THE GEOMETRIC THEORY OF INVERSE SEMIGROUPS I: 
E-UNITARY INVERSE SEMIGROUPS

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We show how the correspondence between inverse semigroups and inductive groupoids (a class of ordered groupoids) leads to a reinterpretation of McAlister’s P-Theorem as a function extension theorem. This result can only be expressed, in general, by working within a larger category: namely that of functorially ordered groupoids. The consequences of this approach for the theory of E-unitary covers of inverse semigroups and its generalisations are worked out in subsequent papers.

Introduction

This article is the first of three, in which we show how the theory of (ordered) groupoids may be used to shed light on certain aspects of the theory of inverse semigroups.

In this paper, we first discuss the correspondence, due to Ehresmann and Schein, between inverse semigroups and a class of ordered groupoids, called inductive groupoids, which will form the basis for all our subsequent work. We then apply this correspondence, by analysing the well-known structure theory of E-unitary inverse semigroups.

Two main points arise:

1. The structure theory of E-unitary semigroups is equivalent to a function extension theorem.

2. In order to describe this extension, it is necessary to work within the larger category of functorially ordered groupoids.

Point (2) fits in with other recent work in semigroup theory in which categories play a role, such as that of Tilson [38]. For example, the simple expedient of

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regarding a monoid as a category with a unique object has turned out to be a surprisingly effective standpoint for developing certain aspects of semigroup theory. A recent application of this idea may be found in Margolis and Pin's generalisation of the \(P\)-theorem to a wide class of semigroups [17].

Our approach is different, however, and depends on some special properties of inverse semigroups which ally them with groupoids. If there are generalisations of our theory, they will depend on a better understanding of the relationship between classes of ordered categories and classes of semigroups. The first steps in this direction are taken in [15].

In our second paper [12], we show how our approach to the structure of \(E\)-unitary inverse semigroups, together with the principle of working within the category of functorially ordered groupoids, enables us not only to develop the theory of \(E\)-unitary covers but also to account for the calculations carried out by McAlister and Reilly [23], who first established this theory. Once again, it will emerge, that the underlying idea involved in the theory of \(E\)-unitary covers may be expressed as an extension theorem for certain kinds of function.

In the final paper in the sequence [13], we show that there is one theorem underlying the previous two papers: Ehresmann's Maximum Enlargement Theorem [2, 3]. We provide an account of this result and then apply it to, amongst other things, O'Carroll's work on idempotent pure homomorphisms [28].

Taken together, these papers are a contribution to the theory of idempotent pure, \(\nu\)-prehomomorphisms (which are described in the first section).

Before explaining in more detail the programme of this paper, it may be useful to provide some general background.

Inverse semigroups were first introduced by Vagner [39] back in 1952 (and independently, a little later, by Preston [30]). On the other hand, we might take the inception of the theory of ordered groupoids as Ehresmann's 1957 paper [3, II-1, p. 47] (note that we shall refer to Ehresmann's papers throughout by giving the volume number followed by the paper number in the Oeuvres Complètes). Both Vagner and Ehresmann were differential geometers and both were motivated by the desire to develop a theory based on pseudogroups of local homeomorphisms (or diffeomorphisms).

There are innumerable definitions of such pseudogroups but, following [7], they are all concerned with families of local homeomorphisms (or local diffeomorphisms) on a topological space (respectively, differentiable manifold) closed under inverses and composition of partial functions.

Pseudogroups have a long, if rather neglected, history. They arose first in the work of Lie, who was interested in pseudogroups of local diffeomorphisms, defined by a family of partial differential equations (what are now called Lie pseudogroups). Pseudogroups, without the differential equations, were later central to Veblen and Whitehead's [40] attempt to provide a foundation for differential geometry, based on a generalisation of Klein's Erlanger Program. This book, and the foundational role of pseudogroups within it, influenced both Vagner and Ehresmann [36], [3, II-1,
The contrasting approaches of Vagner and Ehresmann arise from the fact that there are two natural products defined on the set of partial transformations of a space. The usual composition of partial functions is everywhere defined and associative and so forms a semigroup; this was Vagner's starting point. Ehresmann, on the other hand, took as basic the 'reduced product' of partial functions, which is defined only when the domain of one function matches up with the range of the other. This leads to a groupoid, in the sense of category theory. This groupoid together with the extension ordering of partial functions underlies an ordered groupoid structure.

Inverse semigroups and (classes of) functorially ordered groupoids are then essentially the abstract versions of these two approaches. Indeed, the Vagner–Preston Representation Theorem asserts that every inverse semigroup may be faithfully represented as an inverse semigroup of partial one-to-one mappings on a set \[ \{0,1,\ldots, n\} \].

Note that in this process of abstraction, we have 'forgotten' the underlying topology. Nevertheless, in Ehresmann's work, the underlying topology leaves behind some trace, in that he frequently works with functorially ordered groupoids whose identities form a frame—the lattice of open sets of a topological space being a motivating example.

We now turn to the contents of this paper. In the first section, we give an account of \( \mathcal{V} \)-prehomorphisms and summarise the theory of \( E \)-unitary covers of inverse semigroups. Much more on the latter may be found in Petrich [29]. In Section 2, we define functorially ordered groupoids and their associated morphisms. In Section 3, we set up the correspondence between inverse semigroups and inductive groupoids: this is based on work of Ehresmann and Schein. In the fourth section, we apply the correspondence to study \( E \)-unitary inverse semigroups and, as a consequence, obtain a re-interpretation of the \( P \)-Theorem.

1. Inverse semigroups

In order to make this paper reasonably accessible to those without a background in inverse semigroup theory, we have included a little more in this section than the bare notational conventions: for these we essentially follow Howie [10].

The collection of idempotents of a semigroup \( S \) will be denoted by \( E(S) \) or \( E \), when there is no likelihood of confusion. This work is concerned with inverse semigroups. A semigroup \( S \) is said to be inverse if, for each element \( x \in S \), there is a unique element \( x^{-1} \), called the inverse of \( x \), such that

\[
x - xx^{-1}x \quad \text{and} \quad x^{-1} = x^{-1}xx^{-1}.
\]
More generally, a semigroup $S$ is called *regular* if for each element $x$ of $S$ there exists at least one element $a$ such that $x = xax$. Inverse semigroups are characterised within the class of regular semigroups by the property that their idempotents form a commutative, idempotent subsemigroup [10, Theorem V.1.2]. Commutative, idempotent semigroups are called *commutative bands*.

Commutative bands may also be viewed as ordered structures. Recall that a *meet semilattice* is a partially ordered set in which every pair of elements has a greatest lower bound.

**Theorem 1.1.** (Howie [10, Proposition I.3.3]). (i) Let $(E, \leq)$ be a meet semilattice. Under the operation of greatest lower bound (glb), denoted by $\wedge$, the structure $(E, \wedge)$ is a commutative band, in which for all elements $e, f$ of $E$ we have

$$e \leq f \text{ if and only if } e \wedge f = e.$$  

(ii) Let $(E, .)$ be a commutative band. The relation $\leq$ defined on $E$ by

$$e \leq f \text{ if and only if } e . f = e$$

is a partial order on $E$, with respect to which $(E, \leq)$ is a meet semilattice. Furthermore, for all elements $e$ and $f$ the equality $e \wedge f = e . f$ holds.  

The set of idempotents of an inverse semigroup comes equipped with a partial ordering, denoted $\omega$, with respect to which $(E(S), \omega)$ is a meet semilattice. This ordering may be extended to a partial ordering, defined on the whole of $S$, called the *natural partial order* and denoted by $\leq$, by putting

$$x \leq y \text{ if and only if } x = ey$$

for some idempotent $e$. Standard properties of this ordering may be found in [10, Proposition V.2.4].

Green’s relations will be denoted $R, L, H, D$ and $J$. If $K$ is one of Green’s relations, we shall write $K_x$ for the $K$-equivalence class containing $x$. A categorical interpretation of Green’s relations on inverse semigroups will be given in Section 3.

The following result summarises some standard properties of the notions mentioned.

**Proposition 1.2.** Let $S$ be an inverse semigroup. Then:

(i) If $x \leq y$ and either $xLy$ or $xRy$ then $x = y$.

(ii) $E(S)$ is an order ideal of $(S, \leq)$.

(iii) If $e \leq x^{-1}x$ then $xe$ is characterised by the property that $xe \leq x$ and $(xe)^{-1}(xe) = e$.

(iv) If $e \leq xx^{-1}$ then $ex$ is characterised by the property that $ex \leq x$ and $(ex)(xe)^{-1} = e$.  

□
The geometric theory of inverse semigroups I

The reduced product on an inverse semigroup $S$ is a partial multiplication ' $ defined as follows:

$$x \cdot y = \begin{cases} xy & \text{if } x^{-1}x = yy^{-1}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

When the context is clear, we shall simply represent the reduced product by juxtaposition. The reduced product and the natural partial order together determine the semigroup multiplication, a result which, in one form or another, has probably been known since inverse semigroups were first investigated.

**Proposition 1.3.** Let $x$ and $y$ be elements of an inverse semigroup, then there exist elements $x'$ and $y'$ such that

$$x' \leq x, \quad y' \leq y \quad \text{and} \quad xy = x' \cdot y'.$$

**Proof.** Put $x' = xyy^{-1}$ and $y' = x^{-1}xy$. □

In general, the natural partial order is non-trivial. Inverse semigroups, in which the collection of non-zero elements is unordered under the natural partial order, are said to be primitive.

Homomorphisms between inverse semigroups are semigroup homomorphisms and congruences are semigroup congruences. We shall adopt the convention of [10] that whenever $\tau$ is a congruence on a semigroup $S$, the corresponding natural map $S \to S/\tau$ will be denoted by $\tau^i$. The next result collects together some standard properties of homomorphisms between inverse semigroups.

**Theorem 1.4.** Let $\theta : S \to T$ be a semigroup homomorphism between inverse semigroups. Then:

(i) If $x \in S$, then $\theta(x^{-1}) = \theta(x)^{-1}$.

(ii) The image $\theta(S)$ is an inverse subsemigroup of $T$.

(iii) If $x \leq y$ in $S$, then $\theta(x) \leq \theta(y)$.

(iv) If $\theta(x) = e'$, an idempotent in $T$, then there exists an idempotent $e \in S$ such that $\theta(e) = e'$.

(v) If $\theta(x) \leq \theta(y)$ in $T$, there exists an element $x' \in S$ such that $x' \leq y$ and $\theta(x') = \theta(x)$. □

The properties (iii) and (v) above may be expressed by saying that semigroup homomorphisms preserve and reflect the natural partial orders.

A $\lor$-prehomomorphism between inverse semigroups $S$ and $T$ is a map $\theta : S \to T$ having the following properties:

1. $\theta(xy) \leq \theta(x)\theta(y)$.
2. $\theta(x^{-1}) = \theta(x)^{-1}$.

In this paper, we shall refer to maps, such as $\theta$ above, as prehomomorphisms.
(There is a 'dual' notion, namely the ∧-prehomomorphisms, with which we shall not here be concerned. Both of these terms are due to McAlister and Reilly [23].)

**Proposition 1.5.** Let \( \theta : S \to T \) be a prehomomorphism. Then:
(i) \( \theta \) maps idempotents to idempotents.
(ii) \( \theta \) preserves the natural partial order.
(iii) If \( x \cdot y \) is a reduced product, then \( \theta(x \cdot y) = \theta(x) \cdot \theta(y) \).
(iv) A map \( \theta : S \cdot T \), which preserves the natural partial order, and for which \( \theta(x \cdot y) = \theta(x) \cdot \theta(y) \) whenever \( x \cdot y \) is a reduced product, is a prehomomorphism.

**Proof.** Parts (i) and (ii) are taken from [20, Lemma 2.1] and part (iii) from [20, Lemma 1.4]. Part (iv) is implicit in [27]. □

**Lemma 1.6.** (i) \( \theta : S \to T \) be a map such that \( \theta(xy) \leq \theta(x) \theta(y) \). Then \( \theta \) is a prehomomorphism.
(ii) If \( \theta : S \to T \) is a prehomomorphism such that \( \theta(e f) = \theta(e) \theta(f) \) for all idempotents \( e \) and \( f \) of \( E(S) \), then \( \theta \) is a semigroup homomorphism.

**Proof.** (i) By definition, we have that
\[
\theta(x) = \theta(x x^{-1} x) \leq \theta(x) \theta(x^{-1}) \theta(x)
\]
and
\[
\theta(x^{-1}) = \theta(x^{-1} x x^{-1}) \leq \theta(x^{-1}) \theta(x) \theta(x^{-1}).
\]
Put \( a = \theta(x) \) and \( b = \theta(x^{-1}) \), so that
\[a \leq aba \quad \text{and} \quad b \leq bab.
\]
From \( a \leq aba \) we have \( ab \leq abab \), so that by using one of the equivalent characterisations of the natural partial order, we may write
\[(ab)^2(ab)^{-1} = (ab)(ab)^{-1}.
\]
But then, on multiplying on the right by \( (ab) \) and reducing, we obtain \( (ab)^2 = ab \). We may similarly show that \( (ba)^2 = ba \). Hence
\[a(ba) \leq a \leq aba.
\]
Thus \( a = aba \). Similarly, \( bab = b \); whence \( \theta(x^{-1}) = \theta(x)^{-1} \).

(ii) Let \( x, y \in S \). By Proposition 1.3,
\[xy = (xe) \cdot (ey)
\]
where \( e = x^{-1}xyy^{-1} \). By Proposition 1.5,
\[\theta(xy) = \theta(xe) \cdot \theta(ey),
\]
since prehomomorphisms preserve the reduced product. Again by Proposition 1.5, prehomomorphisms preserve the natural partial order, so that \( xe \leq x \) implies
\( \theta(xe) \leq \theta(x) \) and \( ey \leq y \) implies \( \theta(ey) \leq \theta(y) \). Now note that
\[
\theta(xe)^{-1} \theta(xe) = \theta((xe)^{-1}(xe)) = \theta(e).
\]
Similarly, \( \theta(ey) \theta(ey)^{-1} = \theta(e) \). We now apply Proposition 1.2(iii) and (iv), and obtain that
\[
\theta(xe) = \theta(x) \theta(e) \quad \text{and} \quad \theta(ey) = \theta(e) \theta(y).
\]
By assumption, \( \theta(e) = \theta(x^{-1}x) \theta(yy^{-1}) \). Thus
\[
\theta(xy) = \theta(xe) \theta(ey) = \theta(x) \theta(e) \theta(e) \theta(y) = \theta(x) \theta(e) \theta(y)
\]

\[
= \theta(x) \theta(x^{-1}x) \theta(yy^{-1}) \theta(y) = \theta(xx^{-1}x) \theta(yy^{-1}y)
\]

\[
= \theta(x) \theta(y),
\]
where we apply again the fact that \( \theta \) preserves the reduced product. \( \square \)

The proof of (i) above is due to El-Qallali [4].

**Theorem 1.7** (McAlister [20, Corollary 2.2]). *Inverse semigroups and prehomomorphisms form a category.* \( \square \)

We shall denote the category of inverse semigroups and prehomomorphisms by \( \text{Ip} \) and the category of inverse semigroups and semigroup homomorphisms by \( \text{Ih} \). McAlister [20, Theorem 2.3] proves that \( \text{Ih} \) is a coreflective subcategory of \( \text{Ip} \).

**Lemma 1.8.** If \( \theta : S \rightarrow T \) is a prehomomorphism then for each idempotent \( e \in E(S) \) the map \( \theta \) induces, by restriction, a map \( \theta_e : L_e \rightarrow L_{\theta(e)} \).

**Proof.** If \( xLe \) then \( x^{-1}x = e \) so that \( \theta(x^{-1}x) = \theta(e) \). But \( x^{-1}x \) is a reduced product, so that by applying Proposition 1.5 we obtain \( \theta(x^{-1}x) = \theta(x^{-1}) \theta(x) \). Together with the fact that \( \theta(x^{-1}) = \theta(x)^{-1} \) we have proved \( \theta(x)L\theta(e) \). \( \square \)

With Lemma 1.8 in mind, it will be convenient to introduce some (nonstandard) terminology. A prehomomorphism \( \theta : S \rightarrow T \) will be called an *L-injection* (respectively: *L-surjection*, *L-bijection*) if each of the maps \( \theta_e \), where \( e \) is an idempotent, is an injection (respectively: surjection, bijection) from \( L_e \) to \( L_{\theta(e)} \).

**Proposition 1.9.** Let \( \theta : S \rightarrow T \) be a prehomomorphism. Then the following are equivalent:

(i) Whenever \( \theta(x) \) is an idempotent then \( x \) is an idempotent.

(ii) \( \theta \) is an *L-injection*.

**Proof.** (i) implies (ii). Suppose that \( xLy \) and \( \theta(x) = \theta(y) \). Then the element \( \theta(xy^{-1}) \) is an idempotent. This implies by (i) that \( xy^{-1} \) is an idempotent, \( e \) say. Multiplying
We have $xy^{-1} = e$ on the right by $y$ and noting that $x^{-1}x = y^{-1}y$ from $xLy$, we obtain $x = ey$. But then $x \leq y$ together with $xLy$ gives $x = y$, by an application of Proposition 1.2.

(ii) implies (i). Suppose that $\theta(x) = e$, an idempotent. Then $\theta(x^{-1}x) = e$, which, together with the fact that $xLx^{-1}x$, implies $x = x^{-1}x$ by (ii). \hfill \Box

In semigroup theory, the maps $\theta$ satisfying condition (i) above are usually called idempotent pure (or idempotent determined). As we noted in the introduction, our work will mainly deal with $L$-injective prehomomorphisms.

One early aim of inverse semigroup theory was to describe arbitrary inverse semigroups in terms of groups and semilattices. Theorem 1.11 below, together with Theorem 4.2, may be regarded as one answer to this question. To state this result, we need some further definitions.

Define a relation $\sigma$ on an inverse semigroup $S$ without zero as follows

$x \sigma y \iff$ there exists an element $z \in S$ such that $z \leq x$ and $z \leq y$.

The relation $\sigma$ is a congruence, called the minimum group congruence on $S$. Its properties are described in [10, Theorem V.3.1].

Congruences such as $\sigma$ identify all idempotents. At the other extreme a homomorphism (or, correspondingly, a congruence) $\theta : S \rightarrow T$ such that $\theta(e) = \theta(f)$ implies $e = f$, whenever $e$ and $f$ are idempotents, is said to be idempotent separating.

Note that $\Delta$ below denotes the identity equivalence.

Proposition 1.10. (McAlister [19, Proposition 1.1]). Let $S$ be an inverse semigroup. The following are equivalent:

(i) For all elements $x \in S$ and $e \in E(S)$, the condition $e \leq x$ implies $x \in E(S)$.
(ii) $\sigma \cap R = \Delta$ (equivalently $\sigma \cap L = \Delta$).
(iii) The map $\sigma^3 : S \rightarrow S/\sigma$ is an $R$-injection (equivalently an $L$-injection).
(iv) The map $\phi : S \rightarrow E(S) \times S/\sigma$ defined by $\phi(x) = (x^{-1}x, \sigma(x))$ is one-to-one.

An inverse semigroup satisfying one of the equivalent conditions in the proposition above is called $E$-unitary.

McAlister [18, 19] obtained a structure theorem for $E$-unitary semigroups, in terms of groups and semilattices, which we shall describe in a categorical context in Section 4.

$E$-unitary inverse semigroups in general were first considered by Saitô [33], although, as Schein [37] points out, Golqeb [5] was the first to consider inverse semigroups of partial one-to-one mappings on a set, possessing the defining property of $E$-unitary semigroups. One important class of semigroups which are $E$-unitary are the free inverse semigroups (see [29, Chapter VIII]). However, perhaps the most cogent reason for studying $E$-unitary semigroups is provided by another theorem of McAlister [18]:
**Theorem 1.11** (McAlister [18, Corollary 2.5]). Let $S$ be an inverse semigroup. Then there exists an $E$-unitary inverse semigroup $T$ and an idempotent separating homomorphism from $T$ onto $S$. □

If $T/a = G$, then $T$ is called an $E$-unitary cover of $S$ through $G$. Such covers are by no means unique—see [29, Theorem VII.4.14] for the construction of all $E$-unitary covers of those primitive inverse semigroups $S$, in which all non-zero elements of $S$ are $D$-related, the so-called Brandt semigroups.

Taken together with the description of $E$-unitary semigroups given in Theorem 4.2, Theorem 1.11 may certainly be interpreted as describing arbitrary semigroups in terms of groups and semilattices. However, an equivalent theory, developed by Joubert [11] in the course of his work on topological foliations, adopts a different point of view. The starting point for this approach essentially concerns the problem of describing to what extent local automorphisms of a structure may be extended to global automorphisms. This idea recurs in McAlister and Reilly [23], and more explicitly in McAlister’s survey article [22] (note also [21], which contains important information on the semigroups $K(G)$, which play an important role in this theory).

It is Joubert’s approach, which has provided the motivation for our sequence of papers.

### 2. Ordered groupoids

We begin with some notation concerning ordered sets. Let $(P, \leq)$ and $(Q, \preceq)$ be posets. We shall denote by $[x]$ the set $\{y: y \preceq x\}$, where $x \in P$, the principal order ideal generated by $x$. Let $f: P \rightarrow Q$ be an order preserving map. For each $x \in P$ we have an induced map by restriction $(f|[x]): [x] \rightarrow [f(x)]$. We shall say that $f$ possesses a property principally if for each $x \in P$ the map $(f|[x])$ has that property. We shall say that the poset $(P, \leq)$ itself has a property principally if each principal order ideal has that property.

We next turn to the notation used for categories. All categories will be small, unless we are dealing with large categories of structures. We also adopt the ‘second definition’ of a category given in [8], where a category is regarded as a class equipped with a partial associative operation. Elements of categories will sometimes be called morphisms.

The collection of identities of a category $C$ will be denoted by $C_o$—where ‘$o$’ stands for ‘object’. The collection of composable maps in $C$ is the set

$$C \ast C = \{(x, y) \in C \times C: xy \text{ exists in } C\}.$$  

Define also the following maps:

- $r: C \rightarrow C_o$ given by $x \mapsto r(x)$, the left identity of $x$ (the ‘range of $x$’),
- $d: C \rightarrow C_o$ given by $x \mapsto d(x)$, the right identity of $x$ (the ‘domain of $x$’),
\[ k : C \times C \to C \] given by \((x, y) \mapsto xy\), composition,

\[ [r, d] : C \to C_0 \times C_0 \] given by \(x \mapsto (r(x), d(x))\), the anchor of \(C\).

If \(C\) is, in addition, a groupoid then we have a map

\[ j : C \to C \] given by \(x \mapsto x^{-1}\).

Note that the category product \(xy\) exists if and only if \(d(x) = r(y)\).

The 'hom-sets' of the category \(C\) are defined as follows: if \(e, f \in C_0\) then

\[ \text{hom}(e, f) = \{x \in C : d(x) = e \text{ and } r(x) = f\} \].

With these preliminaries dispensed with, we may now provide the key definitions, which underlie our work. They are all adapted from [3, III-1, p. 63].

A category \(C\) is said to be ordered if it is equipped with a partial ordering \(\leq\) (on its set of morphisms) such that the following axioms hold:

1. \((OC1)\) \(C\) is a category and \((C, \leq)\) a poset.
2. \((OC2)\) if \(x \leq y\) then \(r(x) \leq r(y)\) and \(d(x) \leq d(y)\).
3. \((OC3)\) if \(x' \leq x\) and \(y' \leq y\) and both \(x'y'\) and \(xy\) exist then \(x'y' \leq xy\).
4. \((OC4)\) if \(r(x) = r(y)\), \(d(x) = d(y)\) and \(x \leq y\) then \(x = y\) (this implies that \(\leq\) is trivial on hom-sets).

If \(C\) is a groupoid we require, in addition, the following:

1. \((G)\) if \(x \leq y\) then \(x^{-1} \leq y^{-1}\).

In this paper, as well as in [12] and [13], we shall deal entirely with classes of ordered groupoids. More general classes of ordered categories are discussed in [15].

We shall be interested in more restrictive classes of ordered categories and to that end we list some further axioms, which we shall have occasion to refer to:

1. \((OC5)\) (i) If \(x \in C\) and \(e \leq d(x)\), where \(e \in C_0\), then there exists an element \(x'\) such that \(x' \leq x\) and \(d(x') = e\).
   (ii) If \(x \in C\) and \(e \leq r(x)\), where \(e \in C_0\), then there exists an element \(x'\) such that \(x' \leq x\) and \(r(x') = e\).
2. \((OC7)\) if \((x, y) \in C \times C\) and \(z \leq xy\), then there exists an \(x' \leq x\) and \(y' \leq y\) such that the product \(x'y'\) exists and \(x'y' = z\).
3. \((OC8)\) (i) If \(x \in C\) and \(e \leq d(x)\), where \(e \in C_0\), then there exists a unique element, called the restriction of \(x\) to \(e\), which will be denoted by \((x|e)\), such that \((x|e) \leq x\) and \(d(x|e) = e\).
   (ii) If \(x \in C\) and \(e \leq r(x)\), where \(e \in C_0\), then there exists a unique element, called the corestriction of \(x\) to \(e\), which will be denoted by \((e|x)\), such that \((e|x) \leq x\) and \(r(e|x) = e\).

The reader should note that the 'missing' axiom \((OC6)\), which will play no role here, will make its appearance in a later paper on ordered categories [15].

A groupoid satisfying \((OC1), (G), (OC3)\) and \((OC8)\) will be said to be functorially ordered. Officially, a functorially ordered groupoid is a triple \((G, \leq)\), but we shall usually refer to the 'functorially ordered groupoid \(G\)' etc. When the identities of such a groupoid form a meet semilattice, under the induced order, it will be called
an inductive groupoid. This latter term is rather unsatisfactory, since ‘inductivity’, in any sense of the word, plays no role. But it is sanctioned by use, so we shall abide by it here. We favour the term ‘pseudogroup’, which would agree with Rinow’s terminology [32], except that we do not require the existence of a smallest element.

It is important to stress that inductive groupoids are purely order-theoretic structures, aside from the multiplicative component contributed by the groupoid. This is slightly obscured in [29, p. 625], where the restrictions and corestrictions of an element are regarded as contributing extra products. For this reason, we list below some equivalent formulations of the axioms, which stress their order-theoretic character. It is an attempt at a compromise between Ehresmann’s notation, which is unpopular, and the semigroup usage, which is slightly misleading. In what follows, the partially ordered sets will all be the obvious ones. Let \( C \) be a category satisfying (OC1). Then

1. The map \([r, d] : C \to C \times C \) is order-preserving, and has the property that
   \[ x \leq y \text{ and } [r(x), d(x)] = [r(y), d(y)] \ \text{ implies } \ x = y, \]
   if and only if (OC2) and (OC4) hold.

2. The map \( k : (C \times C, \leq) \to (C, \leq) \) is order-preserving if and only if (OC3) holds.

3. \( j : C \to C \) is order-preserving if and only if (G) holds.

4. The maps \( d, r : C \to C \) are principally onto if and only if (OC5) holds.

5. The map \( k : (C \times C, \leq) \to (C, \leq) \) is principally onto if and only if (OC7) holds.

6. The maps \( d, r : C \to C \) are principally isomorphisms if and only if (OC8) holds.

Note that functorially ordered groupoids are equivalent to certain kinds of double categories [3, III 1, p. 63] and are examples of structured categories.

**Lemma 2.1.** Let \( C \) be a category satisfying (OC1), (OC2), (OC3) and (OC8) and let \( x \) and \( y \) be a composable pair of elements of \( C \). Then:

1. If \( e \leq d(y) \), then \( (xy|e) = (x|r(y|e))(y|e) \).
2. If \( e \leq r(x) \), then \( (e|xy) = (e|x)(d(e|x)|y) \).

**Proof.** We shall prove (i), the proof of (ii) is similar. We must first establish that the elements \((y|e)\) and \((x|r(y|e))\) are well-defined. In the first instance, since the product \(xy\) exists, we have \(d(xy) = d(y)\). But then from \(e \leq d(y)\) we obtain \(e \leq d(xy)\).

Thus by (OC8) the element \((xy|e)\) exists. Similarly, from the fact that \(e \leq d(y)\), the element \((y|e)\) exists and, furthermore, \((y|e) \leq y\). By (OC2) we have \(r(y|e) \leq r(y)\). However, the product \(xy\) exists so that \(d(x) = r(y)\). Thus \(r(y|e) \leq d(x)\). An application of (OC8) shows that the element \((x|r(y|e))\) exists.

It is immediate that \(d(x|r(y|e)) = r(y|e)\), so that the product \((x|r(y|e))(y|e)\) exists. Furthermore, \((x|r(y|e)) \leq x\) and \((y|e) \leq y\) so that by (OC3) we have \((x|r(y|e))(y|e) \leq xy\). We may now conclude the proof for we have both

\[
d((x|r(y|e))(y|e)) = d(y|e) = e \quad \text{and} \quad d(xy|e) = e
\]
and, in addition,

\[(x|r(y|e))(y|e) \leq xy \quad \text{and} \quad (xy|e) \leq xy.\]

But axiom (OC8) provides for a unique element satisfying these conditions. Thus

\[(xy|e) = (x|r(y|e))(y|e). \quad \square\]

**Proposition 2.2.** (i) Let \( C \) be a category satisfying (OC1), (OC2), (OC3) and (OC8). Then (OC4) and (OC7) hold and \( C_0 \) is an order ideal of \( (C, \leq) \) (this latter condition we will refer to as (O1)). If \( C \) is, in addition, a groupoid, then (G) holds.

(ii) Let \( (C, \leq) \) be a groupoid equipped with a partial order satisfying the axioms (OC1), (G), (OC3), (OC8)(i). Then \( (C, \leq) \) satisfies (OC2) and (OC8)(ii).

(iii) Let \( C \) be a groupoid satisfying the axioms (OC1), (G), (OC3), (OC5)(i) and (O1). Then \( C \) satisfies (OC2) and (OC8).

**Proof.** (i) (OC4) holds: Let \( x \leq y \), \( r(x) = r(y) \) and \( d(x) = d(y) \). In particular, \( d(x) \leq d(y) \), so that by (OC8) there exists a unique element \( (y|d(x)) \) such that \( (y|d(x)) \leq y \) and \( d(y|d(x)) = d(x) \). But the element \( x \) has the property \( x \sim y \) and \( d(x) = d(x) \), so that, by the uniqueness guaranteed by (OC8), we must have \( x = (y|d(x)) \). However,

\[d(x) = d(y),\]

so that

\[x = (y|d(x)) = (y|d(y)) = y.\]

(OC7) holds: Let \( k \leq xy \) where the product \( xy \) exists in \( C \). Then \( d(k) \leq d(xy) \). Thus \( (xy|d(k)) \) exists. Now \( (xy|d(k)) \leq xy \) and \( d(xy|d(k)) = d(k) \), so that applying (OC8) we obtain, by uniqueness, \( k = (xy|d(k)) \). We now apply Lemma 2.1 and obtain

\[k = (x|r(y|d(k)))(y|d(k))\]

where \((x|r(y|d(k))) \leq x \) and \((y|d(k)) \leq y \).

(O1) holds: If \( e \in C_0 \) and \( x \leq e \), then \( d(x) \leq d(e) = e \). However, by (OC8), \( x \) is the unique element such that \( x \leq e \) and \( d(x) = d(x) \). Thus, since \( d(x) \) enjoys these properties, we have that \( x = d(x) \).

(G) holds: Now suppose that \( C \) is a groupoid. If \( x \leq y \) then \( r(x) \leq r(y) \) so that \( r(x) \leq d(y^{-1}) \) holds. By (OC8)(i) there is a unique element \((y^{-1}|r(x)) \) such that

\[(y^{-1}|r(x)) \leq y^{-1} \quad \text{and} \quad d(y^{-1}|r(x)) = r(x).\]

It follows that the product \((y^{-1}|r(x))x \) exists. Thus by (OC3)

\[(y^{-1}|r(x))x \preceq y^{-1} \cdot y.\]

But \( y^{-1} \cdot y \) is an identity, so by (O1) the element \((y^{-1}|r(x))x \) is an identity. Thus \( x^{-1} = (y^{-1}|r(x)) \) and we obtain \( x^{-1} \leq y^{-1} \) as required.

(ii) (OC2) holds: From (G) and (OC3) it is easy to see that \( x \leq y \) implies

\[xx^{-1} \leq yy^{-1} \quad \text{and} \quad x^{-1}x \leq y^{-1}y.\]

Thus \( r(x) \leq r(y) \) and \( d(x) \leq d(y) \).
OC8)(ii) holds: If \( f \leq r(x) \), where \( f \) is an idempotent, define
\[
(f | x) = (x^{-1} | f)^{-1}.
\]
It is straightforward to show that \((x^{-1} | f)^{-1} \leq x \) and \( r(x^{-1} | f)^{-1} = f \). Let \( y \leq x \) with \( r(y) = f \). Then \( y^{-1} \leq x^{-1} \) and \( d(y^{-1}) = f \). By (OC8)(i) we obtain \( y^{-1} = (x^{-1} | f) \); thus \( y = (x^{-1} | f)^{-1} \).

(iii) The result will follow by (ii) if we show that (OC8)(i) holds. Let \( x \in C \) and \( e \leq d(x) \) and suppose that there are two elements \( y \) and \( z \) such that \( y, z \in X \) and \( d(y) = d(z) = e \). Then \( r(z^{-1}) = d(z) = e \), so that the product \( yz^{-1} \) exists. But \( z^{-1} \leq x^{-1} \) holds by (G), so that by (OC3) \( yz^{-1} \leq xx^{-1} \). Now \( xx^{-1} \) is an identity so, by the fact that \( C_o \) is an order ideal of \( C \), the element \( yz^{-1} \) is an identity. Thus \( y = z \). □

The above proposition contains results slightly generalising those due to Rinow [32].

**Lemma 2.3.** Let \( G \) be a functorially ordered groupoid and let \( e \) and \( e' \) be identities in \( G \). Then:

(i) If \( e' \leq e \leq d(x) \), then \((x | e') \leq (x | e) \leq x\).

(ii) If \( e' \leq e \leq r(x) \), then \((e' | x) \leq (e | x) \leq x\).

**Proof.** We will prove (i); the proof of (ii) is similar. The element \((x | e)\) is well-defined and \( e' \leq d(x | e) \). Thus \((x | e) | e'\) is a well-defined element. We have that
\[
((x | e) | e') \leq (x | e) \leq x.
\]
But the element \((x | e')\) is well-defined, \((x | e') \leq x \) and \( d(x | e') = e' \). Thus by (OC8),
\[
((x | e) | e') = (x | e').
\]
□

We now turn to the definition of the various kinds of morphisms between ordered groupoids. If \( G \) and \( F \) are functorially ordered groupoids then an **ordered functor** \( \theta \) is just a functor \( \theta : G \to F \) which preserves the orders. If \( G \) and \( F \) are also inductive groupoids and \( \theta \) has, in addition, the property that \((\theta | G_o) : G_o \to F_o \) preserves the meet operation, then \( \theta \) will be called an **inductive functor**.

Note that ordered functors preserve restrictions and corestrictions in the sense that, if \( \theta : G \to H \) is an ordered functor between functorially ordered groupoids, and \( e \leq d(x) \) and \( f \leq r(y) \), then
\[
\theta(x | e) = (\theta(x) | \theta(e)) \quad \text{and} \quad \theta(f | y) = (\theta(f) | \theta(y));
\]
this is proved with the help of (OC8).

An ordered functor \( \theta : G \to H \) is said to **reflect** partial orders if, whenever \( \theta(x) \leq \theta(y) \), there exists an element \( x' \) in \( G \), such that \( \theta(x') = \theta(x) \) and \( x' \leq y \). The functor \( \theta \) is said to be **identity separating** if, whenever \( \theta(e) = \theta(f) \), where \( e \) and \( f \) are identities, it follows that \( e = f \). An **isomorphism** is a bijective, ordered functor whose inverse is an ordered functor.
If \( G \) and \( H \) are functorially ordered groupoids, then a map \( i : G \rightarrow H \), which is an injective, ordered functor, reflecting partial orders, will be called an embedding. Note that in this case \( i(G) \) is a subgroupoid of \( H \) and, when equipped with the induced order from \( H \), is a functorially ordered groupoid order isomorphic to \( G \).

A sub functorially ordered groupoid of \( H \) is a subgroupoid \( G' \) which is functorially ordered under the induced order. This is equivalent to the condition that \( G' \) be a subgroupoid and that if \( x \) is an element of \( G' \) and \( e \) is an identity of \( G' \) such that \( e \leq d(x) \) then \( (x|e) \) belongs to \( G' \). If \( G' \) is a sub functorially ordered groupoid of \( H \) then the identity map \( i : G' \rightarrow H \) is an embedding.

**Lemma 2.4.** Let \( H \) be a functorially ordered groupoid and let \( G \) be a sub functorially ordered groupoid of \( H \). If \( G_0 \) is an order ideal of \((H, \leq)\) then \( G \) is an order ideal of \((H, \leq)\).

**Proof.** Let \( x \in G \) and let \( y \preceq x \) where \( y \in H \). Since \( d(x) \) is an identity of \( G \) and \( d(y) \leq d(x) \) then \( d(y) \) belongs to \( G \). \( G \) is a sub functorially ordered groupoid of \( H \), so that the element \((x|d(y))\) is in \( G \). But by (OC8), \( y=(x|d(y)) \).

Functorially ordered groupoids and ordered functors form a (large) category which we will denote by \( \text{FOof} \). Inductive groupoids and ordered functors form a full subcategory of \( \text{FOof} \) which we will denote by \( \text{IGof} \). The category of inductive groupoids and inductive functors will be denoted by \( \text{IGif} \).

We now review some definitions from Higgins [9]. Let \( G \) be a groupoid and \( e \) an identity of \( G \). The star of \( G \) at \( e \), denoted \( G_e \), is the set of all elements of \( G \) with domain \( e \). If \( \theta : G \rightarrow H \) is a functor between two groupoids, \( \theta \) induces a star map \( \theta_e : G_e \rightarrow H_e \) by restriction. The functor \( \theta \) is said to be star injective (resp. star surjective, star bijective) if each of the star maps \( \theta_e \) is injective (resp. surjective, bijective). We will follow [1] and call star bijective functors covering functors.

An ordered star injective (resp. ordered star surjective, ordered star bijective) functor is an ordered functor, whose underlying functor has the corresponding properties. We will also refer to ordered covering functors.

**Lemma 2.5.** Let \( \theta : G \rightarrow H \) be an ordered covering functor between two functorially ordered groupoids. Then

\[ x \leq y \text{ if and only if } \theta(x) \leq \theta(y) \text{ and } d(x) \leq d(y). \]

**Proof.** Suppose that \( \theta(x) \leq \theta(y) \) and \( d(x) \leq d(y) \). The element \((y|d(x))\) is well-defined and \( \theta(y|d(x)) = (\theta(y)|d(\theta(x))) \). Thus

\[ \theta(y|d(x)) \leq \theta(y) \text{ and } d(\theta(y|d(x))) = d(\theta(x)). \]

But by (OC8) we have that \( \theta(y|d(x)) = \theta(x) \). However, \( d(x) = d(y|d(x)) \) and \( \theta(y|d(x)) = \theta(x) \), together with the fact that \( \theta \) is a covering functor, imply \( x = (y|d(x)) \). Thus \( x \leq y \).
The converse is clear. □

To conclude this section, we record a result, whose proof is trivial, which will be applied in Section 4.

**Lemma 2.6.** Let $H$ be a groupoid and $G$ a subgroupoid of $H$. Then $G$ is a full, coreflective subgroupoid of $H$ if and only if the following two conditions hold:

1. If $x \in H$ and $d(x), r(x) \in G$, then $x \in G$.
2. For each $e \in H_0$ there exists an $x \in H$ such that $r(x) = e$ and $d(x) \in G_0$. □

In the terminology of [2, p. 310], such an $H$ is called an enlargement of $G$. Note that for groupoids the notions of reflective and coreflective subcategory coincide, so that we could equally well require the groupoid $G$ in the above lemma to be a reflective subgroupoid.

3. Inverse semigroups and inductive groupoids

The main theorem of this section, Theorem 3.5, is an amalgamation of results by Ehresmann [3, II-1, pp. 53, 68], Nambooripad and Veeramony [27] and Schein [34]. Proofs of versions of this result have appeared in the books by Hasse and Michler [6] and Petrich [29]. A variation of these ideas was considered by Meakin [24].

Although inductive groupoids have been exploited by Nambooripad and his co-workers (see [26] and [27], for example), the possibility of using them to apply Ehresmann’s work to semigroups appears to have been largely ignored. But, in fact, Schein’s paper [34] effectively showed that the work of Vagner and Ehresmann could be related by a simple dictionary, which would provide the appropriate channel for such an application of Ehresmann’s ideas.

Our version of the proof of Theorem 3.5 will unite the point of view of Schein [34], which stresses the semigroup side of the construction, and that of Ehresmann [3, II-2, p. 75], which stresses the groupoid-theoretic side.

We shall only be interested in functorially ordered groupoids, but we shall give our first definition for ordered categories; this is simply to indicate that there are possibilities for generalisation (which are explored in [15]).

Let $C$ be an ordered category and let $x, y \in C$. Put

$$\langle x, y \rangle = \{ (x', y') \in C\times C : x' \leq x \text{ and } y' \leq y \}$$

equipped with the Cartesian product ordering. If $\langle x, y \rangle$ possesses a maximum element $(x', y')$, then we say that $x$ and $y$ have $x'y'$ for a pseudoproduct in $C$ and we shall write $x \otimes y = x'y'$.

**Proposition 3.1** (Special case of Ehresmann [3, II-2, p. 75, Proposition 9]). Let $G$ be a functorially ordered groupoid. Then
$x \otimes y$ exists iff $e = d(x) \wedge r(y)$ exists in $G_0$,
in which case $x \otimes y = (x|e)(e|y)$.

**Proof.** Suppose that $e = d(x) \wedge r(y)$ is defined. Then the ordered pair $((x|e), (e|y))$ is an element of $\langle x, y \rangle$. By definition, $x' \leq x$ and $y' \leq y$ and $d(x') = r(y') = e'$ (say). But then $e' \leq d(x)$ and $e' \leq r(y)$ so that $e' \leq e$ holds.

Now $x' = (x|e')$ and $y' = (e'|y)$, so we have that $x' = (x|e') \leq (x|e)$ and $y' = (e'|y) \leq (e|y)$ by Lemma 2.3. Thus $(x', y') \leq ((x|e), (e|y))$.

Conversely, suppose that the pseudoproduct $x \otimes y$ exists, where $x \otimes y = x'y'$ and $(x', y')$ is the maximum element of $\langle x, y \rangle$. Then $d(x') = r(y') = e'$ (say), $e' \leq d(x)$ and $e' \leq r(y)$. Thus $d(x)$ and $r(y)$ have $e'$ as a common lower bound. Now let $e''$ be an identity such that $e'' \leq d(x)$ and $e'' \leq r(y)$. Then

$$d(x|e'') \leq x, \quad (e''|y) \leq y \quad \text{and} \quad d(x|e'') = e'' = r(e''|y),$$

which together show that $((x|e''), (e''|y)) \in \langle x, y \rangle$. But then $(x|e'') \leq x'$. Thus $e'' \leq d(x') = e'$.

The following is now immediate:

**Corollary 3.2.** The pseudoproduct in a functorially ordered groupoid is everywhere defined if and only if the identities form a meet semilattice under the induced order.

**Proposition 3.3** (Ehresmann [3, II-2, p. 75, Proposition 5]). Let $G$ be a functorially ordered groupoid. If both $x \otimes (y \otimes z)$ and $(x \otimes y) \otimes z$ exist, they are equal.

**Proof.** We assume that both $x \otimes (y \otimes z)$ and $(x \otimes y) \otimes z$ exist. Let $(x \otimes y) \otimes z = a \cdot z'$ where $(a, z')$ is the maximum element of $\langle (x \otimes y), z \rangle$. In particular,

$$a \leq x \otimes y \quad \text{and} \quad z' \leq z.$$

Let $x \otimes y = x'y'$ where $(x', y')$ is the maximum element of $\langle x, y \rangle$. In particular,

$$x' \leq x \quad \text{and} \quad y' \leq y.$$

At this point, we need to use the fact that (OC7) holds in functorially ordered groupoids, which we established in Proposition 2.2. We have that

$$a \leq x \otimes y = x'y'.$$

By (OC7) there exist elements $x'' \leq x'$ and $y'' \leq y'$ such that the category product $x''y''$ exists and $a = x''y''$. Thus

$$(x \otimes y) \otimes z = az' = (x'y')z'.$$

Having analysed the pseudoproduct above into a groupoid product, we now re-
bracket, thanks to the associativity of the groupoid product, and reconstitute. Firstly, we have the following inequalities

\[ x''x' \leq x, \quad y''y' \leq y \quad \text{and} \quad z''z \leq z. \]

Since the product \( y''z' \) exists, it follows that \( (y'', z') \in \langle y, z \rangle \). Thus \( y''z' \leq y \otimes z \).

Also, it is clear that \( (x'', y''z') \in \langle x, y \otimes z \rangle \). Thus

\[ x''(y''z') \leq x \otimes (y \otimes z). \]

We have proved that

\[ (x \otimes y) \otimes z \leq x \otimes (y \otimes z). \]

A similar argument now yields the inequality in the other direction. □

Most of the work for the following theorem has now been proved on the basis of the above results:

**Theorem 3.4.** (i) Let \( S \) be an inverse semigroup. Denote by \( G(S) \) the triple \((S, ., \leq)\), which is the set \( S \) equipped with the reduced product and the natural partial order \( \leq \). Then \( G(S) \) is an inductive groupoid.

(ii) Let \((G, ., \leq)\) be an inductive groupoid and put \( S(G) = (G, \otimes) \), where \( \otimes \) is the pseudoproduct. Then \( S(G) \) is an inverse semigroup whose natural partial order coincides with \( \leq \) and whose reduced product coincides with \( ' . ' \).

Furthermore, \( S(G(S)) = S \) and \( G(S(G)) = G \).

**Proof.** (i) It is straightforward to show that \((S, .)\) is a groupoid, in which

\[ d(x) = x^{-1}x \quad \text{and} \quad r(x) = xx^{-1}. \]

That \((S, ., \leq)\) is an inductive groupoid follows on the basis of the well known properties of the natural partial order and Proposition 1.2. Note also by Proposition 1.2, that

\[ (x|e) = xe \quad \text{and} \quad (f|y) = fy. \]

(ii) The fact that \((G, \otimes, \leq)\) is a meet semilattice implies, by Corollary 3.2, that the pseudoproduct \( \otimes \) is everywhere defined. Proposition 3.3 implies that \((G, \otimes)\) is a semigroup. If \( x, y \in G \) and their category product \( x \cdot y \) exists then it coincides with \( x \otimes y \). But, for all elements \( x \in G \), we have that

\[ x = x \cdot x^{-1} \cdot x \quad \text{and} \quad x^{-1} = x^{-1} \cdot xx^{-1}. \]

Thus \( G(G) \) is a regular semigroup. If \( e \) is an idempotent in \( S(G) \) then \( e \) is an identity of \( G \). If \( e \) and \( f \) are identities then we have

\[ e \otimes f = (e|d(e) \land r(f))(d(e) \land r(f))|f = (e|e \land f)(e \land f)|f. \]

But

\[ (e|e \land f) = e \land f \quad \text{and} \quad (e \land f)|f = e \land f. \]
Thus $e \otimes f = e \wedge f$. Whence $S(G)$ is an inverse semigroup; the semigroup inverse of an element $x$ is just $x^{-1}$, the inverse in the groupoid.

We now show that the natural partial order in $S(G)$ coincides with the order in the inductive groupoid. Note first that if $e \leq d(x)$ then $(x|e) = x \otimes e$. Let $x \leq y$ in $S(G)$. Then, by definition, $x = e \otimes y$ for some idempotent $e$. But

$$x = e \otimes y = (e|d(e) \wedge r(y))(d(e) \wedge r(y)|y) = (e \wedge r(y)|y).$$

Thus $x \leq y$ in the inductive groupoid $G$.

Now suppose that $x \leq y$ in the inductive groupoid $G$. Then $x = (r(x)|y)$. But $(r(x)|y) = r(x) \otimes y$, which implies that $x \leq y$ in $S(G)$.

We now turn to the reduced product in $S(G)$, which, for the moment, we will denote by $\odot$. It is defined as follows

$$x \odot y = \begin{cases} x \otimes y & \text{iff } x^{-1} \otimes x = y \otimes y^{-1}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

However, $x^{-1} \otimes x = y \otimes y^{-1}$ in $S(G)$ if and only if $x^{-1} \cdot x = y \cdot y^{-1}$ in $G$. If $\odot$ is defined, it is now clear that

$$x \odot y = x \otimes y = x \cdot y.$$

We now prove the last part of the theorem. We continue with $S(G)$ and calculate $G(S(G))$. The underlying set is just $G$ itself. The category product is the reduced product of $S(G)$ which, as we have seen, coincides with the groupoid product of $G$. Finally, the order is the order on $S(G)$ which, again, we have seen is just the order on $G$. Thus $G(S(G)) = (G, \cdot, \leq)$.

Now consider $G(S)$. We shall calculate $G(S(G))$. Denote the semigroup multiplication in $S$ by juxtaposition. If $x, y \in G(S)$ then

$$x \otimes y = (x|e) \cdot (e|y) \quad \text{where } e = d(x) \wedge r(y).$$

By Proposition 1.2, we have, on order theoretic grounds, that

$$(x|e) - xe \quad \text{and} \quad (e|y) = ey$$

where $e = x^{-1} \cdot xy \cdot y^{-1}$. Thus

$$x \otimes y = (xe)(ey) = xy.$$ 

Theorem 3.4 may be strengthened to take account of morphisms:

**Theorem 3.5.** (i) The categories $\mathbf{I}_{p}$ and $\mathbf{I}_{GOF}$ are isomorphic under the inverse functors $G$ and $S$.

(ii) The categories $\mathbf{I}_{h}$ and $\mathbf{I}_{GIF}$ are isomorphic under the inverse functors $G$ and $S$.

**Proof.** (i) Let $\theta : S \rightarrow T$ be a prehomomorphism between two inverse semigroups. By
Proposition 1.5, \( \theta \) maps idempotents to idempotents and preserves both the natural partial order and the reduced product. Thus \( \theta \) induces an order preserving functor \( G(S) \to G(T) \), which we denote by \( G(\theta) \).

Conversely, let \( \psi: G \to F \) be an order preserving functor between two inductive groupoids. By Proposition 1.5, it is clear that \( \psi \) induces a prehomomorphism \( S(G) \to S(F) \), which we denote by \( S(\psi) \).

(ii) Let \( \theta: S \to T \) be a semigroup homomorphism. Then, by (i) above, \( G(\theta) \) is an order preserving functor. If \( e, f \in E(S) \) then \( \theta(ef) = \theta(e)\theta(f) \). We may rewrite this as \( \theta(e \land f) = \theta(e) \land \theta(f) \). But then \( G(\theta) \) is an inductive functor.

Conversely, let \( \theta: G \to K \) be an inductive functor. By (i), the map \( S(\theta) \) is a prehomomorphism. By definition, if \( e \) and \( f \) are identities then \( \theta(e \land f) = \theta(e) \land \theta(f) \). But \( e \land f = e \otimes f \). The result now follows by Proposition 1.6.

Result (i) above seems to have been tacitly obtained for (the more general case of) regular semigroups by Nambooripad and Veeramony [27]. Result (ii) is a special case of the main theorem in [26].

Theorem 3.4 was proved, using a longer, and non-order-theoretic proof, by Schein [34]. Schein was generalising results stated by Ehresmann in [3, II-1, pp. 47, 53], where proofs of associativity of the pseudoproduct are not explicitly given, and where the inductive groupoids considered are required to be conditionally complete (this accounts for the adjective 'inductive'). Arbitrary functorially ordered groupoids and their connection with inductive groupoids are considered in [3, II-1, p. 68]; the proof of the associativity of the pseudoproduct, which makes use of the order-theoretic argument, is contained in seminar notes from the University of Montreal dated 1961 [3, II-1, p. 316, reference (2)]. An explicit proof of associativity is, however, contained in the later paper [3, II-2, p. 75], but by this time the result is proved in the more general context of ordered categories.

As a consequence of Theorem 3.5, we may set up a 'dictionary', translating between inverse semigroups and inductive groupoids. Let \( S \) be an inverse semigroup. Then

- \( x \leq y \) in \( S \) if and only if \( d(x) = d(y) \) in \( G(S) \),
- \( x \leq y \) in \( S \) if and only if \( r(x) = r(y) \) in \( G(S) \),
- \( x \leq y \) in \( S \) if and only if \( x \) and \( y \) are in the same hom-set in \( G(S) \),
- \( x \leq y \) in \( S \) if and only if \( x \) and \( y \) are in the same connected component of \( G(S) \), and

- \( L \)-injective prehomomorphisms correspond to ordered star injections,
- \( L \)-surjective prehomomorphisms correspond to ordered star surjections,
- \( L \)-bijective prehomomorphisms correspond to ordered covering functors.

In the next section, we shall show how ordered star injections and ordered covering functors provide a framework for understanding the structure of \( E \)-unitary inverse semigroups.
4. \(P\)-semigroups

In this section, we shall apply Theorem 3.5 to one of the most important constructions in inverse semigroup theory—that of \(P\)-semigroups. We begin by describing that construction and stating the \('P\)-Theorem'. We shall then show how the construction may be interpreted in a categorical fashion.

A subset \(Y\) of a partially ordered set \(X\) is said to be an \(essential\ ideal\) of \(X\), if \(Y\) is an ideal of \(X\) and if, for each \(x \in X\), there exists an element \(y \in Y\) such that \(y \leq x\). Let \(G\) be a group acting (on the left) by order automorphisms on a partially ordered set \(X\), such that conditions (1) and (2) hold:

1. There exists an essential ideal and subsemilattice \(Y\) of \(X\).
2. \(GY = X\).

Following Petrich [29], we shall call a triple \((Y, G, X)\) satisfying (1) and (2) a \(McAlister\ triple\).

Put 
\[
P = P(Y, G, X) = \{(e, g) \in Y \times G: g^{-1}e \in Y\}.
\]

The set \(P\) may be equipped with a multiplication given by
\[
(e, g)(f, h) = (e \wedge gf, gh),
\]
with respect to which \(P\) is an \(E\)-unitary inverse semigroup. Semigroups of the form \(P(Y, G, X)\) are called \(P\)-semigroups.

**Lemma 4.1** (McAlister [19, Proposition 1.2]). Let \(P = P(Y, G, X)\) be a \(P\)-semigroup. Then:

1. \(E(P) = Y \times \{1\}\).
2. \((e, g)^{-1} = (g^{-1}e, g^{-1})\).
3. \((e, g)(e, g)^{-1} = (e, 1)\) and \((e, g) \cdot (e, g)^{-1} = (g^{-1}e, 1)\).
4. \((e, g) \leq (f, h)\) iff \(g = h\) and \(e \leq f\) in \(Y\).
5. The reduced product of \((e, g)\) and \((f, h)\) exists iff \(gf = e\), in which case \((e, g) \cdot (f, h) = (e, gh)\).
6. \((e, g)\sigma(f, h)\) iff \(g = h\).

**Theorem 4.2** (McAlister [18, Theorem 2.6]). Every \(E\)-unitary inverse semigroup is isomorphic to some \(P\)-semigroup.

McAlister’s original proof of Theorem 4.2 was simplified by Munn [25] and Schein [35]—this latter proof, Schein [37] also traces back to Golqab [5].

The next lemma will be useful later.

**Lemma 4.3.** (Petrich [29, Lemma VII.1.3]). Let \(G\) be a group acting on a partially ordered set \(X\) by order automorphisms on the left. Let \(Y\) be a subsemilattice and an ideal of \(X\) such that \(GY = X\). Then the following are equivalent:
(i) \( gY \cap Y \neq \emptyset \) for all \( g \in G \).
(ii) \( Y \) is an essential ideal of \( X \). \( \square \)

In obtaining a \( P \)-representation of an \( E \)-unitary semigroup \( S \), two of the ingredients—\( G \) and \( Y \)—are easy to determine. The group \( G \) is just \( S/\sigma \), the maximum group homomorphic image of \( S \), and \( Y \) is isomorphic to the semilattice of idempotents of \( S \). The partially ordered set \( X \) is rather harder to obtain. Interpretations of \( X \) were given by Loganathan [16] via cohomology and by Margolis and Pin [17]. We shall give an alternative interpretation, the ramifications of which will be considered in [12] and [13].

We look first at the structure obtained from a group acting on a set.

**Proposition 4.4.** Let \( G \) be a group acting, on the left, on a set \( X \). Then the set \( X \times G \) may be equipped with the following partial multiplication:

\[
(x, g)(y, h) = \begin{cases} 
(x, gh) & \text{if } x = gy, \\
\text{undefined} & \text{otherwise}.
\end{cases}
\]

With respect to this multiplication, the set \( X \times G \) is equipped with the structure of a groupoid, which we will denote by \( G \ltimes X \). In particular, we have the following:

(i) \( (G \ltimes X)_0 = X \times \{1\} \).
(ii) \( (e, g)^{-1} = (g^{-1}e, g^{-1}) \).
(iii) \( d((e, g)) = (g^{-1}e, 1) \) and \( r((e, g)) = (e, 1) \).

**Proof.** A standard construction in groupoid theory, which is elementary and left to the reader. \( \square \)

This construction is much used by Ehresmann [2] but appears to have been first introduced by Reidemeister [31] (I am grateful to Ronnie Brown for this piece of information). Brown [1] calls it the *semidirect product groupoid* (note that in [1] he considers the action of \( G \) on \( X \) on the right and uses the notation \( X \rtimes G \) for the corresponding groupoid). The semidirect product groupoid plays an important role in a number of applications—consult Brown [1] for more information. The interest for us lies in the clear relationship between Lemma 4.1 and Proposition 4.4.

If \( G \) and \( X \) and the action of \( G \) upon \( X \) are equipped with extra structure (for example topological [1]), we might expect \( G \ltimes X \) to acquire extra structure as a consequence. We shall show how this applies when \( X \) is a poset.

**Theorem 4.5.** Let \( G \) act on a partially ordered set \( X \) by order automorphisms. Then \( G \ltimes X \) is a functorially ordered groupoid under the ordering

\[
(e, g) \leq (f, h) \iff g = h \text{ and } e \leq f \text{ in } X.
\]

**Proof.** It is straightforward to verify that \( \leq \) is a partial order. We now verify the other conditions of Proposition 2.2(iii).
(OC3) holds: Let \((e, g) \leq (f, g)\) and \((i, h) \leq (j, h)\) and suppose that the products \((e, g)(i, h)\) and \((f, g)(j, h)\) exist. Then we have \((e, g)(i, h) = (e, gh)\) and \((f, g)(j, h) = (f, gh)\). Thus \((e, gh) \leq (f, gh)\).

(G) holds: From \((e, g) \leq (f, g)\) we obtain \(e \leq f\). Now
\[
(e, g)^{-1} = (g^{-1}e, g^{-1}) \quad \text{and} \quad (f, g)^{-1} = (g^{-1}f, g^{-1}).
\]
From the fact that \(e \leq f\) together with the fact that \(G\) acts on \(X\) by order automorphisms we deduce \(g^{-1}e \leq g^{-1}f\). Thus \((g^{-1}e, g^{-1}) \leq (g^{-1}f, g^{-1})\) as required.

(OL) holds: Let \((e, 1)\) be an identity of \(G \times X\) and \((f, g) \leq (e, 1)\). Then \(g = 1\) and \(f \leq e\), so that \((f, g)\) is an identity.

(OC5)(i) holds: Let \((e, g) \in G \times X\) and let \((f, 1) \leq d((e, g)) = (g^{-1}e, 1)\). Then, in particular, \(f \leq g^{-1}e\). Consider the element \((gf, g)\). From the fact that \(f \leq g^{-1}e\) we obtain \(gf \leq e\) and \(g^{-1}(gf) = f\). Thus \((gf, g) \leq (e, g)\) and \(d((gf, g)) = (f, 1)\). It now follows that we may define
\[
((e, g))(f, 1) = (gf, g).
\]
Similarly if \((f, 1) \leq r(e, g)\) we may define
\[
((f, 1))(e, g) = (f, g).
\]
We shall denote by \(\Pi = \Pi(G, X)\) the groupoid \(G \times X\) considered as a functorially ordered groupoid.

**Lemma 4.6.** The pseudoproduct in \(\Pi = \Pi(G, X)\) of the elements \((e, g)\) and \((f, h)\) exists if and only if the greatest lower bound of \(g^{-1}e\) and \(f\) exists in \(X\): in which case, we have that
\[
(e, g) \otimes (f, h) = (e \wedge gf, gh).
\]

**Proof.** Note that \(d((e, g)) = (g^{-1}e, 1)\) and \(r((f, h)) = (f, 1)\). Thus the pseudoproduct exists if and only if the glb of \((g^{-1}e, 1)\) and \((f, 1)\) exists. If it exists, it is equal to \((g^{-1}e \wedge f, 1)\). Now,
\[
((e, g))(g^{-1}e \wedge f, 1) = (g(g^{-1}e \wedge f), g)
\]
and
\[
((g^{-1}e \wedge f, 1))(f, h) = (g(g^{-1}e \wedge f), h).
\]
Noting that \(g(g^{-1}e \wedge f) = e \wedge gf\) the result follows.

Define a map \(\tau: \Pi(G, X) \rightarrow G\) by \(\tau(e, g) = g\) and regard \(G\) as a totally unordered poset.

**Lemma 4.7.** The map \(\tau\), defined above, is an ordered covering functor.

**Proof.** By Proposition 4.4(i), the set of identitifes of \(\Pi\) is just \(X \times \{1\}\). Thus, from the definition, \(\tau\) maps all identities of \(\Pi\) to the identity of \(G\).
Let \((e, g)\) and \((f, h)\) be elements of \(\Pi\) whose product exists. Then
\[
(e, g)(f, h) = (e, gh).
\]
Now \(\tau(e, g) = g, \quad \tau(e, h) = h\) and \(\tau(e, gh) = gh\). Thus \(\tau\) is a functor.

If \((e, g) \leq (f, h)\) in \(\Pi\) then \(g = h\). Thus \(\tau(e, g) \leq \tau(f, h)\) in \(G\). Whence \(\tau\) is an ordered functor.

Suppose that \(d(e, g) = d(f, h)\) and \(\tau(e, g) = \tau(f, h)\). Then
\[
(g^{-1} e, 1) = (h^{-1} f, 1) \quad \text{and} \quad g = h.
\]
This shows that \((e, g) = (f, h)\). Thus \(\tau\) is a star injection. Finally, let \(g \in G\) and let \((e, 1)\) be an identity of \(\Pi\). The pair \((ge, g)\) belongs to \(\Pi\), \(d(ge, g) = (e, 1)\) and \(\tau(ge, g) = g\). Thus \(\tau\) is a star bijection. \(\Box\)

We may obtain an abstract characterisation of the functorially ordered groupoid \(\Pi(G, X)\).

**Proposition 4.8.** Let \(\Pi\) be a functorially ordered groupoid and let \(\sigma : \Pi \rightarrow G\) be an ordered covering functor onto a group \(G\). Then there is an action of \(G\) on \(X = \Pi_0\) and an isomorphism \(\theta : \Pi \rightarrow \Pi(G, X)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\Pi & \xrightarrow{\theta} & \Pi(G, X) \\
\sigma \downarrow & & \downarrow \tau \\
G & & 
\end{array}
\]

**Proof.** We begin by showing that \(G\) acts on \(X = \Pi_0\) by order automorphisms. If \(e \in X\) and \(g \in G\), then define:
\[
ge e = r(x) \quad \text{where} \quad e = d(x) \quad \text{and} \quad g = \sigma(x).
\]
Note that \(ge\) is (well-)defined for all \(g\) and \(e\): since \(\sigma\) is a covering map there is a unique element \(x\) such that \(e = d(x)\) and \(\sigma(x) = g\). \(h(g(e)) = (hg)e\) holds: let \(\sigma(x) = g\) and \(d(x) = e\), so that we have \(ge = r(x)\). Similarly, let \(\sigma(y) = h\) and \(d(y) = r(x)\), so that we have \(hr(x) = r(y)\). The product \(yx\) is defined in \(\Pi\) since \(d(y) = r(x)\). But then \(\sigma(yx) = hg\) and \(d(yx) = d(x)\), so that \((hg)e\) is defined and equals \(r(y)\) as required.

\(1e = e\) for all \(e \in X\): Note that \(e \in \Pi_0\) is the unique element such that \(d(e) = e\) and \(\sigma(e) = 1\). Thus \(1e = e\).

\(G\) acts on \(X\) by order automorphisms: If \(e \leq f\) in \(X\) and \(g \in G\), then from the definition of the action we have that
\[
ge e = r(x) \quad \text{where} \quad e = d(x) \quad \text{and} \quad g = \sigma(x),
\]
\[
gf = r(y) \quad \text{where} \quad f = d(y) \quad \text{and} \quad g = \sigma(y).
\]
Since \(\sigma(x) = \sigma(y)\) and \(d(x) \leq d(y)\), we may apply Lemma 2.5 to obtain \(x \leq y\). Thus \(ge \leq gf\).
From the action of $G$ on the partially ordered set $X$, we may define the functorially ordered groupoid $\Pi(G, X)$. Define a map $\theta : \Pi \to \Pi(G, X)$ by

$$\theta(x) = (r(x), \sigma(x)).$$

The map $\theta$ is a functor: Let $x, y \in \Pi$ with $xy$ defined. Noting that

$$\sigma(x)^{-1}r(x) = \sigma(x^{-1})d(x^{-1}) = r(x^{-1}) = d(x),$$

we have that $d(\theta(x)) = (d(x), 1)$ and $r(\theta(y)) = (r(y), 1)$, so that the product $\theta(x)\theta(y)$ is defined. It is easy to see that $\theta(x)\theta(y) = \theta(xy)$. If $e$ is an identity then so is $\theta(e) = (e, 1)$.

The map $\theta$ is a bijection: If $\theta(x) = \theta(y)$ then $r(x) = r(y)$ and $\sigma(x) = \sigma(y)$. Thus

$$d(x^{-1}) = d(y^{-1}) \quad \text{and} \quad \sigma(x^{-1}) = \sigma(y^{-1}).$$

But since $\sigma$ is a covering functor $x^{-1} = y^{-1}$, whence $x = y$. If $(e, g) \in \Pi(G, X)$ then it is easy to see that, if $x$ is the unique element such that $d(x) = e$ and $\sigma(x) = g^{-1}$, then $\theta(x^{-1}) = (e, g)$. The map $\theta$ is an order isomorphism: $\theta$ clearly preserves the order relation. Let $(r(x), \sigma(x)) \leq (r(y), \sigma(y))$. Then

$$d(x^{-1}) \leq d(y^{-1}) \quad \text{and} \quad \sigma(x^{-1}) = \sigma(y^{-1}).$$

Thus by Lemma 2.5, we obtain $x^{-1} \leq y^{-1}$, giving $x \leq y$ as required. □

We may now reformulate the theory of $E$-unitary inverse semigroups. In view of Theorem 4.2, we shall assume that our semigroup is some $P$-semigroup $P = P(Y, G, X)$. By Lemma 4.1, the map $\sigma^\dagger : P \to G$ is given by $(x, g) \to g$. By Proposition 1.10, $\sigma^\dagger$ is an $L$-injection. Applying the functor $G$ from Theorem 3.5, we obtain an ordered star injective functor

$$G(\sigma^\dagger) : G(P) \to G.$$

By Theorem 4.5, $\Pi = \Pi(G, X)$ is a well-defined functorially ordered groupoid. $G(P)$ is a subset of $\Pi$. The groupoid product of $G(P)$, which is simply the reduced product of $P$, is, by Lemma 4.1, the same as the (restriction of) the groupoid product in $\Pi(G, X)$. Furthermore, inverses in $G(P)$ are the same as inverses in $\Pi(G, X)$. Thus we have shown that

$$G(P)$$

is a subgroupoid of $\Pi(G, X)$.

It is also evident from Lemma 4.1, that the order relation on $G(P)$, which coincides with the natural partial order of $P$, is the restriction of the order from $\Pi(G, X)$. Thus the identity map

$$i : G(P) \to \Pi(G, X)$$

is an embedding. The set of identities of $G(P)$ is just the set $Y \times \{1\}$. But this is an order ideal of $\Pi$. Thus by Lemma 2.4, $G(P)$ is an order ideal of $(\Pi, \leq)$. $G(P)$ is a full subcategory of $\Pi$: For if $(e, g) \in \Pi$ such that $d(e, g), r(e, g) \in P$, then, from the fact that
The geometric theory of inverse semigroups

\[ d(e, g) = (g^{-1}e, 1) \quad \text{and} \quad r(e, g) = (e, 1), \]

we obtain that \( g^{-1}e, e \in Y \). Thus \((e, g) \in P\).

\( G(P) \) is a coreflective subcategory of \( \Pi \): For if \((e, 1) \in \Pi_o \) then, since \( GY = X \), there exists an element \( f \in Y \) such that \( gf = e \). Consider the element \((gf, g)\). Then

\[ r(gf, g) = (gf, 1) \quad \text{and} \quad d(gf, g) = (f, 1) \in P_o. \]

It is immediate that \( G(\sigma^3) \) is the restriction of \( \tau \) to \( G(P) \).

We may sum up what we have found so far in a commutative diagram:

\[
\begin{array}{ccc}
\Pi & \xrightarrow{i} & G(P) \\
\text{full, coreflective embedding} & & \text{ordered, covering functor}
\end{array}
\]

\[
\begin{array}{ccc}
G(P) & \xrightarrow{G(\sigma^3)} & G \\
\text{ordered star injection} & & \end{array}
\]

Thus, it is a consequence of the \( P \)-Theorem, that we may construct an extension of \( G(\sigma^3) \), within the category of functorially ordered groupoids and ordered functors, which is an ordered covering functor.

**Theorem 4.9.** Let \( H \) be a sub functorially ordered groupoid of \( \Pi = \Pi(G, X) \). Suppose, in addition, that \( H \) is a full, coreflective subcategory of \( \Pi \) and that the following two conditions hold:

1. \( H_0 \) is a meet semilattice under the induced order and an order ideal of \( \Pi \).
2. The map \( \tau' : H \rightarrow G \), the restriction of \( \tau \) to \( H \), is onto.

Put \( Y = \{ e \in X : (e, 1) \in H_o \} \) Then:

(i) \( H = \{ (e, g) \in Y \times G : g^{-1}e \in Y \} \).

(ii) \( (Y, G, X) \) is a McAlister triple.

(iii) \( (H, \otimes) = P(Y, G, X) \) and \( \tau' \) coincides with \( \sigma \).

**Proof.** Since \( H_0 \subseteq \Pi_o \) and \( \Pi_o = X \times \{ 1 \} \), it is clear that \( H_0 \) has the form \( Y \times \{ 1 \} \) for some \( Y \subseteq X \). It is immediate that \( Y \) is a semilattice under the induced order and an order ideal of \( X \). In particular, \( H \subseteq Y \times G \).

(i) Let \((e, g) \in H\). Then \( d(e, g) \in H \), since \( H \) is a subgroupoid of \( \Pi \). Thus \( g^{-1}e \in Y \). Whence

\[ H \subseteq \{ (e, g) \in Y \times G : g^{-1}e \in Y \}. \]

Let \((e, g) \in Y \times G \) be such that \( g^{-1}e \in Y \). Then

\[ d(e, g) = (g^{-1}e, 1) \quad \text{and} \quad r(e, g) = (e, 1), \]

so that \( d(e, g), r(e, g) \in H_0 \). But \( H \) is a full subcategory of \( \Pi \). Thus \((e, g) \in H\).

(ii) It remains to show that \( GY = X \) and that \( Y \) is an essential ideal of \( X \). Let \( e \in X \).

Then \((e, 1) \in \Pi_o \). \( H \) is a coreflective subcategory of \( \Pi \) so that there exists an element \((f, g) \in \Pi\) such that
Thus $f = e$ and $(g^{-1}e, 1) \in H_0$. But then $g^{-1}e = e' \in Y$. Thus, given $e \in X$ we have found elements $g \in G$ and $e' \in Y$ such that $ge' = e$. Finally, the map $\tau'$ is onto. Thus for each $g \in G$ there exists an element $(e, g) \in H$. But then for each $g \in G$ we have that $gY \cap Y \neq \emptyset$. By Lemma 4.3, this is equivalent to $Y$ being an essential ideal of $X$.

(iii) By Lemma 4.6, $P(Y, G, X) = (H, \otimes)$. □

We may now conveniently sum up what we have found.

**Theorem 4.10.** Let $S$ be an inverse semigroup with $S/\sigma = G$. Then the following are equivalent:

(i) $S$ is isomorphic to a $P$-semigroup.

(ii) There is a functorially ordered groupoid $\Pi$, a surjective, ordered, covering functor $\tau : \Pi \to G$ and an embedding $i : G(S) \to \Pi$ such that the following conditions hold:

1. $i(G(S))_0$ is an order ideal of $(\Pi, \leq)$.
2. $i(G(S))$ is a full, coreflective subgroupoid of $\Pi$.
3. $\tau i = G(\sigma_i)$.

**Proof.** (i) implies (ii) has been demonstrated.

(ii) implies (i): Proposition 4.8 and Theorem 4.9. □

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The geometric theory of inverse semigroups