JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 66, 60–81 (1978)

The Riemann Mapping Theorem

LOUIS DE BRANGES

Department of Mathematics, Purdue University, Lafayette, Indiana 47907

Submitted by Ky Fan

A theory of bounded polynomial approximation is obtained on the theme of the Riemann mapping theorem.

Bounded nonnegative measures, with compact support, on the Borel subsets of the complex plane are considered in the weak topology induced by the continuous functions. Two such measures μ and ν are said to be equivalent if the identity

$$\int f(z)\, d\mu(z) = \int f(z)\, d\nu(z)$$

holds for every polynomial $f(z)$. If μ is any such measure, the closure of the set of such measures which are absolutely continuous with respect to μ and which are equivalent to μ is a compact convex set. By the Krein–Milman theorem [1], the set is the closed convex span of its extreme points. A characterization of extremal measures is an underlying principle in connection with the Stone–Weierstrass theorem [2]. The element μ is an extreme point if, and only if, the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are dense in $L^1(\mu)$. Of particular interest are extremal measures for which these same functions are dense in $L^\infty(\mu)$ in its weak topology induced by $L^1(\mu)$. In this case μ is the only element of the convex set.

If μ and ν are bounded nonnegative measures with compact support, define μ to be less than or equal to ν if μ is equivalent to ν and if the inequality

$$\int \log |f(z)|\, d\mu(z) \leqslant \int \log |f(z)|\, d\nu(z)$$

holds for every polynomial $f(z)$.

THEOREM 1. *If μ and ν are bounded nonnegative measures with compact support, then μ is less than or equal to ν if, and only if, the inequality*

$$\int h(z)\, d\mu(z) \leqslant \int h(z)\, d\nu(z)$$

holds for every continuous function $h(z)$ of z which is subharmonic in the complex plane. The inequality then holds for every function $h(z)$ which is subhar-

0022-247X/78/0661–0060$02.00/0

monic in the complex plane, and the least upper bound of $h(z)$ with respect to μ is less than or equal to the least upper bound of $h(z)$ with respect to ν. A weakly continuous homomorphism exists of the weak closure of the polynomials in $L^\infty(\nu)$ into the weak closure of the polynomials in $L^\infty(\mu)$ which is the identity on polynomials. The transformation is bounded by one.

An acquaintance with the theory [3] of square summable power series is presumed. Define $\mathscr{C}(z)$ to be the set of power series $f(z) = \sum a_n z^n$ with complex coefficients such that

$$\|f\|^2 = \sum |a_n|^2 < \infty.$$

Then $\mathscr{C}(z)$ is a Hilbert space which contains the polynomials as a dense vector subspace. The elements of the space are convergent power series in the unit disk. The analytic function represented by an element of the space has a measurable boundary function on the unit circle. The identity

$$\|f\|^2 = \int_0^1 |f(\exp(2\pi it))|^2 \, dt$$

holds for every element $f(z)$ of the space. A characterization [4] of the space is used.

THEOREM 2. *Let μ be a bounded nonnegative measure, with compact support not equal to the set containing only the point λ, such that the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is weakly continuous in $L^\infty(\mu)$. Assume that the weak closure of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ contains all of $L^\infty(\mu)$ and that no nonconstant real element of $L^\infty(\mu)$ belongs to the weak closure of the polynomials. Then a function $\psi(z)$ of z exists, which is bounded and analytic in the unit disk, which has distinct values at distinct points of the disk, and which has value λ at the origin, such that the transformation $f(z)$ into $f(\psi(z))$ is an isometry of the weak closure of the polynomials in $L^\infty(\mu)$ onto the space of functions which are bounded and analytic in the unit disk. An element $W(z)$ of $\mathscr{C}(z)$ exists such that the transformation $f(z)$ into $W(z)f(\psi(z))$ is an isometry of the closure of the polynomials in $L^2(\mu)$ onto $\mathscr{C}(z)$. The support of μ is the boundary of the region onto which ψ maps the unit disk. The μ-measure of every Borel set S is equal to*

$$\int |W(\exp(2\pi it))|^2 \, dt$$

with integration over the real numbers t modulo one such that $\psi(\exp(2\pi it))$ belongs to S.

An elementary approximation theorem is obtained by methods taken from the theory of approximation on a line [5].

THEOREM 3. *If μ is a bounded nonnegative measure with compact support and if the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ for no complex number w, then the polynomials are weakly dense in $L^\infty(\mu)$.*

An orthogonal decomposition is used to study more general measures.

THEOREM 4. *Let μ be a bounded nonnegative measure with compact support such that the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is weakly continuous in $L^\infty(\mu)$ for a complex number λ. Assume that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$. Then $\mu = \alpha + \beta$ for mutually singular nonnegative measures α and β with these properties: An element of $L^\infty(\mu)$ belongs to the weak closure of the polynomials if, and only if, it belongs to the weak closure of the polynomials in $L^\infty(\alpha)$ and in $L^\infty(\beta)$. No nonconstant real element of $L^\infty(\alpha)$ belongs to the weak closure of the polynomials. The linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is weakly continuous in $L^\infty(\alpha)$. The linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\alpha)$ and in $L^\infty(\beta)$ for no complex number w.*

A structure theory follows for measures satisfying the density condition.

THEOREM 5. *Let μ be a bounded nonnegative measure with compact support such that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$. Then a countable number of mutually singular nonnegative measures μ_n exist with these properties: The measure $\mu - \sum \mu_n$ is singular with respect to μ_n for every index n. An element of $L^\infty(\mu)$ belongs to the weak closure of the polynomials if, and only if, it belongs to the weak closure of the polynomials in $L^\infty(\mu_n)$ for every index n. For every index n, the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu_n)$ is a bounded simply connected region Ω_n of zero μ-measure. The regions Ω_n are disjoint.*

A formulation of the balayage principle [4] is used.

THEOREM 6. *Let σ be a bounded nonnegative measure with compact support such that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\sigma)$ and no nonconstant real element of $L^\infty(\sigma)$ belongs to the weak closure of the polynomials. Assume that the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\sigma)$ is a bounded simply connected region Ω. Let Ω^* be the set onto which a Riemann mapping function ψ for Ω maps the closure of the unit disk. If α is a bounded nonnegative measure and if the complement of Ω^* has zero α-measure, then a unique bounded nonnegative measure β exists which is absolutely continuous with respect to σ and which is equivalent to α, and α is less than or equal to β.*

A computation of inequalities [4] holds for measures satisfying the density condition.

THEOREM 7. *Let μ and ν be bounded nonnegative measures with compact support. Assume that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\nu)$, that no nonconstant real element of $L^\infty(\nu)$ belongs to the weak closure of the polynomials, and that the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\nu)$ is a bounded simply connected region Ω. Let Ω^* be the set onto which a Riemann mapping function ψ for Ω maps the closure of the unit disk. Then μ is less than or equal to ν if, and only if, μ is equivalent to ν and the complement of Ω^* has zero μ-measure.*

Nontrivial inequalities do not occur in the presence of a density condition on polynomials.

THEOREM 8. *If μ and ν are bounded nonnegative measures with compact support such that μ is less than or equal to ν and if the polynomials are weakly dense in $L^\infty(\nu)$, then μ and ν are equal.*

Inequalities are well-behaved in orthogonal decompositions.

THEOREM 9. *Let μ and ν be bounded nonnegative measures with compact support. Assume that $\nu = \sum \nu_n$ for mutually singular nonnegative measures ν_n such that an element of $L^\infty(\nu)$ belongs to the weak closure of the polynomials if, and only if, it belongs to the weak closure of the polynomials in $L^\infty(\nu_n)$ for every index n. Then μ is less than or equal to ν if, and only if, $\mu = \sum \mu_n$ for mutually singular nonnegative measures μ_n such that μ_n is less than or equal to ν_n for every index n.*

Some consequences of inequalities follow.

THEOREM 10. *Let μ and ν be bounded nonnegative measures with compact support such that μ is less than or equal to ν. Then a weakly continuous transformation exists of the weak closure in $L^\infty(\nu)$ of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ into the weak closure in $L^\infty(\mu)$ of these functions which is the identity on these functions. The transformation is bounded by one.*

A uniqueness theorem is also noted.

THEOREM 11. *Let μ and ν be bounded nonnegative measures with compact support such that μ is less than or equal to ν and such that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\nu)$. Then μ and ν are equal if the interior of the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\nu)$ has zero μ-measure.*

A semi-lattice structure is present in the partial ordering of measures.

THEOREM 12. *If μ and ν are equivalent bounded nonnegative measures with compact support, then a least bounded nonnegative measure σ with compact support exists which is greater than or equal to μ and to ν.*

The measures satisfying the density condition are closed under decreasing limits.

THEOREM 13. *For each positive integer n, let μ_n be a bounded nonnegative measure with compact support such that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu_n)$. Assume that μ_{n+1} is less than or equal to μ_n for every index n. Then a greatest bounded nonnegative measure μ with compact support exists which is less than or equal to μ_n for every index n. The functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$.*

An example is constructed of a measure for which the density condition is satisfied.

THEOREM 14. *Let μ be a bounded nonnegative measure which is supported in the closure of the unit disk and which is less than or equal to some bounded nonnegative measure which is absolutely continuous with respect to Lebesgue measure on the unit circle. Let $h(z)$ be a function which is bounded and analytic in the unit disk but which is not bounded by one. Let σ be the bounded nonnegative measure defined on a Borel set S by taking $\sigma(S)$ equal to the μ-measure of the set of points w in the closure of the unit disk such that $h(w)$ belongs to S. If σ is absolutely continuous with respect to Lebesgue measure on the unit circle, then the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$.*

A domination principle follows.

THEOREM 15. *If μ is a bounded nonnegative measure with compact support, then a minimal bounded nonnegative measure σ with compact support exists, which is greater than or equal to μ, such that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\sigma)$. A weakly continuous isomorphism exists of the weak closure of the polynomials in $L^\infty(\sigma)$ onto the weak closure of the polynomials in $L^\infty(\mu)$ which takes every polynomial into itself. The transformation is an isometry for the norm metrics and a homeomorphism for the weak topologies. The linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ for a complex number w if, and only if, it is weakly continuous in $L^\infty(\sigma)$.*

The minimal measure is unique. Its construction is placed in the context of the Stone–Weierstrass theorem.

THEOREM 16. *If μ is a bounded nonnegative measure with compact support, then the closure of the set of bounded nonnegative measures which are absolutely* *continuous with respect to μ and which are equivalent to μ contains a greatest element σ. The functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\sigma)$.*

The domination principle is also well-behaved with respect to harmonic functions.

THEOREM 17. *Let μ be a bounded nonnegative measure with compact support and let σ be the greatest element in the closure of the set of bounded nonnegative measures which are absolutely continuous with respect to μ and which are equivalent to μ. A weakly continuous transformation exists of the weak closure in $L^\infty(\sigma)$ of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ into the weak closure in $L^\infty(\mu)$ of these functions which is the identity on these functions. The transformation is an isometry for the norm metrics and a homeomorphism for the weak topologies.*

An approximation theorem follows from the domination principle.

THEOREM 18. *If μ is a bounded nonnegative measure with compact support, then the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$ if, and only if, no interior point of the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ belongs to the support of μ.*

Another approximation theorem follows.

THEOREM 19. *Assume that μ is a bounded nonnegative measure with compact support such that no nonconstant real element of $L^\infty(\mu)$ belongs to the weak closure of the polynomials. Let $h(z)$ be a nonconstant element of the weak closure of the polynomials in $L^\infty(\mu)$ and let ν be the bounded nonnegative measure defined for every Borel set S by taking $\nu(S)$ to be the μ-measure of the set of complex numbers w such that $h(w)$ belongs to S. Then the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$ if they are weakly dense in $L^\infty(\nu)$.*

The domination principle also has consequences for the ideal structure of the weak closure of the polynomials in $L^\infty(\mu)$.

THEOREM 20. *Let μ be a bounded nonnegative measure with compact support such that no nonconstant real element of $L^\infty(\mu)$ belongs to the weak closure of the polynomials. Then every nonzero weakly closed ideal in the weak closure of the polynomials in $L^\infty(\mu)$ contains an element $B(z)$ such that multiplication by $B(z)$ is an isometry of the weak closure of the polynomials in $L^\infty(\mu)$ onto the ideal.*

Further applications of the domination principle to the problem of uniform polynomial approximation on compact subsets of the plane are obtained by

Keith Schwingendorf in his thesis [6]. The author expresses his thanks to him and to Richard Penney for assistance in routing errors from the manuscript.

Proof of Theorem 1. Assume that the inequality

$$\int h(z)\, d\mu(z) \leqslant \int h(z)\, d\nu(z)$$

holds for every continuous subharmonic function $h(z)$. Since the real part of a polynomial is harmonic, the inequality holds whenever $h(z) = \operatorname{Re} f(z)$ for a polynomial $f(z)$. Since the same inequality holds with $f(z)$ replaced by $\lambda f(z)$ for a complex number λ, μ and ν are equivalent. Since

$$h(z) = \max(\log |f(z)|, \log \epsilon)$$

is a continuous subharmonic function of z for every polynomial $f(z)$ and every positive number ϵ, μ is less than or equal to ν.

Assume instead that μ is less than or equal to ν. If $h(z)$ is a function of z which is subharmonic in the complex plane, let σ be a bounded nonnegative measure with compact support such that the function

$$h(z) - \int \log |z - w|\, d\sigma(w)$$

is harmonic in some disk containing the supports of μ and of ν. Choose a sequence of polynomials $f_n(z)$ whose real parts converge uniformly to that function in the disk. The identity

$$\int f_n(z)\, d\mu(z) = \int f_n(z)\, d\nu(z)$$

holds for every index n because μ and ν are assumed to be equivalent. The identity

$$\iint \log |z - w|\, d\sigma(w)\, d\mu(z) = \iint \log |z - w|\, d\mu(z)\, d\sigma(w)$$

holds because the integrand is bounded above by a function which is nonnegative and integrable. Since the same identity holds with μ replaced by ν, the hypothesis that μ is less than or equal to ν implies that

$$\iint \log |z - w|\, d\sigma(w)\, d\mu(z) \leqslant \iint \log |z - w|\, d\sigma(w)\, d\nu(z).$$

The inequality

$$\int h(z)\, d\mu(z) \leqslant \int h(z)\, d\nu(z)$$

now follows by uniform convergence of the approximating polynomials.

If $h(z)$ is a subharmonic function of z, the inequality

$$\int \exp(ph(z))\, d\mu(z) \leqslant \int \exp(ph(z))\, d\nu(z)$$

holds for every positive number p because $\exp(ph(z))$ is a subharmonic function of z. By the arbitrariness of p, the least upper bound of $h(z)$ with respect to μ is less than or equal to the least upper bound of $h(z)$ with respect to ν.

Since the modulus of a polynomial is a subharmonic function, the identity transformation on polynomials is bounded by one from the metric of $L^\infty(\nu)$ into the metric of $L^\infty(\mu)$. Since the transformation is bounded by one from the metric of $L^1(\nu)$ into the metric of $L^1(\mu)$ for the same reason, the transformation is continuous from the weak topology of $L^\infty(\nu)$ into the weak topology of $L^\infty(\mu)$. The theorem follows.

Proof of Theorem 2. Consider first the special case in which the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is continuous in $L^2(\mu)$. Let $K(\lambda, z)$ be the unique element of $L^2(\mu)$ in the closure of the polynomials such that the identity

$$f(\lambda) = \int f(z)\, \bar{K}(\lambda, z)\, d\mu(z)$$

holds for every polynomial $f(z)$. Then

$$K(\lambda, \lambda) = \int |\, K(\lambda, z)|^2\, d\mu(z) > 0.$$

Let ν be the bounded nonnegative measure defined by

$$\nu(S) = \int_S K(\lambda, z)\, K(\lambda, \lambda)^{-1}\, \bar{K}(\lambda, z)\, d\mu(z)$$

for every Borel set S. Then ν is absolutely continuous with respect to μ and it is equivalent to the measure with mass one concentrated at λ. Since the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$, they are weakly dense in $L^\infty(\nu)$.

Argue by contradiction, assuming that ν is a point mass concentrated at λ. Then the point λ carries a positive μ-mass and $K(\lambda, z)$ vanishes away from that point. It will be shown that $K(\lambda, z)$ belongs to the weak closure of the polynomials in $L^\infty(\mu)$. This is done by showing that $K(\lambda, z)$ belongs to the closure of the polynomials in $L^2(\mu_1)$ for every bounded nonnegative measure μ_1, which is absolutely continuous with respect to μ, such that the inequality $\mu(S) \leqslant \mu_1(S)$ holds for every Borel set S. Then the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is continuous in $L^2(\mu_1)$. Define $K_1(\lambda, z)$ and ν_1 for μ_1 in the same way that $K(\lambda, z)$ and ν were defined for μ. Then ν and ν_1 are equivalent bounded nonnegative measures which are absolutely continuous with respect to μ. Since the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are assumed weakly dense in $L^\infty(\mu)$, ν and ν_1 are equal. It follows that $K(\lambda, z)$ and $K_1(\lambda, z)$ are linearly dependent. By the arbitrariness of μ_1, $K(\lambda, z)$ belongs to the weak closure of the polynomials in $L^\infty(\mu)$. Since $K(\lambda, z)$ is a real element of $L^\infty(\mu)$, it is equivalent to a constant with respect to μ by hypothesis. It follows that μ is a point mass concentrated at λ. Since this is contrary to hypothesis, ν cannot be a point mass concentrated at λ.

By Theorem 3 of [4], a function $\psi(z)$ of z exists, which is bounded and analytic in the unit disk, which has distinct values at distinct points of the disk, and which has value λ at the origin, such that the ν-measure of every Borel set S is the Lebesgue measure of the set of real numbers t modulo one such that $\psi(\exp(2\pi it))$ belongs to S. The boundary value function is defined as in the theory of square summable power series [3]. The transformation $f(z)$ into $f(\psi(z))$ is an isometry of the closure of the polynomials in $L^2(\nu)$ onto $\mathscr{C}(z)$. Let $W(z)$ be the element of $\mathscr{C}(z)$ given by

$$W(z) = K(\lambda, \lambda)^{1/2}/K(\lambda, \psi(z)).$$

Let $\mu = \alpha + \beta$ where α is a nonnegative measure which is absolutely continuous with respect to ν and β is a nonnegative measure which is singular with respect to ν. Then the transformation $f(z)$ into $W(z)f(\psi(z))$ is an isometry of the closure of the polynomials in $L^2(\alpha)$ onto $\mathscr{C}(z)$. Since $K(\lambda, z)$ belongs to the closure of the polynomials in $L^2(\mu)$, the element χ of $L^2(\mu)$ which is one almost everywhere with respect to α and zero almost everywhere with respect to β belongs to the closure of the polynomials in $L^2(\mu)$.

It will be shown that χ belongs to the weak closure of the polynomials in $L^\infty(\mu)$. This is done by showing that it belongs to the closure of the polynomials in $L^2(\mu_1)$ for every bounded nonnegative measure μ_1, which is absolutely continuous with respect to μ, such that the inequality $\mu(S) \leqslant \mu_1(S)$ holds for every Borel set S. Then the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu_1)$ and the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is continuous in $L^2(\mu_1)$. Define $K_1(\lambda, z)$, ν_1, and $W_1(z)$ for μ_1 in the same way that $K(\lambda, z)$, ν, and $W(z)$ are defined for μ. Then ν and ν_1 are equivalent bounded nonnegative measures which are absolutely continuous with respect to μ. Since the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$, ν and ν_1 are equal. It follows that χ belongs to the weak closure of the polynomials in $L^2(\mu_1)$.

By the arbitrariness of μ_1, χ is a real element of $L^\infty(\mu)$ in the weak closure of the polynomials. By hypothesis, χ is equivalent to a constant with respect to μ. It follows that μ is absolutely continuous with respect to ν. So the transformation $f(z)$ into $W(z)f(\psi(z))$ is an isometry of the closure of the polynomials in $L^2(\mu)$ onto $\mathscr{C}(z)$. The transformation $f(z)$ into

$$W(\exp(2\pi it))\, f(\psi(\exp(2\pi it)))$$

is an isometry of $L^2(\mu)$ onto the space of square integrable periodic functions of period one. The μ-measure of a Borel set S is equal to

$$\int |W(\exp(2\pi it))|^2 \, dt$$

with integration over the real numbers t modulo one such that $\psi(\exp(2\pi it))$ belongs to S. Another consequence of the arbitrariness of μ_1 is that the transformation $f(z)$ into $f(\psi(z))$ is an isometry of the weak closure of the polynomials in $L^\infty(\mu)$ onto the space of functions which are bounded and analytic in the unit disk.

This completes the proof of the theorem when the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is continuous in $L^2(\mu)$. Now consider the case in which it is only assumed weakly continuous in $L^\infty(\mu)$. Then it is continuous in $L^2(\nu)$ for a bounded nonnegative measure ν which is absolutely continuous with respect to μ, such that μ is absolutely continuous with respect to ν.

Since the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are assumed weakly dense in $L^\infty(\mu)$, they are weakly dense in $L^\infty(\nu)$. Since no nonconstant real element of $L^\infty(\mu)$ exists in the weak closure of the polynomials, no nonconstant real element of $L^\infty(\nu)$ exists in the real closure of the polynomials. By the case of the theorem previously considered, the choice of the measure ν can be made equivalent to the measure with mass one concentrated at λ. As before, a function $\psi(z)$ of z exists, which is bounded and analytic in the unit disk, which has distinct values at distinct points of the disk, and which has value λ at the origin, such that the ν-measure of every Borel set S is the Lebesgue measure of the set of real numbers t modulo one such that $\psi(\exp(2\pi it))$ belongs to S. The theorem now follows.

Proof of Theorem 3. Consider any bounded measure ν, which is absolutely continuous with respect to μ, such that the identity

$$\int f(z) \, d\nu(z) = 0$$

holds for every polynomial $f(z)$. Then the integral

$$\int (z - w)^{-1} \, d\nu(z)$$

is absolutely convergent for all complex numbers w outside of some set of zero plane measure, and it represents a locally integrable function with respect to this measure. For each such complex number w, the identity

$$\int (z - w)^{-1} f(z) \, d\nu(z) = f(w) \int (z - w)^{-1} \, d\nu(z)$$

holds for every polynomial $f(z)$ because $[f(z) - f(w)]/(z - w)$ is a polynomial in z. The left side defines a linear functional on polynomials which is weakly continuous in $L^\infty(\mu)$. Since the linear functional on polynomials defined by $f(z)$ into $f(w)$ is not weakly continuous in $L^\infty(\mu)$,

$$\int (z - w)^{-1}\, d\nu(z) = 0.$$

It follows that

$$\int \log | z - w |\, d\nu(z)$$

is harmonic in the complex plane and that $\nu(S) = 0$ for every Borel set S. The theorem follows.

Proof of Theorem 4. By the proof of Theorem 2, a unique bounded nonnegative measure ν exists, which is absolutely continuous with respect to μ and which is equivalent to the measure with mass one concentrated at the point λ. By the proof of Theorem 2, $\mu = \alpha + \beta$ for nonnegative measures α and β such that α is absolutely continuous with respect to ν, such that β is singular with respect to ν, and such that the element χ of $L^2(\mu)$ which is one almost everywhere with respect to α and zero almost everywhere with respect to β belongs to the weak closure of the polynomials in $L^\infty(\mu)$. Since the weak closure of the polynomials in $L^\infty(\mu)$ is an algebra, an element of $L^\infty(\mu)$ belongs to the weak closure of the polynomials if, and only if, it belongs to the weak closure of the polynomials in $L^\infty(\alpha)$ and in $L^\infty(\beta)$. By the proof of Theorem 2, ν is absolutely continuous with respect to α.

If $h(z)$ is a real element of $L^\infty(\alpha)$ in the weak closure of the polynomials, then it is a real element of $L^\infty(\nu)$ in the weak closure of the polynomials. If ν is not the measure with mass one concentrated at λ, define $\psi(z)$ for ν as in the proof of Theorem 2. By Theorem 3 of [4], $h(\psi(z))$ is an element of $\mathscr{C}(z)$ with real boundary values almost everywhere on the unit circle. Since $h(\psi(z))$ has real values in the unit disk, it is a constant. It follows that $h(z)$ is a constant.

Argue by contradiction assuming that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\alpha)$ and in $L^\infty(\beta)$ for some complex number w. Then the linear functional is weakly continuous in $L^\infty(\mu)$. By the proof of Theorem 2, a unique bounded nonnegative measure σ exists which is absolutely continuous with respect to μ and which is equivalent to the measure with mass one concentrated at w. For the same reasons, a unique bounded nonnegative measure ν exists which is absolutely continuous with respect to α and which is equivalent to the measure with mass one concentrated at w. It follows that σ is equal to ν and is absolutely continuous with respect to α. By a similar argument, σ is absolutely continuous with respect to β. A contradiction is obtained because α and β are mutually singular and σ is nonzero.

Proof of Theorem 5. The theorem follows from Theorems 2, 3, and 4 by an inductive argument.

Proof of Theorem 6. By Theorem 2, an element $\varphi(z)$ of $L^\infty(\mu)$ exists in the weak closure of the polynomials such that $\varphi(\psi(z)) = z$ almost everywhere with respect to Lebesgue measure on the unit circle. The identity also holds in the unit disk if $\varphi(z)$ is defined in Ω by continuity of function values for polynomials. Since $\psi(z)$ has distinct values at distinct points of the unit disk, the identity $\psi(\varphi(z)) = z$ also holds in Ω. Since the μ-measure of every Borel set S is the Lebesgue measure of the set of real numbers t modulo one such that $\psi(\exp(2\pi it))$ belongs to S, the identity also holds almost everywhere with respect to μ. The balayage construction is now made as in the proof of Theorem 5 of [4].

Proof of Theorem 7. If the complement of Ω^* has zero μ-measure, then μ is less than or equal to ν by Theorem 6.

Assume that μ is less than or equal to ν. By Theorem 1, a weakly continuous homomorphism exists of the weak closure of the polynomials in $L^\infty(\nu)$ into the weak closure of the polynomials in $L^\infty(\mu)$ which takes every polynomial into itself. Let $\theta(z)$ be the element of $L^\infty(\mu)$ which corresponds to the element $\varphi(z)$ of $L^\infty(\nu)$. Since $\varphi(z)$ has absolute value one almost everywhere with respect to ν, $\theta(z)$ is bounded by one almost everywhere with respect to μ by Theorem 1. Let σ be the unique bounded nonnegative measure, supported in the closure of the unit disk, such that the identity

$$\int z^m \bar{z}^n \, d\sigma(z) = \int \theta^m(z)\, \bar{\theta}^n(z)\, d\mu(z)$$

holds for all nonnegative integers m and n. Then the identity

$$\int f(z)\, \bar{g}(z)\, d\sigma(z) = \int f(\theta(z))\, \bar{g}(\theta(z))\, d\mu(z)$$

holds for all functions $f(z)$ and $g(z)$ of z which are bounded and analytic in the unit disk. In particular, the identity

$$\int \psi^m(z)\, \bar{\psi}^n(z)\, d\sigma(z) = \int \psi^m(\theta(z))\, \bar{\psi}^n(\theta(z))\, d\mu(z)$$

holds for all nonnegative integers m and n. But $\psi(\theta(z))$ as a function of z in $L^\infty(\mu)$ is the homomorphic image of $\psi(\varphi(z))$ as a function of z in $L^\infty(\nu)$. Since $\psi(\varphi(z)) = z$ by the proof of Theorem 6 and since the homomorphism the identity on polynomials $\psi(\theta(z)) = z$ as a function of z in $L^\infty(\mu)$. It follows that the identity

$$\int f(\psi(z))\, \bar{g}(\psi(z))\, d\sigma(z) = \int f(z)\, \bar{g}(z)\, d\mu(z)$$

holds for all polynomials $f(z)$ and $g(z)$. By the Stone–Weierstrass theorem, the identity holds for all continuous functions $f(z)$ and $g(z)$. It follows that the

μ-measure of every Borel set S is equal to the σ-measure of the set of points w in the closure of the unit disk such that $\psi(w)$ belongs to S. So the complement of Ω^* has zero μ-measure.

Proof of Theorem 8. By Theorem 1, a weakly continuous homomorphism exists of the weak closure of the polynomials in $L^\infty(\nu)$ into the weak closure of the polynomials in $L^\infty(\mu)$ which takes every polynomial into itself. Since the polynomials are assumed to be weakly dense in $L^\infty(\nu)$, the homomorphism is everywhere defined in $L^\infty(\nu)$. The elements χ of $L^\infty(\mu)$ or of $L^\infty(\nu)$ which are characteristic functions of Borel sets are those for which the identity $\chi^2 = \chi$ holds. So the homomorphism takes the characteristic functions of Borel sets into characteristic functions of Borel sets. Since the homomorphism is bounded by one, the induced mapping of sets does not increase measures. The induced mapping of sets also preserves unions and intersections and commutes with sequential monotone limits. Since μ and ν have equal total masses and since the homomorphism is the identity on constants, the induced mapping of sets is measure preserving. A Borel measurable function $\theta(z)$ of complex z exists such that the transformation takes set A into set B where B is the set of numbers w such that $\theta(w)$ belongs to A. It follows that the homomorphism is of the form $f(z)$ into $f(\theta(z))$. Since the homomorphism is the identity on polynomials, $\theta(z)$ can be chosen equal to z. The theorem follows.

Proof of Theorem 9. The measure μ is clearly less than or equal to the measure ν if it is of the form $\mu = \sum \mu_n$ with μ_n less than or equal to ν_n for every index n. Assume that μ is less than or equal to ν. For each index n, let β_n be the element of $L^\infty(\nu)$ which is one almost everywhere with respect to ν_n and which is zero almost everywhere with respect to ν_k for every index k other than n. The hypotheses imply that β_n belongs to the weak closure of the polynomials in $L^\infty(\nu)$. By Theorem 1, a weakly continuous homomorphism exists of the weak closure of the polynomials in $L^\infty(\nu)$ into the weak closure of the polynomials in $L^\infty(\mu)$ which is the identity on polynomials. As in the proof of Theorem 8, the image α_n of β_n is the characteristic function of a Borel set for every n. Let μ_n be the non-negative measure defined by

$$\mu_n(S) = \int_S \alpha_n(z)\, d\mu(z)$$

for every Borel set S. It follows from Theorem 1 that μ_n is less than or equal to ν_n. Since ν_n and ν_k are singular when n and k are unequal, $\beta_n\beta_k = 0$ almost everywhere with respect to ν. Since $\alpha_n\alpha_k = 0$ almost everywhere with respect to μ, μ_n and μ_k are singular. Since

$$1 = \sum \beta_n$$

in the weak topology of $L^\infty(\nu)$, and since the homomorphism is the identity on constants,

$$1 = \sum \alpha_n$$

in the weak topology of $L^\infty(\mu)$. It follows that

$$\mu = \sum \mu_n .$$

Proof of Theorem 10. Consider first the case in which the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\nu)$. By Theorem 5, a countable number of mutually singular nonnegative measures ν_n exists, which satisfy the hypotheses of Theorem 2, such that the measure $\nu - \sum \nu_n$ is singular with respect to every measure ν_n and such that an element of $L^\infty(\nu)$ belongs to the weak closure of the polynomials if, and only if, it belongs to the weak closure of the polynomials in $L^\infty(\nu_n)$ for every n. By Theorems 7 and 8, bounded nonnegative measures μ_n with compact support exist such that μ_n is less than or equal to ν_n for every n and such that $\mu - \sum \mu_n = \nu - \sum \nu_n$. By Theorem 2, the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\nu_n)$ is a bounded simply connected region Ω_n. If Ω_n^* is the set onto which a Riemann mapping function ψ_n for Ω_n maps the closure of the unit disk, then the complement of Ω_n^* has zero μ_n-measure by Theorem 7. Because of Theorem 2, a weakly continuous isometry $h(z)$ into $h(\psi_n(z))$ exists of the weak closure in $L^\infty(\nu_n)$ of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ onto the set of functions which are bounded and harmonic in the unit disk. It follows that a weakly continuous transformation exists of the weak closure in $L^\infty(\nu_n)$ of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ into the weak closure in $L^\infty(\mu_n)$ of the same functions which is the identity on these functions. Also the transformation is bounded by one. The theorem follows in the case that ν satisfies the density condition.

The proof of the theorem in the general case is made using Theorems 16 and 17. Although the present theorem is used in the proof of these later theorems, it is so only in the special case already established.

By Theorem 16, a greatest element σ exists in the closure of the set of bounded nonnegative measures which are absolutely continuous with respect to ν and which are equivalent to ν. The functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\sigma)$. By Theorem 17, a weakly continuous isometry exists of the weak closure in $L^\infty(\sigma)$ of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ into the weak closure in $L^\infty(\nu)$ of these functions which is the identity on these functions. The theorem now follows by composition from the special case already established.

Proof of Theorem 11. By Theorem 5, mutually singular nonnegative measures ν_n exist, which satisfy the hypotheses of Theorem 2, such that $\nu - \sum \nu_n$ is singular with respect to ν_n for every index n and such that an element of $L^\infty(\nu)$ belongs to the weak closure of the polynomials if, and only if, it belongs to the weak closure of the polynomials in $L^\infty(\nu_n)$ for every n. By Theorems 8 and 9, bounded nonnegative measures μ_n with compact support exist such that μ_n

is less than or equal to ν_n for every n and such that $\mu - \sum \mu_n = \mu - \sum \nu_n$. By Theorem 2, the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\alpha(\nu_n)$ is a bounded simply connected region Ω_n. If w belongs to Ω_n, then the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\alpha(\nu)$. Since Ω_n has zero μ-measure by hypothesis, it has zero μ_n-measure. Let Ω_n^* be the set onto which a Riemann mapping function for Ω_n maps the closure of the unit disk. By Theorem 7, the complement of Ω_n^* has zero μ_n-measure. Since μ_n and ν_n are equal by the proof of Theorem 7, μ and ν are equal.

Proof of Theorem 12. Since μ and ν are bounded nonnegative measures with compact support, the functions

$$f(z) = \int \log | z - w | \, d\mu(w) \qquad \text{and} \qquad g(z) = \int \log | z - w | \, d\nu(w)$$

are subharmonic in the complex plane, $f(z)$ is harmonic in each connected component of the complement of the support of μ, and $g(z)$ is harmonic in each connected component of the complement of the support of ν. But for every nonzero complex number w, the expression

$$\log | z - w | = \log | w | + \log | 1 - z/w |$$

is represented by a square summable power series in z/w. Since μ and ν are equivalent, $f(z)$ and $g(z)$ agree in the complement of any disk about the origin which contains the supports of μ and ν. By analytic continuation, $f(z)$ and $g(z)$ agree in the unbounded component of the complement of the union of the supports of μ and ν. It follows that

$$h(z) = \max(f(z), g(z))$$

is a subharmonic function which agrees with $f(z)$ and $g(z)$, and hence is harmonic, in the unbounded component of the complement of the union of the supports of μ and ν. Let σ be the unique bounded nonnegative measure with compact support such that

$$h(z) - \int \log | z - w | \, d\sigma(w)$$

is harmonic in the complex plane. For large $| w |$,

$$\int \log | z - w | \, d\sigma(w) \sim \log | z | \int d\sigma(w)$$

and similarly for μ and ν. It follows that

$$h(z) = \int \log | z - w | \, d\sigma(w).$$

The measure σ clearly has the required properties.

Proof of Theorem 13. By Theorems 7, 8, and 9, the supports of the measures μ_n are contained in a compact set. Since the measures are nonnegative and have the same total mass, they converge in some subsequence to a bounded nonnegative measure μ. An application of Theorem 1 will show that μ is the greatest bounded nonnegative measure with compact support which is less than or equal to μ_n for every index n. The uniqueness of μ implies that it is the limit of the measures μ_n.

Consider any complex number λ such that the linear functional on polynomials defined by $f(z)$ into $f(\lambda)$ is weakly continuous in $L^\infty(\mu)$. By Theorem 1, the linear functional is weakly continuous in $L^\infty(\mu_n)$ for every n. It may be that λ is a point of positive μ_n-mass and that the function which is one at λ and zero elsewhere belongs to the weak closure of the polynomials in $L^\infty(\mu_n)$. By Theorems 8 and 9, λ is then a point of positive μ-mass and the same function belongs to the weak closure of the polynomials in $L^\infty(\nu)$. Otherwise, by Theorems 7 and 9, λ belongs to the closure of the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu_n)$. By Theorem 2, the connected component Ω_n of the set determined by λ is simply connected. A function $\psi_n(z)$ of z exists, which is bounded and analytic in the unit disk, which has distinct values at distinct points of the disk, and which has value λ at the origin, such that Ω_n is the region onto which ψ_n maps the unit disk. Let χ_n be the characteristic function of the set Ω_n^* onto which ψ_n maps the closure of the unit disk. Then $\psi_n(z)$ is the inverse of a function $\varphi_n(z)$ of z in Ω_n^*. If $\varphi_n(z)$ is extended so as to be zero outside of Ω_n^*, then both φ_n and χ_n belong to the weak closure of the polynomials in $L^\infty(\mu_n)$.

By Theorems 7 and 9, Ω_{n+1} is contained in Ω_n whenever Ω_{n+1} is defined. Assume that Ω_n is defined for every n. By Theorem 1, φ_n and χ_n belong to the weak closure of the polynomials in $L^\infty(\mu)$ for every n. By dominated convergence, the function $\chi = \lim \chi_n$ belongs to the weak closure of the polynomials in $L^\infty(\mu)$. It may be that χ is the function which is one at λ and zero elsewhere. Otherwise a function $\psi(z)$ of z exists, which is bounded and analytic in the unit disk, which has distinct values at distinct points of the disk, and which has value λ at the origin, such that $\psi(z)$ is a limit of some subsequence of the functions $\psi_n(z)$. The intersection of the regions Ω_n is the region onto which ψ maps the unit disk and χ is the characteristic function of the set Ω^* onto which ψ maps the closure of the unit disk. Also $\psi(z)$ is the inverse of a function $\varphi(z)$ of z in Ω^*. If $\varphi(z)$ is extended to be zero outside of Ω^*, then $\varphi = \lim \varphi_n$ belongs to the weak closure of the polynomials in $L^\infty(\mu)$. The point λ belongs to Ω.

The desired density property of the measure μ is now verified by a straightforward argument.

Proof of Theorem 14. By hypothesis, the measure μ is less than or equal to some bounded nonnegative measure which is absolutely continuous with respect to Lebesgue measure on the unit circle. Since $h(z)$ belongs to the weak closure of the polynomials in the L^∞ space of such a measure, it belongs to the weak closure of the polynomials in $L^\infty(\mu)$ by Theorems 1 and 7. If μ is not zero, then $h(z)$ is not a constant.

Consider any complex number w in the unit disk at which the value of $h(z)$ is not bounded by one. Then the linear functional on polynomials defined by $f(z)$ into $f(w)$ is unbounded in $L^\infty(\mu)$ because the powers of $h(z)$ are bounded in the space and the powers of $h(w)$ are unbounded numbers. This information is used to show that the function $1/(z-w)$ belongs to the weak closure of the polynomials in $L^\infty(\mu)$. The function is bounded with respect to μ because w does not belong to the support of the measure. Consider any bounded measure ν, which is absolutely continuous with respect to μ, such that the identity

$$\int f(z)\, d\nu(z) = 0$$

holds for every polynomial $f(z)$. Since $[f(z) - f(w)]/(z-w)$ is a polynomial whenever $f(z)$ is a polynomial, the identity

$$\int f(z)/(z-w)\, d\nu(z) = f(w) \int 1/(z-w)\, d\nu(z)$$

holds for every polynomial $f(z)$. The left side of the identity defines a bounded linear functional on polynomials in the metric of $L^\infty(\mu)$. Since the linear functional on polynomials defined by $f(z)$ into $f(w)$ is unbounded in $L^\infty(\mu)$,

$$\int 1/(z-w)\, d\nu(z) = 0.$$

By the arbitrariness of ν, $1/(z-w)$ belongs to the weak closure of the polynomials in $L^\infty(\mu)$.

If α and β are equivalent bounded nonnegative measures which are absolutely continuous with respect to μ, then the subharmonic functions

$$\int \log|z-w|\, d\alpha(w) \qquad \text{and} \qquad \int \log|z-w|\, d\beta(w)$$

are equal and harmonic outside of the unit circle, and they agree on the unit circle. In any component Ω of the subset of the unit disk in which $|h(z)| > 1$, these functions differ by a constant because of the identity

$$\int 1/(z-w)\, d\alpha(w) = \int 1/(z-w)\, d\beta(w)$$

which holds there. If ψ is a Riemann mapping function for the simply connected region Ω, then $h(\psi(z))$ is analytic and bounded in the unit disk, but it is not bounded by one, and it is bounded by one at all points of the unit circle such that $\psi(w)$ does not belong to the unit circle. It follows that the set of points w

on the unit circle such that $\psi(w)$ belongs to the unit circle has positive Lebesgue measure. The functions

$$\int \log |\psi(z) - w| \, d\alpha(w) \qquad \text{and} \qquad \int \log |\psi(z) - w| \, d\beta(w)$$

are bounded and harmonic in the unit disk, and it is easily verified that their boundary values on the unit circle are consistent with the boundary values of $\psi(z)$ on the unit circle. Since these functions differ by a constant in the unit disk, they differ by the same constant almost everywhere with respect to Lebesgue measure on the unit circle. Since the functions agree in a set of positive Lebesgue measure on the unit circle, they agree in the unit disk. By the arbitrariness of Ω, the identity

$$\int \log |z - w| \, d\alpha(w) = \int \log |z - w| \, d\beta(w)$$

holds at all points of the unit disk where $|h(z)| > 1$.

By Theorem 12 a least bounded nonnegative measure γ with compact support exists which is greater than or equal to α and β. By the proof of the theorem, the identity

$$\int \log |z - w| \, d\alpha(w) = \int \log |z - w| \, d\gamma(w)$$

holds at all points z of the unit disk for which $|h(z)| > 1$. The function

$$\int \log |z - w| \, d\gamma(w) - \int \log |z - w| \, d\alpha(w)$$

is nonnegative in the complex plane by the definition of inequality for measures. It is subharmonic in each connected component of the subset of the unit disk in which $|h(z)| < 1$, and it is zero elsewhere in the unit disk. It follows that the function is subharmonic in the unit disk. Since the function is also zero in the complement of the unit disk, it is subharmonic in the complex plane. It follows that the measure $\gamma - \alpha$ is nonnegative. Since α and γ have the same total mass, they are equal. Since β and γ are equal by a similar argument, α and β are equal. By the arbitrariness of the starting equivalent measures α and β, the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$.

Proof of Theorem 15. By Theorem 6, a bounded nonnegative measure σ with compact support exists, which is greater than or equal to μ, such that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\sigma)$. A minimal such measure σ exists because of Theorem 13. In the remainder of the proof, it is assumed that σ satisfies the hypotheses of Theorem 2. The general case then follows immediately from Theorems 3, 5, and 8.

The set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\sigma)$ is now a simply connected region. By Theorems 2 and 7, it can be assumed without loss of generality that this region is the unit disk. The weak closure of the polynomials in $L^\infty(\sigma)$ is now the set of functions which are bounded and analytic in the unit disk. The boundary value function theory [3] for square summable power series applies because σ is absolutely continuous with respect to Lebesgue measure on the unit circle. The weakly continuous homomorphism defined in Theorem 1, which takes the weak closure of the polynomials in $L^\infty(\sigma)$ into the weak closure of the polynomials in $L^\infty(\mu)$, is the identity transformation on functions bounded and analytic in the unit disk, given their boundary values on the unit circle.

Argue by contradiction, assuming that a function $h(z)$ exists, which is bounded and analytic in the unit disk but which is not bounded by one, whose norm in $L^\infty(\mu)$ is less than one. By the balayage construction, a bounded nonnegative measure ν exists, which is supported in the closure of the unit disk, which is less than or equal to σ, which is greater than or equal to μ, and which satisfies the hypotheses of Theorem 14 with respect to $h(z)$. By Theorem 14, the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\nu)$. A contradiction is obtained because ν is not equal to σ.

Since every function which is bounded and analytic in the unit disk has the same norm in $L^\infty(\mu)$ as in $L^\infty(\sigma)$, the functions which are bounded and analytic in the unit disk form a norm closed subspace of $L^\infty(\mu)$. Since the subspace contains the polynomials as a weakly dense subspace, it coincides with the weak closure of the polynomials in $L^\infty(\mu)$. Since the identity transformation on bounded analytic functions is weakly continuous from $L^\infty(\sigma)$ into $L^\infty(\mu)$ and since the unit ball in $L^\infty(\sigma)$ is weakly compact, the transformation is a homeomorphism for the weak topologies. It follows that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ for every point w in the unit disk.

Proof of Theorem 16. Define the measure σ for μ as in Theorem 15. The theorem is proved by showing that σ belongs to the closure of the set of measures which are absolutely continuous with respect to μ and which are equivalent to μ. By Theorems 2, 5, 7, 8, and 9, it is sufficient to give a proof in the case that σ is absolutely continuous with respect to Lebesgue measure on the unit circle.

By Theorem 15, the linear functional on polynomials which assigns the n-th coefficient of the power series expansion is weakly continuous in $L^\infty(\mu)$ and is bounded by one for every nonnegative integer n. It follows that a nonnegative measure ν_n exists, which is absolutely continuous with respect to μ and has total mass one, such that the linear functional is bounded by one in $L^1(\nu_n)$. The bounded nonnegative measure ν defined by

$$\nu = \sum \nu_n/(n+1)^2$$

is absolutely continuous with respect to μ. The linear functional on polynomials which assigns the n-th coefficient of the power series expansion is bounded by

$(n+1)^2$ in $L^1(\nu)$ for every nonnegative integer n. So the linear functional on polynomials defined by $f(z)$ into $f(w)$ is bounded in $L^1(\nu)$ for every point w in the unit disk, and it is bounded independently of w for w in any compact subset of the unit disk. Since it is also bounded in $L^2(\nu)$, a unique element $K(w, z)$ of $L^2(\nu)$ exists in the closure of the polynomials such that the identity

$$f(w) = \int f(z)\, \bar{K}(w, z)\, d\nu(z)$$

holds for every polynomial $f(z)$. It follows that

$$K(w, w) = \int |K(w, z)|^2\, d\nu(z) > 0$$

and that the identity

$$f(w) = \int f(z)\, K(w, z)\, K(w, w)^{-1}\, \bar{K}(w, z)\, d\nu(z)$$

holds for every polynomial $f(z)$. The inequality

$$K(w, w)\, |f(w)|^2 \leqslant \int |f(z)\, K(w, z)|^2\, d\nu(z)$$

follows by the Schwarz inequality. Equality holds only when

$$[f(z) - f(w)]\, K(w, z) = 0$$

almost everywhere with respect to ν.

For each bounded nonnegative measure α with compact support, a bounded nonnegative measure β with compact support is defined by

$$\beta(S) = \alpha(S) - \alpha(S \cap \Delta) + \int_S \int_\Delta K(w, z)\, K(w, w)^{-1}\, \bar{K}(w, z)\, d\alpha(w)\, d\nu(z)$$

for every Borel set S, where Δ denotes the unit disk. The identity

$$\int h(z)\, d\beta(z) = \int h(z)\, d\alpha(z) - \int_\Delta h(z)\, d\alpha(z)$$
$$+ \int_\Delta \int_S h(z)\, K(w, z)\, K(w, w)^{-1}\, \bar{K}(w, z)\, d\nu(z)\, d\alpha(w)$$

holds for every continuous function $h(z)$. It follows that α and β are equivalent. The inequality

$$\int |f(z)|^2\, d\alpha(z) \leqslant \int |f(z)|^2\, d\beta(z)$$

holds for every polynomial $f(z)$. Equality holds if, and only if, $[f(z) - f(w)]\, K(w, z)$ vanishes as a function of z almost everywhere with respect to ν outside of a set of numbers w of zero α-measure. If α is less than or equal to σ, then so is β.

The transformation α into β so defined is continuous and takes measures which are absolutely continuous with respect to μ into measures which are absolutely continuous with respect to μ. The closed invariant subset generated by μ under the action of the transformation contains a unique fixed point γ, and γ belongs to the closure of the set of measures which are absolutely continuous with respect to μ and which are equivalent to μ. The expression $K(w, z)$ vanishes away from w as a function of z almost everywhere with respect to ν outside of a set of numbers w of zero γ-measure.

Argue by contradiction, assuming that a point λ in the unit disk exists such that $K(\lambda, z)$ vanishes away from λ as a function of z almost everywhere with respect to ν. Then $(z - \lambda) K(\lambda, z)$ belongs to the closure of the polynomials in $L^2(\nu)$ and vanishes almost everywhere with respect to ν. Since the linear functional on polynomials defined by $f(z)$ into $f(w)$ is continuous in $L^2(\nu)$ at all points w of the unit disk, $(w - \lambda) K(\lambda, w)$ vanishes at all points w of the unit disk. A contradiction of the positivity of $K(\lambda, \lambda)$ is now obtained because $K(\lambda, w)$ is a continuous function of w in the unit disk.

Since γ is less than or equal to σ and since the unit disk has zero γ-measure, γ and σ are equal by Theorem 11.

Proof of Theorem 17. By Theorems 2, 5, 7, 8, 9, and 16, it is sufficient to give a proof in the case that σ is absolutely continuous with respect to Lebesgue measure on the unit circle. Then the weak closure in $L^\infty(\sigma)$ of the set of functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ is the set of functions which are bounded and harmonic in the unit disk, given their boundary values on the unit circle. The bound of the harmonic function in the disk coincides with its bound with respect to σ. The weakly continuous transformation of $L^\infty(\sigma)$ into $L^\infty(\mu)$ defined in Theorem 10 is the identity transformation on functions which are bounded and harmonic in the unit disk. As in Theorem 10, the transformation is weakly continuous and does not increase norms. It follows from Theorem 16 that the transformation is an isometry. As in the proof of Theorem 15, the transformation is a homeomorphism of $L^\infty(\sigma)$ onto the weak closure in $L^\infty(\mu)$ of the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$.

Proof of Theorem 18. If the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$, then by the proof of Theorem 3 of [4] no interior point of the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ belongs to the support of μ. Conversely, assume that the set contains no interior point in the support of μ. Define the measure σ for μ as in Theorem 16. By Theorems 15 and 16, the set of complex numbers w such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ coincides with the set of complex numbers w for which the linear functional is weakly continuous in $L^\infty(\sigma)$. By Theorem 11, μ and σ are equal. The theorem now follows from Theorem 16.

Proof of Theorem 19. The proof is an application of Theorem 18. Since the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\nu)$, no interior point of the set of complex numbers w for which the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\nu)$ belongs to the support of ν. By Theorem 2, no interior point of the set of complex numbers w for which the linear functional is weakly continuous in $L^\infty(\mu)$ belongs to the support of μ. By Theorem 18, the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$.

Proof of Theorem 20. By Theorems 15 and 16, it can be assumed without loss of generality that the functions of the form $f(z) + \bar{g}(z)$ for polynomials $f(z)$ and $g(z)$ are weakly dense in $L^\infty(\mu)$. It is clearly sufficient to consider the case in which the measure μ is supported at more than one point. By Theorem 3, the linear functional on polynomials defined by $f(z)$ into $f(w)$ is weakly continuous in $L^\infty(\mu)$ for some complex number w. By Theorem 2, a function $\psi(z)$ of z exists, which is bounded and analytic in the unit disk and which has distinct values at distinct points of the disk, such that the transformation $f(z)$ into $f(\psi(z))$ is an isomorphism of the weak closure of the polynomials in $L^\infty(\mu)$ onto the algebra of functions which are bounded and analytic in the unit disk.

Consider any bounded nonnegative measure ν, which is absolutely continuous with respect to μ, such that the linear functional on polynomials defined by $f(z)$ into $f(w)$ is continuous in $L^2(\nu)$ for some complex number w. By Theorem 2, a square summable power series $W(z)$ exists such that the transformation $f(z)$ into $W(z)f(\psi(z))$ is an isometry of the closure of the polynomials in $L^2(\nu)$ onto $\mathscr{C}(z)$. Since the given ideal contains a nonzero element, the closure of the ideal in $L^2(\nu)$ maps onto a closed subspace of $\mathscr{C}(z)$ which is invariant under multiplication by z. By the ideal theorem for square summable power series [3], an element $B(z)$ exists in the weak closure of the polynomials in $L^2(\mu)$ such that the closure of the ideal in $L^2(\nu)$ is the set of elements of the form $B(z)f(z)$ for an element $f(z)$ of the closure of the polynomials in $L^2(\nu)$. The function $B(z)$ has absolute value one almost everywhere with respect to μ. Since it does not depend on ν and since the given ideal is weakly closed, it belongs to the ideal. The theorem follows.

References

1. M. Krein and D. Milman, On extreme points of regular convex sets, *Studia Math.* **9** (1940), 133–138.
2. L. de Branges, The Stone-Weierstrass theorem, *Proc. Amer. Math. Soc.* **10** (1959), 822–824.
3. L. de Branges and J. Rovnyak, "Square Summable Power Series," Holt, Rinehart & Winston, New York, 1966.
4. L. de Branges and D. Trutt, Quantum Cesàro operators, *Advances in Math.*, to appear.
5. L. de Branges, "Espaces Hilbertiens de Fonctions Entières," Masson, Paris, 1972.
6. K. Schwingendorf, "Uniform Polynomial Approximation," Dissertation, Purdue University, 1978.