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On arcs in projective Hjelmslev planes

Thomas Honold[a], Ivan Landjev[b,*]

[a]*Zentrum Mathematik, Technische Universität München, D-80290 München, Germany*
[b]*Institute of Mathematics and Informatics, 8 Acad. G. Bonchev str., 1113 Sofia, Bulgaria*

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Abstract

A (k,n)-arc in the projective Hjelmslev plane $\mathrm{PHG}(R_R^3)$ is defined as a set of k points in the plane such that some n but no $n+1$ of them are collinear. In this paper, we consider the problem of finding the largest possible size of a (k,n)-arc in $\mathrm{PHG}(R_R^3)$. We present general upper bounds on the size of arcs in the projective Hjelmslev planes over chain rings R with $|R| = q^2$, $R/\mathrm{rad}\,R \cong \mathbb{F}_q$. We summarize the known values and bounds on the cardinalities of (k,n)-arcs in the chain rings with $|R| \leqslant 25$ ($|R| = q^2$, $R/\mathrm{rad}\,R \cong \mathbb{F}_q$).

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1. Introduction

Let R be a chain ring and let R_R^3 be the right free module of rank 3 over R. Denote by $\mathscr{P}$ the set all free rank 1 submodules of R_R^3 and by $\mathscr{L}$ the set of all free rank 2 submodules of R_R^3. The incidence structure $(\mathscr{P}, \mathscr{L}, I)$ with incidence relation $I \subset \mathscr{P} \times \mathscr{L}$ given by set-theoretical inclusion is called the right projective Hjelmslev plane over R and is denoted by $\mathrm{PHG}(R_R^3)$.

In this paper, we consider the problem of finding the largest size of a (k,n)-arc in $\mathrm{PHG}(R_R^3)$ for chain rings R with $|R| = q^2$, $R/\mathrm{rad}\,R \cong \mathbb{F}_q$. The interest in this problem comes from coding theory. Multisets of points in projective Hjelmslev geometries are equivalent to fat linear codes over finite chain rings, i.e. codes for which the entries in no coordinate position are contained in a proper ideal of the ring. Thus, the optimal arc problem in a projective Hjelmslev geometry is equivalent to determining the minimal

* Corresponding author. Tel.: +003592-756116; fax: +003592-756116.
E-mail addresses: honold@mathematik.tu-muenchen.de (T. Honold), ivan@moi.math.bas.bg (I. Landjev).

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length of a (left) linear code over R of fixed rank and fixed minimum distance with respect to the Hamming metric [10,13].

Linear codes over finite chain rings can be mapped into codes (not necessarily linear) over a q-ary alphabet. This leads sometimes to codes which are better than any known linear codes, the most striking example being the families of the Kerdock and Preparata codes [5,19].

In what follows, we present some general bounds on the size of an arc in a projective Hjelmslev plane and summarize the known constructions and bounds for arcs in the small Hjelmslev planes over the chain rings with 4, 9, 16 and 25 elements.

2. Basic facts

A ring (associative, with identity $1 \neq 0$; ring homomorphisms preserving the identity) is called a left (right) chain ring if its lattice of left (right) ideals forms a chain. The most important properties of finite chain rings are summarized in the following theorem [2,17,18]).

Theorem 2.1. *For a finite ring R with radical $N \neq 0$ the following conditions are equivalent*:

(i) *R is a left chain ring*;
(ii) *the principal left ideals of R form a chain*;
(iii) *R is a local ring, and $N = R\theta$ for any $\theta \in N \setminus N^2$*;
(iv) *R is a right chain ring.*

Moreover, if R satisfies the above conditions, then every proper left (right) ideal of R has the form $N^i = R\theta^i = \theta^i R$ for some positive integer i.

Henceforth, the symbols R, N, θ will be as in Theorem 2.1. We restrict ourselves to chain rings with index of nilpotency $m = 2$. Thus we will always have $|R| = q^2$, $R/N \cong \mathbb{F}_q$. The chain rings with this property have been classified in [3] (cf. also [20]). If $q = p^r$, then there are exactly $r + 1$ isomorphism classes of such rings. These are:

- for every $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ the rings $R_\sigma = \mathbb{F}_q \oplus \mathbb{F}_q t$ of σ-dual numbers over $\mathbb{F}_q$ with componentwise addition and multiplication $(x_0 + x_1 t)(y_0 + y_1 t) = x_0 y_0 + (x_0 y_1 + x_1 \sigma(y_0)) \cdot t$;
- the Galois ring $\mathrm{GR}(q^2, p^2) = (\mathbb{Z}/p^2\mathbb{Z})[x]/(f(x))$, where $f(x) \in (\mathbb{Z}/p^2\mathbb{Z})[x]$ is monic of degree r and irreducible modulo p.

Note that the ring $\mathrm{GR}(q^2, p^2)$ is commutative, while R_σ is commutative if and only if $\sigma = 1$. We denote the rings $R_{\mathrm{id}} = \mathbb{F}_q \oplus \mathbb{F}_q t$ by $\mathbb{S}_q$, the ring $R_\sigma = \mathbb{F}_4 \oplus \mathbb{F}_4 t$ by $\mathbb{T}_4$ with $\sigma : a \to a^2$ and the ring $\mathrm{GR}(4^2, 2^2) = \mathbb{Z}_4[x]/(x^2 + x + 1)$ by $\mathbb{G}_{16}$.

Let R be a finite (left) chain ring and consider the module $H = R_R^3$. Let $H^* := H \setminus H\theta$. Define the sets $\mathscr{P}$ and $\mathscr{L}$ by

$$\mathscr{P} = \{xR \mid x \in H^*\},$$

$$\mathscr{L} = \{xR + yR \mid x, y \text{ linearly independent}\}$$

as well as an incidence relation $I \subseteq \mathscr{P} \times \mathscr{L}$ by set-theoretical inclusion. For the incidence structure $(\mathscr{P}, \mathscr{L}, I)$ define also the neighbour relation $\bigcirc$ as follows:

(N1) the points $X, Y \in \mathscr{P}$ are neighbours ($X \bigcirc Y$) iff there exist two different lines incident with both of them;

(N2) the lines s, $t \in \mathscr{L}$ are neighbours ($s \bigcirc t$) iff there exist two different points incident with both of them.

Definition 1. The incidence structure $\Pi = (\mathscr{P}, \mathscr{L}, I)$ with the neighbour relation $\bigcirc$ is called the (right) projective Hjelmslev plane over R and is denoted by $\mathrm{PHG}(R_R^3)$.

The relation $\bigcirc$ is an equivalence relation on both $\mathscr{P}$ and $\mathscr{L}$. The class $[X]$ of all points which are neighbours to the point $X = xR$ consists of all free rank 1 submodules contained in $xR + H\theta$. Similarly, the class $[s]$ of all lines which are neighbours to $s = xR + yR$ consists of all free rank 2 submodules contained in $xR + yR + H\theta$. For a rigorous approach to general projective Hjelmslev spaces, see [14–16,22].

The next theorems provide some basic knowledge about the structure of projective Hjelmslev planes over finite chain rings. They are part of more general results [1,4,9,14–16,22].

Theorem 2.2. *Let $\Pi = \mathrm{PHG}(R_R^3)$, where R is a chain ring with $|R| = q^2$, $R/N \cong \mathbb{F}_q$. Then*

(i) $|\mathscr{P}| = |\mathscr{L}| = q^2(q^2 + q + 1)$;
(ii) *every point (line) has q^2 neighbours*;
(iii) *every point (line) is incident with $q(q+1)$ lines (points)*;
(iv) *given a point P and a line l with PIl, there exist exactly q points on l which are neighbours to P and exactly q lines through P which are neighbours to l.*

Let π denote the natural homomorphism $\pi : R^3 \to R^3/R^3\theta$ and $\bar{\pi}$ the mapping induced by π on the submodules of R^3. For every point X and every line l we have

$$[X] = \{Y \in \mathscr{P} \mid \bar{\pi}(Y) = \bar{\pi}(X)\},$$

$$[s] = \{t \in \mathscr{L} \mid \bar{\pi}(t) = \bar{\pi}(s)\}.$$

Let $\mathscr{P}'$ (resp. $\mathscr{L}'$) be the set of all neighbour classes of points (resp. lines).

Theorem 2.3. *The incidence structure $(\mathscr{P}', \mathscr{L}', I')$ with I' defined by*

$$[X]I'[s] \Leftrightarrow \exists X' \in [X], \quad \exists s' \in [s] : X'Is'$$

is isomorphic to the projective plane $\mathrm{PG}(2, q)$.

Let $\Pi = (\mathscr{P}, \mathscr{L}, I) = \mathrm{PHG}(R^3_{\mathrm{R}})$. Given a point P, let $\mathscr{L}(P)$ be the set of all lines of $\mathscr{L}$ incident with points in $[P]$. For two lines $s,\ t \in \mathscr{L}$ we write $s \sim t$ if s and t coincide on $[P]$ and let $\mathscr{L}_1$ be a complete set of representatives of the lines from $\mathscr{L}(P)$.

Theorem 2.4.

$$\Pi_P = ([P], \mathscr{L}_1, I|_{[P]\times\mathscr{L}_1}) \cong \mathrm{AG}(2,q).$$

Let l be a line in Π and $\mathfrak{P} = \{s \cap [X] \,|\, X \bigcirc l,\ s \in \mathscr{L},\ s \bigcirc l\} \cup \{P_\infty\}$. Define an incidence relation $\mathfrak{I} \subseteq \mathfrak{P} \times \mathscr{L}$ by

$$(s \cap [P])\mathfrak{I}t \Leftrightarrow t \cap (s \cap [P]) \neq \emptyset,$$
$$(P_\infty)\mathfrak{I}t \Leftrightarrow t \not\bigcirc l.$$

For two lines $l_1,\ l_2 \in \mathscr{L}$, write $l_1 \sim l_2$ if they are incident under $\mathfrak{I}$ with the same elements of $\mathfrak{P}$. Denote by $\mathfrak{L}$ a set of lines containing exactly one representative from each equivalence class of $\mathscr{L}$ under $\sim$. The following theorem can be found in a different form in [1,6, Satz 1(b); Proposition 3.2].

Theorem 2.5. *The incidence structure* $(\mathfrak{P}, \mathfrak{L}, \mathfrak{I}|_{\mathfrak{P}\times\mathfrak{L}})$ *is isomorphic to* $\mathrm{PG}(2,q)$.

Let the points $X_1, X_2, \ldots, X_s$, $2 \leqslant s \leqslant q$, be collinear points from the same neighbour class, say $[P]$. Then every line incident with two of the points X_i is incident with all of them. There exist exactly q such lines, say $l_1, l_2, \ldots, l_q$, and all these lines are neighbours. The neighbour class $[l] \in \mathscr{L}'$ with $l_i \in [l]$ is said to have the direction of the pointset $\{X_1, X_2, \ldots, X_s\}$.

3. Arcs and blocking multisets in projective Hjelmslev planes

Let $\Pi = (\mathscr{P}, \mathscr{L}, I)$ be a projective Hjelmslev plane.

Definition 2. A multiset in Π is a mapping $\mathfrak{k} : \mathscr{P} \to \mathbb{N}_0$.

The integer $\mathfrak{k}(P)$ is called the *multiplicity* of the point P. The mapping $\mathfrak{k}$ can be extended to the subsets of $\mathscr{P}$ by

$$\mathfrak{k}(\mathscr{Q}) = \sum_{P \in \mathscr{Q}} \mathfrak{k}(P), \quad \text{for } \mathscr{Q} \subseteq \mathscr{P}.$$

The integer $\mathfrak{k}(\mathscr{P}) = \sum_{P\in\mathscr{P}} \mathfrak{k}(P)$ is called the *cardinality* of the multiset $\mathfrak{k}$. The support $\operatorname{Supp}\mathfrak{k}$ of $\mathfrak{k}$ is defined by $\operatorname{Supp}\mathfrak{k} = \{P \in \mathscr{P} \,|\, \mathfrak{k}(P) > 0\}$.

Definition 3. Two multisets $\mathfrak{k}'$ and $\mathfrak{k}''$ in the projective Hjelmslev planes Π' and Π'', respectively, are said to be equivalent if there exists an isomorphism σ: $\Pi' \to \Pi''$ such that $\mathfrak{k}'(P) = \mathfrak{k}''(\sigma(P))$, for every $P \in \mathscr{P}$.

Definition 4. The multiset $\mathfrak{k}:\mathscr{P}\to\mathbb{N}_0$ is called a *(k,n)-arc* if $\mathfrak{k}(\mathscr{P})=k$ and $\mathfrak{k}(l)\leqslant n$ for any line $l\in\mathscr{L}$. A (k,n)-arc is said to be *complete* if there is no (k',n)-arc $\mathfrak{k}$ with $k'>k$ and $\mathfrak{k}'(P)\geqslant\mathfrak{k}(P)$ for every $P\in\mathscr{P}$.

Arcs with $\mathfrak{k}(P)\in\{0,1\}$ are called projective arcs. They can be considered as sets of points by identifying them with their support. In this paper, we consider only projective arcs. We denote by $m_n(R^3_{\mathrm{R}})$ the cardinality of the largest (k,n)-arc in $\mathrm{PHG}(R^3_{\mathrm{R}})$. Similarly, we denote by $m_n(q)$ ($\mu_n(q)$, respectively) the largest size of a (k,n)-arc in $\mathrm{PG}(2,q)$ ($\mathrm{AG}(2,q)$, respectively).

Given a (k,n)-arc in $\mathrm{PHG}(R^3_{\mathrm{R}})$, let λ_i be the number of neighbour classes $[P]$ with $\mathfrak{k}([P])=i$. We have

$$\sum\lambda_i=q^2+q+1, \tag{1}$$

$$\sum i\lambda_i=n. \tag{2}$$

Theorem 3.1. *Let $\mathfrak{k}$ be a (k,n)-arc in $\mathrm{PHG}(R^3_{\mathrm{R}})$, where $|R|=q^2$, $R/N\cong\mathbb{F}_q$. Suppose that there exists a neighbour class $[P]$ with $\mathfrak{k}([P])=u$ and let u_i, $i=0,\dots,q$, be the maximum number of points on a line from the ith parallel class in the affine plane Π_P. Then*

$$k\leqslant q(q+1)n-q\sum_{i=0}^{q}u_i+u.$$

Proof. Let l_i be a line from the ith parallel class in Π_P with $\mathfrak{k}(l_i\cap[P])=u_i$. Denote by $l_j^{(i)}$, $j=1,\dots,q$, the q lines in $\mathrm{PHG}(R^3_{\mathrm{R}})$ containing all points of $l_i\cap[P]$. We have

$$\begin{aligned}k&=\mathfrak{k}([P])+\sum_{i=0}^{q}\sum_{j=1}^{q}\mathfrak{k}(l_j^{(i)}\setminus(l_i\cap[P]))\\&\leqslant u+\sum_{i=0}^{q}q(n-u_i)\\&=u+q(q+1)n-q\sum_{i=1}^{q}u_i.\qquad\square\end{aligned}$$

Remark 3.2. The numbers u_i depend on the restriction of $\mathfrak{k}$ to $[P]$ and are generally unknown. However, we may use some simple estimates to get a more convenient form of the above bound.

(1) Fix a point $Q\in[P]$ with $\mathfrak{k}(Q)=1$. Let $s_0,\dots,s_q$ be the lines in Π_P through Q, arranged in such a way that $s_i\frown l_i$. If we set $b_i=\mathfrak{k}(s_i\cap[P])$ then $1+b_i\leqslant u_i$, $i=0,\dots,q$. We have

$$k\leqslant q(q+1)n-q\sum_{i=0}^{q}u_i+u$$

$$\begin{aligned}&\leqslant q(q+1)n - q\sum_{i=0}^{q}(1+b_i)+u\\&= q(q+1)n - q(q+1) - q(u-1)+u\\&= q^2(n-1)+q(n-u)+u.\end{aligned}$$

(2) We can also use the obvious inequality $u_i \geqslant \lceil u/q \rceil$ to get

$$k \leqslant q(q+1)\left(n - \left\lceil \frac{u}{q} \right\rceil\right) + u.$$

Corollary 3.3.

$$m_n(R^3_{\mathrm{R}}) \leqslant \max_{1\leqslant u\leqslant \min\{\mu_n(q),q^2\}} \min\{u(q^2+q+1),\ q^2(n-1)+q(n-u)+u, q(q+1)(n-\lceil u/q\rceil)+u\}.$$

The exact value of $m_n(R^3_{\mathrm{R}})$ is known [9,12] for values of n close to q^2+q.

Theorem 3.4. *Let R be a chain ring with $|R| = q^2$ and $R/N \cong \mathbb{F}_q$. Then*

$$m_{q^2+s}(R^3_{\mathrm{R}}) = q^4 + q^2 s + qs, \quad 0 \leqslant s \leqslant q-1.$$

For arcs with $n = 2$ we have the following bounds.

Theorem 3.5.

$$m_2(R^3_{\mathrm{R}}) \leqslant \begin{cases} q^2+q+1 & \textit{for } q \textit{ even},\\ q^2 & \textit{for } q \textit{ odd}.\end{cases}$$

If $\mathfrak{k}([X]) = 1$ for all $[X] \in \mathscr{P}'$ and q odd, then the neighbour classes with $\mathfrak{k}([X]) = 0$ are collinear in $\bar{\pi}(\Pi)$.

Remark 3.6. There exists a $(7,2)$-arc in the plane over $\mathbb{Z}_4$, but there is no such arc in the plane over $\mathbb{F}_2[x]/(x^2)$. There exist $(9,3)$-arcs in the projective Hjelmslev planes over both chain rings with 9 elements. For larger chain rings, it is possible to get large $(k,2)$-arcs with more than one point in some of the neighbour classes. A computer search revealed that there exist $(18,2)$-arcs in the planes over the chain rings $\mathbb{S}_4$ and $\mathbb{G}_{16}$. These are obtained by choosing two points in neighbour classes lying on a Hermitian curve in $\bar{\pi}(\Pi)$. Below we give examples for (18,2)-arcs in the planes over each one of the rings $\mathbb{S}_4$ and $\mathbb{G}_{16}$:

$$\begin{array}{llllll}(1,1,0) & (1,1,t) & (1,\alpha,0) & (1,\alpha,\alpha^2 t) & (1,\alpha^2,t) & (1,\alpha^2,\alpha^2 t)\\(1,0,1) & (1,t,1) & (1,0,\alpha) & (1,\alpha^2 t,\alpha) & (1,t,\alpha^2) & (1,\alpha^2 t,\alpha^2)\\(\alpha t,1,1) & (\alpha^2 t,1,1) & (t,1,\alpha+t) & (\alpha t,1,\alpha+t) & (t,1,\alpha^2+\alpha t) & (\alpha^2 t,1,\alpha^2+\alpha t)\end{array}$$

$$\begin{array}{llllll}(1,1,0) & (1,1,2) & (1,x,0) & (1,x,2x+2) & (1,x+1,2) & (1,x+1,2x+2)\\(1,0,1) & (1,2,1) & (1,0,x) & (1,2x+2,x) & (1,2,x+1) & (1,2x+2,x+1)\\(2x,1,3) & (2x+2,1,3) & (0,1,x) & (2x+2,1,x) & (0,1,3x+3) & (2x,1,3x+3).\end{array}$$

Unfortunately, we were unable to construct an (18,2)-arc in the plane over $\mathbb{T}_4$. The best arc we know of is a (12,2)-arc: it is easy to see that we can construct $(2m_2(q), 2)$-arcs in all PHG(R^3_{R}), where $|R| = q^2$.

Some (20,2)-arcs in the plane over $\mathbb{Z}_{25}$ have been constructed by Hemme and Weijand [7]. These arcs have $\mathfrak{k}([P]) \leqslant 1$. One example of such a (20,2)-arc is given below:

$$\begin{array}{lllllll} (1,0,0) & (0,1,0) & (0,0,1) & (1,1,1) & (1,7,21) & (5,1,11) & (1,8,2) \\ (1,9,19) & (1,10,24) & (1,12,13) & (20,1,17) & (1,2,5) & (1,15,22) & (1,4,6) \\ (1,6,23) & (1,14,15) & (1,5,3) & (1,18,9) & (1,23,11) & (1,21,12). & \end{array}$$

For the projective Hjelmslev plane over $\mathbb{S}_5$ there exist (15,2)-arcs. They are easily constructed, for example, by taking the points to lie in three quintuples of collinear (in $\bar{\pi}(\Pi)$) point classes meeting in an empty class.

For projective Hjelmslev planes over chain rings R containing a subring isomorphic to the residue field of R, the following result holds [6].

Theorem 3.7. *Let R be a chain ring with $|R| = q^2$, $R/N \cong \mathbb{F}_q$, q even, which contains a subring isomorphic to the residue field $\mathbb{F}_q$. Then $m_2(R^3_{\mathrm{R}}) \leqslant q^2 + q$.*

4. Constructions for arcs in PHG(R^3_{R})

In this section we give some general procedures for construction of arcs in PHG(R^3_{R}).

Example 4.1. There exist $(q^3 + 2q^2, 2q)$-arcs in all Hjelmslev planes PHG(R^3_{R}), with $|R| = q^2$ and $R/N \cong \mathbb{F}_q$.

Let $[P]$ be a fixed neighbour class of points and $[l_0], [l_1], \ldots, [l_q]$ the neighbour classes of lines incident with $[P]$ in $\bar{\pi}(\Pi)$. In each neighbour class of points incident with $[l_i]$, $i = 1, 2, \ldots, q$, and different from $[P]$ choose q collinear points having the direction of $[l_i]$. Arrange the line segments in the point classes incident with $[l_i]$ in such a way that no more than two of them belong to the same line. This is possible by Theorem 2.5. In each of the neighbour classes of points incident with $[l_0]$ different from $[P]$ choose $2q$ points contained in two parallel lines having direction different from that of $[l_0]$. The neighbour class $[P]$ is empty. It is easy to check that a set so constructed turns out to be a $(q^3 + 2q^2, 2q)$-arc.

Example 4.2. Let $\mathfrak{k}_0$ be a (k_0, n_0)-arc in PG$(2, q)$. If $\operatorname{Supp} \mathfrak{k}_0 = \{X_1, \ldots, X_{k_0}\}$, let $\{l_1, \ldots, l_{k_0}\}$ be a set of different lines in PG$(2, q)$ with $X_i \in l_i$, $i = 1, \ldots, k_0$. Then, for each $1 \leqslant \alpha \leqslant q$ and $1 \leqslant s \leqslant q$ there exists an $(\alpha s k_0, \min_{0 \leqslant i \leqslant k_0} v_i)$-arc in PHG($R^3_{\mathrm{R}}$), where $v_0 = \alpha n_0$, $v_i = s + \alpha |l_i \cap \operatorname{Supp} \mathfrak{k}_0| - \alpha$, for $i = 1, \ldots, k_0$.

Fix k_0 neighbour classes from $\mathscr{P}'$ in $\bar{\pi}(\Pi)$ which form a (k_0, n_0)-arc equivalent to $\mathfrak{k}_0$. Without loss of generality, we can denote the points from the support of this arc by $[X_1], \ldots, [X_{k_0}]$. In each class $[X_i]$ choose αs points so that they are contained in α parallel lines (s points on a line) and the parallel lines have the direction of the neighbour class of lines $[l_i]$. It is straightforward to check that the so-defined multiset has the required parameters.

Example 4.3. Assume that for each $X_i \in \operatorname{Supp} k_0$ there is an 1-secant to $\mathfrak{k}_0$ at X_i. If we take these 1-secants as the lines $l_1, \ldots, l_{k_0}$ in the above construction we get an $(\alpha s k_0, \min\{\alpha n_0, s\})$-arc.

Take $\mathfrak{k}_0$ to be a Hermitian curve, that is, a (9,3)-arc in $\mathrm{PG}(2,4)$. For $\alpha = 1$, $s = 3$ we get a (27,3)-arc and for $\alpha = 2$, $s = 4$ we get a (72,6)-arc in $\mathrm{PHG}(R^3_{\mathrm{R}})$, where $R = \mathbb{S}_4$, $\mathbb{T}_4$ or $\mathbb{G}_{16}$. If we take $\mathfrak{k}_0$ to be an oval, that is, a (5,2)-arc in $\mathrm{PG}(2,4)$ then for $\alpha = 2$, $s = 4$ we get (40,4)-arcs in the planes over all rings with 16 elements.

Example 4.4. Take $\operatorname{Supp} \mathfrak{k}_0$ to be the set of all points in $\mathrm{PG}(2,q)$ and $l_1, \ldots, l_{q^2+q+1}$ the set of all lines. Then, for $s = q$ we get a $(\alpha q(q^2+q+1), q(\alpha+1))$-arc. Deleting the points in the neighbour classes from a fixed line in $\bar{\pi}(\Pi)$ we get an $(\alpha q^3, \alpha q + q - \alpha)$-arc. In particular, for $\alpha = 2$, $q = 4$ this gives a (128,10)-arc in $\mathrm{PHG}(R^3_{\mathrm{R}})$, $R = \mathbb{S}_4, \mathbb{T}_4$ or $\mathbb{G}_{16}$. If we add four points to the five empty neighbour classes in such a way that

(1) the four points in each neighbour class are collinear;
(2) the four points in each neighbour class have the direction of the empty class of lines;
(3) the line segments are arranged in such a way that no three of them lie on the same line in $\mathrm{PHG}(R^3_{\mathrm{R}})$ (cf. Theorem 2.5).

Then we get a (148,11)-arc.

Example 4.5. We can use the $(\alpha q(q^2+q+1), q(\alpha+1))$-arcs form the previous example to get some further construction. Start with such an arc and add β line segments of q points (parallel to the existing ones) to a set of neighbour classes which form a (k_1, n_1)-arc in $\bar{\pi}(\Pi)$. Of course, $\alpha + \beta \leqslant q$. Then we get an $(\alpha q(q^2 + q + 1) + \beta q k_1$, $q(\alpha + 1) + \beta n_1)$-arc.

Example 4.6. We can take $\operatorname{Supp} \mathfrak{k}_0$ to be the union of an oval and an external line to this oval. We allow βs points (resp. γs points) in each of the neighbour classes which are external points (resp. internal points) to the fixed oval. These points are contained in β parallel lines (resp. γ parallel lines) with s points on each line having the direction of another external line. Then we obtain an arc with parameters

$$((\alpha + \beta/2 + \gamma/2)s(q + 1), \min\{2\alpha + \beta, 2\alpha + \gamma, s + \gamma, \beta + \gamma(q + 1)/2\})$$

if q is odd and $q>3$, and an $(\alpha s(q+2)+\beta s(q+1), \min\{2\alpha+\beta, \beta(q+1)\})$-arc if q is even, $q>2$.

Example 4.7. There exist $(q^4-(s+1)q^2-\binom{s+1}{2}, q^2-s)$-arcs in $\mathrm{PHG}(R_R^3)$ for all $1\leqslant s\leqslant q-1$ for q odd and for all $1\leqslant s\leqslant q$ for q even.

Choose lines $l, l_1, \ldots, l_s$, $s\leqslant q$ for q odd and $s\leqslant q+1$ for q even, to obtain an oval in the dual plane of $\bar{\pi}(\Pi)$. Define $\mathfrak{k}$ by

$$\mathfrak{k}(X)=\begin{cases}0 & X\,I\,l_i \text{ or } X\,I\,l'\in[l],\\ 1 & \text{otherwise.}\end{cases} \tag{3}$$

Then $\mathfrak{k}$ is an arc with the desired parameters.

5. Nonexistence results for arcs in $\mathrm{PHG}(R_R^3)$

In this section, we improve the general bound of Corollary 3.3 for some small values of n. From Corollary 3.3, we get the following two inequalities for the largest size of an arc with $n=3$ or 4:

$$m_3(R_R^3)\leqslant 2q^2+q+2,$$

$$m_4(R_R^3)\leqslant 3q^2+q+3.$$

Theorem 5.1. *Let R be a chain ring with $|R|=q^2$, $R/N\cong\mathbb{F}_q$, $q\geqslant 4$. Then*

$$m_3(R_R^3)\leqslant 2q^2.$$

Proof. Suppose on the contrary that $\mathfrak{k}$ is a $(2q^2+1,3)$-arc in $\mathrm{PHG}(R_R^3)$. Then by Theorem 3.1 we have $\mathfrak{k}([P])\leqslant 3$ for all $[P]\in\mathscr{P}'$. The same theorem implies also that for every neighbour class $[P]$ with $\mathfrak{k}([P])=3$ the points of $\operatorname{Supp}\mathfrak{k}\cap[P]$ are collinear. Otherwise, we would have

$$2q^2+1=|\mathfrak{k}|\leqslant 3+3q1+(q-2)q2=2q^2-q+3,$$

a contradiction.

If $[P]$ is a neighbour class of points with $\mathfrak{k}([P])=3$ and $[l]$ is the neighbour class of lines incident with $[P]$ having the direction of the points of $\operatorname{Supp}\mathfrak{k}\cap[P]$, then $\mathfrak{k}([l])=3$. If $[P]$ has $\mathfrak{k}([P])=2$ and $[l]$ is the neighbour class having the direction of the points of $\operatorname{Supp}\mathfrak{k}\cap[P]$ then $\mathfrak{k}([l])\leqslant q+2$. For all other classes of lines $[l]$, we have $\mathfrak{k}([l])\leqslant m_3(q)\leqslant 2q+1$ (cf. [21].

Counting the flags $(P,[l])$ in two ways, where $P\in\operatorname{Supp}\mathfrak{k}$, $[l]\in\mathscr{L}'$ and $P\,I\,l'\in[l]$, we get

$$3\lambda_3+(q+2)\lambda_2+(q^2+q+1-\lambda_2-\lambda_3)(2q+1)\geqslant(2q^2+1)(q+1),$$

whence

$$(2q-2)\lambda_3+(q-1)\lambda_2\leqslant q^2+2q. \tag{4}$$

On the other hand, identities (1) and (2) yield $2\lambda_3+\lambda_2=(q^2-q)+\lambda_0$, which together with (4) gives

$$(q-1)(q^2-q)\leqslant 2(q-1)\lambda_3+(q-1)\lambda_2\leqslant q^2+2q,$$

a contradiction to $q\geqslant 4$. $\square$

In case of $q=4$ we can further improve on this bound.

Theorem 5.2. *There is no* $(31,3)$*-arc in* $\mathrm{PHG}(R_R^3)$ *with* $R=\mathbb{S}_4,\mathbb{T}_4$ *or* $\mathbb{G}_{16}$.

Proof. Suppose that $\mathfrak{k}$ is a (31,3)-arc in $\mathrm{PHG}(R_R^3)$. By Theorem 3.1 we have $\mathfrak{k}([P])\leqslant 2$ for each class $[P]\in\mathscr{P}'$. It is easy to check that $\mathfrak{k}([l])\leqslant 8$ if no two neighbour points are incident with a line from $[l]$ and that $\mathfrak{k}([l])\leqslant 6$ otherwise. We have

$$31=|\mathfrak{k}|\leqslant\tfrac{1}{5}(6\lambda_2+8(21-\lambda_2)),$$

whence $\lambda_2\leqslant 6$. On the other hand, by (1) and (2), $\lambda_2\geqslant 10$, a contradiction. $\square$

Theorem 5.3. *Let R be a chain ring with $|R|=q^2$, $R/N\cong\mathbb{F}_q$. Then*

$$m_4(R_R^3)\leqslant\begin{cases}3q^2+3 & \textit{for } q \textit{ odd},\\ 3q^2+4 & \textit{for } q \textit{ even}.\end{cases}$$

Proof. Assume that q is odd and that $\mathfrak{k}$ is a $(3q^2+4,4)$-arc in $\mathrm{PHG}(R_R^3)$. By Theorem 3.1 we have $\mathfrak{k}([P])\leqslant 4$ for every $[P]\in\mathscr{P}'$. Let $[P]$ be neighbour class with $\mathfrak{k}([P])=4$ and let $\{X_1,\dots,X_4\}\subset\operatorname{Supp}\mathfrak{k}\cap[P]$. If exactly three among the X_i's are collinear then the numbers u_i from Theorem 3.1 are $(3,2,2,2,1,\dots,1)$ and

$$|\mathfrak{k}|\leqslant 4+q1+3q2+(q-3)q3=3q^2-2q+4<3q^2+4,$$

a contradiction. If no three of the X_i's are collinear, then the u_i's are $(\underbrace{2,\dots,2}_{\geqslant 4},1,\dots,1)$. (Note that the number of parallel classes of lines in $[P]$, which have lines with two points on them, is at least 4 due to the fact that the diagonals of a quadrangle cannot be collinear in a projective geometry over a field of odd characteristic.) Now we have

$$|\mathfrak{k}|\leqslant 4+4q2+(q-3)q3=3q^2-q+4<3q^2+4.$$

Thus we conclude that the X_i's are collinear. Using similar arguments, we can prove that for a neighbour class $[P]$ with $\mathfrak{k}([P])=3$ the three points in $\operatorname{Supp}\mathfrak{k}\cap[P]$ are collinear.

If $[P]$ is a neighbour class of points with $\mathfrak{k}([P])=4$ and if $[l]$ is the neighbour class of lines incident with $[P]$ and having the direction of the points of $\operatorname{Supp}\mathfrak{k}\cap[P]$, then $\mathfrak{k}([l])=4$. If $[P]$ has $\mathfrak{k}([P])=3$ and $[l]$ is the neighbour class having the direction of $\operatorname{Supp}\mathfrak{k}\cap[P]$ then $\mathfrak{k}([l])\leqslant q+3$. For all other classes of lines $[l]$, we have $\mathfrak{k}([l])\leqslant m_4(q)\leqslant 3q+4$.

Counting the number of pairs $(P,[l])$ with $P \in \operatorname{Supp}\mathfrak{k}$, $P\,I\,l' \in [l]$, we get

$$4\lambda_4 + (q+3)\lambda_3 + (q^2+q+1-\lambda_3-\lambda_4)(3q+4) \geqslant (q+1)(3q^2+4),$$

whence

$$3q\lambda_4 + (2q+1)\lambda_3 \leqslant 4q^2+3q. \tag{5}$$

By (1) and (2), we have $\lambda_3+2\lambda_4=(q^2-2q+2)+\lambda_1+2\lambda_0$. If $\lambda_4>0$, then $\lambda_0 \geqslant q$ and $\lambda_3+2\lambda_4 \geqslant q^2+2$; while, if $\lambda_4=0$, then $\lambda_1+2\lambda_0 \geqslant q$ and $\lambda_3+2\lambda_4 \geqslant q^2-q+2$. By (5) we obtain

$$4q^2+3q \geqslant 3q\lambda_4+(2q+1)\lambda_3 \geqslant 3q\lambda_4+\tfrac{3}{2}q\lambda_3 \geqslant \tfrac{3}{2}q(q^2-q+2),$$

which is impossible unless $q=3$.

If $q=3$, then $\lambda_4=0$ and (5) becomes $7\lambda_3 \leqslant 45$, i.e. $\lambda_3 \leqslant 6$. This implies:
(i) $\lambda_3=6, \lambda_2=6, \lambda_1=1, \lambda_0=0$;
(ii) $\lambda_3=5, \lambda_2=8, \lambda_1=0, \lambda_0=0$.
Consider a class of lines $[l]$ which has the same direction as the points in $\operatorname{Supp}\mathfrak{k} \cap [P]$, where $[P]$ is a point class with $\mathfrak{k}([P])=3$. Denote by $[P]_i$ the other three point classes on $[l]$. Since in both cases $\lambda_0=0$, we must have $\mathfrak{k}([P_i])=1$. This is a contradiction since $\lambda_1 \leqslant 1$.

Now assume that q is even and that $\mathfrak{k}$ is a $(3q^2+5,4)$-arc. Then $\mathfrak{k}([P]) \leqslant 3$ for all $[P] \in \mathscr{P}'$. As before, for a class $[P]$ with $\mathfrak{k}([P])=3$, the points of $\operatorname{Supp}\mathfrak{k} \cap [P]$ are collinear. This gives

$$(q+3)\lambda_3+(q^2+q+1-\lambda_3)(3q+4) \geqslant (q+1)(3q^2+5),$$

whence $\lambda_3 \leqslant 2q-1$. Now

$$3q^2+5=|\mathfrak{k}| \leqslant 3(2q-1)+2(q^2-q+2)=2q^2+4q+1,$$

i.e. $(q-2)^2 \leqslant 0$ and $q=2$. By Theorem 3.4, there is no (17,4)-arc for $q=2$. This completes the proof. □

There is a special interest in arcs with the numbers $\mathfrak{k}(\operatorname{Supp}\mathfrak{k} \cap [P])$ constant for all $[P] \in \mathscr{P}'$ which meet the bound of Corollary 3.3. The next theorem demonstrates the nonexistence of such an arc.

Theorem 5.4. *There is no* $(52,6)$*-arc in* $\mathrm{PHG}(R^3_R)$ *with* $R=\mathbb{S}_3$ *or* $\mathbb{Z}_9$.

Proof. Suppose that $\mathfrak{k}$ is a (52,6)-arc in $\mathrm{PHG}(R^3_R)$. By Theorem 3.1, each neighbour class of points contains at most four points from $\mathfrak{k}$. Therefore $\mathfrak{k}([P])=4$ for all $[P] \in \mathscr{P}'$. No three points in the same neighbour class are collinear, for it would imply

$$|\mathfrak{k}| \leqslant 4+33+334=49,$$

a contradiction. Hence the points in each neighbour class of points form an oval in the affine plane described in Theorem 2.4.

Let $[P]$ be a neighbour class of points and $[l]$ a neighbour class of lines such that $[P]I'[l]$. The lines of $[l]$ either coincide or are disjoint on the points of $[P]$. Let $l_1, l_2, l_3 \in [l]$ be lines such that

$$l_1 \cap l_2 \cap [P] = l_1 \cap l_3 \cap [P] = l_2 \cap l_3 \cap [P] = \emptyset,$$

in other words, these lines form a parallel class of lines in the affine plane defined on the points of $[P]$ (cf. Theorem 2.4). We call the (unordered) triple

$$(\mathfrak{k}(l_1 \cap [P]), \mathfrak{k}(l_2 \cap [P]), \mathfrak{k}(l_3 \cap [P])) \tag{6}$$

the type of $[P]$ with respect to $[l]$. In our case, (6) is either (2,2,0) or (2,1,1). Moreover, the type of $[P]$ is (2,2,0) or (2,1,1) with respect to exactly two classes of lines $[l]$ incident with $[P]$ in $(\mathscr{P}', \mathscr{L}', I')$. This follows from the fact that an external line to an oval in $\mathrm{PG}(2,3)$ is incident with two internal and two external points (cf. [8]).

Now consider a neighbour class of lines $[l] \in \mathscr{L}'$ and let $[P_i]$, $i = 0, 1, 2, 3$, be the point classes incident with $[l]$ in $(\mathscr{P}', \mathscr{L}', I')$. For the types of the classes $[P_i]$ with respect to $[l]$ we have the following cases:

(i) 4 classes of type (2,2,0);
(ii) 1 classes of type (2,2,0) and 3 classes of type (2,1,1);
(iii) 4 classes of type (2,1,1).

It is impossible to have 3 classes of type (2,2,0) and one of type (2,1,1) or two classes of type (2,2,0) and two of type (2,1,1) since then there would exist a line from $[l]$ containing more than 6 points of $\operatorname{Supp} \mathfrak{k}$.

Let A (resp. B, C) the number of neighbour classes of lines for which the possibility (i) (resp. (ii) and (iii)) occurs. We have

$$A + B + C = 13,$$

$$4A + B = 26.$$

The second equality is obtained by counting the flags $([P], [l])$ $[P]I'[l]$, such that the type of $[P]$ with respect to $[l]$ is (2,2,0). This implies $3A = 13 + C$ and $A \geqslant 5$. There is neighbour class of points, say $[P]$, which lies on at least three neighbour classes of lines for which (i) occurs. This is a contradiction since the type of $[P]$ is (2,2,0) with respect to exactly two line classes. □

6. Table of exact values and bounds for $m_n(R^3_{\mathrm{R}})$

In Table 1 we summarize our knowledge about the values of $m_n(R^3_{\mathrm{R}})$ for the chain rings R with $|R| = q^2 \leqslant 25$, $R/N \cong \mathbb{F}_q$.

Table 1
Values and bounds for $m_n(R^3_{\mathrm{R}})$[a]

R n	$\mathbb{S}_2$	$\mathbb{Z}_4$	$\mathbb{S}_3$	$\mathbb{Z}_9$	$\mathbb{S}_4$	$\mathbb{T}_4$	$\mathbb{G}_{16}$	$\mathbb{S}_5$	$\mathbb{Z}_{25}$
2	$^{a}6^{af}$	7^{ae}	$^{a}9^{ae}$	$^{a}9^{ae}$	$^{H}18–20^{f}$	$^{H}12–20^{f}$	$^{H}18–21^{ae}$	$^{H}15–25^{e}$	$^{H}20–25^{e}$
3	$^{a}10^{a}$	$^{a}10^{a}$	$^{a}18–19^{a}$	$^{a}18–19^{a}$	$^{D}27–30^{h}$	$^{D}27–30^{h}$	$^{D}27–30^{h}$	$^{G}27–50^{g}$	$^{G}27–50^{g}$
4	16^{ab}	16^{ab}	$^{a}24–30^{ai}$	$^{a}24–30^{ai}$	$^{D}40–52^{i}$	$^{D}40–52^{i}$	$^{D}40–52^{i}$	$^{B}48–83^{i}$	$^{B}48–83^{i}$
5	22^{ab}	22^{ab}	$^{a}31–40^{a}$	$^{a}31–40^{a}$	$^{G}56–72^{d}$	$^{G}56–72^{d}$	$^{G}56–72^{d}$	$^{G}75–109^{d}$	$^{G}75–109^{d}$
6	28^{ab}	28^{ab}	$^{A}45–51^{j}$	$^{A}45–51^{j}$	$^{D}72–84^{d}$	$^{D}72–84^{d}$	$^{D}72–84^{d}$	$^{B}90–135^{d}$	$^{B}90–135^{d}$
7			$^{C}51–62^{d}$	$^{C}51–62^{d}$	$^{D}76–106^{d}$	$^{D}76–106^{d}$	$^{D}76–106^{d}$	$^{G}105–157^{d}$	$^{G}105–157^{d}$
8			$^{C}64–75^{d}$	$^{C}64–75^{d}$	$^{A}96–126^{d}$	$^{A}96–126^{d}$	$^{A}96–126^{d}$	$^{B}120–187^{d}$	$^{B}120–187^{d}$
9			81^{b}	81^{b}	$^{G}116–143^{d}$	$^{G}116–143^{d}$	$^{G}116–143^{d}$	$^{G}150–217^{d}$	$^{G}150–217^{d}$
10			93^{b}	93^{b}	$^{E}128–160^{f}$	$^{E}128–160^{f}$	$^{E}128–160^{f}$	$^{A}175–243^{d}$	$^{A}175–243^{d}$
11			105^{b}	105^{b}	$^{E}148–171^{d}$	$^{E}148–171^{d}$	$^{E}148–171^{d}$	$^{G}180–269^{d}$	$^{G}180–269^{d}$
12			117^{b}	117^{b}	$^{C}186–191^{d}$	$^{C}186–191^{d}$	$^{C}186–191^{d}$	$^{D}220–295^{d}$	$^{D}220–295^{d}$
13					$^{C}198–211^{d}$	$^{C}198–211^{d}$	$^{C}198–211^{d}$	$^{E}250–313^{d}$	$^{E}250–313^{d}$
14					$^{C}211–231^{d}$	$^{C}211–231^{d}$	$^{C}211–231^{d}$	$^{E}275–343^{d}$	$^{E}275–343^{d}$
15					$^{C}225–248^{d}$	$^{C}225–248^{d}$	$^{C}225–248^{d}$	$^{E}310–373^{d}$	$^{E}310–373^{d}$
16					256^{b}	256^{b}	256^{b}	$^{F}315–403^{d}$	$^{F}315–403^{d}$
17					276^{b}	276^{b}	276^{b}	$^{F}340–429^{d}$	$^{F}340–429^{d}$
18					296^{b}	296^{b}	296^{b}	$^{F}365–455^{d}$	$^{F}365–455^{d}$
19					316^{b}	316^{b}	316^{b}	$^{F}390–469^{d}$	$^{F}390–469^{d}$
20					336^{b}	336^{b}	336^{b}	$^{C}465–499^{d}$	$^{C}465–499^{d}$
21								$^{C}510–529^{d}$	$^{C}510–529^{d}$
22								$^{C}531–559^{d}$	$^{C}531–559^{d}$
23								$^{C}553–589^{d}$	$^{C}553–589^{d}$
24								$^{C}576–615^{d}$	$^{C}576–615^{d}$

[a]Note: ^{a} [10,13], ^{b} Theorem 3.4, ^{c} Theorem 3.1, ^{d} Corollary 3.3, ^{e} Theorem 3.5, ^{f} Theorem 3.7, ^{g} Theorem 5.1, ^{h} Theorem 5.2, ^{i} Theorem 5.3, ^{j} Theorem 5.4, ^{A} Example 4.1, ^{B} Example 4.2, ^{C} Example 4.7, ^{D} Example 4.3, ^{E} Example 4.4, ^{F} Example 4.5, ^{G} Example 4.6, ^{H} Remark 3.6.

7. Uncited reference

[11]

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