An implicit degree condition for long cycles in 2-connected graphs

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Abstract

Let id(v) denote the implicit degree of a vertex v. In this work we prove that: If G is a 2-connected graph with max{id(u), id(v)} ≥ c/2 for each pair of nonadjacent vertices u and v that are vertices of an induced claw or an induced modified claw of G, then G contains either a Hamilton cycle or a cycle of length at least c. This extends several previous results on the existence of long cycles in graphs.

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1. Introduction

Let G be a simple graph with vertex set V and edge set E. For a vertex v ∈ V, its neighborhood, denoted by N(v), and its degree, denoted by d(v), are defined as the set and the number of vertices in G that are adjacent to v, respectively. In the past few decades, there have been many results obtained on the existence of Hamilton cycles and long cycles in graphs in terms of degrees of vertices. Among them, the following is well known.

**Theorem 1** (Bermond [2], Linial [6], Pósa [7]). Let G be a 2-connected graph such that d(u) + d(v) ≥ c for each pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of length at least c.

It was proved by Fan [5] that the condition of Theorem 1 can be weakened as follows.

**Theorem 2** (Fan [5]). Let G be a 2-connected graph such that max{d(u), d(v)} ≥ c/2 for each pair of vertices u and v at distance 2. Then G contains either a Hamilton cycle or a cycle of length at least c.

In [1], Bedrossian et al. gave a further generalization of Fan’s theorem. They imposed one more restriction on the pair of vertices u and v: they must be vertices of an induced claw (the bipartite graph K_{1,3}) or an induced modified claw (K_{1,3} plus an edge).

**Theorem 3** (Bedrossian et al. [1]). Let G be a 2-connected graph such that max{d(u), d(v)} ≥ c/2 for each pair of nonadjacent vertices u and v that are vertices of an induced claw or an induced modified claw of G. Then G contains either a Hamilton cycle or a cycle of length at least c.

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For a vertex \( v \) in graph \( G \), we use \( N_2(v) \) to denote the set of vertices in \( G \) which are at distance 2 from \( v \). In [8], the authors introduced the concept of implicit degrees of vertices.

**Definition 1 (Zhu et al. [8]).** Let \( v \) be a vertex of a graph \( G \). If \( N_2(v) \neq \emptyset \) and \( d(v) \geq 2 \), then set \( k = d(v) - 1 \), \( m_2 = \min\{d(u) | u \in N_2(v)\} \) and \( M_2 = \max\{d(u) | u \in N_2(v)\} \). Suppose \( d_1 \leq d_2 \leq \cdots \leq d_{k+2} \leq \cdots \) is the degree sequence of the vertices of \( N(v) \cup N_2(v) \). Let

\[
d^*(v) = \begin{cases} 
m_2, & \text{if } d_k < m_2; \\
d_{k+1}, & \text{if } d_{k+1} > M_2; \\
d_k, & \text{if } d_k \geq m_2 \text{ and } d_{k+1} \leq M_2.
\end{cases}
\]

Then the implicit degree of \( v \), denoted as \( id(v) \), is defined as \( id(v) = \max\{d(v), d^*(v)\} \). If \( N_2(v) = \emptyset \) or \( d(v) \leq 1 \), then we define \( id(v) = d(v) \).

From the definition of implicit degrees, it is clear that \( id(v) \geq d(v) \) for every vertex \( v \). Zhu et al. [8] gave a generalization of Theorem 1.

**Theorem 4 (Zhu et al. [8]).** Let \( G \) be a 2-connected graph such that \( id(u) + id(v) \geq c \) for each pair of nonadjacent vertices \( u \) and \( v \) in \( G \). Then \( G \) contains either a Hamilton cycle or a cycle of length at least \( c \).

This theorem was extended by Chen [4] as follows.

**Theorem 5 (Chen [4]).** Let \( G \) be a 2-connected graph such that \( \max\{id(u), id(v)\} \geq c/2 \) for each pair of vertices \( u \) and \( v \) at distance 2. Then \( G \) contains either a Hamilton cycle or a cycle of length at least \( c \).

In this work we show that, for this result, we can also impose a restriction on the pair of vertices \( u \) and \( v \) as in Theorem 3. Our result is a common generalization of Theorems 3 and 5.

**Theorem 6.** Let \( G \) be a 2-connected graph such that \( \max\{id(u), id(v)\} \geq c/2 \) for each pair of nonadjacent vertices \( u \) and \( v \) that are vertices of an induced claw or an induced modified claw of \( G \). Then \( G \) contains either a Hamilton cycle or a cycle of length at least \( c \).

An \((x, y)\)-path is a path connecting two vertices \( x \) and \( y \). A \( y \)-path is a path which has \( y \) as one of its end-vertices. If a longest path has \( y \) as one of its end-vertices, then we call this path a \( y \)-longest path. Our proof of Theorem 6 is based on the following two lemmas.

**Lemma 1 (Bondy [3]).** Let \( G \) be a non-hamiltonian 2-connected graph and \( P = v_1v_2\cdots v_p \) be a longest path in \( G \). Then \( G \) contains a cycle of length at least \( d(v_1) + d(v_p) \).

**Lemma 2.** Let \( G \) be a non-hamiltonian 2-connected graph satisfying the condition of Theorem 6. Suppose that there is a \( y \)-longest path in \( G \). Then there exists a \( y \)-longest path such that the other end-vertex of the path has degree at least \( c/2 \).

We postpone the proof of Lemma 2 to next section.

**Proof of Theorem 6.** Suppose that \( G \) does not contain a Hamilton cycle. By using Lemma 2 twice, we obtain a longest path with both end-vertices having degree at least \( c/2 \). Then by Lemma 1, we can find a cycle of length at least \( c \).

2. **Proof of Lemma 2**

Let \( P = v_1v_2\cdots v_p \) be a path of a graph \( G \). In the following we let \( N^-(v_1) = \{v_i | v_iv_{i+1} \in E(G)\} \), and let \( k(P) \) denote the maximum index \( i \) with \( v_iv_i \in E(G) \). To prove Lemma 2, we need the following two lemmas.

**Lemma 3 (Zhu et al. [8]).** Let \( G \) be a 2-connected graph and let \( P = v_1v_2\cdots v_p \), with \( v_1 = x \) and \( v_p = y \), be a longest path of \( G \). If \( d(x) < id(x) \) and \( v_1v_p \notin E(G) \), then either

1. there is some \( v_j \in N^-(x) \) such that \( d(v_j) \geq id(x) \); or
2. \( N(x) = \{v_2, v_3, \ldots, v_{d(x)}\} \) and \( id(x) = \min\{d(u) | u \in N_2(x)\} \).
Lemma 4. Let $G$ be a non-hamiltonian 2-connected graph satisfying the condition of Theorem 6. Suppose that there is a $y$-longest path in $G$. Then there exists a $y$-longest path such that the other end-vertex has implicit degree at least $c/2$.

Proof. Let $P = v_1v_2 \cdots v_p$ ($v_p = y$) be a $y$-longest path. Assume $P$ was chosen among all $y$-longest paths such that $k(P)$ is as large as possible.

Suppose $id(v_1) < c/2$. From the choice of $P$, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$, and there exists no cycle of length $p$. Since $G$ is 2-connected, $v_1$ is adjacent to at least one vertex on $P$ other than $v_2$. So, $3 \leq k(P) < p$. Now let $k = k(P)$.

Case 1. $N(v_1) = \{v_2, \ldots, v_k\}$.

Since $P$ is a longest path, $N(v_i) \subseteq V(P)$ for $i = 2, \ldots, k - 1$. By the fact that $G$ is non-hamiltonian and 2-connected, we have $k + 2 \leq p$. Furthermore, since $G - v_k$ is connected, there must be an edge $v_jv_s \in E(G)$ with $j < k < s$.

We assume that such an edge $v_jv_s$ was chosen so that $s$ is as large as possible. Clearly we have $s \leq p - 1$. Now $P' = v_jv_{j-1} \cdots v_jv_{j+1}v_{j+2} \cdots v_p$ is a $y$-longest path with $k(P') = s > k = k(P)$, a contradiction.

Case 2. $N(v_1) \neq \{v_2, \ldots, v_k\}$.

Choose $v_r \notin N(v_1)$ with $r < k$ such that $r$ is as large as possible. Then $v_1v_i \in E(G)$ for every $i$ with $r < i \leq k$. Let $j$ be the smallest index such that $j > r$ and $v_j \notin N(v_1) \cap N(v_r)$. Since $v_{r+1} \in N(v_1) \cap N(v_r)$, we have $j \geq r + 2$.

On the other hand, it is obvious that $j \leq k + 1$.

By the choice of $v_r, v_j \notin E(G)$. Then it follows from the choice of $v_j$ that $\{v_1, v_r, v_{j-1}, v_j, \}$ induces a claw or a modified claw. Since $id(v_1) < c/2$, we have $id(v_r) \geq c/2$. Now $P'' = v_rv_{r-1} \cdots v_1v_{r+1}v_{r+2} \cdots v_p$ is a $y$-longest path with $id(v_r) \geq c/2$. \hfill $\square$

Proof of Lemma 2. By contradiction. Suppose that for any $y$-longest path $P$, the end-vertex of $P$ other than $y$ has degree less than $c/2$.

From Lemma 4, we can choose a longest path $P = v_1v_2 \cdots v_p$ ($v_p = y$) in $G$ such that $id(v_1) \geq c/2$. We assume that $P$ was chosen such that $k(P) = k$ is as large as possible.

If there is some $v_j \in N^-(v_1)$ such that $d(v_j) \geq id(v_1)$, then $P' = v_jv_{j-1} \cdots v_1v_{j+1}v_{j+2} \cdots v_p$ is a $y$-longest path with $d(v_j) \geq c/2$. Thus we may restrict our attention to the case $d(v_j) \leq id(v_1)$ for every vertex $v_j \in N^-(v_1)$.

Since $G$ is non-hamiltonian and 2-connected, from Lemma 3 we have $N(x) = \{v_2, v_3, \ldots, v_k\}$ and $d(u) \geq c/2$ for every vertex $u \in N_2(v_1)$.

Since $G - v_k$ is connected, there must be an edge $v_rv_s \in E(G)$ with $r < k < s$. We assume such an edge $v_rv_s$ was chosen so that

(i) $s$ is as large as possible;

(ii) $r$ is as large as possible, subject to (i).

Clearly we have $s \leq p - 1$.

Case 1. $s \geq k + 2$.

Now $P' = v_rv_{r-1} \cdots v_1v_{r+1}v_{r+2} \cdots v_p$ is a $y$-longest path different from $P$ and $k(P') = k = k(P)$. At the same time, $P'' = v_{s-1}v_{s-2} \cdots v_{s+1}v_{s+2} \cdots v_p$ is a $y$-longest path different from $P$ and $k(P'') = k = k(P)$.

So we get two longest paths, one with $v_r$ and $v_p$ as its end-vertices, and the other with $v_{s-1}$ and $v_p$ as its end-vertices.

Claim 1. $v_rv_{s-1} \in E(G)$.

Proof. Suppose $v_rv_{s-1} \notin E(G)$. Then $v_r$ and $v_{s-1}$ are nonadjacent vertices in a claw or a modified claw induced by $\{v_r, v_{s-1}, v_s, v_{s+1}\}$. So max$\{id(v_r), id(v_{s-1})\} \geq c/2$, contradicting the choice of $P$. \hfill $\square$

From Claim 1, we get $d(v_1, v_{s-1}) = 2$. So $d(v_{s-1}) \geq c/2$ and $id(v_{s-1}) \geq d(v_{s-1}) \geq c/2$, contradicting the choice of $P$.

Case 2. $s = k + 1$.

By the choice of $P$, $N(v_k) \subseteq V(P)$. Since $G - v_{k+1}$ is connected, there exists an edge $v_kv_{t} \in E(G)$ with $k + 2 \leq t \leq p - 1$. Choose $v_kv_t$ such that $t$ is as small as possible. Now $d(v_1, v_t) = 2$ and $d(v_t) \geq c/2$. 

Claim 2. \( t \geq k + 3 \).

**Proof.** Suppose \( t = k + 2 \). Then \( P' = v_{k+1}v_r v_{r-1} \cdots v_1 v_{r+1}v_{r+2} \cdots v_k v_{k+2} \cdots v_p \) is a \( y \)-longest path different from \( P \). Now \( id(v_{k+1}) \geq d(v_{k+1}) \geq c/2 \) and \( k(P') > k = k(P) \), a contradiction. \( \Box \)

Suppose \( v_kv_{t+1} \not\in E(G) \). By the choice of \( v_t \), \( v_kv_{t-1} \not\in E(G) \). Now \( \{v_k, v_{t-1}, v_t, v_{t+1}\} \) induces a claw or a modified claw. So \( \max\{id(v_k), id(v_{t-1})\} \geq c/2 \). At the same time, both \( P' = v_kv_{k-1} \cdots v_{r+1}v_1 \cdots v_r v_{k+1} \cdots v_p \) and \( P'' = v_{t-1}v_{t-2} \cdots v_{k+1}v_r v_{r-1} \cdots v_1 v_{r+1} \cdots v_k v_t v_{t+1} \cdots v_p \) are \( y \)-longest paths. Note that \( k(P') > k = k(P) \) and \( k(P'') > k = k(P) \), contradicting the choice of \( P \).

Suppose \( v_kv_{t+1} \in E(G) \). Then \( P' = v_{t+1}v_{t-1} \cdots v_{k+1}v_r v_{r-1} \cdots v_1 v_{r+1} \cdots v_k v_{t+1}v_{t+2} \cdots v_p \) is a \( y \)-longest path different from \( P \). Now \( d(v_1, v_t) = 2 \), so \( id(v_t) \geq d(v_t) \geq c/2 \). Moreover, \( k(P') > k = k(P) \), a contradiction to the choice of \( P \).

This completes the proof of **Lemma 2**.

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**References**