Optimal control of a competitive system with age-structure

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Abstract

We investigate optimal control of a first order partial differential equation (PDE) system representing a competitive population model with age structure. The controls are the proportions of the populations to be harvested, and the objective functional represents the profit from harvesting. The existence and unique characterization of the optimal control pair are established.

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1. Introduction

We consider optimal control of a competitive population system with age structure. The solutions of the state system represent population densities of the two interacting, competing species. The controls are the proportions of the populations to be harvested.

For the age-time domain, \( Q = (0, A) \times (0, T) \), the control set is defined as

\[ U = \left\{ (f, g) \in \left( L^\infty(Q) \right)^2 \mid 0 \leq f(a, t) \leq N_1, \ 0 \leq g(a, t) \leq N_2 \ \text{a.e. in} \ Q \right\}. \]

Given a control pair \((f, g) \in U\), the corresponding state variables, \((u, v) = (u, v)(f, g)\), satisfy the state system
\begin{align*}
&u_t + u_a = -\mu_1(a,u)u - fu - u \int_0^A c_1(x,a)v(x,t) \, dx \quad \text{in } Q, \\
v_t + v_a = -\mu_2(a,v)v - gv - v \int_0^A c_2(x,a)u(x,t) \, dx \quad \text{in } Q, \\
u(a,0) = u_0(a), \quad v(a,0) = v_0(a) \quad \text{for } a \in [0,A], \\
u(0,t) = \int_0^A \beta_1(\bar{a})u(\bar{a},t) \, d\bar{a} \quad \text{for } t \in [0,T], \\
v(0,t) = \int_0^A \beta_2(\bar{a})v(\bar{a},t) \, d\bar{a} \quad \text{for } t \in [0,T].
\end{align*}

In this system, $\mu_1(a,u)$ and $\mu_2(a,v)$ represent age and density specific mortality which is the death rate of each species in an infinitesimally small age interval. Similarly, $\beta_1(a)$ and $\beta_2(a)$ represent age specific fertility [21]. Each of the control terms, $fu$ and $gv$, depict the proportion of the species that is harvested. The coupling terms represent the interaction between the species with kernels $c_1(x,a)$, $c_2(x,a)$; the population $u$ at age $a$ can interact with the population $v$ over a range of ages, with the kernel $c_1$ specifying that range. For further background on age structured models, see [11,21,29].

We seek to maximize a profit functional

\begin{align*}
J(f, g) &= \int_0^T \int_0^A \left[ K_1(a)f(a,t)u(a,t) + K_2(a)g(a,t)v(a,t) \\
&\quad - \frac{1}{2}(B_1f^2(a,t) + B_2g^2(a,t)) \right] \, da \, dt
\end{align*}

over $(f, g) \in U$, where $K_1(a)$ and $K_2(a)$ are selling price factors and $B_1$, $B_2$ are weight factors. The functional represents the revenue from harvesting less the cost of harvesting. The revenue terms are the proportion of the species harvested multiplied by the selling price dependent on age. Our optimal control pair $(f^*, g^*)$ in $U$ will satisfy

\[ J(f^*, g^*) = \max_{(f,g) \in U} J(f, g). \]


In the context of optimal control of age-structured populations, Brokate [8,9], Barbu and Iannelli [5,6], Anita [1,2], Anita et al. [3], and Murphy and Smith [25] considered optimal

In Section 2, the existence of the state system (1.1) and Lipschitz properties of the solution in terms of the controls are shown. Section 3 starts with the differentiability of the solution map and the existence of solutions of the adjoint system. The upper semi-continuity of the objective functional with respect to strong $L^1$ convergence is established. In the last section, the existence of a unique optimal control pair is obtained with the use of Ekeland’s principle [4,14].

2. Estimates on the state system

We make the following assumptions:

- $\beta_i$ is a nonnegative function in $L^\infty(0, A)$ with $\|\beta_i\|_\infty \leq \beta$ for $i = 1, 2$ and $\beta$, a positive constant, (2.1)
- $\mu_i(a, w)$ is a nonnegative bounded function on $(0, A) \times L^\infty(Q)$ with bounded positive derivative with respect to the second variable, $i = 1, 2$, (2.2)
- $K_i(a) \in L^\infty(0, A)$ and $\|K_i(a)\|_\infty \leq K$ for $i = 1, 2$ and $K$, a positive constant, (2.3)
- $c_1, c_2$ are positive functions in $L^\infty((0, A) \times (0, A))$ with $c_i(x, a) \leq C_1$ for all $(x, a) \in (0, A) \times (0, A)$, $i = 1, 2$. (2.4)

Let $M$ be chosen such that

$$0 \leq u_0(a) \leq M, \quad 0 \leq v_0(a) \leq M. \quad \text{(2.5)}$$

We define our state solution space as

$$X = \{(u, v) \in L^\infty(Q)^2 \mid 0 \leq u(a, t) \leq M, \quad 0 \leq v(a, t) \leq M \text{ a.e. on } Q \}.$$ 

From Theorem 2.1 in [29], we have existence and uniqueness results for the state system in the above solution space and we have the following representation for the solutions:
\[
\begin{aligned}
\mathcal{U}(a,t) &= \begin{cases}
    - \int_t^a [\mu_1(s+a-t, u(s+a-t, s)) + f(s+a-t, s)] u(s+a-t, s) \, ds \\
    + \int_a^t [\mu_1(s+a-t, u(s+a-t, s)) + f(s+a-t, s)] u(s+a-t, s) \, ds \\
    - \int_{t-a}^t [\mu_1(s+a-t, u(s+a-t, s)) + f(s+a-t, s)] u(s+a-t, s) \, ds \\
    - \int_{t-a}^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    + \int_0^A c_1(x, s+a-t) v(x, s) \, dx \, ds \\
    + u_0(a-t) & \text{if } a > t,
    \\
    - \int_t^t [\mu_1(s+a-t, u(s+a-t, s)) + f(s+a-t, s)] u(s+a-t, s) \, ds \\
    - \int_{t-a}^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    + \int_0^A c_2(x, s+a-t) u(x, s) \, dx \, ds \\
    + v_0(a-t) & \text{if } a < t,
\end{cases}
\end{aligned}
\]

(2.6)

and

\[
\begin{aligned}
\mathcal{V}(a,t) &= \begin{cases}
    - \int_t^a [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    + \int_a^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    - \int_{t-a}^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    - \int_{t-a}^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    + \int_0^A c_2(x, s+a-t) u(x, s) \, dx \, ds \\
    + v_0(a-t) & \text{if } a > t,
    \\
    - \int_t^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    - \int_{t-a}^t [\mu_2(s+a-t, v(s+a-t, s)) + g(s+a-t, s)] v(s+a-t, s) \, ds \\
    + \int_0^A c_2(x, s+a-t) u(x, s) \, dx \, ds \\
    + v_0(a-t) & \text{if } a < t.
\end{cases}
\end{aligned}
\]

The formulas for \( u, v \) were derived by the method of characteristics. Lipschitz properties are developed for the solutions in terms of the controls in Theorem 2.1. These estimates are used to prove the existence and uniqueness of the optimal control pair in Section 4.

**Theorem 2.1.** For \( T \) sufficiently small, the map

\[
(f, g) \in U \rightarrow (u, v) = (u, v)[(f, g)] \in X
\]

is Lipschitz in the following ways:

\[
\int_Q \left( |\mu_1 - \mu_2| + |v_1 - v_2| \right)(a,t) \, da \, dt
\]

\[\leq C_2 T \int_Q \left( |f_1 - f_2| + |g_1 - g_2| \right)(a,t) \, da \, dt
\]

and

\[
\|u_1 - u_2\|_{L^\infty(Q)} + \|v_1 - v_2\|_{L^\infty(Q)}
\]

\[\leq C_3 T \left( \|f_1 - f_2\|_{L^\infty(Q)} + \|g_1 - g_2\|_{L^\infty(Q)} \right),
\]

where \((u_1, v_1) = (u, v)[(f_1, g_1)]\) and \((u_2, v_2) = (u, v)[(f_2, g_2)]\) for \((f_1, g_1), (f_2, g_2) \in U\).
Proof. From the representation of the state solution (2.6), (2.7) as a fixed point, we first do an $L^1$ estimate on $Q$, 

$$
\int_{Q \cap \{a < t\}} |u_1 - u_2| \, da \, dt \leq TM \int_{Q} |f_1 - f_2|(a, t) \, da \, dt + T C_4 \int_{Q} \left(|u_2 - u_1| + |v_2 - v_1|\right)(a, t) \, da \, dt.
$$

We used the Lipschitz property of $\mu_i$ and boundedness of $c_i, f_i, \beta_i$ for $i = 1, 2$. If one estimates the integral over $Q \cap \{a > t\}$, then the constant would depend on the bound of the initial condition $u_0$. Similar estimates hold for $v_1 - v_2$. Collecting terms, we obtain 

$$
\int_{Q} \left(|u_1 - u_2| + |v_1 - v_2|\right) \, da \, dt \leq C_5 \int_{Q} \left(|f_1 - f_2| + |g_1 - g_2|\right) \, da \, dt + C_T \int_{Q} \left(|u_1 - u_2| + |v_1 - v_2|\right) \, da \, dt,
$$

where $C_T < 1$, when $T$ is small.

Then we estimate the integral in the age variable only to use that result in the $L^\infty$ estimate, 

$$
\int_0^A |u_1 - u_2|(a, t) \, da \leq C_6 T \int_0^A \left(|u_1 - u_2| + |v_1 - v_2|\right)(a, t) \, da + C_T \|f_1 - f_2\|_{L^\infty}.
$$

For $T$ sufficiently small, the estimate becomes, 

$$
\sup_t \int_0^A |u_1 - u_2| + |v_1 - v_2|(a, t) \, da \leq C_8 \left\{\|f_1 - f_2\|_{L^\infty} + \|g_1 - g_2\|_{L^\infty}\right\}.
$$

In the $L^\infty$ estimate, for $t > a$, we have 

$$
|u_1(a, t) - u_2(a, t)| \leq C_\beta \int_0^A |u_1 - u_2|(s, t-a) \, ds + C_9 T \left(\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^\infty}\right) + \|f_1 - f_2\|_{L^\infty},
$$

where the $C_\beta$ coefficient depends on the bound of the birth term. The $t < a$ case is simpler. Thus using the estimate of the integral in the age variable, we obtain the $\|u_1 - u_2\|_{L^\infty}$ estimate and similarly the $\|v_1 - v_2\|_{L^\infty}$ estimate. Combining these results gives 

$$
\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^\infty} \leq C_{10} \left(\|f_1 - f_2\|_{L^\infty} + \|g_1 - g_2\|_{L^\infty}\right) + C_{11} T \left(\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^\infty}\right).
$$

If $T$ is small, we obtain the $L^\infty$ estimate. □
3. Optimality conditions

In order to differentiate our functional \( J \) with respect to the controls \( (f, g) \), we first need the differentiability of the solution map \( (f, g) \rightarrow (u, v) = (u, v)[(f, g)] \).

**Theorem 3.1.** The map \((f, g) \in U \rightarrow (u, v) = (u, v)[(f, g)] \in X\) is differentiable in the following sense:

\[
\frac{(u, v)[(f + \varepsilon l_1, g + \varepsilon l_2)] - (u, v)[(f, g)]}{\varepsilon} \rightarrow (\psi, \phi)
\]

in \( (L^\infty(Q))^2 \), for \((f + \varepsilon l_1, g + \varepsilon l_2)[(f, g)] \in U \) and \( \varepsilon \rightarrow 0 \) with \( l_1, l_2 \in L^\infty(Q) \). The sensitivities \( (\psi, \phi) \) satisfy

\[
\begin{align*}
\psi_t + \psi_a &= -\mu_1(u)\psi - \mu_1'(u)u\psi - f\psi - l_1u - \psi \int_0^A c_1(x, a)v(x, t)\,dx \\
&\quad - u \int_0^A c_1(x, a)\phi(x, t)\,dx, \\
\phi_t + \phi_a &= -\mu_2(v)\phi - \mu_2'(v)v\phi - g\phi - l_2v - \phi \int_0^A c_2(x, a)u(x, t)\,dx \\
&\quad - v \int_0^A c_2(x, a)\psi(x, t)\,dx \quad \text{in } Q,
\end{align*}
\]

(3.1)

\[
\psi = \phi = 0 \quad \text{for } a \in (0, A) \text{ and } t = 0,
\]

\[
\psi(0, t) = \int_0^A \beta_1(\tilde{a})\psi(\tilde{a}, t)\,d\tilde{a} \quad \text{for } t \in [0, T],
\]

\[
\phi(0, t) = \int_0^A \beta_2(\tilde{a})\phi(\tilde{a}, t)\,d\tilde{a} \quad \text{for } t \in [0, T],
\]

where \( \mu_i' \) means the partial derivative of \( \mu_i \) with respect to its second argument \( (u \text{ or } v) \).

**Proof.** Since the solution map

\[(f, g) \rightarrow (u, v)\]

is Lipschitz in \( L^\infty \) by Theorem 2.1, we have the existence of the Gateaux derivative \( \psi, \phi \) by [4, p. 17]. Passing to the limit in the representation of the quotients, gives that \( \psi, \phi \) satisfy system (3.1). \( \square \)

Our adjoint system corresponding to controls \( (f, g) \) and states \( (u, v) = (u, v)[(f, g)] \) is
\[-(p_t + p_a) = -\mu_1(u)p - \mu'_1(u)up - fp - p \int_0^A c_1(x, a)v(x, t) \, dx\]
\[-\int_0^A c_2(x, a)qv(x, t) \, dx + \beta_1(a)p(0, t) + K_1(a)f,\]
\[-(q_t + q_a) = -\mu_2(v)q - \mu'_2(v)vq - gq - q \int_0^A c_2(x, a)u(x, t) \, dx\]
\[-\int_0^A c_1(x, a)pu(x, t) \, dx + \beta_2(a)q(0, t) + K_2(a)g,\]  
(3.2)

\[p(a, T) = q(a, T) = 0 \text{ for } a \in [0, A]\]

The weak solution of the adjoint system satisfies
\[\int_Q (p\alpha_1 + q\alpha_2 - K_1fy_1 - K_2gy_2) \, da \, dt = 0\]
for any functions \(\alpha_1, \alpha_2 \in L^\infty(Q)\), where \(y_1\) and \(y_2\) satisfy

\[(y_1)_t + (y_1)_a + \left(\mu_1(u) + \mu'_1(u)u + f + \int_0^A c_1(x, a)v(x, t) \, dx\right)y_1\]
\[+ u \int_0^A c_2(x, a)y_2(x, t) \, dx = \alpha_1,\]

\[(y_2)_t + (y_2)_a + \left(\mu_2(v) + \mu'_2(v)v + g + \int_0^A c_2(x, a)u(x, t) \, dx\right)y_2\]
\[+ v \int_0^A c_1(x, a)y_1(x, t) \, dx = \alpha_2\]  
(3.3)

with

\[y_1(a, 0) = y_2(a, 0) = 0 \text{ for } a \in [0, A],\]

\[y_1(0, t) = \int_0^A \beta_1(\bar{a})y_1(\bar{a}, t) \, d\bar{a} \text{ for } t \in [0, T],\]
y_2(0,t) = \int_0^A \beta_2(\bar{a}) y_2(\bar{a},t) d\bar{a}.

Methods similar to Theorem 2.1 of [29] can be used to prove the existence of solutions to the adjoint system through the existence of solutions y_1, y_2 to the above system. See also [4] for similar existence results for the adjoint system. The solution of the adjoint system satisfies a Lipschitz condition with respect to the controls, which is needed for proving the existence of an optimal control pair. This Lipschitz dependence can be shown as in Theorem 2.1.

**Theorem 3.2.** For \((f,g) \in U\), the adjoint system (3.2) has a weak solution \((p,q)\) in \(L^\infty(Q) \times L^\infty(Q)\) such that

\[ \|p_1 - p_2\|_\infty + \|q_1 - q_2\|_\infty \leq C_1 T \left( \|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty \right), \]

where adjoint solutions \(p_i, q_i\) correspond to control pairs \((f_i, g_i), i = 1, 2\).

The characterization and uniqueness of the optimal control pair \((f^*, g^*)\) is dependent on the use of Ekeland’s principle [4,14]. To employ this principle, we embed our functional in the space \(L^1(Q)\) by defining

\[ J(f,g) = \begin{cases} J(f,g) & \text{if } (f,g) \in U, \\ -\infty & \text{if } (f,g) \notin U. \end{cases} \] (3.4)

**Theorem 3.3.** If \((f^*, g^*)\) in \(U\) is an optimal control pair maximizing (3.4) and \((u^*, v^*)\) and \((p,q)\) are the corresponding state and adjoint solutions, then

\[ f^*(a,t) = \mathcal{L}_1 \left( \frac{(K_1 - p)u^*}{B_1} \right), \]
\[ g^*(a,t) = \mathcal{L}_2 \left( \frac{(K_2 - q)v^*}{B_2} \right) \quad \text{a.e. in } L^\infty(Q), \] (3.5)

where

\[ \mathcal{L}_i(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq N_i, \\ N_i & \text{if } x > N_i \text{ for } i = 1, 2. \end{cases} \]

**Proof.** Since \((f^*, g^*)\) is an optimal control pair, then we have

\[ 0 \geq \lim_{\varepsilon \to 0^+} \frac{J(f^* + \varepsilon l_1, g^* + \varepsilon l_2) - J(f^*, g^*)}{\varepsilon} \]
\[ = \int_0^T \int_0^A \left( K_1 f^* \psi + K_2 g^* \phi + K_1 u^* l_1 + K_2 v^* l_2 - B_1 f^* l_1 - B_2 g^* l_2 \right) da dt \]
\[ = \int_0^T \int_0^A \left( p(-l_1 u^*) + q(-l_2 v^*) + K_1 u^* l_1 + K_2 v^* l_2 - B_1 f^* l_1 - B_2 g^* l_2 \right) da dt \]
\[ T \int_0^A l_1 [(K_1 - p)u^* - B_1 f^*] + l_2 [(K_2 - q)v^* - B_2 g^*] \, da \, dt, \]

where we used \( \alpha_1 = -l_1 u^* \) and \( \alpha_2 = -l_2 v^* \) in the weak solution definition of \( p, q \) and used the system satisfied by the sensitivities \((\psi, \phi)\) in (3.1). By standard optimality arguments, we obtain the representations in (3.5).

For notational purposes, define \( L(x_1, x_2) = (L_1(x_1), L_2(x_2)) \). The upper semi-continuity of the functional with respect to \( L^1(\mathbb{Q}) \) convergence is needed to prove the existence of the optimal control pair.

**Theorem 3.4.** The functional \( J(f, g) \) is upper semicontinuous with respect to \( L^1(\mathbb{Q}) \) convergence.

**Proof.** We suppose that
\[(f_n, g_n) \to (f, g) \quad \text{in} \quad L^1(\mathbb{Q}) \times L^1(\mathbb{Q}).\]

On a subsequence (using same notation),
\[f_n^2 \to f^2 \quad \text{a.e. on} \quad \mathbb{Q} \text{ by [15, p. 21].}\]

By Lebesgue’s dominated convergence theorem,
\[\lim_{n \to \infty} \int_{\mathbb{Q}} f_n^2 \, da \, dt = \int_{\mathbb{Q}} f^2 \, da \, dt.\]

We have a similar result for \( \int_{\mathbb{Q}} g_n^2 \, da \, dt \). These results handle the convergence of the squared terms in our functional.

Next, we illustrate the convergence of one term in the functional, denote \((u_n, v_n)\) as state solutions corresponding to \((f_n, g_n)\), and \((u, v)\) corresponding to \((f, g)\),
\[\left| \int_{\mathbb{Q}} K_1(f_n u_n - f u) \, da \, dt \right| \leq \int_{\mathbb{Q}} K_1|f_n - f|u_n a \, da \, dt + \int_{\mathbb{Q}} K_1|u_n - u|f \, da \, dt \]
\[\leq KM(\|f_n - f\|_{L^1(\mathbb{Q})} + \|g_n - g\|_{L^1(\mathbb{Q})}) + KN(\|u_n - u\|_{L^1(\mathbb{Q})} + \|v_n - v\|_{L^1(\mathbb{Q})}) \]
\[\leq K(M + N(C_{13}T))(\|f_n - f\|_{L^1(\mathbb{Q})} + \|g_n - g\|_{L^1(\mathbb{Q})}),\]

where \( K = \max(\|K_1\|_{\infty}, \|K_2\|_{\infty}) \).

In conclusion, we have the upper semi-continuity, \( J(f, g) \geq \limsup_{n \to \infty} J(f_n, g_n) \).

We now are ready to prove our main result, the existence of an optimal control pair.
4. Existence of the optimal control pair

The functional $\mathcal{J}(f, g)$ is upper semi-continuous with respect to strong $L^1$ convergence not with respect to weak $L^1$ convergence. We form “approximate” maximizing sequences which are strong $L^1$ convergent, by applying Ekeland’s principle [4,14]: For $\varepsilon > 0$, there exists $(f_\varepsilon, g_\varepsilon)$ in $L^1(Q) \times L^1(Q)$ such that

(i) $J(f_\varepsilon, g_\varepsilon) > \sup_{(f,g) \in U} J(f, g) - \varepsilon$,

(ii) $J(f_\varepsilon, g_\varepsilon) = \max \{ J(f, g) - \sqrt{\varepsilon} \| f_\varepsilon - f \|_{L^1(Q)} - \sqrt{\varepsilon} \| g_\varepsilon - g \|_{L^1(Q)} \mid (f, g) \in U \}$.

Note that $(f_\varepsilon, g_\varepsilon)$ is a maximizing pair for $J_\varepsilon$ defined by

$J_\varepsilon(f, g) = J(f, g) - \sqrt{\varepsilon} (\| f_\varepsilon - f \|_{L^1(Q)} + \| g_\varepsilon - g \|_{L^1(Q)})$.

**Theorem 4.1.** If $(f_\varepsilon, g_\varepsilon)$ is an optimal pair maximizing the functional $J_\varepsilon(f, g)$, then

$J_{\varepsilon}(f_\varepsilon, g_\varepsilon) = L \left( \frac{(K_1 - p_1)u_\varepsilon - \sqrt{\varepsilon} \theta_{1\varepsilon}}{B_1}, \frac{(K_2 - q_2)v_\varepsilon - \sqrt{\varepsilon} \theta_{2\varepsilon}}{B_2} \right)$,

where the functions $\theta_{1\varepsilon}, \theta_{2\varepsilon}$ belong to $L^\infty(Q)$ and $|\theta_i(a,t)| \leq 1$ for $i = 1, 2$, for all $(a,t) \in Q$.

This result can be obtained by the same proof technique as in Theorem 3.3.

**Theorem 4.2.** If $T(1/B_1 + 1/B_2)$ is sufficiently small, there exists one and only one optimal control pair $(f^*, g^*)$ in $U$ maximizing $J(f, g)$.

**Proof.** First we prove uniqueness. Define $F : U \rightarrow U$ by

$F(f, g) = L \left( \frac{(K_1 - p_1)u}{B_1}, \frac{(K_2 - q_2)v}{B_2} \right)$,

with $u, v$ and $p, q$ being the state and adjoint solutions corresponding to $(f, g)$. Using the Lipschitz properties of $u, v$ and $p, q$ from Theorems 2.1 and 3.2, we have

$\| F(f_1, g_1) - F(f_2, g_2) \|_\infty \leq \frac{\| (K_1 - p_1)u_1 - (K_1 - p_2)u_2 \|_\infty}{B_1} + \frac{\| (K_2 - q_2)v_1 - (K_2 - q_2)v_2 \|_\infty}{B_2}$

$\leq C_{14} T \left( \frac{1}{B_1} + \frac{1}{B_2} \right) (\| f_1 - f_2 \|_\infty + \| g_1 - g_2 \|_\infty)$, (4.1)

where the constant $C_{14} T$ depends on the $L^\infty$ bounds on the state and adjoint solutions and the Lipschitz constants. If $C_{14} T (1/B_1 + 1/B_2)$, then the map $F$ has a unique fixed point $(f^*, g^*)$.

To prove this fixed point is an optimal control pair, we use the approximate maximizers $(f_\varepsilon, g_\varepsilon)$ from Ekeland’s principle and corresponding states $u_\varepsilon, v_\varepsilon$ and adjoints $p_\varepsilon, q_\varepsilon$. From Theorem 4.1 and the contraction property of $F$, we have
\[
\begin{align*}
\left\| F(f_\varepsilon, g_\varepsilon) - \mathcal{L}\left( \frac{(K_1 - p_\varepsilon)u_\varepsilon - \sqrt{\varepsilon \theta_1^\varepsilon}}{B_1}, \frac{(K_2 - q_\varepsilon)v_\varepsilon - \sqrt{\varepsilon \theta_2^\varepsilon}}{B_2} \right) \right\|_\infty
\leq & \quad \left\| \mathcal{L}\left( \frac{(K_1 - p_\varepsilon)u_\varepsilon - \sqrt{\varepsilon \theta_1^\varepsilon}}{B_1}, \frac{(K_2 - q_\varepsilon)v_\varepsilon - \sqrt{\varepsilon \theta_2^\varepsilon}}{B_2} \right) \right\|_\infty \\
\leq & \quad \frac{\varepsilon}{B_1} + \frac{\varepsilon}{B_2} \leq \varepsilon \left( \frac{1}{B_1} + \frac{1}{B_2} \right). \quad (4.2)
\end{align*}
\]

Next, we use (4.1) and (4.2) to show that \((f_\varepsilon, g_\varepsilon) \rightarrow (f^*, g^*)\) in \(L^\infty(Q) \times L^\infty(Q)\). This gives

\[
\left\| (f^*, g^*) - (f_\varepsilon, g_\varepsilon) \right\|_\infty = \left\| f^* - f_\varepsilon \right\|_\infty + \left\| g^* - g_\varepsilon \right\|_\infty
\leq \left\| F(f^*, g^*) - F(f_\varepsilon, g_\varepsilon) \right\|_\infty
\leq \left\| F(f_\varepsilon, g_\varepsilon) - \mathcal{L}\left( \frac{(K_1 - p_\varepsilon)u_\varepsilon - \sqrt{\varepsilon \theta_1^\varepsilon}}{B_1}, \frac{(K_2 - q_\varepsilon)v_\varepsilon - \sqrt{\varepsilon \theta_2^\varepsilon}}{B_2} \right) \right\|_\infty
\leq \frac{\varepsilon}{B_1} + \frac{\varepsilon}{B_2}
\leq C_{14}T \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \left( \left\| f^* - f_\varepsilon \right\|_\infty + \left\| g^* - g_\varepsilon \right\|_\infty \right) + \sqrt{\varepsilon} \left( \frac{1}{B_1} + \frac{1}{B_2} \right).
\]

For \(T(1/B_1 + 1/B_2)\) small enough, we obtain

\[
\left\| (f^*, g^*) - (f_\varepsilon, g_\varepsilon) \right\|_\infty \leq \frac{\varepsilon}{1 - C_{14}T(1/B_1 + 1/B_2)},
\]

which gives the desired convergence. Using property (i) of Ekeland’s principle, the inequality

\[
\mathcal{J}(f_\varepsilon, g_\varepsilon) > \sup_{(f,g) \in U} \mathcal{J}(f, g) - \varepsilon
\]

implies (as \(\varepsilon \to 0\))

\[
\mathcal{J}(f^*, g^*) \geq \sup_{(f,g) \in U} \mathcal{J}(f, g). \quad \Box
\]

Combining Theorems 3.3 and 4.2, we obtain a characterization of the optimal control pair in terms of the state system, adjoint system, and the relationship (3.5).

References