. & . & . & . & \\
. & . & . & . & \\
. & . & . & . & \\
. & . & . & . & z^{-\mu_p}
\end{array}\right) \tag{211}
$$

With respect to the standard basis in $X_{D_{11}}$, given by the basis matrix in (198), we have

$$
K = \left(\begin{array}{ccccc|cccc|ccccc}
K_{11}^{(1)} & K_{11}^{(2)} & . & . & K_{11}^{(\mu_1)} & . & . & . & . & K_{1\lambda_1}^{(1)} & K_{1\lambda_2}^{(2)} & . & . & K_{1\lambda_{\mu_1}}^{(\mu_1)} \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
K_{k1}^{(1)} & K_{k1}^{(2)} & . & . & K_{k1}^{(\mu_1)} & . & . & . & . & K_{k\lambda_1}^{(1)} & K_{k\lambda_2}^{(2)} & . & . & K_{k\lambda_{\mu_1}}^{(\mu_1)}
\end{array}\right), \tag{212}
$$

which leads to

$$
Z_K\mathbf{B} = K(zI - A)^{-1}\mathbf{B} = \sum_{\nu=1}^{\mu_1} \frac{K^{(\nu)}}{z^{\nu}}, \tag{213}
$$

i.e. (202) holds.

Finally, since with respect to the basis given by the columns of Θ_{11}, we have

$$C = \left(\begin{array}{ccccc|ccc|cccccc} 1 & 0 & . & . & 0 & . & . & . & . & . & . & . & . \\ \hline . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ \hline . & . & . & . & . & . & . & . & 1 & 0 & . & . & 0 \end{array}\right), \tag{214}$$

so that, using (211), equality (203) follows.

2. Assume Z_1, Z_2 are given by the representation (204) with $W \in \Gamma_K$. Part 1 implies that they solve Eq. (145). Since we have $Z_2 = Z_K - Z_1 Z_C$, the strict properness of Z_1 implies that of Z_2. Thus, it suffices to check the strict properness of Z_1. As $W \in \Gamma_K$, we have $W = \sum_{\nu=1}^{\mu_1} \frac{K^{(\nu)}}{z^\nu} + \hat{W} D_{11}^{-1}$ for some, strictly proper, $\hat{W}$. Using (202), we conclude that $Z_1 = -D_{21} - W D_{11} = [-D_{21} D_{11}^{-1} - W] D_{11}$ is strictly proper.
   Conversely, assume Z_1, Z_2 is a strictly proper rational solution of equation (145). By part 3.6, we have $Z_1 = -D_{21} - W D_{11} = [-D_{21} D_{11}^{-1} - W] D_{11}$, for some rational function W. Since Z_1 is assumed to be strictly proper, we have that $W = -Z_1 D_{11}^{-1} - D_{21} D_{11}^{-1}$ is strictly proper too. The strict properness of $-D_{21} D_{11}^{-1} - W$ shows that W is a strictly proper partial realization of $\sum_{\nu=1}^{\mu_1} \frac{K^{(\nu)}}{z^\nu} = -D_{21} D_{11}^{-1}$, i.e. $W \in \Gamma_K$.
3. Assume $Z_1 = -D_{22}^{-1} D_{21} - W D_{11} = (-D_{22}^{-1} D_{21} D_{11}^{-1} - W) D_{11}$ is strictly proper. Note that $D_{22}^{-1} D_{21} D_{11}^{-1}$ is strictly proper as, by construction, $D_{21} D_{11}^{-1}$ is strictly proper and D_{22} being row proper is properly invertible. Thus W is a solution to the $-D_{22}^{-1} D_{21} D_{11}^{-1}$-induced $(\mu_1, \ldots, \mu_p)$ partial realization problem.
   Conversely, if W solves the $-D_{22}^{-1} D_{21} D_{11}^{-1}$-induced $(\mu_1, \ldots, \mu_p)$ partial realization problem, then Z_1, Z_2, defined by (203), are strictly proper solutions of (145). □

3.7. *Observer construction*

By Lemma 3.1, we may assume that the observable pair (C, A) is in dual Brunovsky form. Using the left coprime factorization $D_{11}^{-1}\Theta_1 = C(zI - A)^{-1}$, the shift realization (19) provides a functional representation for the pair (C, A) in the polynomial model state space $X_{D_{11}}$, with $D_{11}(z) = \mathrm{diag}(z^{\mu_1}, \ldots, z^{\mu_p})$. By Theorem 3.6 in [8], any conditioned invariant subspace $\mathscr{V} \subset X_{D_{11}}$ has a representation $\mathscr{V} = X_{D_{11}} \cap T\mathbb{F}[z]^p$ for some nonsingular polynomial matrix for which all left Wiener-Hopf factorization indices of $D_{11}^{-1}T$ are nonpositive. The polynomial matrix T is not necessarily uniquely defined (even after identifying polynomial matrices differing by a right unimodular factor). Thus, one expects that if $\mathscr{V} = X_{D_{11}} \cap T\mathbb{F}[z]^p \subset \mathrm{Ker}\, K$, the corresponding observer would have a nice functional representation. Moreover, by choosing an appropriate basis in X_T, simple state space formulas can be derived. The next proposition is based on [14]. The state space formulas for the tracking observer are taken from [27, Proposition 5.29], though the derivation is different.

Proposition 3.7. *Given the observable linear system*

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx, \end{cases} \tag{215}$$

with, $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$, $C \in \mathbb{F}^{p\times n}$, the pair (C, A) in dual Brunovsky form, and we assume the coprime factorization (89) with $D_{11}(z) = \operatorname{diag}(z^{\mu_1}, \dots, z^{\mu_p})$ and further that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is observable. Then

1. *Let $T \in \mathbb{F}[z]^{p\times p}$ be nonsingular. Then the system defined in the state space X_T by*

$$\Sigma_{est} : \sigma\xi = F\xi + Gy + Hu \tag{216}$$

with

$$\begin{cases} Fg = S_T g, & g \in X_T, \\ G\eta = -\pi_T D_{11}\eta, & \eta \in \mathbb{F}^p, \\ H\omega = \pi_T B\omega & \omega \in \mathbb{F}^m \end{cases} \tag{217}$$

is a strongly tracking observer for $K = \pi_T | X_{D_{11}}$. Clearly, $\mathscr{V} = \operatorname{Ker} K = X_{D_{11}} \cap T(z)\mathbb{F}[z]^p$ is a conditioned invariant subspace.

2. *Let $\{e_1, \dots, e_p\}$ be the standard basis in $\mathbb{F}^p$. Let $b_1, \dots, b_p$ be defined by*

$$b_i = \pi_T e_i, \quad i = 1, \dots, p. \tag{218}$$

Fixing a basis $\mathscr{B}$ for X_T and using the standard basis in $X_{D_{11}}$, given by the basis matrix (198), we have

$$\begin{aligned} F &= [S_T]^{\mathscr{B}}_{\mathscr{B}}, \\ G &= -[\pi_T D_{11}]^{\mathscr{B}} = -\begin{pmatrix} F^{\mu_1} b_1 & . & . & . & F^{\mu_p} b_p \end{pmatrix} \end{aligned} \tag{219}$$

then

$$K = [\pi_T | X_{D_{11}}]^{\mathscr{B}} = \mathscr{R}_\mu(F, G) = (b_1, Fb_1, \dots, F^{\mu_1 - 1}b_1, \dots, b_p, \dots, F^{\mu_p - 1}b_p), \tag{220}$$

where $\mathscr{R}_\mu(F, G)$ is the μ-partial reachability matrix defined in Section 3.6.

3. *If $\mathscr{V} = X_{D_{11}} \cap T(z)\mathbb{F}[z]^p \subset \operatorname{Ker} K$, then there exists polynomial matrices L_1, L_2 for which the polynomial system matrix $\begin{pmatrix} T & D_{11} \\ -L_1 & L_2 \end{pmatrix}$ leads, via the shift realization, to a minimal realization of a tracking observer. In particular, we have*

$$Z_1 = L_2 + L_1 T^{-1} D_{11}, \tag{221}$$

or equivalently,

$$Z_1 = J(zI - F)^{-1} G = \pi_- L_1 T^{-1} D_{11}. \tag{222}$$

4. *Let W be a strictly proper rational function for which $Z_1 = -D_{21} - WD_{11}$ is strictly proper. Let $W = L_1 T^{-1}$ be a right coprime factorization and let $L_2 = -D_{21}$. Then the shift realization associated with the polynomial system matrix*

$$\begin{pmatrix} T & D_{11} \\ L_1 & -D_{21} \end{pmatrix}, \tag{223}$$

i.e. the system given, in the state space X_T, by

$$\begin{aligned}F &= S_T,\\ G &= -\pi_T D_{11}\cdot,\\ J &= (L_1 T^{-1}\cdot)_{-1}\end{aligned} \tag{224}$$

is a tracking observer for (215).

Proof

1. By Theorem 3.3, a tracking observer is characterized by the solvability of the Sylvester equations (127). Without loss of generality, for Σ given by (215), we can assume that the pair (C, A) is given by means of the shift realization, i.e. $A = S_{D_{11}}$ and $Cf = (D_{11}^{-1} f)_{-1} = \xi_f$. So, in this case, $X = X_{D_{11}}$ and $\mathscr{V} = X_{D_{11}} \cap T\mathbb{F}[z]^p$, where $D^{-1}T$ has nonpositive factorization indices. We take X_T as the state space of the tracking observer and define a map $Z : X_{D_{11}} \longrightarrow X_T$ by $Z = \pi_T | X_{D_{11}}$. Clearly, $\operatorname{Ker} Z = \operatorname{Ker} \pi_T | X_{D_{11}} = X_{D_{11}} \cap T(z)\mathbb{F}[z]^p$. By Theorem 3.3 in [14], Z is surjective. Next we define $F = S_T$ and compute
$$\begin{aligned}(ZA - FZ)f &= (\pi_T S_{D_{11}} - S_T \pi_T)f = \pi_T \pi_{D_{11}} zf - \pi_T z \pi_T f\\ &= \pi_T(zf - D_{11}(z)\xi_f) - \pi_T zf = -\pi_T D_{11}(z)\xi_f.\end{aligned}$$
We define the map $G : \mathbb{F}^p \longrightarrow X_T$ by $G\xi = -\pi_T D_{11}\xi$. Thus we have $ZA - FZ = GC$. By construction, we have $K = Z$, so $J = 1$. Finally, we define $H = ZB$ and so we have obtained a strongly tracking observer equations.
2. We compute
$$-\pi_T D_{11} e_j = -\pi_T z^{\mu_j} e_j = -\pi_T z^{\mu_j} \pi_T e_j = -S_T^{\mu_j} b_j.$$
Taking the matrix representation with respect to the basis $\mathscr{B}$, we get (219). The proof of (220) follows along similar lines.
3. It follows from the representation of linear functionals on the polynomial model X_T, that any map $J : X_T \longrightarrow \mathbb{F}^k$ has a representation of the form
$$Jf = (L_1 T^{-1} f)_{-1}$$
for some $k \times p$ polynomial matrix L_1. Note that if (J, F) is an observable pair, then necessarily L_1, T are right coprime. We compute, for $f \in X_T$
$$\begin{aligned}JF^k f &= (L_1 T^{-1} \pi_T z^k f)_{-1} = (L_1 T^{-1} T \pi_- T^{-1} z^k f)_{-1}\\ &= (L_1 \pi_- T^{-1} z^k f)_{-1} = (L_1 T^{-1} z^k f)_{-1}\\ &= (L_1 T^{-1} f)_{-k-1}.\end{aligned}$$
This shows that $J(zI - F)^{-1} = \pi_- L_1 T^{-1}$ and, using the representation of G given in (219), we compute
$$\begin{aligned}J(zI - F)^{-1} G &= -\pi_- L_1 T^{-1} \pi_T D_{11} = -\pi_- L_1 T^{-1} T \pi_- T^{-1} D_{11}\\ &= -\pi_- L_1 \pi_- T^{-1} D_{11} = -\pi_- L_1 T^{-1} D_{11}\end{aligned}$$
and (222) follows. Defining $L_2 = -\pi_+ L_1 T^{-1} D_{11}$, (221) follows.

4. By Theorem 3.5, assuming the system Σ to be observable, all tracking observers have the representation $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \begin{pmatrix} -D_{21} & \Theta_{21} \end{pmatrix} - W \begin{pmatrix} D_{11} & \Theta_{11} \end{pmatrix}$. In particular, $Z_1 = -D_{21} - WD_{11} = (-D_{21}D_{11}^{-1} - W)D_{11}$, i.e. $\pi_+(-D_{21}D_{11}^{-1} - W)D_{11} = 0$. Thus, recalling Definition 3.4 and using (202), W is a partial realization of the nice sequence defined by (178). With the right coprime factorization $W = L_1T^{-1}$, we have $Z_1 = -D_{21} - L_1T^{-1}D_{11}$ to which correspond both the polynomial system matrix $\begin{pmatrix} T & D_{11} \\ L_1 & -D_{21} \end{pmatrix}$ as well as the associated shift realization (19). Hence (224) follows. $\square$

At the risk of being repetitious, we make the following remark. Given a tracking observer in the representation (221), it determines a conditioned invariant subspace $\mathscr{V} \subset X_{D_{11}}$ simply by defining $\mathscr{V} = X_{D_{11}} \cap T\mathbb{F}[z]^p$. One should note however that the same conditioned invariant subspace may arise out of different representations, so generally T is not uniquely determined by $\mathscr{V}$. For more on this including the analysis of spectral assignability for observers, see [16].

Given a tracking observer for the system (262), we have several different representations for it, given by

$$Z_1 = Q^{-1}P = \overline{P}\,\overline{Q}^{-1} = L_2 - L_1T^{-1}D_{11}$$

with all factorizations coprime. We have the following.

Lemma 3.2. *Let D_{11} be defined via the coprime factorization (89). Given two minimal polynomial system matrices*

$$\begin{pmatrix} T & D_{11} \\ L_1 & L_2 \end{pmatrix}, \begin{pmatrix} S & D_{11} \\ M_1 & M_2 \end{pmatrix}$$

with T, S nonsingular. If the associated transfer functions are equal, i.e.

$$Z_1 = L_2 - L_1T^{-1}D_{11} = M_2 - M_1S^{-1}D_{11}, \tag{225}$$

then

1. *T, S are unimodularly equivalent, i.e. without loss of generality we can assume $S = T$.*
2. *There exists a unique minimal representation $Z_1 = L_2 - L_1T^{-1}D_{11}$ for which $L_2D_{11}^{-1}$ is strictly proper.*
3. *If Z_1 is strictly proper, D_{11} row proper, then $L_2D_{11}^{-1}$ is strictly proper if and only if L_1T^{-1} is.*

Proof

1. By our assumption of minimality, the shift realizations associated with the polynomial system matrices are Fuhrmann system equivalent (FSE), see [7,18]. In particular, there exist polynomial matrices U, V with U, S left coprime and V, T right coprime for which $UT = SV$. Since T and S have the same size $p \times p$, it follows that they are unimodularly equivalent. So, without loss of generality, we can assume $S = T$. By strict system equivalence, we have
$$\begin{pmatrix} U & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} T & -D_{11} \\ L_1 & L_2 \end{pmatrix} = \begin{pmatrix} T & -D_{11} \\ M_1 & M_2 \end{pmatrix} \begin{pmatrix} V & Y \\ 0 & I \end{pmatrix}.$$
2. Equality (225) implies $\pi_-(L_1 - M_1)T^{-1} = 0$ or $L_1 - M_1 = XT$. Substituting back in (225), we obtain
$$L_2 - (M_1 + XT)T^{-1}D_{11} = (L_2 - XD_{11}) - M_1T^{-1}D_{11} = M_2 - M_1T^{-1}D_{11}$$

and so $L_2 = M_2 + XD_{11}$. Therefore it is clear that there is a unique pair L_1, L_2 for $L_2D_{11}^{-1}$ is strictly proper.

3. If D_{11} is row proper, then D_{11}^{-1} is proper. Thus, from the representation $Z_1 = L_2 - L_1T^{-1}D_{11}$, we have $Z_1D_{11}^{-1} = L_2D_{11}^{-1} - L_1T^{-1}D_{11}$. This shows that $L_2D_{11}^{-1}$ is strictly proper if and only if L_1T^{-1} is. $\square$

4. Observers in the behavioral framework

We move now the focus of our attention to the behavioral setting. The analysis of observers in the behavioral framework was initiated in [28]. As in the state space framework, we define observers, discuss the various degrees of observability and analyze how these are reflected in a suitable class of observers.

Following Valcher and Willems [28], we begin by defining the concepts of trackability, detectability, reconstructibility and observability in the behavioral context. Since behaviors have many different representations, we will concentrate on two representations.

The first is the AR or kernel representation. This representation is given by the dynamic equations

$$R_2(\sigma)z = R_1(\sigma)y. \tag{226}$$

Here y, z are the **manifest variables**, split into two subgroups. y are the measured variables, whereas z are the **to be estimated variables** that is the internal variables for which no direct measurement is available. The problem is to design a system that, based on the measurements, produces an estimate $\hat{z}$ for the variables z.

The second representation is the latent variable, or ARMA, representation given by the dynamic equations

$$R_2(\sigma)z - R_1(\sigma)y = L(\sigma)\xi, \tag{227}$$

where y, z are the manifest variables of which only the variables y are measured, and ξ the **latent variables**. As before, our aim is to estimate z, based on the measurements of y.

It has been argued, see [32], that the behavioral framework is useful in analysing interconnections of subsystems. It seems that, as far as observation and observers are concerned, the behavioral framework is particularly suited for analysing the relation between subsets of variables which are part of a larger system. The tool for this is the all important elimination theorem, see Theorem 2.5 and the references preceeding it.

As in the case of state space systems, there are many possible observers for a given dynamical system. The quality of the estimation given by a particular observer depends on the quality of the information on the system obtained by the measurement, or observation, process. Therefore we begin our discussion by listing, following Willems [31], the most useful observation properties.

Definition 4.1

1. Given a system $\Sigma = (\mathbb{Z}_+, \mathbb{F}^{\omega_1+\omega_2}, \mathscr{B}_{sys})$ with the behavior defined by

$$\mathscr{B}_{sys} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in z^{-1}\mathbb{F}^{\omega_1+\omega_2}[[z^{-1}]] | R_2(\sigma)z = R_1(\sigma)y \right\}. \tag{228}$$

We say that

(a) z is **trackable** from y if there exists an integer N such that $(y, z), (y, \bar{z}) \in \mathscr{B}_{sys}$ and $z_k = \bar{z}_k$ for $1 \leqslant k \leqslant N$ implies $z = \bar{z}$.

(b) z is **detectable** from y if $(y,z),(y,\bar z)\in\mathscr{B}_{sys}$ implies $\lim_{k\to\infty}(z_k-\bar z_k)=0$.
(c) z is **reconstructible** from y if $(y,z),(y,\bar z)\in\mathscr{B}_{sys}$ implies there exists an integer N such that $(z_k-\bar z_k)=0$ for $k>N$.
(d) z is **observable** from y if $(y,z),(y,\bar z)\in\mathscr{B}_{sys}$ implies $z=\bar z$.

2. Given a behavior $\mathscr{B}_{fsys}$ in the DV representation (227). We say that
   (a) z is **trackable** from y if there exists an integer N such that $(y,z,\xi),(y,\bar z,\bar\xi)\in\mathscr{B}_{fsys}$ and $z_k=\bar z_k$ for $1\leqslant k\leqslant N$ implies $z=\bar z$.
   (b) z is **detectable** from y if $(y,z,\xi),(y,\bar z,\bar\xi)\in\mathscr{B}_{fsys}$ implies $\lim_{k\to\infty}(z_k-\bar z_k)=0$.
   (c) z is **reconstructible** from y if $(y,z,\xi),(y,\bar z,\bar\xi)\in\mathscr{B}_{fsys}$ implies $(z_k-\bar z_k)=0$ for $k>N$.
   (d) z is **observable** from y if $(y,z,\xi),(y,\bar z,\bar\xi)\in\mathscr{B}_{fsys}$ implies $z=\bar z$.

   We will say that a system Σ given by (228) is **trackable** if z is **trackable** from y. Similarly, we define **detectability**, **reconstructibility** and **observability**.

Now it seems that we have, for a behavior given by the latent variable representation (227), two definitions of observability. The first is given by Definition 4.1.2. On the other hand, we can start with (227), use elimination theory to eliminate the variable ξ and consider observability according to Definition 4.1.1. The next proposition shows that these two definitions of observability coincide.

Proposition 4.1. *Given a behavior in the DV representation*

$$\mathscr{B}_{fsys}=\left\{\begin{pmatrix}y\\z\\\xi\end{pmatrix}\,\middle|\,R_2(\sigma)z-R_1(\sigma)y=L(\sigma)\xi\right\},\tag{229}$$

let the behavior $\mathscr{B}_{sys}$ be the one obtained from $\mathscr{B}_{fsys}$ by elimination of the latent variables ξ, i.e.

$$\mathscr{B}_{sys}=\left\{\begin{pmatrix}y\\z\end{pmatrix}\,\middle|\,\exists\xi,\begin{pmatrix}y\\z\\\xi\end{pmatrix}\in\mathscr{B}_{fsys}\right\}.\tag{230}$$

Then

1. *z is trackable from y in the DV representation* (229) *if and only if z is trackable from y in the AR representation* (230).
2. *z is detectable from y in the DV representation* (229) *if and only if z is detectable from y in the AR representation* (230).
3. *z is reconstructible from y in the DV representation* (229) *if and only if z is reconstructible from y in the AR representation* (230).
4. *z is observable from y in the DV representation* (229) *if and only if z is observable from y in the AR representation* (230).

Proof

1. Assume $\mathscr{B}_{fsys}$ is trackable. Let now $\begin{pmatrix}y\\z\end{pmatrix},\begin{pmatrix}y\\\bar z\end{pmatrix}\in\mathscr{B}_{sys}$, and assume $z(k)=\bar z(k)$ for $k\leqslant N$. Thus there exist $\xi,\bar\xi$ such that $\begin{pmatrix}y\\z\\\xi\end{pmatrix},\begin{pmatrix}y\\\bar z\\\bar\xi\end{pmatrix}\in\mathscr{B}_{fsys}$. We conclude $z=\bar z$, i.e. $\mathscr{B}_{sys}$ is trackable.

Conversely, assume $\mathscr{B}_{sys}$ is trackable. Let now $\begin{pmatrix} y \\ z \\ \xi \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \\ \bar{\xi} \end{pmatrix} \in \mathscr{B}_{fsys}$ and assume $z_k = \bar{z}_k$ for $k \leqslant N$. Clearly, this implies $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys}$, and $z_k = \bar{z}_k$ for $k \leqslant N$, which implies $z = \bar{z}$, i.e. z is trackable from y.

2. Assume $\mathscr{B}_{fsys}$ is detectable. Let now $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys}$. Thus there exist $\xi, \bar{\xi}$ such that $\begin{pmatrix} y \\ z \\ \xi \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \\ \bar{\xi} \end{pmatrix} \in \mathscr{B}_{fsys}$. This implies $\lim_{k\to\infty}(z_k - \bar{z}_k) = 0$, i.e. $\mathscr{B}_{sys}$ is detectable.
   Conversely, assume $\mathscr{B}_{sys}$ is detectable. If $\begin{pmatrix} y \\ z \\ \xi \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \\ \bar{\xi} \end{pmatrix} \in \mathscr{B}_{fsys}$, then $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys}$. and, in turn, this implies $\lim_{k\to\infty}(z_k - \bar{z}_k) = 0$, $\mathscr{B}_{fsys}$ is detectable.
3. As in the previous part, interpreting convergence to zero in the discrete topology.
4. Assume $\mathscr{B}_{fsys}$ is observable. Let $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys}$. Thus there exist $\xi, \bar{\xi}$ such that $\begin{pmatrix} y \\ z \\ \xi \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \\ \bar{\xi} \end{pmatrix} \in \mathscr{B}_{fsys}$. By our assumption, this implies $z = \bar{z}$, i.e. $\mathscr{B}_{sys}$ is observable.
   Conversely, assume $\mathscr{B}_{sys}$ is observable. Let us assume $\begin{pmatrix} y \\ z \\ \xi \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \\ \bar{\xi} \end{pmatrix} \in \mathscr{B}_{fsys}$, then $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys}$. and, in turn, this implies $z = \bar{z}$, i.e. $\mathscr{B}_{fsys}$ is observable. □

Our next step is to give a polynomial characterization of trackability, detectability and observability. Valcher and Willems [28] also make an (implicit) distinction between trackability and observability but they do not have a separate name for trackability. Our terminology is however slightly different inasmuch as we make this distinction.

Proposition 4.2. *Consider the dynamical system $\Sigma = (\mathbf{Z}_+, \mathbb{F}^{\omega_1+\omega_2}, \mathscr{B})$ with behavior given by* (228). *Then*

1. *z is trackable from y if and only if the polynomial matrix $R_2(z)$ has full column rank.*
2. *z is detectable from y if and only if the polynomial matrix $R_2(z)$ is right stable.*
3. *z is reconstructible from y if and only if the polynomial matrix $R_2(z)$ is right monomic.*
4. *z is observable from y if and only if the polynomial matrix $R_2(z)$ is right prime.*

Proof

1. Assume $R_2(z)$ has full column rank. This implies the existence of a factorization $R_2 = R_2'E$, where R_2' is right prime, i.e. has a polynomial left inverse, and E is nonsingular. In such a factorization E is uniquely determined up to a left unimodular factor. Assume now $(y, z), (y, \bar{z}) \in \mathscr{B}$. This implies $R_2(\sigma)z = R_2(\sigma)\bar{z}$ which leads to $E(\sigma)(z - \bar{z}) = 0$. We apply Lemma 2.1 with $N = \deg E$.
   If $R_2(z)$ does not have full column rank then there exists a unimodular polynomial matrix $U(z)$ for which $R_2(z) = \begin{pmatrix} R_2'(z) & 0 \end{pmatrix} U(z)$. Clearly, for any integer N, there exists an $h \neq 0$ in $\operatorname{Ker} \begin{pmatrix} R_2'(\sigma) & 0 \end{pmatrix}$ satisfying $h_k = 0$ for $k \leqslant N + \deg U(z)^{-1}$. Since $U(\sigma)$ is invertible, we have $0 \neq U(\sigma)^{-1}h \in \operatorname{Ker} R_2(\sigma)$. Thus, if $(y, z) \in \mathscr{B}$ we have also $(y, z + h) \in \mathscr{B}$. This rules out trackability.
2. Assume $R_2(z)$ is right stable, i.e. has a stable rational left inverse. Thus there exists a factorization $R_2(z) = R_2'(z)E(z)$, where $R_2'(z)$ is right prime, i.e. has a polynomial left inverse, and E is a nonsingular, stable polynomial matrix. As before, in such a factorization E is uniquely

determined up to a left unimodular factor. Assume now $(y, z), (y, \bar{z}) \in \mathscr{B}$. This implies $R_2(\sigma)z = R_2(\sigma)\bar{z}$ and hence also $E(\sigma)(z - \bar{z}) = 0$, i.e. $z - \bar{z} \in X^E$. Thus $\lim_{k\to\infty}(z_k - \bar{z}_k) = 0$.
If $R_2(z)$ is not right stable, we can assume without loss of generality that $R_2(z)$ has full column rank but in the factorization $R_2(z) = R_2'(z)E(z)$ the polynomial matrix $E(z)$ is not stable. Thus there exists an $h \in \operatorname{Ker} E(\sigma)$ for which h_k does not converge to zero as $k \to \infty$. Thus if $(y, z) \in \mathscr{B}$, then also $(y, z + h) \in \mathscr{B}$, contradicting detectability.
3. The proof follows the lines of statement 2 with a different definition of stability.
4. Assume $R_2(z)$ is right prime and let $R_2^{\#}(z)$ be any polynomial left inverse. This implies $R_2^{\#}(\sigma)R_2(\sigma) = I$. $(y, z), (y, \bar{z}) \in \mathscr{B}$ implies $R_2(\sigma)(z - \bar{z}) = 0$, which, by the left invertibility of $R_2(\sigma)$, implies $z = \bar{z}$.
Finally, if $R_2(z)$ is not right prime, there exists a nonzero element $h \in \operatorname{Ker} R_2(\sigma)$. Thus if $(y, z) \in \mathscr{B}$, then also $(y, z + h) \in \mathscr{B}$, contradicting observability. □

Corollary 4.1. *Given a dynamical system Σ with behavior (228). Then*

1. *Observability implies both detectability and reconstructibility.*
2. *Detectability and reconstructibility both imply trackability.*

If the system Σ is trackable, then it can be transformed to a more convenient form. The result is taken from [28].

Lemma 4.1. *Given a trackable system Σ with behavior (228). Then*

1. *It can be rewritten equivalently as*
$$\mathscr{B}_{sys} = \left\{ \begin{pmatrix} v \\ z \end{pmatrix} \in z^{-1}\mathbb{F}^{\omega_1+\omega_2}[[z^{-1}]] \;\middle|\; \begin{cases} D_1(\sigma)v = 0 \\ M_1(\sigma)v + D_2(\sigma)z = 0 \end{cases} \right\}. \tag{231}$$
with D_2 nonsingular and D_1 of full row rank and both row proper.
2. *There exists a splitting of the variables v as, up to ordering, $v = \begin{pmatrix} y \\ u \end{pmatrix}$ so that the system equations can be written as*
$$\begin{cases} D_{11}(\sigma)y = N_1(\sigma)u, \\ D_{21}(\sigma)y + D_{22}(\sigma)z = N_2(\sigma)u \end{cases} \tag{232}$$
with D_{11}, D_{22} nonsingular and row proper, $D_{11}^{-1}N_1$ proper and $D_{21}D_{11}^{-1}$ strictly proper.

Proof

1. To the system equation (226), we apply a unimodular polynomial matrix U so that $U(z)R_2(z) = \begin{pmatrix} 0 \\ D_2(z) \end{pmatrix}$ with $D_2(z)$ of full row rank. By Proposition 4.2, our assumption of trackability implies that $R_2(z)$ has full column rank. It follows that $D_2(z)$ is necessarily nonsingular. Finally, we let $U(z)R_1(z) = \begin{pmatrix} D_1(z) \\ N_1(z) \end{pmatrix}$ and we are done.
2. By Part 1, D_1 is of full row rank and row proper. It is standard that, by reordering the variables v, we can assume, without loss of generality, that $v = \begin{pmatrix} y \\ u \end{pmatrix}$ and the polynomial matrices D_1, M_1 can be written correspondingly as $D_1 = \begin{pmatrix} D_{11} & -N_1 \end{pmatrix}$ and $M_1 = \begin{pmatrix} D_{21} & -N_2 \end{pmatrix}$ with D_{11} nonsingular and $D_{11}^{-1}N_1$ proper. The system equations (231) can be rewritten in the form (232).

By applying an appropriate unimodular transformation on the left, we can assume without loss of generality that $D_{21}D_{11}^{-1}$ is strictly proper. □

Remark. It follows from Proposition 4.2 that the system (231) is detectable if and only if $D_2(z)$ is stable and observable if and only if $D_2(z)$ is unimodular.

To illustrate the previous results, we consider a few examples.

Example 1 (*State estimation*). We consider the state space system Σ

$$\Sigma := \begin{cases} \sigma x = Ax, \\ y = Cx, \end{cases} \tag{233}$$

where our aim is to estimate the state x from the observations y. We do not assume the pair (C, A) to be observable. We rewrite the system as

$$\begin{pmatrix} 0 \\ I \end{pmatrix} y = \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} x. \tag{234}$$

Let $D_{11}^{-1}\Theta_{11}$ be a left coprime factorization of $C(zI - A)^{-1}$. Thus $\Theta_{11}(z)(zI - A) - D_{11}(z)C = 0$. Let $\left(\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right)$ be a MRA of $\begin{pmatrix} -\Theta_{11} & D_{11} \end{pmatrix}$. In particular, it is a right prime polynomial matrix. Thus, we have a factorization

$$\begin{pmatrix} zI - A \\ C \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} D_{22}$$

for some, necessarily nonsingular, polynomial matrix D_{22}. Clearly, D_{22} is a g.c.r.d. of $(zI - A)$, C. Let $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$ be an arbitrary polynomial solution of the Bezout equation $L_1N_1 + L_2N_2 = I$. Clearly, we have

$$\begin{pmatrix} -\Theta_{11} & D_{11} \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} 0 & zI - A \\ I & C \end{pmatrix} = \begin{pmatrix} D_{11} & 0 \\ L_2 & D_{22} \end{pmatrix}.$$

We note that necessarily $\left(\begin{smallmatrix} -\Theta_{11} & D_{11} \\ L_1 & L_2 \end{smallmatrix}\right)$ is a unimodular polynomial matrix. Eqs. (234) are equivalent therefore to

$$\begin{pmatrix} -\Theta_{11}(\sigma) & D_{11}(\sigma) \\ L_1(\sigma) & L_2(\sigma) \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} y = \begin{pmatrix} D_{11}(\sigma) & 0 \\ L_2(\sigma) & D_{22}(\sigma) \end{pmatrix} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} x,$$

and hence to

$$\begin{pmatrix} D_{11}(\sigma) \\ L_2(\sigma) \end{pmatrix} y = \begin{pmatrix} 0 \\ D_{22}(\sigma) \end{pmatrix} x. \tag{235}$$

Thus, applying Proposition 4.2, x is trackable, reconstructible, detectable or observable from y if and only if D_{22} is nonsingular, monomic, stable or unimodular respectively.

Example 2. Partial state estimation: We consider the state space system

$$\begin{cases} \sigma x = Ax, \\ y = Cx, \\ z = Kx, \end{cases} \tag{236}$$

where we do not assume the pair (C, A) to be observable but assume, recalling Proposition 3.2, that the pair $\left(\left(\begin{smallmatrix} C \\ K \end{smallmatrix}\right), A\right)$ is. Our aim is to estimate z from the observed variables y. The state variables are considered as latent variables.

We rewrite the system equations as

$$\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \sigma I - A \\ C \\ K \end{pmatrix} x. \tag{237}$$

We use the left coprime factorization (89), i.e.

$$\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} C \\ K \end{pmatrix} (zI - A)^{-1}$$

to eliminate the state variables from (237). Thus we get

$$\begin{pmatrix} 0 \\ D_{22}(\sigma) \end{pmatrix} z = -\begin{pmatrix} D_{11}(\sigma) \\ D_{21}(\sigma) \end{pmatrix} y.$$

Thus z is trackable, reconstructible, detectable or observable from y if and only if D_{22} is nonsingular, monomic, stable or unimodular respectively. This is in perfect agreement with Theorem 3.1.

We go on to analyse the mechanism of estimation, i.e. the construction of observers. In analogy with the case of state space systems, we define observers for a dynamical system as a corresponding dynamical system that uses the measured data to give an estimate of the to be estimated variables of the original system. Thus an **estimator** is a dynamical system Σ_{est} given by

$$Q(\sigma)\hat{z} = P(\sigma)\hat{y}, \tag{238}$$

with manifest behavior $\mathscr{B}_{est}$. As a minimum requirement, we want Σ_{est} to be able to process the observed data. A further requirement is that the true trajectory of the system manifest variables would not be rejected by the estimator. To make these requirements precise, we define the projections π_y and $\hat{\pi}_{\hat{y}}$ acting on $\mathscr{B}$ and $\mathscr{B}_{est}$ respectively, by

$$\begin{aligned} \pi_y \begin{pmatrix} y \\ z \end{pmatrix} &= y, \\ \hat{\pi}_{\hat{y}} \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} &= \hat{y}. \end{aligned} \tag{239}$$

The next definition is based on Definition 3.2 which in turn was inspired by [28].

Definition 4.2. Given a discrete time system $\Sigma = (\mathbf{Z}_+, \mathbb{F}^{\omega_1+\omega_2}, \mathscr{B})$ with behavior

$$\mathscr{B}_{sys} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} | R_2(\sigma)z = R_1(\sigma)y \right\}. \tag{240}$$

Here y are the measured variables and z the relevant, or to be estimated, variables.

An **acceptor** is a dynamical system (238)

$$Q(\sigma)\hat{z} = P(\sigma)\hat{y} \tag{241}$$

with behavior $\mathscr{B}_{est}$

$$\mathscr{B}_{est} = \left\{ \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} | Q(\sigma)\hat{z} = P(\sigma)\hat{y} \right\} \tag{242}$$

which satisfies

$$\pi_y \mathscr{B}_{sys} \subset \hat{\pi}_{\hat{y}} \mathscr{B}_{est}. \tag{243}$$

An **estimator** is an acceptor (238) with z and $\hat{z}$ taking values in the same space $\mathbb{F}^{\omega_r}$. A **consistent estimator** is an estimator with behavior $\mathscr{B}_{est}$ which satisfies

$$\mathscr{B}_{sys} \subset \mathscr{B}_{est}. \tag{244}$$

The **error trajectory** of a consistent estimator is defined by $e = z - \hat{z}$ and the **estimation error behavior** $\mathscr{B}_{err}$ by

$$\mathscr{B}_{err} = \left\{ e = z - \hat{z} \,\middle|\, \begin{pmatrix} y \\ z \end{pmatrix} \in \mathscr{B}_{sys} \text{ and } \begin{pmatrix} y \\ \hat{z} \end{pmatrix} \in \mathscr{B}_{est} \right\}. \tag{245}$$

A consistent estimator (238) is said to be

1. a **consistent tracking observer** of z from y if there exists an integer T such that given $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathscr{B}_{sys}$ and $\begin{pmatrix} y \\ \hat{z} \end{pmatrix} \in \mathscr{B}_{est}$ then $z_k = \hat{z}_k$ for $k = 1, \ldots, T$ implies $z = \hat{z}$.
2. a **consistent asymptotic observer** of z from y if $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathscr{B}_{sys}$ and $\begin{pmatrix} y \\ \hat{z} \end{pmatrix} \in \mathscr{B}_{est}$ implies $\lim_{k\to\infty} z_k - \bar{z}_k = 0$.
3. a **consistent exact observer** of z from y if $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathscr{B}_{sys}$ and $\begin{pmatrix} y \\ \hat{z} \end{pmatrix} \in \mathscr{B}_{est}$ implies $z = \hat{z}$.

In the sequel we will assume always that observers are consistent and refer to consistent observers as simply observers. The question of consistency was discussed first in [3].

The following proposition relates the properties of a behavior to the properties of the estimator.

Proposition 4.3. *Given a system with behavior* (231). *Then*

1. *A tracking observer exists if and only if $\mathscr{B}_{sys}$ is trackable.*
2. *An asymptotic observer exists if and only if $\mathscr{B}_{sys}$ is detectable.*
3. *An exact observer exists if and only if $\mathscr{B}_{sys}$ is observable.*

Proof

1. Assume a (consistent) tracking observer exists for $\mathscr{B}_{sys}$, given by (238). Consistency implies the inclusion (272). Let $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{sys}$, hence we also have $\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ \bar{z} \end{pmatrix} \in \mathscr{B}_{est}$. Now $\mathscr{B}_{est}$ is the behavior of a tracking observer, hence there exists an integer T such that $z_k = \bar{z}_k$ for $k = 1, \ldots, T$ implies $z = \bar{z}$ This shows that $\mathscr{B}_{sys}$ is trackable.
   Conversely, assume $\mathscr{B}_{sys}$ is trackable. Applying Lemma 4.1, we rewrite the dynamic equation (226) as
   $$\begin{pmatrix} D_1(\sigma) & 0 \\ N_1(\sigma) & D_2(\sigma) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{246}$$
   with D_2 nonsingular. We construct now consistent tracking observers. Let
   $$\begin{pmatrix} -P & Q \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ N_1 & D_2 \end{pmatrix} \tag{247}$$
   with Y taken nonsingular, i.e. we have
   $$\begin{aligned} -P &= XD_1 + YN_1, \\ Q &= YD_2. \end{aligned} \tag{248}$$

Let the estimator equations be given by

$$Q(\sigma)\bar{z} = P(\sigma)y. \tag{249}$$

The factorization (247) shows that

$$\operatorname{Ker}\begin{pmatrix} D_1(\sigma) & 0 \\ N_1(\sigma) & D_2(\sigma) \end{pmatrix} \subset \operatorname{Ker}\begin{pmatrix} -P(\sigma) & Q(\sigma) \end{pmatrix},$$

i.e. we have consistency. Let now $\binom{y}{z} \in \mathscr{B}_{sys}$ and $\binom{y}{\bar{z}} \in \mathscr{B}_{est}$. From Eq. (246) we have $(X(\sigma)D_1(\sigma) + Y(\sigma)N_1(\sigma))y + Y(\sigma)D_2(\sigma)z = 0$, while (249) can be rewritten as

$$Y(\sigma)D_2(\sigma)\bar{z} = X(\sigma)D_1(\sigma) + Y(\sigma)N_1(\sigma)y.$$

Subtracting, we get $Q(\sigma)(z - \bar{z}) = 0$, i.e. $(z - \bar{z}) \in \operatorname{Ker} Q(\sigma)$. We apply Lemma 2.1 to conclude that Σ_{est} is a tracking observer.
Note that a special choice is $X = 0$, $Y = I$ which shows that $D_2(\sigma)\bar{z} = N_1(\sigma)y$ is a tracking observer.

2. Assume there exists a consistent asymptotic observer for $\mathscr{B}_{sys}$. Let $\binom{y}{z}, \binom{y}{\bar{z}} \in \mathscr{B}_{sys}$ which, by the inclusion (272), shows that $\binom{y}{z}, \binom{y}{\bar{z}} \in \mathscr{B}_{est}$. Since the observer is asymptotic, we have $z_k - \bar{z}_k \to 0$, i.e. Σ_{sys} is detectable.
Conversely, assume Σ_{sys} is detectable, i.e. D_2 is stable. As before, (247) defines a consistent tracking observer which is going to be asymptotic if and only if Y is chosen to be a nonsingular stable polynomial matrix.
As in the proof of the existence of a tracking observer, $D_2(\sigma)\bar{z} = -N_1(\sigma)y$ is an asymptotic tracking observer.
3. Assume $Q(\sigma)\bar{z} = P(\sigma)y$ is a consistent exact observer. The inclusion (272) implies the factorization (247). In particular, we have $Q(z) = Y(z)D_2(z)$. The error behavior is given by $\mathscr{B}_{err} = \operatorname{Ker} Q(\sigma)$ and, by assumption, $\mathscr{B}_{err} = \{0\}$. Thus $Q(z)$ is unimodular which forces the unimodularity of D_2, i.e. Σ_{sys} is observable.
Conversely, assume Σ_{sys} is observable. The system equations can be written as (246) with D_2 unimodular. Without loss of generality, we may assume $D_2 = I$. Thus $\bar{z} = -N_1(\sigma)y$ is a consistent exact observer. □

The next theorem gives characterizations of the various estimators.

Theorem 4.1. *Given a trackable system Σ_{sys} described by the equations*

$$\Sigma_{sys} := \begin{cases} D_1(\sigma)y = 0, \\ N_1(\sigma)y + D_2(\sigma)z = 0, \end{cases} \tag{250}$$

with D_2 nonsingular, D_1 of full row rank and manifest behavior

$$\mathscr{B}_{sys} = \operatorname{Ker}\begin{pmatrix} D_1(\sigma) & 0 \\ N_1(\sigma) & D_2(\sigma) \end{pmatrix}. \tag{251}$$

Let an estimator Σ_{est} be given by

$$Q(\sigma)\hat{z} = P(\sigma)\hat{y} \tag{252}$$

with manifest behavior $\mathscr{B}_{est}$.

1. *Σ_{est} is a tracking observer for Σ_{sys} if and only if*
$$\mathscr{B}_{sys} \subset \mathscr{B}_{est}, \tag{253}$$
i.e. if and only if the polynomial matrix Q is nonsingular and there exist polynomial matrices X, Y for which the following factorization holds
$$\begin{pmatrix} -P & Q \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ N_1 & D_2 \end{pmatrix}. \tag{254}$$
The **error behavior** *$\mathscr{B}_{err}$ is given by*
$$\mathscr{B}_{err} = \operatorname{Ker} Q(\sigma), \tag{255}$$
i.e. the **error dynamics** *are given by the nonsingular polynomial matrix $Q(z)$.*
2. *Let Σ_{sys} be detectable. Then Σ_{est} is an asymptotic observer for Σ_{sys} if and only if there exist polynomial matrices X, Y, with Y nonsingular and stable, for which the factorization* (254) *holds.*
3. *Let Σ_{sys} be observable. Then Σ_{est} is an exact observer for Σ_{sys} if and only if there exist polynomial matrices X, Y, with Y unimodular, for which the factorization* (254) *holds.*

Proof

1. Assume first that Σ_{est} is a tracking observer. Clearly, $\binom{0}{0} \in \mathscr{B}_{sys}$. Choose $\hat{z} \in \operatorname{Ker} Q(\sigma)$ arbitrarily. Thus $\binom{0}{\hat{z}} \in \mathscr{B}_{est}$ and we have
$$\operatorname{Ker} Q(\sigma) \subset \mathscr{B}_{err}. \tag{256}$$
By the assumption of trackability, $\hat{z}(k) = 0$ for $k = 1, \ldots, N$ implies $\hat{z} = 0$ which shows that $Q(z)$ is necessarily nonsingular. Now we always have $0 \in \mathscr{B}_{err}$. This shows that $\binom{y}{z} \in \mathscr{B}_{sys}$ implies $\binom{y}{z} \in \mathscr{B}_{est}$. So $\mathscr{B}_{sys}$ is a subbehavior of $\mathscr{B}_{est}$ and therefore we obtain the inclusion (253).
Conversely, assume the inclusion (253). So $\mathscr{B}_{sys}$ is a subbehavior of $\mathscr{B}_{est}$ and therefore there exist a factorization (254). If $\binom{y}{z} \in \mathscr{B}_{sys}$ then also $Q(\sigma)z = P(\sigma)y$. But for any estimate $\binom{y}{\hat{z}} \in \mathscr{B}_{est}$ we also have $Q(\sigma)\hat{z} = P(\sigma)y$. Thus for the error trajectory $e = z - \hat{z}$ we have $Q(\sigma)e = 0$, i.e.
$$\mathscr{B}_{err} \subset \operatorname{Ker} Q(\sigma). \tag{257}$$
From inclusions (256) and (257) we conclude the equality (255). By Lemma 2.1, Σ_{est} is a tracking observer if and only if $Q(z)$ is a nonsingular polynomial matrix.
Since subbehaviors are determined by factorizations, we infer that, for some polynomial matrices $X, Y, \overline{X}, \overline{Y}$, there exists a factorization of the form
$$\begin{pmatrix} D_1 & 0 \\ -P & Q \end{pmatrix} = \begin{pmatrix} \overline{Y} & \overline{X} \\ X & Y \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ N_1 & D_2 \end{pmatrix}.$$
This implies that $\overline{Y} = I$, $\overline{X} = 0$ and (254) holds.
2. Assume that Σ_{est} is an asymptotic observer. By Proposition 4.2, this implies $\mathscr{B}_{sys}$ is detectable. It follows from (253) that a factorization of the form (254) exists. The error behavior is given by $\mathscr{B}_{err} = \operatorname{Ker} Q(\sigma)$. Hence, necessarily $Q(z)$ is right stable. The detectability of $\mathscr{B}_{sys}$ implies that $Q(z)$ is right stable and in particular of full column rank. Thus $Q(z)$ is nonsingular and

stable. Since $Q(z) = Y(z)D_2(z)$ and $D_2(z)$ is stable by the detectability of $\mathscr{B}_{sys}$, it follows that $Y(z)$ is also stable.
Conversely, assume Σ_{sys} is detectable and the factorization (254) holds with Y stable. Detectability implies $D_2(z)$ is stable and hence that $Q(z)$ is stable. Since $\mathscr{B}_{err} = \operatorname{Ker} Q(\sigma)$, we conclude that Σ_{est} is an asymptotic observer.

3. Assume that Σ_{est} is an exact observer, given by (238). By Proposition 4.2, this implies $\mathscr{B}_{sys}$ is observable. It follows from (253) that a factorization of the form (254) exists. The error behavior is given by $\mathscr{B}_{err} = \operatorname{Ker} Q(\sigma)$. Hence, necessarily $Q(z)$ is right prime and in particular of full column rank. Thus $Q(z)$ is unimodular. Since $Q(z) = Y(z)D_2(z)$ and $D_2(z)$ is unimodular by the observability of $\mathscr{B}_{sys}$, it follows that $Y(z)$ is also unimodular.
Conversely, assume Σ_{sys} is observable and the factorization (254) holds with Y unimodular. Observability implies $D_2(z)$ is unimodular and hence $\mathscr{B}_{err} = \operatorname{Ker} Q(\sigma) = \{0\}$. This shows that Σ_{est} is an exact observer. $\square$

Remarks. It should be noted that, though we have defined consistent tracking observers, i.e. assuming the inclusion (244) that defines consistency, this is actually not necessary. In fact, the proof of Theorem 4.1.1 shows that a tracking observer is automatically consistent.

Theorem 4.1 does not address the issue of causality of the estimator, i.e. the question of whether $Q^{-1}P$ is proper or strictly proper. This is addressed next.

Proposition 4.4. *Given a system Σ with dynamic equations (232) with D_{11}, D_{22} nonsingular and row proper and $D_{21}D_{11}^{-1}$ strictly proper. Then, a sufficient condition for the existence of a strictly proper, consistent tracking observer is that $\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is strictly proper.*

Proof. Assume that $\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is strictly proper. Let $\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$ be a basis matrix for the polynomial model $X_{\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}}$. Thus there exists a constant matrix B for which $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} B$. This implies the existence of an observable pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ for which $\begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} C \\ K \end{pmatrix}(zI - A)^{-1}$. We claim that the system

$$\Sigma_{st} := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx \end{cases} \tag{258}$$

is a state representation of the behavior associated with the dynamic equations (232) with the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ observable. Now, by the remarks following Proposition 3.3, a tracking observer for Σ_{st} always exists and we assume it is given by

$$\Sigma_{est} := \begin{cases} \sigma \xi = F\xi + Gy + Hu, \\ \zeta = J\xi. \end{cases} \tag{259}$$

The observer equation can be rewritten as

$$\begin{pmatrix} \sigma I - F \\ J \end{pmatrix} \xi = \begin{pmatrix} G & H & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} y \\ u \\ \bar{z} \end{pmatrix}. \tag{260}$$

Let $Q^{-1}\Pi$ be a left coprime factorization of $J(zI-F)^{-1}$ and define $P=\Pi G$, $R=\Pi H$. Applying Theorem 2.5 to (260), we get the observer equation

$$Q(\sigma)\bar{z}=P(\sigma)y+R(\sigma)u. \tag{261}$$

Since $Q^{-1}P=J(zI-F)^{-1}\begin{pmatrix}P & R\end{pmatrix}$ the observer is strictly proper. Consistency will follow from Theorem 5.1. □

The question of whether the condition in Proposition 4.4 is also necessary remains open.

5. Conventional observers from behavioral point of view

In order to gain insight into the process of observer construction in the behavioral framework, we shall restate the standard observer theory, as outlined in Theorem 3.3 in behavioral terms. The central tool in this procedure is elimination theory. By rewriting the state space equations of both the system and observer in behavioral form, we obtain two ARMA representations of the full system behavior $\mathscr{B}_{fsys}$ and the full observer behavior $\mathscr{B}_{fobs}$, both containing latent variables, namely the respective state variables. A simple rewriting of the intertwining equations (127) between the state space representations of the system and the observer leads to an injective behavior homomorphism $Z:\mathscr{B}_{fsys}\longrightarrow\mathscr{B}_{fobs}$. In the next step, we use left coprime factorizations to eliminate the state variables in $\mathscr{B}_{fsys}$ and $\mathscr{B}_{fobs}$ respectively and obtain the behaviors $\mathscr{B}_{sys}$ and $\mathscr{B}_{obs}$ in AR form, i.e. in a kernel representation. The behavior homomorphism Z induces an injective homomorphism $\overline{Z}:\mathscr{B}_{sys}\longrightarrow\mathscr{B}_{obs}$ that is actually the natural embedding of $\mathscr{B}_{sys}$ in $\mathscr{B}_{obs}$ as a subbehavior. So we have the following schematic commutative diagram:

$$\begin{tikzcd}
\mathcal{B}_{fsys} \arrow[r, "\text{elimination}"] \arrow[d, "Z"'] & \mathcal{B}_{sys} \arrow[d, "\overline{Z}"] \\
\mathcal{B}_{fobs} \arrow[r, "\text{elimination}"] & \mathcal{B}_{obs}
\end{tikzcd}$$

As $\mathscr{B}_{sys}$ is contained in $\mathscr{B}_{obs}$ as a subbehavior, we can call upon factorization theory to provide the link to the Valcher-Willems results.

Theorem 5.1. *Given the state space system*

$$\Sigma_{sys}:=\begin{cases}\sigma x=Ax+Bu,\\ y=Cx,\\ z=Kx,\end{cases} \tag{262}$$

where we assume that the pair $\left(\begin{pmatrix}C\\K\end{pmatrix},A\right)$ *is observable and that for the observer*

$$\Sigma_{est}:=\begin{cases}\sigma\xi=F\xi+Gy+Hu,\\ \zeta=J\xi\end{cases} \tag{263}$$

the pair (J, F) *is observable. Then*

1. *The full behavior of the system* (262) *is given by*

$$\mathscr{B}_{fsys} = \left\{ \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} \in z^{-1}\mathbb{F}^{n+m+p+k}[[z^{-1}]] \mid \begin{pmatrix} \sigma I - A \\ C \\ K \end{pmatrix} x - \begin{pmatrix} B & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \operatorname{Ker} \begin{pmatrix} \sigma I - A & -B & 0 & 0 \\ C & 0 & -I & 0 \\ K & 0 & 0 & -I \end{pmatrix}. \tag{264}$$

Similarly, the full behavior of the observer (263) *is given by*

$$\mathscr{B}_{fest} = \left\{ \begin{pmatrix} \xi \\ u \\ y \\ \zeta \end{pmatrix} \middle| \begin{pmatrix} \sigma I - F \\ J \end{pmatrix} \xi - \begin{pmatrix} H & G & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} u \\ y \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \operatorname{Ker} \begin{pmatrix} \sigma I - F & -H & -G & 0 \\ J & 0 & 0 & -I \end{pmatrix}. \tag{265}$$

2. (a) *The Sylvester equations*

$$\begin{cases} ZA - FZ = GC, \\ H = ZB, \\ K = JZ \end{cases} \tag{266}$$

characterizing observers, can be rewritten in matrix form as

$$\begin{pmatrix} Z & G & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} zI - A & -B & 0 & 0 \\ C & 0 & -I & 0 \\ K & 0 & 0 & -I \end{pmatrix}$$

$$= \begin{pmatrix} zI - F & -H & -G & 0 \\ J & 0 & 0 & -I \end{pmatrix} \begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{267}$$

(b) *Defining*

$$\mathscr{B}_{sys} = \left\{ \begin{pmatrix} u \\ y \\ z \end{pmatrix} \middle| \exists x, \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} \in \mathscr{B}_{fsys} \right\}$$

$$\mathscr{B}_{obs} = \left\{ \begin{pmatrix} u \\ y \\ \zeta \end{pmatrix} \middle| \exists \xi, \begin{pmatrix} \xi \\ u \\ y \\ \zeta \end{pmatrix} \in \mathscr{B}_{fobs} \right\} \tag{268}$$

Then the map $\tau : \mathscr{B}_{fsys} \longrightarrow \mathscr{B}_{fobs}$, *defined by*

$$\tau \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} = \begin{pmatrix} Zx \\ u \\ y \\ z \end{pmatrix} \tag{269}$$

is an injective continuous behavior homomorphism.

3. *Using the coprime factorization* (89), *the manifest behavior* $\mathscr{B}_{sys}$ *of* $\mathscr{B}_{fsys}$ *is given by*

$$\begin{aligned} \mathscr{B}_{sys} &= \left\{ \begin{pmatrix} u \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} -\Theta_1(\sigma)B & D_{11}(\sigma) & 0 \\ -\Theta_2(\sigma)B & D_{21}(\sigma) & D_{22}(\sigma) \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \operatorname{Ker} \begin{pmatrix} -\Theta_1(\sigma)B & D_{11}(\sigma) & 0 \\ -\Theta_2(\sigma)B & D_{21}(\sigma) & D_{22}(\sigma) \end{pmatrix}. \end{aligned} \tag{270}$$

4. *Let* $Q^{-1}\Pi$ *be a left coprime factorization of* $J(zI - F)^{-1}$ *and define* $P_1 = \Pi G$, $P_2 = \Pi H$. *Then the manifest observer behavior* $\mathscr{B}_{obs}$, *is given by*

$$\begin{aligned} \mathscr{B}_{obs} &= \left\{ \begin{pmatrix} u \\ y \\ \zeta \end{pmatrix} \middle| \begin{pmatrix} -P_2(\sigma) & -P_1(\sigma) & Q(\sigma) \end{pmatrix} \begin{pmatrix} u \\ y \\ \zeta \end{pmatrix} = 0 \right\} \\ &= \operatorname{Ker} \begin{pmatrix} -P_2(\sigma) & -P_1(\sigma) & Q(\sigma) \end{pmatrix}. \end{aligned} \tag{271}$$

5. *We have*
   (a) $$\mathscr{B}_{sys} \subset \mathscr{B}_{obs}. \tag{272}$$
   (b) *There exist polynomial matrices* X, Y, *with* Y *nonsingular, for which we have the following factorization*
   $$\begin{pmatrix} -P_2(z) & -P_1(z) & Q(z) \end{pmatrix} = \begin{pmatrix} X(z) & Y(z) \end{pmatrix} \begin{pmatrix} -\Theta_1(z)B & D_{11}(z) & 0 \\ -\Theta_2(z)B & D_{21}(z) & D_{22}(z) \end{pmatrix}, \tag{273}$$
   i.e.
   $$\begin{aligned} Q(z) &= Y(z)D_{22}(z), \\ -P_1(z) &= X(z)D_{11}(z) + Y(z)D_{21}(z), \\ -P_2(z) &= -X(z)\Theta_1(z)B - Y(z)\Theta_2(z)B. \end{aligned} \tag{274}$$

6. *For the system* Σ_{sys}, *given by* (262), *we have*
   (a) z *is trackable from* $\binom{u}{y}$ *if and only if* D_{22} *is nonsingular.*
   (b) z *is detectable from* $\binom{u}{y}$ *if and only if* D_{22} *is stable.*
   (c) z *is reconstructible from* $\binom{u}{y}$ *if and only if* D_{22} *is monomic.*
   (d) z *is observable from* $\binom{u}{y}$ *if and only if* D_{22} *is unimodular.*

Proof

1. The kernel representations given by (264) and (265) are just a rewriting of (262) and (263) respectively.
2. (a) That Eqs. (266) and (267) are equivalent is trivial to check.
   (b) By the characterization of behavior homomorphisms given in Theorem 2.2, the map $\tau : \mathscr{B}_{fsys} \longrightarrow \mathscr{B}_{fobs}$, defined by (269) is indeed a behavior homomorphism. By the assumption that the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ is observable, it follows that $\begin{pmatrix} zI-A & -B & 0 & 0 \\ C & 0 & -I & 0 \\ K & 0 & 0 & -I \end{pmatrix}$ and $\begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ are right coprime. By Theorem 2.3, we conclude that the homomorphism τ is injective.
3. The system equations (262) can be rewritten as
$$\begin{pmatrix} \sigma I - A \\ C \\ K \end{pmatrix} x = \begin{pmatrix} B & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix}. \tag{275}$$
Using the coprime factorization (89), the MLA of $\begin{pmatrix} zI-A \\ C \\ K \end{pmatrix}$ is given by $\begin{pmatrix} -\Theta_1(z) & D_{11}(z) & 0 \\ -\Theta_2(z) & D_{21}(z) & D_{22}(z) \end{pmatrix}$, hence, by the elimination theorem, the manifest behavior $\mathscr{B}_{sys}$ is given by
$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\Theta_1(\sigma) & D_{11}(\sigma) & 0 \\ -\Theta_2(\sigma) & D_{21}(\sigma) & D_{22}(\sigma) \end{pmatrix} \begin{pmatrix} B & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} -\Theta_1(\sigma)B & D_{11}(\sigma) & 0 \\ -\Theta_2(\sigma)B & D_{21}(\sigma) & D_{22}(\sigma) \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix}, \end{aligned}$$
which is equivalent to (270).
4. The observer equations (263) can be rewritten as
$$\begin{pmatrix} \sigma I - F \\ J \end{pmatrix} \xi = \begin{pmatrix} H & G & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix}. \tag{276}$$
As the MLA of $\begin{pmatrix} zI-F \\ J \end{pmatrix}$ is $\begin{pmatrix} -\Pi & Q \end{pmatrix}$, another application of the elimination theorem yields
$$\begin{aligned} 0 &= \begin{pmatrix} -\Pi(\sigma) & Q(\sigma) \end{pmatrix} \begin{pmatrix} H & G & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} -P_2(\sigma) & -P_1(\sigma) & Q(\sigma) \end{pmatrix} \begin{pmatrix} u \\ y \\ \zeta \end{pmatrix}, \end{aligned}$$
which is equivalent to (271).
5. (a) Follows from the injectivity of the map τ, defined by (269).

   We can verify this directly. Let $\begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} \in \mathscr{B}_{fsys}$, and Z satisfy Eqs. (266). Defining $\xi = Zx$, we have

$$\sigma\xi = Z\sigma x = Z(Ax + Bu) = (FZ + GC)x + Hu = F\xi + Gy + Hu$$
$$z = Kx = JZx = J\xi,$$

i.e. $\begin{pmatrix} Zx \\ u \\ y \\ z \end{pmatrix} \in \mathscr{B}_{fest}$.

(b) Follows from the inclusion (272), the kernel representations (270) and (271) of $\mathscr{B}_{sys}$ and $\mathscr{B}_{obs}$ respectively and the fact that subbehaviors relate to factorizations.

1. The system equations, after elimination of the state variable are given by

$$\begin{pmatrix} 0 \\ D_{22}(\sigma) \end{pmatrix} z = \begin{pmatrix} \Theta_1(\sigma)B & -D_{11}(\sigma) \\ \Theta_2(\sigma)B & -D_{21}(\sigma) \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}.$$

Note that $\begin{pmatrix} 0 \\ D_{22}(z) \end{pmatrix}$ is of full column rank, right stable, right monomic or right prime if and only if $D_{22}(z)$ is nonsingular, stable, monomic or unimodular respectively. Applying Proposition 4.2, the result follows. □

Remarks

1. It is worth pointing out that the inclusion (272) of Theorem 5.1 can be interpreted as a manifestation of the internal model principle, see [33]. In fact, it seems that the concepts of behavior and behavior homomorphisms provide the right context and language in which to formulate general internal model principles.
2. Of course, Part 6 of Theorem 5.1 is in full agreement with Proposition 3.3.

5.1. State maps and observers

State maps have been introduced in [23], and studied further in [15], as a tool for obtaining first order representations of behaviors, i.e. for establishing a realization theory for behaviors. We will show next how state maps are related to the construction of observers.

We consider first the autonomous system Σ given by

$$D(\sigma)w = 0, \tag{277}$$

with $D(z) \in \mathbb{F}[z]^{p\times p}$ nonsingular. By [11] and [15], there exists a unique, up to similarity, observable pair (C, A) such that the latent variable, state to output system

$$\Sigma := \begin{cases} \sigma x = Ax, \\ w = Cx, \\ z = Ix = x, \end{cases} \tag{278}$$

is a first order representation of the behavior $X^D = \operatorname{Ker} D(\sigma)$. Our aim is to construct an observer for the state x, based on the observations w. Applying Theorem 3.5, we have the coprime factorizations

$$\begin{pmatrix} C \\ I \end{pmatrix} (zI - A)^{-1} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & I \end{pmatrix}^{-1} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \tag{279}$$

where $D_{11}^{-1}\Theta_1$ is a left coprime factorization of $C(zI - A)^{-1}$. Furthermore, we can assume D_{11} to be row proper and $D_{21}D_{11}^{-1}$ to be strictly proper. Eliminating the state variable from (278), leads to

$$\begin{pmatrix} -\Theta_1 & D_{11} & 0 \\ -\Theta_2 & D_{21} & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. to

$$\begin{pmatrix} D_{11} & 0 \\ D_{21} & I \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{280}$$

Now $\begin{pmatrix} -\Theta_1 & D_{11} & 0 \\ -\Theta_2 & D_{21} & I \end{pmatrix}$ is a MLA of $\begin{pmatrix} zI - A \\ C \\ I \end{pmatrix}$. In particular, this implies the Bezout equation

$$\Theta_2(zI - A) - D_{21}C = I. \tag{281}$$

We claim that the polynomial matrix $\begin{pmatrix} -\Theta_1 & D_{11} \\ -\Theta_2 & D_{21} \end{pmatrix}$ is unimodular. To see this, note that the left coprimeness of Θ_1, D_{11} implies the existence of a solution of the Bezout equation $-\Theta_1 M_1 + D_{11}M_2 = I$. Clearly, with $Q = -\Theta_2 M_1 + D_{21}M_2$, we have

$$\begin{pmatrix} -\Theta_1 & D_{11} \\ -\Theta_2 & D_{21} \end{pmatrix} \begin{pmatrix} M_1 & zI - A \\ M_2 & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}.$$

Multiplying by $\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}$ on the right and redefining M_1, M_2 appropriately, we can assume without loss of generality that $Q = 0$ and unimodularity is proved.

Next, we compute

$$\begin{pmatrix} -\Theta_1 & D_{11} \\ -\Theta_2 & D_{21} \end{pmatrix} \begin{pmatrix} 0 & zI - A \\ I & C \end{pmatrix} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & I \end{pmatrix}.$$

In terms of behaviors, we conclude that

$$\operatorname{Ker} \begin{pmatrix} D_{11}(\sigma) & 0 \\ D_{21}(\sigma) & I \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} 0 & \sigma I - A \\ I & C \end{pmatrix}. \tag{282}$$

This shows that

$$\hat{x} = -D_{21}(\sigma)w \tag{283}$$

is an exact state observer.

On the other hand, by Theorem 4.3 in [15], we conclude that $-D_{21}(z)$, and hence also $D_{21}(z)$, is a state map for the autonomous dynamical system $D_{11}(\sigma)w = 0$.

We proceed to discuss a general system, given in the kernel representation

$$\mathscr{B} = \operatorname{Ker} R(\sigma), \tag{284}$$

We will assume that this kernel representation of $\mathscr{B}$ is minimal, i.e. that $R(z)$ is a $p \times m$ full row rank, row proper polynomial matrix with row indices $\nu_1 \geqslant \cdots \geqslant \nu_p \geqslant 0$. We let $n = \sum_{i=1}^{p} \nu_i$. Let $D_{11}(z)$ be any nonsingular $p \times p$, row proper polynomial matrix for which $D_{11}(z)^{-1}R(z)$ is proper and has a proper inverse. Let $\Theta_1(z)$ be an arbitrary basis matrix for the polynomial model $X_{D_{11}}$. Let (C, A) be the unique observable pair for which

$$D_{11}(z)^{-1}\Theta_1(z) = C(zI - A)^{-1}. \tag{285}$$

holds. With R_∞, R_1 be defined by

$$R_\infty = \pi_+ D_{11}^{-1} R, \tag{286}$$

and

$$R_1(z) = R(z) - D(z) R_\infty \tag{287}$$

respectively, there exists a unique, constant, $n \times m$ matrix B for which

$$R_1(z) = -\Theta_1(z) B. \tag{288}$$

Thus we have the representation

$$R(z) = D_{11}(z) R_\infty - \Theta_1(z) B. \tag{289}$$

From the coprime factorization (285), we obtain the intertwining relation

$$\Theta_1(z)(zI - A) = D_{11}(z) C \tag{290}$$

which in turn can be embedded in a doubly coprime factorization

$$\begin{gathered}\begin{pmatrix} D_{11}(z) & -\Theta_1(z) \\ D_{21}(z) & -\Theta_2(z) \end{pmatrix} \begin{pmatrix} \overline{Y}_2(z) & C \\ -\overline{Y}_1(z) & zI - A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} \overline{Y}_2(z) & C \\ -\overline{Y}_1(z) & zI - A \end{pmatrix} \begin{pmatrix} D_{11}(z) & -\Theta_1(z) \\ D_{21}(z) & -\Theta_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{gathered} \tag{291}$$

In particular, we have

$$-\Theta_2(z)(zI - A) + D_{21}(z) C = I. \tag{292}$$

We compute

$$\begin{pmatrix} D_{11}(z) & -\Theta_1(z) \\ D_{21}(z) & -\Theta_2(z) \end{pmatrix} \begin{pmatrix} R_\infty & -C \\ B & -(zI - A) \end{pmatrix} = \begin{pmatrix} R(z) & 0 \\ X(z) & -I \end{pmatrix}, \tag{293}$$

where

$$X(z) = D_{21}(z) R_\infty - \Theta_2(z) B. \tag{294}$$

Since, by (292), $\begin{pmatrix} D_{11}(z) & -\Theta_1(z) \\ D_{21}(z) & -\Theta_2(z) \end{pmatrix}$ is unimodular, it follows that

$$\operatorname{Ker} \begin{pmatrix} R_\infty & -C \\ B & -(\sigma I - A) \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} R(\sigma) & 0 \\ X(\sigma) & -I \end{pmatrix}. \tag{295}$$

The coprime factorizations (285) show that $\begin{pmatrix} D_{11}(z) & -\Theta_1(z) \end{pmatrix}$ is a MLA of $\begin{pmatrix} C \\ zI - A \end{pmatrix}$, hence we can eliminate the latent variable x from the equation

$$\begin{pmatrix} R_\infty \\ B \end{pmatrix} w = \begin{pmatrix} C \\ \sigma I - A \end{pmatrix} x \tag{296}$$

to obtain, using the representation (289),

$$\begin{aligned} \begin{pmatrix} D_{11}(\sigma) & -\Theta_1(\sigma) \end{pmatrix} \begin{pmatrix} R_\infty \\ B \end{pmatrix} w &= D_{11}(\sigma) R_\infty - \Theta_1(\sigma) B \\ &= R(\sigma) w = 0 \end{aligned}$$

we obtain the behavior Ker $R(\sigma)$. Thus (296) is a first order representation of the behavior Ker $R(\sigma)$ and $X(z)$, given by (294), is a corresponding state map for the dynamical system given by the equation $R(\sigma)w = 0$.

We proceed now to show the connection of the state map with a state observer, starting with the state representation (296). Using the identity (293) and applying Proposition 4.3.3, we conclude that

$$\hat{x} = -X(\sigma)w \tag{297}$$

is an exact, though singular, state observer. This could also be derived directly from the state representation (296), using the Bezout equation (292) and the definition of $X(z)$ given in (294).

6. Examples

We end this paper by the analysis of some simple examples to illustrate the theory.

Example 1. Observable case: Consider the state space system in $\mathbb{F}^3$ given by

$$\Sigma := \begin{cases} \sigma x = Ax + Bu, \\ y = Cx, \\ z = Kx, \end{cases} \tag{298}$$

with (C, A) given in dual Brunovsky form, i.e.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

and

$K : \mathbb{F}^3 \longrightarrow \mathbb{F}$ defined by

$$K = \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}.$$

Clearly

$$\begin{aligned} C(zI - A)^{-1} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ -1 & z & 0 \\ 0 & -1 & z \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-3} & z^{-2} & z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} z^{-3} & z^{-2} & z^{-1} \end{pmatrix} = z^{-3} \begin{pmatrix} 1 & z & z^2 \end{pmatrix}. \end{aligned}$$

Furthermore, we compute

$$\begin{aligned} \begin{pmatrix} C \\ K \end{pmatrix} (zI - A)^{-1} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-3} & z^{-2} & z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} z^{-3} & z^{-2} & z^{-1} \\ z^{-1} + \alpha z^{-1} + \alpha^2 z^{-3} & \alpha z^{-1} + \alpha^2 z^{-2} & \alpha^2 z^{-1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & z^2 \\ z^2+\alpha z+\alpha^2 & \alpha z^2+\alpha^2 z & \alpha^2 z^2 \end{pmatrix}$$

$$= \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ z^2+\alpha z+\alpha^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & z & z^2 \\ 0 & -z^3 & -\alpha z^3 - z^4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ z^2+\alpha z+\alpha^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} z^3 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & z^2 \\ 0 & -1 & -\alpha - z \end{pmatrix}$$

$$= \begin{pmatrix} z^3 & 0 \\ -(z^2+\alpha z+\alpha^2) & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & z^2 \\ 0 & -1 & -\alpha - z \end{pmatrix}$$

So we have obtained

$$\begin{aligned} \Theta_1(z) &= \begin{pmatrix} 1 & z & z^2 \end{pmatrix}, \\ \Theta_2(z) &= \begin{pmatrix} 0 & -1 & -\alpha - z \end{pmatrix}, \\ D_{11}(z) &= z^3, \\ D_{21}(z) &= -(z^2+\alpha z+\alpha^2), \\ D_{22}(z) &= 1. \end{aligned}$$

In both cases the factorizations are left coprime. Thus the shift realization allows us to consider the polynomial model X_{z^3} as te state space and the pair (C, A) results from the shift realization (20) and taking the matrix representation with respect to the standard basis $\{1, z, z^2\}$ in X_{z^3}. The map $K : X_{z^3} \longrightarrow \mathbb{F}$ has the functional representation

$$Kf = f(\alpha), \quad f \in X_{z^3}.$$

That $D_{22}(z)$ is unimodular is the result of the observability assumption, see Theorem 5.1.6. Similarly,

$$\begin{aligned} P_1(z) &= -D_{22}(z)^{-1} D_{21}(z) = (z^2+\alpha z+\alpha^2), \\ P_2(z) &= D_{22}(z)^{-1}\Theta_2(z) = \begin{pmatrix} 0 & -1 & -\alpha - z \end{pmatrix}, \end{aligned}$$

$P_1(z), P_2(z)$ are in this case polynomial solutions of equation (104). Indeed, we can check that

$$\begin{aligned} P_1(z)C + P_2(z)(zI - A) &= (z^2+\alpha z+\alpha^2)\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & -1 & -\alpha - z \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ -1 & z & 0 \\ 0 & -1 & z \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & z^2+\alpha z+\alpha^2 \end{pmatrix} + \begin{pmatrix} 1 & \alpha & -\alpha z - z^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} = K. \end{aligned}$$

Since the P_i are polynomial, an exact observer exists. This is a consequence of the observability of (C, A).

Construction of observers is, by Theorem 3.3, related to conditioned invariant subspaces contained in Ker K. We discuss several cases.

Case I (*Maximal conditioned invariant subspace*). We use the characterization of conditioned invariant subspaces derived in [8]. Consider first the conditioned invariant subspace $\mathscr{V} = X_{z^3} \cap (z-\alpha)\mathbb{F}[z] \subset X_{z^3}$, i.e. $T(z) = (z-\alpha)$. Note that this is a maximal conditioned invariant subspace in Ker K, as Ker K is itself a conditioned invariant subspace. In this case we have $F = (\alpha)$. We have $Z = [\pi_T | X_{z^3}]_{st}^{st} = \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}$ which is the partial reachability matrix in this case. We compute

$$\begin{aligned} ZA - FZ &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \alpha \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \\ &= -\begin{pmatrix} 0 & 0 & \alpha^3 \end{pmatrix} = GC \end{aligned}$$

So we get

$$\begin{aligned} F &= (\alpha), \\ G &= -\alpha^3, \\ Z &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}, \\ J &= 1. \end{aligned}$$

A solution $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ is given by

$$\begin{aligned} \begin{pmatrix} Z_1 & Z_2 \end{pmatrix} &= J(zI - F)^{-1} \begin{pmatrix} G & Z \end{pmatrix} \\ &= 1 \cdot \frac{1}{z-\alpha} \begin{pmatrix} -\alpha^3 & 1 & \alpha & \alpha^2 \end{pmatrix}, \end{aligned}$$

Alternatively, we could apply the Antoulas parametrization (204). To do this, we need to find (minimal) partial realizations for the nice sequence $1, \alpha, \alpha^2$. In this case the minimal partial realization is unique, has McMillan degree 1 and has the functional representation $W = \frac{1}{z-\alpha}$. With this, we compute

$$\begin{aligned} Z_1(z) &= (z^2 + \alpha z + \alpha^2) - \frac{1}{z-\alpha} z^3 = \frac{(z-\alpha)(z^2+\alpha z+\alpha^2) - z^3}{z-\alpha} \\ &= \frac{(z^3-\alpha^3) - z^3}{z-\alpha} = \frac{-\alpha^3}{z-\alpha}, \\ Z_2(z) &= \begin{pmatrix} 0 & -1 & -\alpha - z \end{pmatrix} + \frac{1}{z-\alpha} \begin{pmatrix} 1 & z & z^2 \end{pmatrix} \\ &= \frac{1}{z-\alpha} \left[\begin{pmatrix} 0 & -(z-\alpha) & -(z^2-\alpha^2) \end{pmatrix} + \begin{pmatrix} 1 & z & z^2 \end{pmatrix} \right] \\ &= \frac{1}{z-\alpha} \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}, \end{aligned} \tag{299}$$

which is in complete agreement with the previous computations.

We check

$$\begin{aligned} Z_1(z)Z_c(z) + Z_2(z) &= -\alpha^3 (z-\alpha)^{-1} \begin{pmatrix} z^{-3} & z^{-2} & z^{-1} \end{pmatrix} + (z-\alpha)^{-1} \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \\ &= z^{-3}(z-\alpha)^{-1} [-\alpha^3 \begin{pmatrix} 1 & z & z^2 \end{pmatrix} + \begin{pmatrix} z^3 & \alpha z^3 & \alpha^2 z^3 \end{pmatrix} \\ &= z^{-3}(z-\alpha)^{-1} \begin{pmatrix} z^3 - \alpha^3 & \alpha z^3 - \alpha^3 z & \alpha^2 z^3 - \alpha^3 z^2 \end{pmatrix} \end{aligned}$$

$$= z^{-3}\begin{pmatrix} z^2+\alpha z+\alpha^2 & \alpha z^2+\alpha^2 z & \alpha^2 z^2\end{pmatrix}$$
$$= \begin{pmatrix} 1 & \alpha & \alpha^2\end{pmatrix}\begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-3} & z^{-2} & z^{-1}\end{pmatrix} = Z_K(z).$$

From $J(zI-F)^{-1} = Q^{-1}\Pi$ and $P_1 = \Pi G$ and $P_2 = \Pi Z$, we have $P_1(z) = -\alpha^3$ and $P_2(z) = \begin{pmatrix} 1 & \alpha & \alpha^2\end{pmatrix}$. We proceed to the factorization (273), i.e.

$$\begin{pmatrix} -P_2(z) & -P_1(z) & Q(z)\end{pmatrix} = \begin{pmatrix} X(z) & Y(z)\end{pmatrix}\begin{pmatrix} -\Theta_1(z) & D_{11}(z) & 0 \\ -\Theta_2(z) & D_{21}(z) & I\end{pmatrix},$$

with $W(z) = \frac{1}{z-\alpha} = Y(z)^{-1}X(z)$, and check that

$$\begin{aligned}&\begin{pmatrix} -\begin{pmatrix} 1 & \alpha & \alpha^2\end{pmatrix} & \alpha^3 & z-\alpha\end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & z-\alpha\end{pmatrix}\begin{pmatrix} -\begin{pmatrix} 1 & z & z^2\end{pmatrix} & z^3 & 0 \\ -\begin{pmatrix} 0 & -1 & -\alpha-z\end{pmatrix} & -(z^2+\alpha z+\alpha^2) & 1\end{pmatrix}.\end{aligned}$$

Case II (*Non maximal conditioned invariant subspace*). We use the same system as in (298), with K defined by

$$Kf = f(\alpha), \quad f \in X_{z^3}.$$

However, we consider the conditioned invariant subspace $\mathscr{V} = X_{z^3} \cap (z-\alpha)(z-\beta)\mathbb{F}[z] \subset X_{z^3}$, i.e. $T(z) = (z-\alpha)(z-\beta)$. Since $(z-\alpha)(z-\beta)\mathbb{F}[z] \subset (z-\alpha)\mathbb{F}[z]$, it clearly follows that $\mathscr{V} \subset \operatorname{Ker} K$. Using the standard basis $\{1, z, z^2\}$ in X_{z^3}, we have the matrix representations

$$Z = [\pi_T | X_{z^3}]_{st}^{st} = \begin{pmatrix} 1 & 0 & -\alpha\beta \\ 0 & 1 & \alpha+\beta\end{pmatrix},$$
$$F = [S_T]_{st}^{st} = \begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta\end{pmatrix}.$$

The representation of Z is the partial reachability matrix in this case. We compute

$$\begin{aligned} ZA - FZ &= \begin{pmatrix} 1 & 0 & -\alpha\beta \\ 0 & 1 & \alpha+\beta\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix} - \begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta\end{pmatrix}\begin{pmatrix} 1 & 0 & -\alpha\beta \\ 0 & 1 & \alpha+\beta\end{pmatrix} \\ &= \begin{pmatrix} 0 & -\alpha\beta & 0 \\ 1 & \alpha+\beta & 0\end{pmatrix} - \begin{pmatrix} 0 & -\alpha\beta & -\alpha\beta(\alpha+\beta) \\ 1 & \alpha+\beta & -\alpha\beta+(\alpha+\beta)^2\end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \alpha\beta(\alpha+\beta) \\ 0 & 0 & \alpha\beta-(\alpha+\beta)^2\end{pmatrix} = GC,\end{aligned}$$

so it follows that

$$G = \begin{pmatrix} \alpha\beta(\alpha+\beta) \\ \alpha\beta-(\alpha+\beta)^2\end{pmatrix}. \tag{300}$$

On the other hand, using Theorem 3.7, we can compute G directly via (224) as follows:

$$G = -\pi_T D_{11} = -\pi_{(z-\alpha)(z-\beta)} z^3 = (\alpha\beta-(\alpha+\beta)^2)z + \alpha\beta(\alpha+\beta),$$

which agrees with (300). We know $K = JZ$ and indeed,

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & -\alpha\beta \\ 0 & 1 & \alpha+\beta \end{pmatrix},$$

i.e. $J = \begin{pmatrix} 1 & \alpha \end{pmatrix}$.

We compute

$$\begin{aligned}
\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} &= \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} z & \alpha\beta \\ -1 & z-(\alpha+\beta) \end{pmatrix}^{-1} \begin{pmatrix} \alpha\beta(\alpha+\beta) & 1 & 0 & -\alpha\beta \\ \alpha\beta-(\alpha+\beta)^2 & 0 & 1 & \alpha+\beta \end{pmatrix} \\
&= \frac{\begin{pmatrix} 1 & \alpha \end{pmatrix}}{(z-\alpha)(z-\beta)} \begin{pmatrix} z-(\alpha+\beta) & -\alpha\beta \\ 1 & z \end{pmatrix} \begin{pmatrix} \alpha\beta(\alpha+\beta) & 1 & 0 & -\alpha\beta \\ \alpha\beta-(\alpha+\beta)^2 & 0 & 1 & \alpha+\beta \end{pmatrix} \\
&= [(z-\alpha)(z-\beta)]^{-1} \begin{pmatrix} z-\beta & \alpha(z-\beta) \end{pmatrix} \begin{pmatrix} \alpha\beta(\alpha+\beta) & 1 & 0 & -\alpha\beta \\ \alpha\beta-(\alpha+\beta)^2 & 0 & 1 & \alpha+\beta \end{pmatrix} \\
&= (z-\alpha)^{-1} \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} \alpha\beta(\alpha+\beta) & 1 & 0 & -\alpha\beta \\ \alpha\beta-(\alpha+\beta)^2 & 0 & 1 & \alpha+\beta \end{pmatrix} \\
&= (z-\alpha)^{-1} \begin{pmatrix} -\alpha^3 & 1 & \alpha & \alpha^2 \end{pmatrix}. \qquad (301)
\end{aligned}$$

So

$$\begin{aligned}
Z_1 &= \frac{-\alpha^3}{z-\alpha}, \\
Z_2 &= \frac{\begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}}{z-\alpha}, \qquad (302)
\end{aligned}$$

which is in agreement with (299).

Note that the nonminimality of the realization

$$\left(\begin{array}{c|c} F & G \\ \hline J & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -\alpha\beta & \alpha\beta(\alpha+\beta) \\ 1 & \alpha+\beta & \alpha\beta-(\alpha+\beta)^2 \\ \hline 1 & \alpha & 0 \end{array}\right) \qquad (303)$$

is a consequence of having used a nonmaximal conditioned invariant subspace $\subset \operatorname{Ker} K$. That we have obtained a strongly tracking observer follows from the fact that $\operatorname{Ker} K$ is itself a conditioned invariant subspace.

The observer given by (301) is an asymptotic tracking observer if and only if α and β are stable. However, for the existence of an asymptotic tracking observer a necessary and sufficient condition is the stability of α.

That $Z_k = Z_1 Z_C + Z_2$ follows from a previous calculation.

We check now directly that (303), although not a minimal realization, is indeed a tracking observer for the system. To this end, we compute

$$\begin{pmatrix} \sigma\xi_1 \\ \sigma\xi_2 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \alpha\beta(\alpha+\beta) \\ \alpha\beta-(\alpha+\beta)^2 \end{pmatrix} x_3$$

For the estimation error we have

$$e = z - \zeta = Kx - J\xi = x_1 + \alpha x_2 + \alpha^2 x_3 - \xi_1 - \alpha\xi_2$$

and hence

$$\begin{aligned}
\sigma e &= K\sigma x - J\sigma\xi = \sigma x_1 + \alpha\sigma x_2 + \alpha^2\sigma x_3 - \sigma\xi_1 - \alpha\sigma\xi_2 \\
&= \alpha x_1 + \alpha^2 x_2 - [-\alpha\beta\xi_2 + \alpha\beta(\alpha+\beta)x_3]
\end{aligned}$$

$$
\begin{aligned}
&\quad -\alpha[\xi_1 + (\alpha+\beta)\xi_2 + (\alpha\beta - (\alpha+\beta)^2)x_3] \\
&= \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 - \alpha\xi_1 - \alpha^2\xi_2 = \alpha[x_1 + \alpha x_2 + \alpha^2 x_3 - \xi_1 - \alpha\xi_2] \\
&= \alpha e
\end{aligned}
$$

This shows that we have constructed a strongly tracking observer as $e_1 = 0$ implies $e_t = 0$.

Case III. Finally, we consider the trivial conditioned invariant subspace $\mathscr{V} = \{0\}$. Clearly, for any polynomial $t(z) = z^3 + t_2 z^2 + t_1 z + t_0$, we have $\mathscr{V} = X_{z^3} \cap t(z)\mathbb{F}[z] \subset X_{z^3}$. By the observability of the pair (C, A), $\mathscr{V}$ is an outer observability subspace. In this case we have $F = \begin{pmatrix} 0 & 0 & -t_0 \\ 1 & 0 & -t_1 \\ 0 & 1 & -t_2 \end{pmatrix}$. The characteristic polynomial of F is freely assignable, via the choice of t_0, t_1, t_2. We have $Z = [\pi_T | X_{z^3}]_{st}^{st} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which is the partial reachability matrix in this case. We compute

$$
ZA - FZ = A - F = \begin{pmatrix} 0 & 0 & t_0 \\ 0 & 0 & t_1 \\ 0 & 0 & t_2 \end{pmatrix} = GC.
$$

So $G = \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}$.

Again, we can compute G directly via (224)

$$
G = -\pi_T D_{11} = -\pi_{z^3+t_2z^2+t_1z+t_0} z^3 = t_2 z^2 + t_1 z + t_0,
$$

which, with respect to the standard basis has the matrix representation $G = \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}$. We know $K = JZ$, so necessarily $J = \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}$.

We compute

$$
\begin{aligned}
\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} z & 0 & t_0 \\ -1 & z & t_1 \\ 0 & -1 & z + t_2 \end{pmatrix}^{-1} \begin{pmatrix} t_0 & 1 & 0 & 0 \\ t_1 & 0 & 1 & 0 \\ t_2 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{\begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}}{z^3 + t_2 z^2 + t_1 z + t_0} \begin{pmatrix} z^2 + t_2 z + t_1 & -t_0 & -t_0 z \\ z + t_2 & z^2 + t_2 z & -t_1 z - t_0 \\ 1 & z & z^2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} t_0 & 1 & 0 & 0 \\ t_1 & 0 & 1 & 0 \\ t_2 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{\begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}}{z^3 + t_2 z^2 + t_1 z + t_0} \\
&\quad \times \begin{pmatrix} t_0 z^2 & z^2 + t_2 z + t_1 & -t_0 & -t_0 z \\ t_1 z^2 + t_0 z & z + t_2 & z^2 + t_2 z & -t_1 z - t_0 \\ t_2 z^2 + t_1 z + t_0 & 1 & z & z^2 \end{pmatrix}.
\end{aligned} \tag{304}
$$

By an arduous direct computation, which is not reproduced here, one can verify that indeed $Z_K = Z_1 Z_C + Z_2$ holds.

Example 2 (*Unobservable case*). We consider next the unobservable case. Start from the state space system

$$\begin{aligned} A &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \gamma \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \\ K &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \end{aligned} \tag{305}$$

so the pair (C, A) is not observable. We assume that $\alpha \neq 0$ which is the condition for the pair $\left(\begin{pmatrix} C \\ K \end{pmatrix}, A\right)$ to be observable.

We begin our analysis from the state space viewpoint. Let a conditioned invariant subspace be given in the kernel representation $\mathscr{V} = \operatorname{Ker} Z$ with Z of full row rank. Since we need the inclusion $\mathscr{V} \subset \operatorname{Ker} K$, we have $\operatorname{Ker} Z \subset \operatorname{Ker} K$. Now, for K given by (305), we have $\dim \operatorname{Ker} K = 2$, which implies $\dim \operatorname{Ker} Z = 0, 1, 2$. The case $\operatorname{Ker} Z = 0$ is trivial and we focus on the other two cases.

Case I. $\dim \operatorname{Ker} Z = 2$

In this case $\operatorname{Ker} Z = \operatorname{Ker} K$ and we may assume that $Z = K$ and $J = I$. The Sylvester equation (266) reduces to $KA - FK = GC$. We compute

$$\begin{aligned} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \gamma \end{pmatrix} - F \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \\ &= \begin{pmatrix} \alpha - F & \alpha^2 - F\alpha & \alpha^2\gamma - F\alpha^2 \end{pmatrix}. \end{aligned}$$

Equating terms, we obtain $F = \alpha$, $G = 0$ and $\gamma = \alpha$. Thus

$$\begin{aligned} \begin{pmatrix} Z_1 & Z_2 \end{pmatrix} &= \left(\begin{array}{c|cccc} \alpha & 0 & 1 & \alpha & \alpha^2 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right) \\ &= \begin{pmatrix} 0 & (z-\alpha)^{-1} & (z-\alpha)^{-1}\alpha & (z-\alpha)^{-1}\alpha^2 \end{pmatrix} \end{aligned} \tag{306}$$

Now $Z_1 = 0$ implies $Z_2 = Z_K$. We check that, with $\gamma = \alpha$,

$$\begin{aligned} Z_K &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ -1 & z & 0 \\ 0 & -1 & z-\alpha \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-2}(z-\alpha)^{-1} & z^{-1}(z-\alpha)^{-1} & (z-\alpha)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} z^{-1} + \alpha z^{-2} + \alpha^2 z^{-2}(z-\alpha)^{-1} & \alpha z^{-1} + \alpha^2 z^{-1}(z-\alpha)^{-1} & \alpha^2(z-\alpha)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (z-\alpha)^{-1} & (z-\alpha)^{-1}\alpha & (z-\alpha)^{-1}\alpha^2 \end{pmatrix}, \end{aligned}$$

which is compatible with (306).

Case II. $\dim \operatorname{Ker} Z = 1$ In this case we can assume that

$$Z = \begin{pmatrix} 0 & \lambda & \mu \\ 1 & \alpha & \alpha^2 \end{pmatrix}, \tag{307}$$

with λ and μ not both zero. Let $\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ be a basis vector for $\mathscr{V} = \operatorname{Ker} Z$. Using the representation (307) of Z, we have

$$\xi_1 = \alpha\mu - \alpha^2\lambda,$$
$$\xi_2 = -\mu,$$
$$\xi_3 = \lambda.$$

Clearly, $\begin{pmatrix} \alpha\mu - \alpha^2\lambda \\ -\mu \\ \lambda \end{pmatrix} \in \operatorname{Ker} C$ if and only if $\mu = 0$. $\mathscr{V}$ being conditioned invariant translates into

$$Av = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \gamma \end{pmatrix} \begin{pmatrix} -\alpha^2\lambda \\ 0 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha^2\lambda \\ \gamma\lambda \end{pmatrix} = \sigma \begin{pmatrix} \alpha\mu - \alpha^2\lambda \\ -\mu \\ \lambda \end{pmatrix},$$

which shows that $\mu = \alpha\lambda$, $\sigma = \gamma$, and $\gamma = \alpha$. This shows that $\begin{pmatrix} 0 \\ -\alpha \\ 1 \end{pmatrix}$ is a basis vector for $\mathscr{V}$ and that $Z = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix}$. The Sylvester equations (266) lead to

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \alpha \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1-b & \alpha - (a+b\alpha) & (\alpha - a)\alpha - b\alpha^2 \\ \alpha - d & \alpha^2 - (c + d\alpha) & \alpha^3 - (c\alpha + d\alpha^2) \end{pmatrix},$$

to get $a = 0$, $b = 1$, $c = 0$, $d = \alpha$, $g_1 = g_2 = 0$. So we have the realization

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 1 & \alpha \\ 0 & \alpha & 0 & 1 & \alpha & \alpha^2 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \tag{308}$$

or

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = (z-\alpha)^{-1} \begin{pmatrix} 0 & 1 & \alpha & \alpha^2 \end{pmatrix}, \tag{309}$$

i.e.

$$\begin{cases} Z_1 = 0, \\ Z_2 = (z-\alpha)^{-1} \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix}. \end{cases} \tag{310}$$

It follows from the analysis of the two cases that a necessary and sufficient condition for the existence of a reduced order tracking observer for the system given by (305) is $\gamma = \alpha$. A tracking observer of full McMillan degree always exists. See the remarks following Proposition 3.3.

Our next step is to analyse the same system from the functional point of view. We begin by computing the coprime factorization for the system.

$$
\begin{aligned}
C(zI-A)^{-1} &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ -1 & z & 0 \\ 0 & -1 & z-\gamma \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-2}(z-\gamma)^{-1} & z^{-1}(z-\gamma)^{-1} & (z-\gamma)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} z^{-2} & z^{-1} & 0 \end{pmatrix} = z^{-2} \begin{pmatrix} 1 & z & 0 \end{pmatrix}.
\end{aligned}
$$

With K as above, we compute

$$
\begin{aligned}
&\begin{pmatrix} C \\ K \end{pmatrix} (zI-A)^{-1} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-2}(z-\gamma)^{-1} & z^{-1}(z-\gamma)^{-1} & (z-\gamma)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} z^{-2} & z^{-1} & 0 \\ z^{-1}+\alpha z^{-2}+\alpha^2(z-\gamma)^{-1}z^{-2} & \alpha z^{-1}+\alpha^2(z-\gamma)^{-1}z^{-1} & \alpha^2(z-\gamma)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} z^2 & 0 \\ 0 & (z-\gamma)z^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & 0 \\ z(z-\gamma)+\alpha(z-\gamma)+\alpha^2 & \alpha z(z-\gamma)+\alpha^2 z & \alpha^2 z^2 \end{pmatrix} \\
&= \begin{pmatrix} z^2 & 0 \\ 0 & (z-\gamma)z^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -z(z-\gamma)-\alpha(z-\gamma)-\alpha^2 & 1 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} 1 & 0 \\ -z(z-\gamma)-\alpha(z-\gamma)-\alpha^2 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & z & 0 \\ z(z-\gamma)+\alpha(z-\gamma)+\alpha^2 & \alpha z(z-\gamma)+\alpha^2 z & \alpha^2 z^2 \end{pmatrix} \\
&= \left[\begin{pmatrix} 1 & 0 \\ -[z(z-\gamma)+\alpha(z-\gamma)+\alpha^2] & 1 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & (z-\gamma)z^2 \end{pmatrix} \right]^{-1} \\
&\quad \times \begin{pmatrix} 1 & 0 \\ -[z(z-\gamma)+\alpha(z-\gamma)+\alpha^2] & 1 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} 1 & z & 0 \\ (z(z-\gamma)+\alpha(z-\gamma)+\alpha^2) & -\alpha z(z-\gamma)+\alpha^2 z & \alpha^2 z^2 \end{pmatrix} \\
&= \begin{pmatrix} z^2 & 0 \\ -[z^3(z-\gamma)+\alpha z^2(z-\gamma)+\alpha^2 z^2] & (z-\gamma)z^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & 0 \\ 0 & -z^2(z-\gamma) & \alpha^2 z^2 \end{pmatrix} \\
&= \begin{pmatrix} z^2 & 0 \\ -[z(z-\gamma)+\alpha(z-\gamma)+\alpha^2] & z-\gamma \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & 0 \\ 0 & -(z-\gamma) & \alpha^2 \end{pmatrix} \\
&= \begin{pmatrix} z^2 & 0 \\ (\gamma-\alpha)(z+\alpha) & z-\gamma \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & 0 \\ 1 & \gamma & \alpha^2 \end{pmatrix}
\end{aligned}
$$

In both cases we have left coprime factorizations. So, in this case, the coprime factorization (89) is given by

$$\Theta_1 = \begin{pmatrix} 1 & z & 0 \end{pmatrix},$$
$$\Theta_2 = \begin{pmatrix} 1 & \gamma & \alpha^2 \end{pmatrix},$$
$$D_{11} = z^2,$$
$$D_{21} = (\gamma - \alpha)(z + \alpha),$$
$$D_{22} = z - \gamma.$$

By Theorem 3.5, a rational solution of equation (104) is given by

$$Z_1 = -D_{22}^{-1} D_{21} = -(z-\gamma)^{-1}(\gamma - \alpha)(z+\alpha),$$
$$Z_2 = D_{22}^{-1}\Theta_2 = (z-\gamma)^{-1}\begin{pmatrix} 1 & \gamma & \alpha^2 \end{pmatrix}.$$

Indeed, we can check that

$$\begin{aligned} D_{22}[Z_1 C + Z_2(zI - A)] &= -(\gamma-\alpha)(z+\alpha)\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & \gamma & \alpha^2 \end{pmatrix}\begin{pmatrix} z & 0 & 0 \\ -1 & z & 0 \\ 0 & -1 & z-\gamma \end{pmatrix} \\ &= \begin{pmatrix} (z-\gamma) & (z-\gamma)\alpha & (z-\gamma)\alpha^2 \end{pmatrix} \\ &= D_{22}K, \end{aligned}$$

which is equivalent to (104).

By the Antoulas parametrization (203), the general rational solution of equation (104) is given by

$$Z_1 = -D_{22}^{-1} D_{21} - W D_{11},$$
$$Z_2 = D_{22}^{-1}\Theta_2 + W\Theta_1,$$

which leads to

$$\begin{aligned} Z_1 D_{11}^{-1} &= -D_{22}^{-1} D_{21} D_{11}^{-1} - W \\ &= -(z-\gamma)^{-1}(\gamma-\alpha)(z+\alpha)z^{-2} - W \\ &= -\frac{(\gamma-\alpha)}{z^2} - \frac{(\gamma^2-\alpha^2)}{z^2(z-\gamma)} - W. \end{aligned}$$

For Z_1 to be strictly proper, W has to be a partial realization of the sequence $0, -(z-\gamma)$. So we take $W = -\frac{(\gamma-\alpha)}{z^2}$ and, with this choice, we have

$$Z_1 = -\frac{(\gamma^2-\alpha^2)}{z^2(z-\gamma)}.$$

In the same way

$$\begin{aligned} Z_2 &= D_{22}^{-1}\Theta_2 + W\Theta_1 \\ &= (z-\gamma)^{-1}\begin{pmatrix}1 & \gamma & \alpha^2\end{pmatrix} - \frac{(\gamma-\alpha)}{z^2}\begin{pmatrix}1 & z & 0\end{pmatrix} \\ &= \frac{1}{z^2(z-\gamma)}\begin{pmatrix}z^2-(\gamma-\alpha)(z-\gamma) & \gamma z^2(\gamma-\alpha)z(z-\gamma) & \alpha^2 z^2\end{pmatrix} \end{aligned}$$

So

$$\begin{pmatrix}Z_1 & Z_2\end{pmatrix} = \frac{1}{z^2(z-\gamma)}\Big(-(\gamma^2-\alpha^2)z^2 \quad z^2-(\gamma-\alpha)(z-\gamma) \\ \gamma z^2-(\gamma-\alpha)z(z-\gamma) \quad \alpha^2 z^2\Big).$$

Generally, this is a strictly proper function, having McMillan degree 3. The only case where McMillan degree drops is that of $\gamma = \alpha$. In that case, we obtain

$$\begin{aligned} \begin{pmatrix}Z_1 & Z_2\end{pmatrix} &= \frac{1}{z^2(z-\alpha)}\begin{pmatrix}0 & z^2 & \alpha z^2 & \alpha^2 z^2\end{pmatrix} \\ &= \frac{1}{(z-\alpha)}\begin{pmatrix}0 & 1 & \alpha & \alpha^2\end{pmatrix}, \end{aligned}$$

which is in full agreement with (306) and (310). Note that if the pair (C, A) is detectable and $\gamma = \alpha$ is stable, then the constructed observer is asymptotic.

Next, we seek a functional representation

$$\begin{aligned} Q^{-1}\Pi &= J(zI-F)^{-1} \\ &= \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}z & \alpha\gamma \\ -1 & z-(\alpha+\gamma)\end{pmatrix}^{-1} \\ &= \begin{pmatrix}0 & 1\end{pmatrix}\frac{1}{(z-\alpha)(z-\gamma)}\begin{pmatrix}z-(\alpha+\gamma) & -\alpha\gamma \\ 1 & z\end{pmatrix} \\ &= \frac{1}{(z-\alpha)(z-\gamma)}\begin{pmatrix}1 & z\end{pmatrix} \end{aligned}$$

So $Q = (z-\alpha)(z-\gamma)$, $\Pi = \begin{pmatrix}1 & z\end{pmatrix}$ and $P = \Pi G = \alpha^2(\gamma-\alpha)$. We compute now X, Y, using the factorization (273),

$$\begin{aligned} &\begin{pmatrix}\alpha^2(\gamma-\alpha) & (z-\alpha)(z-\gamma)\end{pmatrix} \\ &= \begin{pmatrix}X(z) & Y(z)\end{pmatrix}\begin{pmatrix}z^2 & 0 \\ -[z(z-\gamma)+\alpha(z-\gamma)+\alpha^2] & z-\gamma\end{pmatrix}. \end{aligned}$$

This leads to $Y = z-\alpha$, $X = \alpha$ and finally $W = D_{22}^{-1}Y^{-1}X = \frac{\alpha}{z(z-\alpha)}$ which agrees with the previous derivation for W.

Example 3 (*A state observer*). Assume the observable system (298) is in dual Brunovsky form and our aim is to construct a state observer. So we have

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$
$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} C \\ K \end{pmatrix} (zI - A)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 \\ z^{-2} & z^{-1} & 0 \\ z^{-3} & z^{-2} & z^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} z^3 & 0 & 0 & 0 \\ -z^2 & 1 & 0 & 0 \\ -z & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z & z^2 \\ 0 & -1 & -z \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and this is a left coprime factorization. Thus we have

$$\Theta_1 = \begin{pmatrix} 1 & z & z^2 \end{pmatrix},$$
$$\Theta_2 = \begin{pmatrix} 0 & -1 & -z \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$D_{11} = z^3,$$
$$D_{21} = -\begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix},$$
$$D_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also

$$P_1 = \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -1 & -z \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

As in a previous example, we have $F = \begin{pmatrix} 0 & 0 & -t_0 \\ 1 & 0 & -t_1 \\ 0 & 1 & -t_2 \end{pmatrix}$ and $J = I$. Since $J(zI - F)^{-1} = I(zI - F)^{-1} = (zI - F)^{-1} I$, it follows that

$$Q(z) = zI - F = \begin{pmatrix} z & 0 & t_0 \\ -1 & z & t_1 \\ 0 & -1 & z + t_2 \end{pmatrix},$$
$$P(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $G = \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}$, $-PG = -\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}$. Now the equation

$$\begin{pmatrix} -P(z)G & Q(z) \end{pmatrix} = \begin{pmatrix} X(z) & Y(z) \end{pmatrix} \begin{pmatrix} D_{11}(z) & 0 \\ D_{21}(z) & D_{22}(z) \end{pmatrix}$$

is clearly solvable with

$$X(z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y(z) = Q(z) = \begin{pmatrix} z & 0 & t_0 \\ -1 & z & t_1 \\ 0 & -1 & z + t_2 \end{pmatrix}. \tag{311}$$

Indeed

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} z^3 - \begin{pmatrix} z & 0 & t_0 \\ -1 & z & t_1 \\ 0 & -1 & z + t_2 \end{pmatrix} \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix} = -\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}.$$

In this case we have

$$\begin{aligned} WD_{11} &= Z_1 - P_1 = (zI - F)^{-1}G - P_1 \\ &= \frac{1}{z^3 + t_2 z^2 + t_1 z + t_0} \\ &\quad \times \begin{pmatrix} z^2 + t_2 z + t_1 & -t_0 & -t_0 z \\ z + t_2 & z^2 + t_2 z & -t_1 z - t_0 \\ 1 & z & z^2 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} - \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix} \\ &= \frac{-z^3}{z^3 + t_2 z^2 + t_1 z + t_0} \begin{pmatrix} z^2 + t_2 z + t_1 \\ z + t_2 \\ 1 \end{pmatrix} \end{aligned}$$

Note that if the polynomial $t(z)$ is stable, then

$$\frac{1}{z^3 + t_2 z^2 + t_1 z + t_0} \begin{pmatrix} z^2 + t_2 z + t_1 \\ z + t_2 \\ 1 \end{pmatrix} \tag{312}$$

parametrizes all minimal, of McMillan degree 3 in this case, stable partial realizations of $z^{-3}\begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix}$.

We digress now to show the connection with the partial realization problem as it appears in [14]. Assuming observability, we have $C(zI - A)^{-1} = D_{11}(z)^{-1}\Theta_1(z)$. Thus $\Theta_1(z)(zI - A) = D_{11}(z)C$ and so $\xi \mapsto \Theta_1\xi$ is a bijective map from $\mathbb{F}^n$ to X_D. Now $K : X_{z^3} \longrightarrow \mathbb{F}^3$ induces a map $\overline{K} \longrightarrow \mathbb{F}^3$ such that $K\xi = \overline{K}\Theta_1\xi$. Since $K = I$ and $\theta_1(z) = \begin{pmatrix} 1 & z & z^2 \end{pmatrix}$, we have, for $f \in X_{z^3}$, $\overline{K} f = \begin{pmatrix} [f, z^{-1}] \\ [f, z^{-2}] \\ [f, z^{-3}] \end{pmatrix}$, i.e.

$$\begin{pmatrix} z^{-1} \\ z^{-2} \\ z^{-3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} z^{-1} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} z^{-2} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z^{-3}$$

So

$$K^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, K^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, K^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and we encounter the same partial realization problem as before for which Eq. (312) gives the parametrization of all minimal solutions.

Going back to the computation of Z_2, we have

$$\begin{aligned} Z_2 - P_2 &= (zI - F)^{-1} - P_2 \\ &= \frac{1}{z^3 + t_2 z^2 + t_1 z + t_0} \\ &\quad \times \begin{pmatrix} z^2 + t_2 z + t_1 & -t_0 & -t_0 z \\ z + t_2 & z^2 + t_2 z & -t_1 z - t_0 \\ 1 & z & z^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & z \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{z^3 + t_2 z^2 + t_1 z + t_0} \\ &\quad \times \begin{pmatrix} z^2 + t_2 z + t_1 & z^3 + t_2 z^2 + t_1 z & z^4 + t_2 z^3 + t_1 z^2 \\ z + t_2 & z^2 + t_2 z & z^3 + t_2 z^2 \\ 1 & z & z^2 \end{pmatrix}. \end{aligned}$$

We can check that

$$-WZ_C = \frac{-z^3}{z^3 + t_2 z^2 + t_1 z + t_0} \begin{pmatrix} z^2 + t_2 z + t_1 \\ z + t_2 \\ 1 \end{pmatrix} \begin{pmatrix} z^{-3} & z^{-2} & z^{-1} \end{pmatrix},$$

i.e. $Z_2 - P_2 = -WZ_C$ holds.

Of course, in this case we can simplify the computations. We have $Z = J = I$ and $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = J(zI - F)^{-1} \begin{pmatrix} G & H \end{pmatrix} = (zI - F)^{-1} \begin{pmatrix} G & I \end{pmatrix}$, so

$$\begin{aligned} Z_1 Z_C + Z_2 &= (zI - F)^{-1} GC(zI - A)^{-1} + (zI - F)^{-1} \\ &= (zI - F)^{-1}(GC + zI - A)(zI - A)^{-1} \\ &= (zI - A)^{-1} = Z_K. \end{aligned}$$

With X, Y given by (311), we compute

$$\begin{aligned} -\overline{S} &= XD_{11} + YD_{21} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} z^3 - \begin{pmatrix} z & 0 & t_0 \\ -1 & z & t_1 \\ 0 & -1 & z + t_2 \end{pmatrix} \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix} = -\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}, \end{aligned}$$

i.e.

$$\overline{S} = \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix}. \tag{313}$$

Similarly,

$$\overline{T} = Y = \begin{pmatrix} z & 0 & t_0 \\ -1 & z & t_1 \\ 0 & -1 & z + t_2 \end{pmatrix}.$$

It is easily checked that $\overline{S}T = \overline{T}S$.

Example 4 (*No observations*). We work out an extreme case where $C = 0$, i.e. there are no observations. Let $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \left(\begin{array}{c|cc} F & G & Z \\ \hline J & 0 & 0 \end{array}\right)$, with the realization being minimal. The equality $Z_K = Z_1 Z_C + Z_2$ implies $Z_2 = Z_K$, and we can take without loss of generality $F = A$, $Z = I$ and $J = K$. The equality $ZA - FZ = GC$ shows that G can be arbitrarily chosen. Thus $\left(\begin{array}{c|c} A & G \\ \hline K & 0 \end{array}\right)$ is a tracking observer. The observer equation reduces to $\sigma\zeta = A\zeta$, i.e. the error dynamics is given, with $\epsilon = x - \zeta$, by $\sigma\epsilon = A\epsilon$, $e = K\epsilon$. This is a tracking observer, but basically it reduces estimation to guesswork.

Alternatively, from the point of view of geometric control, there exists a tracking observer if and only if there exists a conditioned invariant subspace $\mathscr{V} \subset \operatorname{Ker} K$. The assumption that $\left(\begin{pmatrix} 0 \\ K \end{pmatrix}, A\right)$ is observable is equivalent to the observability of the pair (K, A). The fact that $C = 0$ implies that a subspace is conditioned invariant if and only if it is invariant for A. We show that necessarily $\mathscr{V} = 0$. Indeed, if $x \in \mathscr{V}$ then $A^j x \in \mathscr{V} \subset \operatorname{Ker} K$. This implies that for all $j \geqslant 0$, we have $KA^j x = 0$ and, by observability, it follows that $\mathscr{V} = 0$. Thus a minimal order observer has the dimension of the original state space. As before, $ZA - FZ = GC = 0$, the surjectivity of Z and the equality of dimension show that F is necessarily similar to A and, without loss of generality, we can take $F = A$ and $Z = I$.

References

[1] A.C. Antoulas, New results on the algebraic theory of linear systems: the solution of the cover problem, Linear Algebra Appl. 50 (1983) 1–43.

[2] G. Basile, G. Marro, Controlled and conditioned invariant subspaces in linear system theory, J. Optim. Theory Appl. 3 (1973) 306–316.

[3] M. Bisiacco, M.E. Valcher, J.C. Willems, A behavioral approach to estimation and dead-beat observer design with applications to state space models, IEEE Trans. Aut. Contr. 51 (2006) 1787–1797.

[4] J.J. Bongiorno, D.C. Youla, On observers in multi-variable control systems, Int. J. Control 8 (1968) 221–243.

[5] E. Emre, M.L.J. Hautus, A polynomial characterization of (*A*, *B*)-invariant and reachability subspaces, SIAM J. Contr. Optim. 18 (1980) 420–436.

[6] P.A. Fuhrmann, Algebraic system theory: An analyst's point of view, J. Franklin Inst. 301 (1976) 521–540.

[7] P.A. Fuhrmann, On strict system equivalence and similarity, Int. J. Contr. 25 (1977) 5–10.

[8] P.A. Fuhrmann, Duality in polynomial models with some applications to geometric control theory, IEEE Trans. Aut. Contr. AC-26 (1981) 284–295.

[9] P.A. Fuhrmann, Functional models in linear algebra, Linear Algebra Appl. 162–164 (1992) 107–151.

[10] P.A. Fuhrmann, On behavior homomorphisms and system equivalence, Syst. Control Lett. 44 (2001) 127–134.

[11] P.A. Fuhrmann, A study of behaviors, Linear Algebra Appl. 351–352 (2002) 303–380.

[12] P.A. Fuhrmann, On continuous behavior homomorphisms, Syst. Control Lett. 49 (2003) 359–363.

[13] P.A. Fuhrmann, On duality in some problems of geometric control, Acta Appl. Math. 91 (2006) 207–251.

[14] P.A. Fuhrmann, U. Helmke, On the parametrization of conditioned invariant subspaces and observer theory, Linear Algebra Appl. 332–334 (2001) 265–353.

[15] P.A. Fuhrmann, P. Rapisarda, Y. Yamamoto, On the state of behaviors, Linear Algebra Appl. 424 (2007) 570–614.

[16] P.A. Fuhrmann, J. Trumpf, On observability subspaces, Int. J. Contr. 79 (2006) 1157–1195.

[17] P.A. Fuhrmann, J.C. Willems, A study of (*A*,*B*)-invariant subspaces via polynomial models, Int. J. Contr. 31 (1980) 467–494.

[18] T. Kailath, Linear Systems, Prentice Hall, Englewood Cliffs, NJ, 1980.

[19] R.E. Kalman, P. Falb, M. Arbib, Topics in Mathematical System Theory, McGraw Hill, 1969.

[20] M. Kuijper, First order Representations of Linear Systems, Birkhauser Verlag, 1994.

[21] D.G. Luenberger, Observers for multivariable systems, IEEE Trans. Aut. Control AC-11 (1966) 190–197.

[22] J.W. Polderman, Proper elimination of latent variables, Proceedings of 12th world congress IFAC, 1992, pp. 73–76.

[23] P. Rapisarda, J.C. Willems, State maps for linear systems, SIAM J. Contr. Optim. 35 (1997) 1053–1091.

[24] P. Rocha, Structure and representation of 2-D Systems, Ph.D. Thesis, University of Groningen, The Netherlands, 1990.

[25] H.H. Rosenbrock, State-space and Multivariable Theory, John Wiley, New York, 1970.

[26] G.C. Rota, On models for linear operators, Comm. Pure Appl. Math. 13 (1960) 469–472.

[27] J. Trumpf, On the Geometry and Parametrization of Almost Invariant Subspaces and Observer Theory, Ph.D. Thesis, University of Würzburg, Germany, 2002.

[28] M.E. Valcher, J.C. Willems, Observer synthesis in the behavioral framework, IEEE Trans. Aut. Contr. 44 (1999) 2297–2307.

[29] J.C. Willems, From time series to linear systems. Part I: Finite-dimensional linear time invariant systems, Automatica 22 (1986) 561–580.

[30] J.C. Willems, Models for dynamics, in: U. Kirchgraber, H.O. Walther (Eds.), Dynamics Reported, vol. 2, Wiley-Teubner, 1989, pp. 171–269.

[31] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Aut. Contr. AC-36 (1991) 259–294.

[32] J.C. Willems, On interconnections, control and feedback, IEEE Trans. Aut. Contr. AC-42 (1997) 326–337.

[33] W.M. Wonham, Linear Multivariable Control, 2nd ed., Springer Verlag, New York, 1979.

[34] W.M. Wonham, A.S. Morse, Decoupling and pole assignment in linear multivariable systems: a geometric approach, SIAM J. Contr. 8 (1970) 1–18.