Pairwise weakly and pairwise strongly irresolute functions

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Abstract In this paper we consider a new weak and strong forms of irresolute functions in bitopological spaces, namely, i j -quasi-irresolute functions and strongly irresolute functions. Several characterizations and basic properties of these functions are given. We investigate the relationships among some weak forms of continuity and other generalizations of continuous mappings in bitopological spaces.

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1. Introduction

The study of bitopological spaces was first initiated by Kelly [3] and thereafter a large number of papers have been done to generalized the topological concepts to bitopological space. Irresolute mappings in bitopological spaces was defined by Mukherjee [11]. In 1991 Khedr [4] introduced and investigate a class of mappings in bitopological spaces called pairwise 0- irresolute mappings. Khedr [7] defined the concept of quasi- irresolute mappings in these spaces and studied some of its properties. The concepts of strongly irresolute mappings in bitopological spaces was defined by Khedr in [6] and he showed that quasi-irresoluteness and semi-continuity are independent of each other.

The aim of this paper is to introduce basic properties of quasi- irresolute and strongly irresolute functions in bitopological spaces. We study these functions and some of results on s-closed spaces and semi-compact spaces in bitopological spaces. Also, we investigate the relationships among some weak forms of continuity, irresoluteness, quasi-irresoluteness and strong irresoluteness.

Throughout this paper (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, v_1, v_2) (or briefly X, Y and Z) denote bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, we shall denote the closure of A and the interior of A with respect to τ_i (or σ_i) by i-cl(A) and i-int(A) respectively for i = 1, 2. Also i, j = 1, 2 and i ≠ j.

A subset A is said to be i j -semi-open [1], if there exists a τ_j-open set U of X such that U ⊂ A ⊂ j-cl(U), or equivalently if A ⊂ j-cl(i-int(A)). The complement of an i j -semi-open set is said to be i j -semi-closed. An i j -semi-interior [1] of A, denoted by i j -sint(A), is the union of all i j -semi-open sets contained in A. The intersection of all i j -semi-closed sets containing A is called the i j -semi-closure [1] of A and denoted by i j -scl(A).
A subset $A$ of $X$ is said to be \emph{ij-semi-regular} [7] if it is both \emph{ij}-semi-open and \emph{ij}-semi-closed in $X$. The family of all \emph{ij}-semi-open (resp. \emph{ij}-semi-closed, \emph{ij}-semi-regular) sets of $X$ is denoted by $j\text{-SO}(X)$ (resp. $j\text{-SC}(X)$, $j\text{-SR}(X)$) and for $x \in X$, the family of all \emph{ij}-semi-open sets containing $x$ is denoted by $j\text{-SO}(X,x)$.

A point $x \in X$ is said to be \emph{ij}-semi-\theta-adherent point of $A$ [2] if $j\text{-scl}(A) \cap A \neq \emptyset$ for every \emph{ij}-semi-open set $U$ containing $x$. The set of all \emph{ij}-semi-\theta-adherent points of $A$ is called the \emph{ij}-semi-\theta-closure of $A$ and denoted by $j\text{-scl}(A)$. A subset $A$ is called \emph{ij}-semi-$\theta$-closed if $j\text{-scl}(A) = A$. The set $\{x \in X : j\text{-scl}(U) \subset A\}$ for some $U$ is \emph{ij}-semi-open is called the \emph{ij}-semi-$\theta$-interior of $A$ and is denoted by $j\text{-sint}(A)$. A subset $A$ is called \emph{ij}-semi-$\theta$-open if $A = j\text{-sint}(A)$.

Now, we mention the following definitions and results:

**Definition 1.1.** A bitopological space $(X, \tau_1, \tau_2)$ is said to be:

(i) Pairwise semi-$T_0$ [8] (briefly P-semi-$T_0$) if for each distinct points $x, y \in X$, there exists either an \emph{ij}-semi-open set containing $x$ but not $y$ or a \emph{ij}-semi-open set containing $y$ but not $x$.

(ii) Pairwise semi-$T_1$ [8] (briefly P-semi-$T_1$) if for every two distinct points $x$ and $y$ in $X$, there exists an \emph{ij}-semi-open set $U$ containing $x$ but not $y$ and a \emph{ij}-semi-open set $V$ containing $y$ but not $x$.

(iii) Pairwise semi-$T_2$ [6] (briefly P-semi-$T_2$) if for every two distinct points $x$ and $y$ in $X$, there exists either $U \in j\text{-SO}(X,x)$ and $V \in j\text{-SO}(X,y)$ such that $U \cap V = \emptyset$.

**Lemma 1.3.** [8] For every subset $A$ of a space $X$, we have the following:

(i) $X \setminus j\text{-scl}(A) = j\text{-sint}(X \setminus A)$.

(ii) $X \setminus j\text{-sint}(A) = j\text{-scl}(X \setminus A)$.

**Lemma 1.4.** [7] Let $A$ be a subset of a space $X$. Then we have:

(i) If $U \in j\text{-SO}(X)$, then $j\text{-scl}(U) \subset j\text{-SR}(X)$.

(ii) If $A \in j\text{-SO}(X)$, then $j\text{-scl}(A) = j\text{-scl}(A)$.

**Lemma 1.5.** [9] Let $A$ be a subset of a space $X$. Then we have if $A \in j\text{-SR}(X)$, then $A$ is both \emph{ij}-semi-$\theta$-closed and \emph{ij}-semi-$\theta$-open.

**Lemma 1.6.** [8] If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an \emph{ij}-pre-semi-continuous, then for each subset $S \subset Y$ and each $U \in j\text{-SO}(X)$ containing $f^{-1}(S)$, there exists $V \in j\text{-SO}(Y)$ such that $S \subset V$ and $f^{-1}(V) \subset U$.

**Lemma 1.7.** [7] A bitopological space $(X, \tau_1, \tau_2)$ is \emph{ij}-semi-regular (resp. \emph{ij}-s-regular) if and only if for each \emph{ij}-semi-open (resp. \emph{ij}-open) set $G$ and each point $x \in G$, there exists an \emph{ij}-semi-open set $U$ such that $x \in U$, $F \subset V$ and $j\text{-scl}(U) \subset G$.

**Lemma 1.8.** [7] A bitopological space $(X, \tau_1, \tau_2)$ is \emph{ij}-s-closed if and only if for every cover of $X$ by \emph{ij}-semi-regular sets has a finite subcover.

**Lemma 1.9.** [9]

(i) Every \emph{ij}-semi-$\theta$-closed set is \emph{ij}-\theta-$\text{sg}$-closed.

(ii) A bitopological space $(X, \tau_1, \tau_2)$ is an \emph{P-semi-} $T_{1/2}$-space if and only if every \emph{ij}-\theta-$\text{sg}$-closed set is \emph{ij}-semi-closed.

**Definition 1.10.** [9] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:

(i) \emph{ij}-\theta-semigeneralized continuous (briefly \emph{ij}-\theta-$\text{sg}$-continuous) if $f^{-1}(V)$ is \emph{ij}-\theta-$\text{sg}$-closed in $X$ for every \emph{ij}-semi-closed $V$ of $Y$.

(ii) \emph{ij}-\theta-semigeneralized irresolute (briefly \emph{ij}-\theta-$\text{sg}$-irresolute) if $f^{-1}(V)$ is \emph{ij}-\theta-$\text{sg}$-closed in $X$ for every \emph{ij}-\theta-$\text{sg}$-closed set $V$ of $Y$.

(iii) \emph{ij}-\theta-$\text{sg}$-closed if for every \emph{ij}-semi-closed set $U$ of $X$, $f(U)$ is an \emph{ij}-\theta-$\text{sg}$-closed in $Y$.

(iv) \emph{ij}-semi-generalized closed (briefly \emph{ij}-$\text{sg}$-closed) if for each $\tau_2$-closed set $F$ of $X$, $f(F)$ is an \emph{ij}-\theta-$\text{sg}$-closed set in $Y$.

2. Characterization of pairwise quasi-irresolute functions.

**Definition 2.1.** [7] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be \emph{ij}-quasi-irresolute if for each $x \in X$ and each $V \in j\text{-SO}(Y, f(x))$, there exists $U \in j\text{-SO}(X,x)$ such that $f(U) \subset j\text{-scl}(V)$. If $f$ is 12-quasi-irresolute and 21-quasi-irresolute, then $f$ is called pairwise quasi-irresolute.

**Definition 2.2.** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be \emph{ij}-irresolute [11] (resp. \emph{ij}-semi-continuous) if $f^{-1}(V)$ is an \emph{ij}-semi-open set of $X$ for every \emph{ij}-semi-open (resp. \emph{ij}-open) set $V$ of $Y$.

**Theorem 2.3.** The following statements are equivalent for a function $f : X \to Y$:

(i) $f$ is \emph{ij}-quasi-irresolute.

(ii) $f^{-1}(V) \subset j\text{-scl}(B)$ for every subset $B$ of $Y$.

(iii) $f^{-1}(A) \subset j\text{-scl}(f(A))$ for every subset $A$ of $X$.

(iv) $f^{-1}(f^{-1}(B)) \subset j\text{-S}(X)$ for every \emph{ij}-semi-$\theta$-open set $F$ in $Y$.

(v) $f^{-1}(V) \subset j\text{-SO}(X)$ for every \emph{ij}-semi-$\theta$-open set $V$ in $Y$.

**Proof.** (i) $\Rightarrow$ (ii): Let $B \subset Y$ and $x \notin f^{-1}(j\text{-scl}(B))$. Then $f(x) \notin j\text{-scl}(B)$ and there exists $V \in j\text{-SO}(Y, f(x))$ such that $j\text{-scl}(V) \cap B = \emptyset$. By (i), there exists $U \in j\text{-SO}(X,x)$ such that $f(U) \subset j\text{-scl}(V)$. Hence $f(U) \cap B = \emptyset$ and $f^{-1}(B) = \emptyset$. Consequently, we obtain $x \notin j\text{-scl}(f^{-1}(B))$.

(ii) $\Rightarrow$ (iii): For any subset $A$ of $X$, the inclusion $j\text{-scl}(A) \subset j\text{-scl}(f^{-1}(f(A)))$ holds. By (ii), we have $j\text{-scl}(f^{-1}(f(A))) \subset f^{-1}(j\text{-scl}(f(A)))$ and hence $f^{-1}(j\text{-scl}(A)) \subset j\text{-scl}(f(A))$.

(iii) $\Rightarrow$ (ii): For any subset $B$ of $Y$, we have $j\text{-scl}(f^{-1}(B)) \subset j\text{-scl}(B)$. By (iii), we obtain $f^{-1}(j\text{-scl}(f^{-1}(B))) \subset j\text{-scl}(f^{-1}(B))$ and hence $j\text{-scl}(f^{-1}(B)) \subset f^{-1}(j\text{-scl}(B))$. 

(ii) \(\Rightarrow\) (iv): Let \(F\) be an \(ij\)-semi \(\theta\)-closed set in \(Y\). By (ii), we have \(\text{i}-\text{scl}(f^{-1}(F)) \subseteq f^{-1}(\text{i}-\text{scl}(A)) = f^{-1}(A)\). Therefore, \(f^{-1}(F)\) is \(ij\)-semi-closed in \(X\).

(iv) \(\Rightarrow\) (v): If \(Y\) is \(ij\)-semi \(\theta\)-open in \(X\), then \(f^{-1}(Y)\) is \(ij\)-semi \(\theta\)-closed. By (iv), \(f^{-1}(f^{-1}(Y)) = X\) is \(ij\)-semi-closed in \(X\). Thus \(f^{-1}(Y) = \text{i}-\text{scl}(X)\).

(v) \(\Rightarrow\) (i): Let \(x \in X\) and \(Y \in \text{i}-\text{SO}(Y, f(x))\). It follows from Lemmas 1.4 and 1.5 that \(\text{i}-\text{scl}(\text{i}-\text{SO}(Y)) \subseteq \text{i}-\text{scl}(\text{i}-\text{SO}(X))\). By (v), we observe that \(U \in \text{i}-\text{SO}(X)\) and \(f(U) \subseteq \text{i}-\text{scl}(V)\). The proof is complete.

The next theorem contains an unexpected result.

**Theorem 2.4.** The following statements are equivalent for a function \(f : X \to Y\):

(i) \(f\) is \(ij\)-quasi-irresolute

(ii) For each \(x \in X\) and each \(Y \in \text{i}-\text{SO}(Y, f(x))\), there exists \(U \in \text{i}-\text{SO}(X)\) such that \(\text{i}-\text{scl}(U) \subseteq \text{i}-\text{scl}(V)\).

(iii) \(f^{-1}(F) \subseteq \text{i}-\text{SR}(Y)\) for every \(F \subseteq \text{i}-\text{SR}(Y)\).

**Proof.** Suppose that \(V\) is a \(ij\)-semi \(\theta\)-closed set in \(Y\). By Lemma 1.9(i), \(V\) is \(ij\)-\(\theta\)-sg-closed in \(Y\). Since \(f\) is \(ij\)-\(\theta\)-sg-irresolute, \(f^{-1}(V)\) is \(ij\)-\(\theta\)-sg-closed in \(X\). By Lemma 1.9(ii), \(f^{-1}(Y)\) is \(ij\)-semi-closed.

Thus this shows that \(f\) is \(ij\)-quasi-irresolute, by Theorem 2.3.

**Theorem 2.7.** If \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is an \(ij\)-quasi-irresolute and \(Y\) is \(ij\)-semi-regular, then \(f\) is \(ij\)-irresolute.

**Proof.** Let \(V \in \text{i}-\text{SO}(Y)\) and \(x \in f^{-1}(V)\), there exists \(W \in \text{i}-\text{SO}(Y)\) such that \(f(V) \subseteq W \subseteq \text{i}-\text{scl}(W)\). Since \(f\) is \(ij\)-quasi-irresolute, then there exists \(U \subseteq \text{i}-\text{SO}(X)\) such that \(f(U) \subseteq \text{i}-\text{scl}(W)\) and hence \(f^{-1}(V) \subseteq \text{i}-\text{SO}(X)\). Therefore, \(f\) is \(ij\)-irresolute.

**Theorem 2.8.** If \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is an \(ij\)-quasi-irresolute and \(Y\) is \(ij\)-semi-regular, then \(f\) is \(ij\)-semi-continuous.

**Proof.** Similar to that of Theorem 2.7.

**Lemma 2.9.** Let \(f : X \to Y\) and \(g : X \to X \times Y\) be the graph function of \(f\) where \(g(x) = (x, f(x))\) for each \(x \in X\). If \(g\) is \(ij\)-quasi-irresolute, then \(f\) is \(ij\)-quasi-irresolute.

**Proof.** If \(x \in X\) and \(V \in \text{i}-\text{SO}(f(x))\). Then \(X \times Y\) is an \(ij\)-semi-open set in \(X \times Y\) containing \(g(x)\). Since \(g\) is \(ij\)-quasi-irresolute there exists \(U \subseteq \text{i}-\text{SO}(X)\) such that \(g(U) \subseteq \text{i}-\text{scl}(X \times Y)\). Then we obtain \(f(U) \subseteq \text{i}-\text{SO}(V)\).

The converse of Lemma 2.9, is not true as the next example shows.

**Example 10.** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\), and \(\tau_2 = \{\emptyset, \{b\}, \{a, b\}\}\). Define a function \(f : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2)\) by \(f(a) = b, f(b) = a\), and \(f(c) = c\). Then \(f\) is \(12\)-irresolute and hence \(12\)-quasi-irresolute but \(g\) is not \(12\)-quasi-irresolute. It is apparent that \(12\)-SO contains \(\{\emptyset, \{a\}, \{b\}, \{a, b\}\}\). Then \(g\) is not \(12\)-quasi-irresolute at \(c\).

The converse of Lemma 2.9, is not true as the next example shows.

**Theorem 2.5.** The following statements are equivalent for a function \(f : X \to Y\):

(i) \(f\) is \(ij\)-quasi-irresolute

(ii) \(\text{i}-\text{scl}(f^{-1}(B)) \subseteq f^{-1}(\text{i}-\text{scl}(B))\) for every subset \(B\) of \(Y\).

(iii) \(f^{-1}(\text{i}-\text{scl}(A)) \subseteq \text{i}-\text{scl}(f^{-1}(A))\) for every subset \(A\) of \(X\).

(iv) \(f^{-1}(\text{i}-\text{SR}(X, f(x))) \subseteq \text{i}-\text{SR}(Y, f(x))\) for every \(x \in X\).

(v) \(f^{-1}(V) \subseteq \text{i}-\text{SR}(X, f(x))\) for every \(x \in X\).

**Proof.** By making use of Theorem 2.4, we can prove this Theorem in the similar way to the proof of Theorem 2.3.

**Theorem 2.6.** Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be \(ij\)-\(\theta\)-sg-irresolute. If \((X, \tau_1, \tau_2)\) is \(ij\)-\(\theta\)-sg-closed, then \(f\) is \(ij\)-quasi-irresolute.
Theorem 2.13. If a function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pair-wise quasi-irresolute and ij-pre-semi-closed, then for every ij-\( \theta \)-sg-closed set \( F \) of \( Y, f^{-1}(F) \) is ij-\( \theta \)-sg-closed set of \( X \).

Proof. Suppose that \( F \) is an ij-\( \theta \)-sg-closed set of \( Y \). Assume \( f^{-1}(F) \subseteq U \) where \( U \subseteq \text{ij-SO}(X) \). Since \( f \) is ij-pre-semi-closed and by Lemma 1.6, there is an ij-semi-open set \( V \) such that \( F \subseteq V \) and \( f^{-1}(V) \subseteq U \). Therefore, \( f^{-1}(U) \subseteq \text{ij-SO}(X) \). Consequently, \( f^{-1}(F) \) is ij-\( \theta \)-sg-closed set in \( X \). \( \square \)

Theorem 2.14. If a space \( X \) is pairwise semi-\( T_{1/2} \) and \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is surjective, quasi-irresolute and ij-pre-semi-closed, then \( Y \) is pairwise semi \( T_{1/2} \).

Proof. Assume that \( A \) is an ij-\( \theta \)-sg-closed subset of \( Y \). Then by Theorem 2.13, we have \( f^{-1}(A) \) is an ij-\( \theta \)-sg-closed subset of \( X \). By Theorem 2.12, \( f^{-1}(A) \) is ij-\( \theta \)-closed and hence, \( A \) is ij-\( \theta \)-closed. It follows that \( Y \) is pairwise semi \( T_{1/2} \). \( \square \)

Definition 2.15. Let \( X \) and \( Y \) be bitopological spaces. A subset \( S \) of \( X \times Y \) is called ij-strongly semi \( \theta \)-closed if for each \((x, y) \in S \setminus \text{Scl}(S)\), there exist \( U \subseteq \text{ij-SO}(X, x) \) and \( V \subseteq \text{ij-SO}(Y, y) \) such that \( (\text{ij-scl}(U) \times \text{ij-scl}(V)) \cap S = \emptyset \).

Definition 2.16. A function \( f: X \to Y \) is said to be an ij-\( \theta \)-strongly semi \( \theta \)-closed graph if its graph \( G(f) \) is ij-\( \theta \)-strongly semi \( \theta \)-closed in \( X \times Y \) where \( G(f) = \{(x, f(x)) : x \in X\} \).

Theorem 2.17. If a function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is a ij-\( \theta \)-quasi-irresolute and Y is pairwise semi \( T_2 \), then \( G(f) \) is ij-\( \theta \)-semiclosed.

Proof. Let \((x, y) \not\in G(f)\), then we have \( y \neq f(x) \). Since \( Y \) is pairwise semi \( T_2 \), by Lemma 2.11, there exist \( V \subseteq \text{ij-SO}(Y, y) \) and \( W \subseteq \text{ij-SO}(X, x) \) such that \( \text{ij-scl}(W) \cap \text{ij-scl}(V) = \emptyset \). Since \( f \) is ij-\( \theta \)-quasi-irresolute, by Theorem 2.4, there exists \( U \subseteq \text{ij-SO}(X, x) \) such that \((f \times f)(U) \subseteq \text{ij-scl}(W)\). Therefore, we have \( f(\text{ij-scl}(U)) \cap \text{ij-scl}(V) = \emptyset \). This shows that \( G(f) \) is ij-\( \theta \)-semi \( \theta \)-closed. \( \square \)

Theorem 2.18. If a function \( f: X \to Y \) has an ij-\( \theta \)-strongly semi \( \theta \)-closed graph and \( g: X \to Y \) is an ij-\( \theta \)-quasi-irresolute function, then the set \( A = \{(x_1, x_2) \in X \times X : f(x_1) = g(x_2)\} \) is an ij-\( \theta \)-semiclosed \( \theta \)-closed in \( X \times X \).

Proof. Let \((x_1, x_2) \not\in (X \times X) \setminus A\). Then we have \( f(x_1) \neq g(x_2) \) and hence \((x_1, f(x_1)) \not\in (X \times Y) \setminus G(f)\). Since \( G(f) \) is ij-\( \theta \)-semiclosed \( \theta \)-closed, there exists \( U \subseteq \text{ij-SO}(X, x_1) \) such that \( U \subseteq f^{-1}(f(U)) \subseteq \text{ij-scl}(W) \) and hence \((f \times f)(U) \subseteq \text{ij-scl}(W)\). Consequently, we obtain \((f \times f)(U) \subseteq \text{ij-scl}(W)\) and hence \( f(\text{ij-scl}(U)) \cap \text{ij-scl}(W) = \emptyset \). Hence \( A \) is ij-\( \theta \)-semi \( \theta \)-closed in \( X \times X \). \( \square \)

Corollary 2.19. If \( f: X \to Y \) is an ij-\( \theta \)-quasi-irresolute function and \( Y \) is pairwise semi \( T_2 \), then the set \( A = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\} \) is ij-\( \theta \)-semi \( \theta \)-closed in \( X \times X \).
Theorem 3.3. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is ij-strongly irresolute, then $f$ is ij-$\sigma$-$\delta$-continuous.

Proof. Let $V$ be ij-semi-closed set of $Y$. Since $f$ is ij-strongly irresolute, then by Theorem 3.2, $f^{-1}(V)$ is ij-semi $\theta$-closed. By lemma 1.9(i), $f^{-1}(V)$ is ij-$\theta$-sg-closed. Thus $f$ is ij-$\theta$-sg-continuous. □

The converse of above theorem need not be true the following example show that.

Example 3.4. Let $X = \{a, b, c, d\}$, $Y = \{x, y, z\}$, $\tau_1 = \{\phi, \{c\}, \{b, c, d\}, X\}$, $\tau_2 = \{\phi, \{c, d\}, X\}$, $\sigma_1 = \{\phi, \{z\}, Y\}$ and $\sigma_2 = \{\phi, \{y, z\}, Y\}$. Define a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by setting $f(a) = f(b) = f(d) = x$ and $f(c) = z$. Then $f$ is 12-$\theta$-sg-continuous, since $A = \{a, b, d\} = f^{-1}(\{x\})$ is 12-$\theta$-sg-closed. But $A$ is not 21-semi $\theta$-closed. Hence $f$ is not 21-strong irresolute, since $\{x\}$ is 21-semi-closed in $Y$.

Theorem 3.5. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \nu_1, \nu_2)$ are two functions, then:

(i) If $f$ is ij-strongly irresolute and $g$ is ij-irresolute, then $gof : X \rightarrow Z$ is ij-irresolute.

(ii) If $f$ is ij-quasi-irresolute and $g$ is ij-strongly irresolute, then $gof$ is ij-strongly irresolute.

Proof.  
(i) Let $V \in ijSO(Z)$. Then $g^{-1}(V) \in ijSO(Y)$, since $g$ is ij-irresolute. By Theorem 3.2, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is ij-semi $\theta$-open in $X$. Thus $gof$ is ij-strong irresolute.

(ii) This follows immediately from Theorem 2.5 and 3.2. □

Theorem 3.6. An ij-irresolute function $f : X \rightarrow Y$ is ij-strongly irresolute if and only if $X$ is ij-semi-regular.

Proof. Let $f : X \rightarrow X$ be the identity function. Then $f$ is ij-irresolute and ij-strongly irresolute by hypothesis. For any $x \in X$ and any $F \in ijSC(X)$ not containing $x$, $f(x) = x \in X \setminus F \in ijSO(X)$ and there exists $U \in ijSO(X)$ such that $f(ij-scl(U)) \subseteq X \setminus F$. Therefore, we obtain $x \in U \in ijSO(X)$, $F \subseteq X \setminus ij-scl(U) \subseteq ijSO(X)$ and $U \subseteq (X \setminus ij-scl(U))$. This obvious that $X$ is ij-semi-regular. □

Conversely, suppose that $f : X \rightarrow Y$ is ij-irresolute and $X$ is ij-semi-regular. For any $x \in X$ and any $F \in ijSO(f(x))$, $f^{-1}(V) \in ijSO(X)$ and there exists $U \in ijSO(X)$ such that $x \in U \in ij-scl(U) \subseteq f^{-1}(V)$ by Lemma 1.7. Therefore, we have $f(ij-scl(U)) \subseteq V$. This shows that $f$ is ij-strongly irresolute.

Theorem 3.7. Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ the graph function of $f$. If $g$ is ij-irresolute, then $f$ is ij-semi-irresolute and $X$ is ij-semi-regular.

Proof. First, we show that $f$ is ij-strongly irresolute. Let $x \in X$ and $V \in ijSO(f(x))$. Then $X \times V$ is an ij-semi-open set of $X \times Y$ containing $g(x)$. Since $g$ is ij-irresolute, there exists $U \in ijSO(X)$ such that $g(ij-scl(U)) \subseteq X \times V$. Therefore, we obtain $f(ij-scl(U)) \subseteq V$. Next, let $x \in X$ and $U \in ijSO(X)$. Since $g(x) \in U \times X \in ijSO(X \times Y)$, there exists $U_0 \in ijSO(X)$ such that $g(ij-scl(U_0)) \subseteq U \times Y$. Therefore, we obtain $x \in U_0 \subseteq ij-scl(U_0) \subseteq U$ and hence $X$ is ij-semi-regular. □

Remark 3.8. The converse to Theorem 3.7, is not true because in Example 2.10, $f$ is 12-strongly irresolute and $X$ is 12-semi-regular but $g$ is not 12-strongly irresolute.

Theorem 3.9. If $f : X \rightarrow Y$ is a $P$-strongly irresolute injection and $Y$ is $P$-semi $T_0$, then $X$ is $P$-semi $T_2$.

Proof. Let $x$ and $y$ be any pair of distinct points of $X$. Since $f$ is injective it follows that $f(x) \neq f(y)$. Since $Y$ is $P$-semi $T_0$, there exists $V \in ijSO(f(x))$ not containing $f(y)$ or $W \in ijSO(f(y))$ not containing $f(x)$. If it holds that $f(x) \notin V \in ijSO(f(x))$ and since $f$ is $P$-strongly irresolute then there exists $U \in ijSO(X)$ such that $f(ij-scl(U)) \subseteq V$. Therefore, we obtain $g(f(x)) \notin f(ij-scl(U))$ and hence $y \in X \setminus ij-scl(U) \in ijSO(X)$. If the other case holds, then we obtain the similar result. Therefore, $X$ is $P$-semi $T_2$. □

4. $ij$-semi-compact and $ij$-$s$-closed spaces.

Definition 4.1. Let $A$ be a subset of a space $X$, then:

(i) A subset $A$ is said to be $ij$-semi-compact relative to $X$ (resp. $ij$-s-closed relative to $X$ [7]) if for every cover $\{V : z \in V\}$ of $A$ by $ij$-semi-open sets of $X$, there exists a finite subset $V_0$ of $\{V : z \in V\}$ (resp. $A \subseteq V$) $\subseteq \{V : z \in V\}$.

(ii) A space $X$ is said to be $ij$-semi-compact [5] (resp. $ij$-s-closed [7]) if $X$ is $ij$-semi-compact relative to $X$ (resp. $ij$-s-closed relative to $X$).

(iii) A subset $A$ is called $ij$-semi-compact if the subspace $A$ is $ij$-semi-compact.

Theorem 4.2. Let $f : X \rightarrow Y$ be an ij-strongly irresolute function. If $A$ is ij-$s$-closed relative to $X$, then $f(A)$ is ij-semi-compact.

Proof. Let $A$ be ij-$s$-closed relative to $X$ and $\{V : z \in V\}$ any cover of $f(A)$ by ij-$s$-open sets of $Y$. For each $x \in A$, there exists $V(x) \in \{V : z \in V\}$ such that $f(x) \in V(x)$. Since $f$ is ij-strongly irresolute, there exists $U_x \in ijSO(X)$ such that $f(ij-scl(U_x)) \subseteq V(x)$. The family $\{U_x : x \in A\}$ form an ij-$s$-open cover of $A$ and there exists a finite number of points $x_1, x_2, \ldots, x_n$ in $A$ such that $A \subseteq \bigcup_i \{ij-scl(U_{x_i}) : i = 1, 2, \ldots, n\}$. Thus $f(A)$ is ij-semi-compact relative to $Y$. □

Corollary 4.3. If $X$ is ij-$s$-closed and $f : X \rightarrow Y$ is an ij- quasi-irresolute (resp. ij-strongly irresolute) surjection, then $Y$ is ij-$s$-closed (resp. ij-semi-compact).

Proof. The second case follows from Theorem 4.2. We shall shows the first. Let $\{V : z \in V\}$ be an ij-$s$-open cover of $Y$. By Lemma 1.4, the family $\{ij-scl(V) : z \in V\}$ is a cover of $Y$ by ij-$s$-regular sets of $Y$. It follows from Theorem 2.4,
Lemma 4.4. A surjection $f : X \to Y$ is $ij$-pre-semi-closed if and only if for each point $y \in Y$ and each $U \in ij$-$SO(X)$ containing $f^{-1}(y)$, there exists $V \in ij$-$SO(Y)$ such that $f^{-1}(V) \subset U$.

Proof. The first side follows from Lemma 1.6. On the other hand, let $A$ be an $ij$-semi-open set of $X$. Suppose that $y \in Y \setminus f(A)$ where $X \setminus A$ is $ij$-semi-closed set of $X$. By hypostasis, there exists an $ij$-semi-open set $V \subset Y$ such that $f^{-1}(V) \subset X \setminus A$. Thus $A \subset f^{-1}(Y \setminus V)$, this implies $f(A) \subset Y \setminus V$. Hence $y \in V \cap f(A)$ and $V \cap f(A)$ is $ij$-semi-open set of $Y$. It follows that $f(A)$ is $ij$-semi-closed set in $Y$ and hence $f$ is an $ij$-pre-semi-closed. □

Theorem 4.5. Let $f : X \to Y$ be an $ij$-pre-semi-closed surjection and $f^{-1}(y)$ be $ij$-closed relative to $Y$ (resp. $ij$-semi-compact relative to $Y$) for each $y \in Y$. If $K$ is an $ij$-semi-compact relative to $Y$, then $f^{-1}(K)$ is $ij$-semi-closed relative to $X$ (resp. $ij$-semi-compact relative to $Y$).

Proof. Suppose that for each $y \in Y$, $f^{-1}(y)$ is $ij$-closed relative to $Y$ and $K$ is $ij$-semi-compact relative to $Y$. Let $\{V_x : x \in \mathcal{V}\}$ be a cover of $f^{-1}(K)$ by $ij$-semi-open sets of $X$. For each $y \in K$, there exists a finite subset $\mathcal{V}(y)$ of $\mathcal{V}$ such that $f^{-1}(y) \subset \bigcup \{ij$-$scl(U_x) : x \in \mathcal{V}(y)\}$. By Lemma 1.4, $ij$-$scl(U_x) \subset ij$-$SO(X)$ for each $x \in \mathcal{V}$ and hence $\bigcup \{ij$-$scl(U_x) : x \in \mathcal{V}(y)\} \subset ij$-$SO(X)$. By Lemma 4.4, there exists $V_y \in ij$-$SO(Y)$ such that $f^{-1}(V_y) \subset \bigcup \{ij$-$scl(U_x) : x \in \mathcal{V}(y)\}$. Since $\{V_y : y \in K\}$ is an $ij$-semi-open cover of $K$, for a finite number of points $y_1, y_2, \ldots, y_n$ in $K$, we have $K \subset \bigcup V_{y_i} : i = 1, 2, \ldots, n$ and hence $f^{-1}(K) \subset \bigcup f^{-1}(V_{y_i}) \subset \bigcup \mathcal{V}(y_i)(ij$-$scl(U_x))$. Therefore, $f^{-1}(K)$ is $ij$-semi-closed relative to $X$. The proof of the case $ij$-semi-compact relative to $Y$ is similar. □

Corollary 4.6. Let $f : X \to Y$ be an $ij$-pre-semi-closed surjection and $f^{-1}(y)$ be $ij$-closed relative to $Y$ (resp. $ij$-semi-compact relative to $Y$) for each $y \in Y$. If $Y$ is $ij$-semi-compact, then $X$ is $ij$-closed (resp. $ij$-semi-compact).

Proof. This follows immediately from Theorem 4.5. □

5. Comparisons.

Definition 5.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $ij$-semi-weakly continuous if for each $x \in X$ and each $\sigma_2$-open neighborhood $V$ of $f(x)$, there exists $U \in ij$-$SO(X)$ such that $f(U) \subset ccl(V)$.

Remark 5.2. $ij$-strongly irresolute implies $ij$-irresolute and $ij$-irresolute implies $ij$-quasi-irresolute. However, $ij$-strongly irresolute and $i$-continuous are independent of each other as the following two examples show.

Example 5.3. Let $X = \{a, b, c\}$, $\tau_1 = \{\{\}, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\{\}, \{a\}, \{a, b\}, X\}$, $\sigma_1 = \{\{a\}, \{b\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\{a\}, \{b\}, \{a, b\}, Y\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $ij$-strongly irresolute but not $ij$-continuous.

Example 5.4. Let $X = \{a, b, c\}$, $\tau_1 = \{\{\}, \{a\}, \{b\}, \{a, b\}, Y\}$, $\tau_2 = \{\{\}, \{a\}, \{a, b\}, X\}$, $\sigma_1 = \{\{\}, \{a\}, \{b\}, Y\}$ and $\sigma_2 = \{\{\}, \{a\}, \{b\}, Y\}$. Then the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f(a) = c, f(b) = b$ and $f(c) = a$. It is evident that $f$ is $ij$-continuous. Since $b \in f(U)$ and $12$-$SO(a) = \{\{a\}, X\}$, then for each $U \in 12$-$SO(a)$, we have $f(U)a = 21$-$scl(a)$ for every $U \in 12$-$SO(a)$. This shows that $f$ is not $ij$-quasi-irresolute.

Theorem 5.5. An $ij$-irresoluteness implies both $ij$-quasi-irresolute and $ij$-semi-continuous.

Proof. Straightforward from the fact that every $ij$-open set is $ij$-semi-open and [[6], Remark 5.1]. □

The converse of Theorem 5.5 is not true as Example 5.4 and the following example show.

Example 5.6. Let $X = \{a, b, c\}$, $\tau_1 = \{\{\}, \{a\}, \{b\}, \{a, b\}, Y\}$, $\sigma_1 = \{\{\}, \{a\}, \{b\}, Y\}$ and $\sigma_2 = \{\{\}, \{a\}, \{b\}, Y\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is $ij$-quasi-irresolute. However, $f$ is not $ij$-semi-continuous and hence not $ij$-irresolute.

Theorem 5.7. An $ij$-quasi-irresolute implies $ij$-semi-weak continuity.

Proof. It follows from definition. □

The converse of the above theorem is not true, since in Example 5.4, $f$ is $ij$-semi-weakly continuous but not $ij$-quasi-irresolute.

Remark 5.8. Every $ij$-semi-continuous function is $ij$-semi-weakly continuous but the converse is not true, the following example shows that.

Example 5.9. Let $X = \{a, b\}$, $\tau_1 = \{\{\}, \{a\}, X\}$, $\tau_2 = \{\{\}, \{a\}, \{b\}, X\}$, $\sigma_1 = \{\{\}, \{a\}, \{b\}, X\}$ and $\sigma_2 = \{\{\}, \{a\}, \{b\}, X\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is $ij$-weakly irresolute but not $ij$-semi-continuous.

Remark 5.10. Every $ij$-strongly irresolute function is $ij$-irresolute. The converse need not be true, the following example shows that.

Example 5.11. Let $X = \{a, b, c\}$, $\tau_1 = \{\{\}, \{a, b\}, X\}$, $\tau_2 = \{\{\}, \{a\}, \{b\}, X\}$, $\sigma_1 = \{\{\}, \{a\}, \{b\}, X\}$ and $\sigma_2 = \{\{\}, \{a\}, \{b\}, X\}$. Then the function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Then $f$ is $ij$-irresolute but $f$ is not $ij$-strongly irresolute, since $\{b, c\} \in 12$-$SO(X)$ and $12$-$SO(X) = \{\{\}, \{a\}, \{b\}, X\}$ such that $21$-$scl(a, b) = X$. Thus $f(x) \{b, c\}$ and hence $f$ is not $ij$-strongly irresolute.

By [[6], Remark 5.1] and for remarks in this section, we obtain the following example, where none of the implication is reversible.
Pairwise weakly and pairwise strongly irresolute functions

References