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Indecomposable Laplacian integral graphs

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Abstract

A graph that can be constructed from isolated vertices by the operations of union and complement is decomposable. Every decomposable graph is Laplacian integral. i.e., its Laplacian spectrum consists entirely of integers. An indecomposable graph is not decomposable. The main purpose of this note is to demonstrate the existence of infinitely many indecomposable Laplacian integral graphs.

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1. Introduction

If $G = (V, E)$ and $H = (W, F)$ are graphs on disjoint sets of vertices, their union is the graph $G \oplus H = (V \cup W, E \cup F)$. The complement of G is the graph $G^c = (V, V^{(2)} \backslash E)$, i.e., the graph with vertex set $V = V(G)$ such that vertices u and v are adjacent in G^c if and only if they are not adjacent in G. If the order of G is $\mathrm{o}(V) = n$, the Laplacian of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix. Denote the

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Laplacian spectrum of G by $s(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n = 0$. Graph G is Laplacian integral if $s(G)$ consists entirely of integers. Because $L(G) + L(G^c) = nI_n - J_n$, where I_n is the identity matrix and J_n the n-by-n matrix each of whose entries is 1

$$s(G^c) = (n - \lambda_{n-1}, \ldots, n - \lambda_2, n - \lambda_1, 0). \tag{1}$$

It follows that G is Laplacian integral if and only if G^c is Laplacian integral.

Graphs that can be constructed from isolated vertices by means of unions and complements are called decomposable by some, and cographs by others. Because $L(G \oplus H)$ is the (matrix) direct sum of $L(G)$ and $L(H)$, it follows from Eq. (1) that decomposable graphs are Laplacian integral. Indeed, of the 57 connected, Laplacian integral graphs on $n \leqslant 6$ vertices, only two complementary pairs (four graphs) on six vertices are indecomposable (not decomposable). The relative scarcity of indecomposable Laplacian integral graphs of small order, coupled with their conspicuous absence from the Laplacian literature (see, e.g. [9–12]) led the second author, in his June 2007 course for the Lisbon Summer School on Algebra and Combinatorics, to wonder about the existence of infinite families of indecomposable Laplacian integral graphs. In fact, such families are not hard to find.

2. Decomposable graphs

A graph is decomposable if and only if its complement is decomposable. Moreover, with the exception of K_1 (the one-vertex complete graph) , the complement of a connected decomposable graph is necessarily disconnected. Thus, apart from K_1, no decomposable graph is self-complementary (isomorphic to its complement). Can a self-complementary graph be Laplacian integral? Yes, and then some!

Graphs with the same Laplacian spectrum are said to be isospectral. We are indebted to Haemers and Spence [6,7] for sharing the nonisomorphic, self-complementary graphs G_1 and G_2 in Fig. 1. Because $s(G_1) = (7, 6^3, 3^3, 2, 0) = s(G_2)$, with superscripts indicating eigenvalue multiplicities, these graphs are not only Laplacian integral, but isospectral.

Perhaps the nicest characterization is that a graph is decomposable if and only if it does not contain an induced subgraph isomorphic to the four-vertex path P_4 [15, p. 184]. Using this criterion, the bipartite graph G_3 in Fig. 2 is easily seen to be indecomposable. Because

$$s(G_3) = (6, 5, 4^2, 3^2, 2^2, 1, 0), \tag{2}$$

it is also Laplacian integral.

The join of graphs G and H is defined by

$$GVH = (G^c \oplus H^c)^c.$$

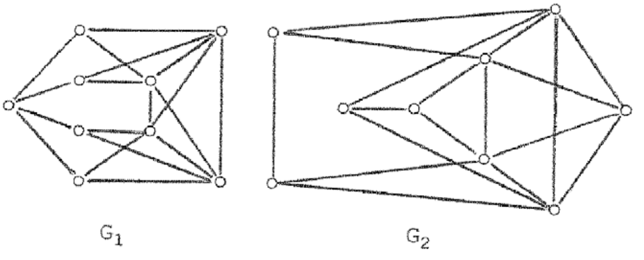

Fig. 1. Isospectral, self-complementary, Laplacian integral graphs.

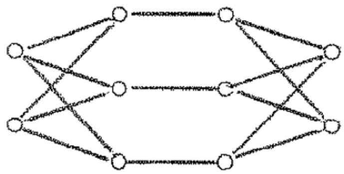

Fig. 2. The graph G_3.

An alternative definition of decomposable graphs is that they can be constructed from isolated vertices by means of unions and joins. Thus, e.g., the complete bipartite graph $K_{r,s} = K_r^c V K_s^c = (K_r \oplus K_s)^c$. Recipes for constructing decomposable graphs can be simpler to express using join notation, e.g., in the case of $G_4 = [(K_1 \oplus K_1)V(K_1 \oplus K_{1,2})] \oplus K_2 \oplus [K_1 V(K_1 \oplus K_2)]$. It follows from Eqs. (1) and (2) that

$$s(G_4) = (6, 5, 4^2, 3^2, 2^2, 1, 0^3) = s(G_3 \oplus K_2^c).$$

Because G_4 is decomposable but not bipartite, while $G_3 \oplus K_2^c$ is bipartite but not decomposable, we obtain the following.

Proposition 2.1. *Neither decomposability nor chromatic number can be determined from the Laplacian spectrum alone.*

A similar example can be found in [13].

3. Infinite families of indecomposable, Laplacian integral graphs

Let $B = J_r \oplus [J_{r+1} - I_{r+1}]$ where, recall, J_r is the r-by-r matrix each of whose entries is 1, and I_n is the n-by-n identity matrix. If $C = J_{2r+1} - B$, then

$$A = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$$

is the adjacency matrix of an $(r + 1)$-regular bipartite graph H_r with Laplacian matrix $L(H_r) = (r + 1)I_{4r+2} - A$ and spectrum $s(H_r) = (2r + 2, 2r + 1, [r + 2]^r, [r + 1]^{2r-2}, r^r, 1, 0)$. For example, $H_1 \cong C_6$ and $s(H_1) = (4, 3^2, 1^2, 0)$; $H_2 \cong G_3$, the graph illustrated in Fig. 2 whose spectrum is given in Eq. (2). The general H_r, illustrated in Fig. 3, might be described as two copies of the complete bipartite graph $K_{r,r+1}$ joined together with an $(r + 1)$-matching of the larger parts.

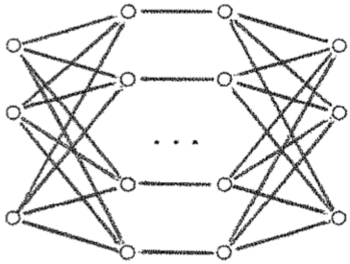

Fig. 3. The graph H_r.

Theorem 3.1. *$\{H_r : r \geqslant 1\}$ is an infinite family of indecomposable, $(r+1)$-regular, bipartite, Lapiacian integral graphs.*

Proof. The result follows from the previous discussion and the abundance of induced P_4's visible in Fig. 3. □

Corollary 3.2. *$\{H_r^c : r \geqslant 1\}$ is an infinite family of nonbipartite, 3r-regular, indecomposable, Laplacian integral graphs.*

Corollary 3.3. *$\{H_r : r \geqslant 1\}$ and $\{H_r^c : r \geqslant 1\}$ are infinite families of bipartite regular and regular adjacency integral graphs, respectively.*

Proof. The adjacency spectrum of an r-regular graph is a translation (by r) of its Laplacian spectrum. □

The Cartesian product of graphs (elsewhere called "product" [8, p. 22], or even "sum" [4, p. 65], [2]) is defined as follows: If $G = (V, E)$ and $H = (W, F)$ are graphs on n and k vertices, respectively, then $G \square H$ is the graph with vertex set $V \times W$ such that (v_1, w_1) is adjacent to (v_2, w_2) if and only if (i) $v_1 = v_2$ and $w_1 w_2 \in F$, or (ii) $v_1 v_2 \in E$ and $w_1 = w_2$. The graph $K_3 \square K_2$, isomorphic to the complement of C_6, is shown in Fig. 4.

If $A = (a_{ij})$ is a k-by-k matrix, and $B = (b_{ij})$ is n-by-n, the Kronecker product $A \otimes B$ is the kn-by-kn partitioned matrix with (i, j)-block equal to $a_{ij}B$, $1 \leqslant i, j \leqslant k$ [14, p.138].

If $s(G) = (\lambda_1, \lambda_2, \ldots \lambda_n)$, and $s(H) = (\mu_1, \mu_2, \ldots \mu_k)$ then, because $L(G \square H) = I_k \otimes L(G) + L(H) \otimes I_n$, it follows that the eigenvalues of $L(G \square H)$ are $\lambda_i + \mu_j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$ [4, p. 70]; [14, p. 149]. Thus, from $s(K_3) = (3^2, 0)$ and $s(K_2) = (2, 0)$, it follows that $s(K_3 \square K_2) = (5^2, 3^2, 2, 0)$. More generally,

$$s(K_n \square K_2) = ([n+2]^{n-1}, n^{n-1}, 2, 0). \tag{3}$$

Lemma 3.4. *If G is a connected graph on $n \geqslant 3$ vertices, then $G \square K_2$ is indecomposable.*

Proof. Suppose $G = (V, E)$, where $V = \{v_1 v_2, \ldots, v_n\}$. Let $H = (W, F)$ be isomorphic to G and let $f : V \to W$ be an isomorphism. Then $G \square K_2 \cong (V \cup W, E \cup F \cup \{v_i f(v_i) : 1 \leqslant i \leqslant n\})$, i.e., $G \square K_2$ is the graph obtained from $G \oplus H$ by adding new edges joining v_i and $f(v_i)$, $1 \leqslant i \leqslant n$.

It remains to observe that if x and y are neighbors of some vertex z of G, then the subgraph of $G \square K_2$ induced on $\{x, z, f(z), f(y)\}$ is isomorphic to P_4. □

Theorem 3.5. *If G is a connected Lapiacian integral graph on $n \geqslant 3$ vertices, then $G \square K_2$ is an indecomposable Lapiacian integral graph.*

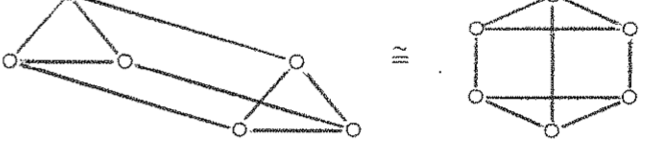

Fig. 4. $K_3 \square K_2 \cong C_6^c$.

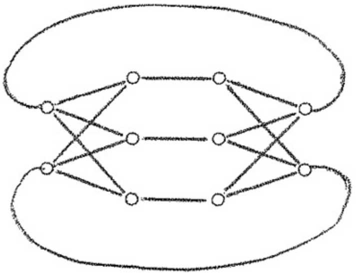

Fig. 5. $G_5 = K_{2,3} \square K_2$.

Proof. Indecomposability follows from the Lemma.

If $s(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ then, $s(G \square K_2)$ consists of the elements of the multiset $\{\lambda_i + 2 : 1 \leqslant i \leqslant n\} \cup \{\lambda_i : 1 \leqslant i \leqslant n\}$ arranged in nonincreasing order. $\square$

Corollary 3.6. *If G is a connected decomposable graph on $n \geqslant 3$ vertices, then $G \square K_2$ is a connected indecomposable Lapiacian integral graph.*

Proof. Decomposable graphs are Lapiacian integral. $\square$

4. Concluding remarks

Consider the graph $G_5 = K_{2,3} \square K_2$ illustrated in Fig. 5. Because $K_{2,3} \cong (K_2 \oplus K_3)^c$ it follows from Eq. (1) that $s(K_{2,3}) = (5, 3, 2^2, 0)$, hence

$$s(G_5) = (7, 5^2, 4^2, 3, 2^3, 0).$$

Comparing with G_3 from Fig. 2 and $s(G_3) = (6, 5, 4^2, 3^2, 2^2, 1, 0)$ from Eq. (2), and following Kirkland [9,10], it is natural to wonder whether the intermediate graph H (obtained by deleting one of the looping edges in Fig. 5) is Laplacian integral. Computations show that, to two decimal places, $s(H) = (6.65, 5, 4^3, 3, 2^2, 1.35, 0)$.

The Cartesian product $K_3 \square K_2$ and its complement, C_6, account for one pair of indecomposable Lapiacian integral graphs on $n = 6$ vertices; $K_{1,2} \square K_2$ and its complement account for the other.

The graphs G_1 and G_2 in Fig. 1 are neither decomposable nor "factorable" as Cartesian products of smaller graphs. On the other hand, $K_3 \square K_3$ is a 4-regular, self-complementary, Laplacian integral graph on nine vertices, one of two such graphs given in [1].

Finally, $G_4^c \square K_2$ and $(G_3 \oplus K_2^c)^c \square K_2$ are isospectral, indecomposable Laplacian integral graphs.

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