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Lie Algebraic Approaches to Classical Partition Identities

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1. INTRODUCTION

In this paper we study new relationships between Lie algebra theory and certain partition formulas which are important in combinatorial analysis. Specifically, we interpret Macdonald's identities [14] as multivariable vector partition theorems and we relate the Rogers–Ramanujan partition identities to the Weyl–Kac character formula. We would like to emphasize that the rather technical Lie theory that we need has already appeared in earlier papers. This work began when one of us, a combinatorialist (S.M.), asked certain questions of the other one, a Lie theorist. At the end of the Introduction we shall say how the key ideas developed.

Our starting point is the two-variable "Jacobi theta-function identity"

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1}t)(1 - q^{2n-1}t^{-1}) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} t^n. \tag{1.1}$$

Cheema [4] makes the substitution $q^2 = uv$, $t^2 = u/v$ and obtains

$$\prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^n v^{n-1})(1 - u^{n-1} v^n) = \sum_{n\in\mathbb{Z}} (-1)^n u^{n(n+1)/2} v^{n(n-1)/2}. \tag{1.2}$$

He then interprets (1.2) combinatorially and deduces:

THEOREM 1.3. *The excess of the number of partitions of (m, n) into an even number of distinct parts of the type $(a, a-1)$, $(b-1, b)$ or (c, c) over those into an odd number of such parts is $(-1)^r$ or 0 according as (m, n) is of the type $(r(r+1)/2, r(r-1)/2)$, or not.*

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If we set $u = q$ and $v = q^2$ in (1.2) we obtain Euler's identity

$$\varphi(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n-1)/2}, \tag{1.4}$$

where by definition, $\varphi(q) = \prod_{n=1}^{\infty} (1 - q^n)$.

This identity relates the partitions of a natural integer into an odd or an even number of distinct positive integers. A combinatorial proof of (1.4) was first given by Franklin [8]. A recent exposition may be found in [1b]. The essential notion is the definition of a "dot map," or array of dots, which corresponds to a partition, and associated operations to be performed on these arrays.

Cheema suggests looking for a direct proof of Theorem 1.3 using dot maps. Zolnowsky [19], supplies just such a proof which extends the notion of dot maps to two-dimensional vector partitions.

Theorem 1.3 has an elegant analog that may be deduced from the "quintuple product identity"

$$\prod_{n=1}^{\infty} (1 - q^n)(1 - q^n t)(1 - q^{n-1}t^{-1})(1 - q^{2n-1}t^2)(1 - q^{2n-1}t^{-2})$$
$$= \sum_{n\in\mathbb{Z}} q^{(3n^2+n)/2}(t^{3n} - t^{-3n-1}). \tag{1.5}$$

This identity, the origin of which may be traced to an elliptic sigma formula of Weierstrass, has a very interesting history, which can be found in the last section of [3]. If we set $t = v^{-1}$ and $q = uv^2$ we find that

$$\prod_{n=1}^{\infty} (1 - u^n v^{2n})(1 - u^n v^{2n-1})(1 - u^{n-1} v^{2n-1})(1 - u^{2n-1} v^{4n-4})(1 - u^{2n-1} v^{4n})$$
$$= \sum_{n\in\mathbb{Z}} u^{n(3n+1)/2} v^{n(3n-2)} - \sum_{n\in\mathbb{Z}} u^{n(3n+1)/2} v^{(n+1)(3n+1)}. \tag{1.6}$$

A combinatorial interpretation of (1.6) immediately gives:

THEOREM 1.7. *The excess of the number of partitions of (m, n) into an even number of distinct parts of the type $(a, 2a)$, $(b, 2b - 1)$, $(c - 1, 2c - 1)$, $(2d - 1, 4d - 4)$, or $(2e - 1, 4e)$ over those into an odd number of such parts is* 1 *or* -1 *if (m, n) is of the type $(r(3r + 1)/2, r(3r - 2))$ or $(r(3r + 1)/2, (r + 1)(3r + 1))$, respectively, and* 0 *otherwise.*

Cheema [4] states similar results. Using Zolnowsky's dot map proof of Theorem 1.3 as a guide, we give in [15] a direct combinatorial proof of Theorem 1.7.

One main stimulus for our work has been to look for a generalization of Theorems 1.3 and 1.7 to higher-dimensional vector partitions. Ideally, this generalization would result from extending (1.2) and (1.6) to any number of variables in such a way that the coefficients of the monomials on the sum side are always 1, -1, or 0. In the last few years, there has been a remarkable generalization of Jacobi's θ-function identity, in the setting of Lie algebras, which enables us to carry out our extension of Theorems 1.3 and 1.7.

To motivate this recent development, consider Euler's identity (1.4) and Jacobi's identity

$$\varphi(q)^3 = \sum_{n\in\mathbb{Z}_+} (-1)^n(2n+1)\, q^{n(n+1)/2}. \tag{1.8}$$

($\mathbb{Z}_+$ is the set of nonnegative integers.) These are both obtained by one-variable specialization of (1.2).

Dyson discovered a family of multivariable identities generalizing (1.1). From these, he derived formulas, generalizing (1.8), for a certain infinite set of powers of $\varphi(q)$. But as he describes amusingly in his article "Missed Opportunities" [6], he did not recognize these powers as the dimensions of the simple "classical" Lie algebras, so he missed the connection with Lie theory. Independently, Macdonald [14] found formulas for $\varphi^{\dim \mathfrak{g}}$ for all complex simple Lie algebras $\mathfrak{g}$, and in fact he obtained these identities as one-variable specializations of respective multivariable identities. He also obtained multivariable identities generalizing (1.5).

Kac [11b] and Moody [17b] independently recognized Macdonald's unspecialized identities as the precise analogs of Weyl's denominator formulas, for the infinite-dimensional "Euclidean" GCM Lie algebras (defined by generators and relations using a symmetrizable "generalized Cartan matrix"). The GCM Lie algebras were introduced independently by Kac [11a] and Moody [17a], and they are alternatively known as the Kac–Moody Lie algebras. Kac [11b] also proved Weyl's character and denominator formulas for *all* GCM Lie algebras. (For a bibliography on GCM Lie algebras, see [9, 12b].)

As the reader may be unfamiliar with these ideas, we give a brief survey of GCM Lie algebras, and the Weyl–Kac denoninator and character formulas in Section 2.

Macdonald's unspecialized identities are just what one needs to generalize Theorems 1.3 and 1.7. Recall, however, that to obtain Theorem 1.3, we must rewrite (1.1) in the form (1.2). The same kind of reformulation is needed for the quintuple product identity (recall (1.5) and (1.6)). Similarly, Macdonald's unspecialized identities must first be analogously rewritten before we may regard them as higher-dimensional vector partition generating function identities and "read off" analogs of Theorems 1.3 and 1.7. The relevant rewriting procedure has already been introduced in [12b]. The method is to use as new variables the formal exponentials $e(-\alpha)$, where α ranges through the simple roots of the GCM Lie algebra. When applied to the Lie algebras $\mathfrak{sl}(2, \mathbb{C})^\wedge$ and $A_2^{(2)}$ (see Section 3), this procedure gives (1.2) and (1.6), respectively. ($\mathfrak{sl}(2, \mathbb{C})^\wedge$ is also known as $A_1^{(1)}$.) The situation for these two Lie algebras is typical. For example, a direct application of a general Lie algebra principle implies that the minomials in the sum side of every Macdonald identity (using as variables e(-simple roots)) always have as coefficients 1, -1, or 0. When viewed combinatorially, this fact says that in the context of our general multivariable vector partition theorem, the excess of the number of suitably restricted partitions of a vector into an even number of allowable parts over those into an odd number of such parts is 1, -1 or 0 (cf. [17b]). The types of parts allowed in the partition are controlled by the roots of the Lie algebra. Any number of variables can be achieved.

In Section 4 we state and prove an abstract vector partition theorem. In addition, we study the important special cases $A_l^{(1)}$.

In [9], Macdonald's identities are shown to be a consequence of the Euler–Poincaré principle and certain involved computations of homology. A vector partition interpretation of the constructions used suggests that there should be a conceptually simpler "combinatorial" proof of the denominator formula. Such a proof is given in [16] for the case of $A_1^{(1)}$. This simpler proof of the denominator formula for $A_1^{(1)}$ suggests that the Euler–Poincaré principle combined with elementary homology computations provides an alternative to using direct one-to-one correspondences to show that two collections of objects are the same size.

To this point, we have used Lie algebra theory to greatly extend the scope of classical results dealing with higher-dimensional vector partittions. Now we reverse the situation, and exploit what are perhaps the two most famous partition identities in combinatorics to discover new results about Lie algebras.

Consider the Rogers–Ramanujan identities

$$\frac{1}{\prod_{n\geqslant 1}(1-q^{5n-1})(1-q^{5n-4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)}, \tag{1.9}$$

$$\frac{1}{\prod_{n\geqslant 1}(1-q^{5n-2})(1-q^{5n-3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}. \tag{1.10}$$

These identities have a long and colorful history. For a combinatorial interpretation and background, see [1b, Chap. 7].

So far, only the denominator formula for infinite-dimensional GCM Lie algebras has been mentioned. While this formula gives a product expansion of the denominator in the multivariable Weyl–Kac character formula, there is in general no such expansion for the numerator. However, the search for a Lie algebraic context in which to study the Rogers–Ramanujan identities leads to a product expansion for the numerator *suitably specialized*, of the character formula of every "standard" irreducible module for $A_1^{(1)}$ and $A_2^{(2)}$. (The standard modules were introduced by Kac in [11b].) Indeed, when we set all the variables equal in the numerator, it turns out that we obtain the denominator with each variable replaced by a certain nonnegative integral power of a single variable. Thus, by the denominator formula, the appropriately specialized numerator factors into an infinite product. The exact formulation and proof of this "numerator formula" are given in Sections 5 and 6. In particular, when all the variables are set equal, the character of the module may be written as a single infinite product. The process of setting all the variables equal is called "principal specialization," as this specialization arises in a canonical way from Kostant's "principal" automorphism; see [12b, Sect. 17].

With our numerator formula in mind we can state:

THEOREM 1.11. *After multiplication by*

$$\frac{1}{\prod_{n\geqslant 0}(1-q^{2n+1})}=\prod_{n\geqslant 1}(1+q^n), \tag{1.12}$$

the left-hand sides of the Rogers–Ramanujan identities (1.9) *and* (1.10) *become the principally specialized characters for a certain pair of irreducible modules for* $A_1^{(1)}$. *Furthermore, expression* (1.12) *is itself the principally specialized character for a certain module for* $A_1^{(1)}$.

We do not have a Lie algebraic proof of the Rogers–Ramanujan identities; what we do know is that the product sides are essentially principally specialized characters. Thus, the Rogers–Ramanujan identities are related to Kostant's principal automorphism.

There are many generalizations and analogs of the Rogers–Ramanujan identities, due to Gordon, Göllnitz and Gordon, and Andrews (see Andrew's book [1b]). "Most" of these are "explained" just as above, using different modules for $A_1^{(1)}$. We present in detail these examples and indicate which modules they correspond to in Section 5.

In [1a] two identities of Rogers and Slater are quoted that are not covered by Andrews' theory. They turn out to be "explained" just as the Rogers–Ramanujan identities, but this time, by a pair of modules for $A_2^{(2)}$. Now Slater [18] has a list of identities among which there are 21 of Rogers–Ramanujan type but not included in Andrews' theory. All 21 (including the Rogers–Slater pair) are "explained" by $A_2^{(2)}$; see Section 6.

$A_2^{(2)}$ and $A_1^{(1)}$ are not the only GCM Lie algebras for which there is a numerator formula; in [12c], an abstract argument generalizes our numerator formula to all GCM Lie algebras. Kac has pointed out that this abstract argument is in fact classical.

The statements of many of the results considered here do not really concern the GCM Lie algebras, but concern instead only the "affine root systems" (see [14]), together with the imaginary roots (introduced in [11a, 17a]; see Section 2) and ρ (introduced in [11b]).

The Rogers–Ramanujan identities focus attention on the standard modules for $A_1^{(1)}$. These modules are studied in [7], where it is discovered that the weight multiplicities for a certain modules for $A_1^{(1)}$ are precisely given by the classical partition function. This fact leads to certain ideas which illuminate the structure of certain of these Lie algebras [13]. These theorems appear to shed new light on the original Rogers–Ramanujan identities. These connections between Lie algebras and combinatorics are leading to results in both directions.

Now we would like to say how the key ideas in this paper evolved. S.M. raised the question whether we could generalize the vector partition Theorem 1.3 to any number of variables. J.L. recognized the corresponding formula, which was similar to (1.2), as the $A_1^{(1)}$ special case of something he had just used in [12a, b]. This was a general rewriting procedure for the denominator formula of a general GCM Lie algebra. The resulting vector partition identities and examples were then straightforward.

Independently, both S.M. and J.L. were trying to understand the Rogers–Ramanujan identities and relate them to their own work. After the work on multivariable vector partitions, S.M. suggested that it would be interesting to study those cases in which the numerator of the character formula factors into a product. J.L. later suggested trying to factor a *suitably specialized* numerator. This first specialization did not work. S.M. suggested trying a sequence of specializations, one of which was to set all the variables equal. Robert L. Wilson suggested using yet a different specialization .The specialization in which we set all the variables equal turns out to work. At this point both S.M. and J.L. independently discovered and proved that for every standard module for $A_1^{(1)}$, the numerator of the character formula factors after all of the variables are set equal.

2. An Exposition of GCM Lie Algebras and Statements of the Denominator and Character Formulas

After introducing the reader to the affine GCM Lie algebras, we shall present the general context in which we set up the denominator and character formulas. The theory of GCM, or Kac–Moody, Lie algebras originated in [11a, 17a], which the reader should consult for details. The reader is also referred to [10] for elementary background on Lie algebras in general and complex semisimple Lie algebras in particular. For further bibliography on GCM Lie algebras see [9, 12b]. Unless otherwise noted, all vector spaces, Lie algebras, and modules are over $\mathbb{C}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be the Cartan subalgebra, of dimension, say, l. The restriction to $\mathfrak{h}$ of the Killing form of $\mathfrak{g}$ is nonsingular, inducing a nonsingular, symmetric bilinear form $(\cdot, \cdot)$ on the dual $\mathfrak{h}^*$. This form is real-valued and positive definite on the real span of the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a positive system of roots, and let $\alpha_1, ..., \alpha_l \in \mathfrak{h}^*$ be the corresponding simple roots. The Cartan matrix of $\mathfrak{g}$ is the $l \times l$ integral matrix A given by

$$A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i).$$

The matrices which arise in this way (as $\mathfrak{g}$ varies) are called the *classical Cartan matrices of finite type.*

Assume that $\mathfrak{g}$ is simple. Define $\tilde{\mathfrak{g}}$ to be the infinite-dimensional complex Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

obtained by tensoring $\mathfrak{g}$ with the algebra of finite Laurent series in one indeterminate t.

The decomposition

$$\tilde{\mathfrak{g}} = \coprod_{j \in \mathbb{Z}} \mathfrak{g} \otimes t^j$$

is a Lie algebra grading of $\tilde{\mathfrak{g}}$. Let D be the corresponding degree derivation of $\tilde{\mathfrak{g}}$; i.e., D acts as multiplication by j on $\mathfrak{g} \otimes t^j$. Let $\tilde{\mathfrak{g}}_1$ be the natural semidirect product Lie algebra $\mathbb{C}D \oplus \tilde{\mathfrak{g}}$, and let $\mathfrak{h}_1$ be the $(l+1)$-dimensional Abelian subalgebra $\mathbb{C}D \oplus \mathfrak{h}$, so that $\tilde{\mathfrak{g}}_1$ has a natural "root space decomposition" with respect to $\mathfrak{h}_1$. Specifically, for each $\mu \in \mathfrak{h}_1{}^*$, define the *root space* $\tilde{\mathfrak{g}}_1{}^\mu \subset \tilde{\mathfrak{g}}_1$ corresponding to μ to be $\{x \in \tilde{\mathfrak{g}}_1 \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}_1\}$. We call μ a *root* of $\tilde{\mathfrak{g}}_1$ if $\mu \neq 0$ and if $\tilde{\mathfrak{g}}_1{}^\mu \neq 0$. Let $\Delta \subset \mathfrak{h}_1{}^*$ be the set of roots. The Lie algebra $\tilde{\mathfrak{g}}_1$ has the *root space decomposition*

$$\tilde{\mathfrak{g}}_1 = \mathfrak{h}_1 \oplus \coprod_{\varphi \in \Delta} \tilde{\mathfrak{g}}_1{}^\varphi.$$

Identify $\mathfrak{h}^*$ with the subspace of $\mathfrak{h}_1{}^*$ consisting of the functionals which vanish on D. The roots fall mutually into two classes: those which do not vanish on $\mathfrak{h}$ and those which do. The former comprise the set Δ_R of *real roots*, and the latter, the set Δ_I of *imaginary roots*. The set Δ_0 of real roots which vanish on D are just the ordinary roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Define $\gamma \in \mathfrak{h}_1{}^*$ to be the functional which is 1 on D and which vanishes on $\mathfrak{h}$. Then the real roots of $\tilde{\mathfrak{g}}_1$ are exactly the functionals of the form $j\gamma + \varphi$, where $j \in \mathbb{Z}$ and $\varphi \in \Delta_0$, and the imaginary roots of $\tilde{\mathfrak{g}}_1$ are the functionals of the form $j\gamma$, where $j \in \mathbb{Z}$, $j \neq 0$. Note that for $j \in \mathbb{Z}$ and $\varphi \in \Delta_0$,

$$\tilde{\mathfrak{g}}_1^{j\gamma+\varphi} = \mathfrak{g}^\varphi \otimes t^j$$

(where $\mathfrak{g}^\varphi \subset \mathfrak{g}$ is the ordinary root space of $\mathfrak{g}$ with respect to $\mathfrak{h}$); and for $j \in \mathbb{Z}$, $j \neq 0$,

$$\tilde{\mathfrak{g}}_1^{j\gamma} = \mathfrak{h} \otimes t^j.$$

Also,

$$\mathfrak{g} \otimes t^j = \tilde{\mathfrak{g}}_1^{j\gamma} \oplus \coprod_{\varphi \in \Delta_0} \tilde{\mathfrak{g}}_1^{j\gamma+\varphi}.$$

The root spaces for the real roots are all one-dimensional, while the root spaces for the imaginary roots are all l-dimensional.

Let $\Delta_+ \subset \Delta$ be the union of a fixed positive system in Δ_0 with the set of roots in Δ which are positive on D. Call Δ_+ the set of *positive roots* of $\tilde{\mathfrak{g}}_1$. Those positive roots which cannot be written as the sum of two positive roots are called the *simple roots* of $\tilde{\mathfrak{g}}_1$. There are exactly $l + 1$ of them: the l simple roots $\alpha_1, \ldots, \alpha_l$ of $\mathfrak{g}$, together with the functional $\alpha_0 = \gamma - \psi$, where $\psi \in \Delta$ is the highest root of $\mathfrak{g}$. Every positive root is a nonnegative integral linear combination of the simple roots, which form a basis of $\mathfrak{h}_1{}^*$. The restrictions $\alpha_0 \mid \mathfrak{h}, \ldots, \alpha_l \mid \mathfrak{h}$ are of course linearly dependent over $\mathbb{Z}$. The $(l + 1) \times (l + 1)$ matrix $\tilde{A}$ given by

$$\tilde{A}_{ij} = 2(\alpha_i \mid \mathfrak{h}, \alpha_j \mid \mathfrak{h})/(\alpha_i \mid \mathfrak{h}, \alpha_i \mid \mathfrak{h})$$

$(i, j = 0, \ldots, l)$ is called the *Cartan matrix* of $\tilde{\mathfrak{g}}$ (or of $\tilde{\mathfrak{g}}_1$). As $\mathfrak{g}$ varies, the resulting matrices are called the *affine* (*Cartan*) matrices, and the Lie algebra $\tilde{\mathfrak{g}}$ (or their central extensions $\hat{\mathfrak{g}} = \mathfrak{l}(\tilde{A})$ to be defined below) are called the *affine Lie algebras*. For $\mathfrak{g}$ of Cartan type X_l, $\hat{\mathfrak{g}}$ is denoted $X_l^{(1)}$ (cf. [11b]). For example, if $\mathfrak{g} = A_l = \mathfrak{sl}(l + 1, \mathbb{C})$, then $\hat{\mathfrak{g}} = A_l^{(1)}$.

For each $i = 0, \ldots, l$, there is a unique element $x_i \in \mathfrak{h}$ such that $(x_i, h) = \alpha_i(h)$ for all $h \in \mathfrak{h}$; here $(\cdot, \cdot)$ denotes the Killing form of $\mathfrak{g}$. There is a unique rational multiple h_i' of x_i such that $\alpha_i(h_i') = 2$. We may choose elements $e_i, f_i \in \tilde{\mathfrak{g}}$ such that e_i lies in the root space $\tilde{\mathfrak{g}}_1^{\alpha_i}$, $f_i \in \tilde{\mathfrak{g}}_1^{-\alpha_i}$, and $[e_i, f_i] = h_i'$. Then $[h_i', e_j'] = A_{ij} e_j$, $[h_i', f_j] = -A_{ij} f_j$ and $[e_i, f_j] = \delta_{ij} h_i'$ for all $i, j = 0, \ldots, l$. The elements h_i', e_i and f_i generate $\tilde{\mathfrak{g}}$ as i ranges from 0 to l. We shall call them *canonical generators* of $\tilde{\mathfrak{g}}$.

We now present the GCM Lie algebras, which considerably generalize the finite-dimensional semisimple and the affine Lie algebras.

Let $l \in \mathbb{Z}_+$, and let $A = (A_{ij})_{i,j \in \{0,\ldots,l\}}$ be an $(l + 1) \times (l + 1)$ (generalized) Cartan matrix. This means that $A_{ij} \in \mathbb{Z}$ (the set of integers) for all i and j, $A_{ii} = 2$ for all i, $A_{ij} \leqslant 0$ whenever $i \neq j$, and $A_{ji} = 0$ whenever $A_{ij} = 0$. (Later we shall assume that A is symmetrizable.)

We define the (possibly infinite-dimensional) Lie algebra $\mathfrak{l} = \mathfrak{l}(A)$ by the following procedure: Take $\mathfrak{l}_1$ to be the Lie algebra defined by $3(l + 1)$ "canonical" generators h_i, e_i, f_i $(0 \leqslant i \leqslant l)$ subject to the relations $[h_i, h_j] = 0$, $[e_i, f_j] = \delta_{ij} h_i$, $[h_i, e_j] = A_{ij} e_j$, $[h_i, f_j] = -A_{ij} f_j$ for all $i, j \in \{0, \ldots, l\}$, and $(\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0 = (\operatorname{ad} f_i)^{-A_{ij}+1} f_j$ whenever $i \neq j$. For every $(l + 1)$-tuple $(n_0, \ldots, n_l)$ of nonnegative (respectively, nonpositive) integers not all zero, define $\mathfrak{l}_1(n_0, \ldots, n_l)$ to be the (finite-dimensional) subspace of $\mathfrak{l}_1$ spanned by the elements

$$[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{r-1}}, e_{i_r}] \cdots]]$$

(respectively,

$$[f_{i_1}, [f_{i_2}, ..., [f_{i_{r-1}}, f_{i_r}] ...]],$$

where e_j (respectively, f_j) occurs $| n_j |$ times. Also, let $\mathfrak{l}_1(0, ..., 0)$ be the Abelian subalgebra spanned by $h_0, ..., h_l$, and take $\mathfrak{l}_1(n_0, ..., n_l) = 0$ for any other $(l + 1)$-tuple of integers. Then

$$\mathfrak{l}_1 = \coprod_{(n_0, ..., n_l) \in \mathbb{Z}^{(l+1)}} \mathfrak{l}_1(n_0, ..., n_l);$$

this is a Lie algebra gradation of $\mathfrak{l}_1$; and the elements $h_0, ..., h_l$, $e_0, ..., e_l$, $f_0, ..., f_l$ are linearly independent in $\mathfrak{l}_1$. The space $\mathfrak{l}_1(0, ..., 0, 1, 0, ..., 0)$ (respectively, $\mathfrak{l}_1(0, ..., 0, -1, 0, ..., 0)$) is nonzero and is spanned by e_i (respectively, f_i): here ± 1 is in the ith position.

There is a unique graded ideal $\mathfrak{r}_1$ in $\mathfrak{l}_1$ maximal among those graded ideals not intersecting the span of h_i, e_i and f_i $(0 \leqslant i \leqslant l)$. Let $\mathfrak{l} = \mathfrak{l}(A)$ be the $\mathbb{Z}^{(l+1)}$-graded Lie algebra $\mathfrak{l}_1(A)/\mathfrak{r}_1$. The images in $\mathfrak{l}$ of h_i, e_i, f_i and $\mathfrak{l}_1(n_0, ..., n_l)$ shall be denoted h_i, e_i, f_i, and $\mathfrak{l}(n_0, ..., n_l)$, respectively. Let $\mathfrak{h}$ be the span of $h_0, ..., h_l$ in l. The space $\mathfrak{l}(n_0, ..., n_l)$ has the same (finite) dimension as $\mathfrak{l}(-n_0, ..., -n_l)$.

If A is a classical Cartan matrix of finite type, then $\mathfrak{r}_1 = 0$ and $\mathfrak{l}(A)$ is the finite-dimensional semisimple Lie algebra whose Cartan matrix is A.

Let D_i $(0 \leqslant i \leqslant l)$ be the ith-degree derivation of $\mathfrak{l}$, that is, the derivation which acts as scalar multiplication by n_i on $\mathfrak{l}(n_0, ..., n_l)$. Let $\mathfrak{d}_0$ be the ($(l + 1)$-dimensional) Abelian Lie algebra of derivations of $\mathfrak{l}$ spanned by $D_0, ..., D_l$, and let $\mathfrak{d}$ be a subspace of $\mathfrak{d}_0$. Form the natural semidirect product Lie algebra $\mathfrak{l}^e = \mathfrak{d} \times \mathfrak{l}$, and let $\mathfrak{h}^e$ be the Abelian subalgebra $\mathfrak{d} \oplus \mathfrak{h}$. Define $\alpha_0, ..., \alpha_l \in (\mathfrak{h}^e)^*$ by the conditions $[h, e_i] = \alpha_i(h)e_i$ for all $h \in \mathfrak{h}^e$ and $i \in \{0, ..., l\}$. Note that $\alpha_j(h_i) = A_{ij}$. Call $\mathfrak{d}$ an *admissible* subspace of $\mathfrak{d}_0$ if $\alpha_0, ..., \alpha_l$ are linearly independent. Admissible subspaces exist because $\mathfrak{d}_0$ is admissible. Fix an admissible subspace $\mathfrak{d}$.

If A is classical of finite type, then we may choose $\mathfrak{d} = 0$, so that $\mathfrak{l}^e = \mathfrak{l}$, and then the roots, Weyl group and other concepts discussed below simply reduce to the usual classical ones for $\mathfrak{l}$.

For all $\varphi \in (\mathfrak{h}^e)^*$, define

$$\mathfrak{l}^\varphi = \{x \in \mathfrak{l} \mid [h, x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}^e\}.$$

Then $[\mathfrak{l}^\varphi, \mathfrak{l}^\psi] \subset \mathfrak{l}^{\varphi+\psi}$ for all φ, $\psi \in (\mathfrak{h}^e)^*$; $\mathfrak{l}^0 = \mathfrak{h}$; $e_i \in \mathfrak{l}^{\alpha_i}$ and $f_i \in \mathfrak{l}^{-\alpha_i}$ for all $i \in \{0, ..., l\}$; $\dim \mathfrak{l}^\varphi = \dim \mathfrak{l}^{-\varphi}$ for all $\varphi \in (\mathfrak{h}^e)^*$; and the decomposition

$$\mathfrak{l} = \coprod_{(n_0, ..., n_l) \in \mathbb{Z}^{(l+1)}} \mathfrak{l}(n_0, ..., n_l)$$

coincides with the decomposition

$$\mathfrak{l} = \coprod_{\varphi \in (\hat{\mathfrak{h}}^e)^*} \mathfrak{l}^\varphi,$$

with $\mathfrak{l}^\varphi = \mathfrak{l}(n_0, ..., n_l)$ when $\varphi = \sum_{i=0}^{l} n_i \alpha_i$.

Let $\Delta = \{\varphi \in (\hat{\mathfrak{h}}^e)^* \mid \varphi \neq 0$ and $\mathfrak{l}^\varphi \neq 0\}$, the set of *roots* of $\mathfrak{l}$. Let Δ_+ (the set of *positive* roots) be the set of roots which are nonnegative integral linear combinations of $\alpha_0, ..., \alpha_l$, and let $\Delta_- = -\Delta_+$ (the set of *negative* roots). Then $\Delta = \Delta_+ \cup \Delta_-$, and

$$\mathfrak{l} = \hat{\mathfrak{h}} \oplus \coprod_{\varphi \in \Delta_+} \mathfrak{l}^\varphi \oplus \coprod_{\varphi \in \Delta_-} \mathfrak{l}^\varphi.$$

The center of $\mathfrak{l}$ is the subspace of $\hat{\mathfrak{h}} \subset \hat{\mathfrak{h}}^e$ on which all the roots of $\mathfrak{l}$ vanish.

For each $i \in \{0, ..., l\}$, define the linear automorphism r_i of $(\hat{\mathfrak{h}}^e)^*$ by the condition $r_i\varphi = \varphi - \varphi(h_i)\alpha_i$ for all $\varphi \in (\hat{\mathfrak{h}}^e)^*$. Then $r_i\alpha_i = -\alpha_i$, and r_i acts as the identity on the codimension 1 subspace consisting of all $\varphi \in (\hat{\mathfrak{h}}^e)^*$ such that $\varphi(h_i) = 0$. Also, $r_i\alpha_j = \alpha_j - A_{ij}\alpha_i$. Let W (the *Weyl group*) be the group of automorphisms of $(\hat{\mathfrak{h}}^e)^*$ generated by $r_0, ..., r_l$. Then W is a Coxeter group with generators r_i and relations which can be given in terms of the Cartan matrix A. Each element of W preserves Δ, and W is naturally isomorphic to the group of linear automorphisms it induces on the span of Δ.

Define the set Δ_R of *real* roots to be the set of Weyl group transforms of $\alpha_0, ..., \alpha_l$, and define the set Δ_I of *imaginary* roots to be $\Delta - \Delta_R$. Then dim $\mathfrak{l}^\varphi = 1$ for all $\varphi \in \Delta_R$, but this need not be true for $\varphi \in \Delta_I$. We have $W\Delta_R = \Delta_R$, $W\Delta_I = \Delta_I$, $\Delta_R = -\Delta_R$, $\Delta_I = -\Delta_I$, and $W(\Delta_I \cap \Delta_+) = \Delta_I \cap \Delta_+$.

For all $w \in W$, define

$$\Phi_w = \Delta_+ \cap w\Delta_- = \{\varphi \in \Delta_+ \mid w^{-1}\varphi \in \Delta_-\},$$

so that $\Phi_w \subset \Delta_R \cap \Delta_+$. Let $n(w)$ be the number of elements in Φ_w. Let $l(w)$ be the *length* of w, that is, the smallest nonnegative integer j such that w can be written as $r_{i_1} r_{i_2} \cdots r_{i_j}$ $(0 \leqslant i_m \leqslant l)$. Then $n(w) = l(w)$ (a finite number).

Define $\rho \in (\hat{\mathfrak{h}}^e)^*$ to be any fixed element satisfying the conditions $\rho(h_i) = 1$ for all $i \in \{0, ..., l\}$. For every finite subset Φ of Δ, define $\langle \Phi \rangle \in (\hat{\mathfrak{h}}^e)^*$ to be the sum of the elements of Φ.

We now list some useful facts from [9, Sect. 2]:

LEMMA 2.1. *For all $w \in W$ and $i \in \{0, ..., l\}$, $\langle \Phi_{r_i w} \rangle = r_i \langle \Phi_w \rangle + \langle \Phi_{r_i} \rangle$.*

LEMMA 2.2. *Let $w \in W$, let Φ be a finite subset of Δ_+, and let $\gamma \in$ span Δ be a finite sum of not necessarily distinct positive imaginary roots. If $\langle \Phi_w \rangle = \langle \Phi \rangle + \gamma$, then $\gamma = 0$ and $\Phi = \Phi_w$. In particular, Φ consists of real roots.*

LEMMA 2.3. *For all* $i \in \{0,..., l\}$, $r_i\rho = \rho - \alpha_i$. *For all* $w \in W$, $\langle \Phi_w \rangle = \rho - w\rho$.

LEMMA 2.4. *The only Weyl group element which fixes* ρ *is the identity. Equivalently, if* $w_1\rho = w_2\rho$ ($w_1, w_2 \in W$), *then* $w_1 = w_2$.

LEMMA 2.5. *If* $w_1, w_2 \in W$ *and* $\Phi_{w_1} = \Phi_{w_2}$, *or even if* $\langle \Phi_{w_1} \rangle = \langle \Phi_{w_2} \rangle$, *then* $w_1 = w_2$.

Define $\mathfrak{n} = \coprod_{\varphi \in \Delta_+} \mathfrak{l}^\varphi$; $\mathfrak{n}^- = \coprod_{\varphi \in \Delta_-} \mathfrak{l}^\varphi$. Then $\mathfrak{l} = \mathfrak{n}^- \oplus \hat{\mathfrak{h}} \oplus \mathfrak{n}$.

Let V be an $\hat{\mathfrak{h}}^e$-module (for example, an $\mathfrak{l}^e$-module regarded as an $\hat{\mathfrak{h}}^e$-module by restriction), and let $\mu \in (\hat{\mathfrak{h}}^e)^*$. Define the *weight space* $V_\mu \subset V$ corresponding to μ to be $\{v \in V \mid h \circ v = \mu(h)v$ for all $h \in \hat{\mathfrak{h}}^e\}$. Call μ a *weight* of V if $V_\mu \neq 0$, and call the nonzero elements of V_μ *weight vectors* with weight μ.

An $\mathfrak{l}^e$-module V is called a ***highest weight module*** if it is generated by an $\mathfrak{n}$-annihilated weight vector v. In this case, the *highest weight vector* v is uniquely determined up to nonzero scalar multiple, its weight is called the *highest weight* of V, and its weight space is the *highest weight space* of V. The highest weight space is one-dimensional, V is the direct sum of its weight spaces, which are all finite-dimensional, and the weights of V are all of the form $\mu - \sum_{i=0}^{l} n_i\alpha_i$ ($n_i \in \mathbb{Z}_+$), where $\mu \in (\hat{\mathfrak{h}}^e)^*$ is the highest weight. (Note that this last condition determines the highest weight.)

An $\mathfrak{l}^e$-module R is called *standard* if R is a highest weight module with a highest weight vector x such that there exists $n \in \mathbb{Z}_+$ with $f_i^n \circ x = 0$ for all $i \in \{0,..., l\}$. The trivial one-dimensional module is standard; its highest weight is 0. Let P be the set of *dominant integral linear forms*, that is the set of all $\lambda \in (\hat{\mathfrak{h}}^e)^*$ such that $\lambda(h_i) \in \mathbb{Z}_+$ for all $i \in \{0,..., l\}$. Then the highest weight of a standard $\mathfrak{l}^e$-module lies in P, and for all $\mu \in P$, there exists a standard $\mathfrak{l}^e$-module with highest weight μ. If A is a classical Cartan matrix of finite type and $\mathfrak{d} = 0$ (see above), then the standard $\mathfrak{l}^e$-modules are just the finite-dimensional irreducible $\mathfrak{l}$-modules.

Assume now that the Cartan matrix A is symmetrizable, i.e., that there are positive rational numbers $q_0,..., q_l$ such that $\mathrm{diag}(q_0,..., q_l)A$ is a symmetric matrix.

For $\mu \in P$, there is exactly one (up to equivalence) standard $\mathfrak{l}^e$-module with highest weight μ, and it is irreducible. Thus P bijectively indexes the set of equivalence classes of standard $\mathfrak{l}^e$-modules.

The affine Lie algebras $\tilde{\mathfrak{g}}$ defined above are not quite GCM Lie algebras. We must elaborate on this point:

Recall the affine matrix $\tilde{A}$, where A is the classical Cartan matrix of the simple Lie algebra $\mathfrak{g}$. $\tilde{A}$ is a symmetrizable Cartan matrix. Let $\mathfrak{l}$ be the corresponding GCM Lie algebra $\mathfrak{l}(\tilde{A})$, and call this algebra $\hat{\mathfrak{g}}$. Since the rank of $\tilde{A}$ is l, the center $\mathfrak{c}$ of $\hat{\mathfrak{g}}$ is a one-dimensional subspace of $\hat{\mathfrak{h}}$. There is an exact sequence

$$0 \rightarrow \mathfrak{c} \rightarrow \hat{\mathfrak{g}} \xrightarrow{\pi} \tilde{\mathfrak{g}} \rightarrow 0,$$

in which π is determined by the condition that it send the canonical generators h_i, e_i, f_i for $\hat{\mathfrak{g}}$ to the respective canonical generators h_i', e_i, f_i for $\tilde{\mathfrak{g}}$. The map π sends $\hat{\mathfrak{h}}$ onto $\mathfrak{h}$, and the kernel $\mathfrak{c}$ of π lies in $\hat{\mathfrak{h}}$. There is a natural identification between the set Δ of roots of $\hat{\mathfrak{g}}$ and the set Δ of roots of $\tilde{\mathfrak{g}}$ (or rather of $\tilde{\mathfrak{g}}_1$) so that for each root φ, π maps the root space $\hat{\mathfrak{g}}^\varphi$ isomorphically onto the corresponding root space $\tilde{\mathfrak{g}}_1{}^\varphi \subset \tilde{\mathfrak{g}}$. Under this identification, the real and imaginary roots identify properly, as do the positive and simple roots.

The reason for introducing the central extension $\hat{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}$ is that we need the standard modules, and $\tilde{\mathfrak{g}}$ does not in general act on the standard modules for $\hat{\mathfrak{g}}$.

The identification between the set of roots of $\hat{\mathfrak{g}}$ and the set of roots of $\tilde{\mathfrak{g}}$ extends to a natural linear isomorphism between the span of Δ in $(\hat{\mathfrak{h}}^e)^*$ and $\mathfrak{h}_1{}^*$. We use this isomorphism to identify these two spaces. Now the Weyl group W acts on $(\hat{\mathfrak{h}}^e)^*$ and preserves Δ, and we want to see the action of W on $\mathfrak{h}_1{}^*$. Note that $\mathfrak{h}_1{}^*$ is equipped with a singular symmetric bilinear form $\{\cdot, \cdot\}$ defined by the condition

$$\{\varphi, \psi\} = (\varphi \mid \mathfrak{h}, \psi \mid \mathfrak{h}),$$

for all $\varphi, \psi \in \mathfrak{h}_1{}^*$. The radical of $\{\cdot, \cdot\}$ is one-dimensional and is spanned by γ. Also, this form is real-valued and positive semidefinite on the real span of Δ. Recall that for $i, j = 0,..., l$,

$$\begin{aligned} r_i\alpha_j &= \alpha_j - \alpha_j(h_i)\alpha_i = \alpha_j - \tilde{A}_{ij}\alpha_i \\ &= \alpha_j - \frac{2\{\alpha_i, \alpha_j\}}{\{\alpha_i, \alpha_i\}}\alpha_i . \end{aligned}$$

Hence, r_i acts on $\mathfrak{h}_1{}^*$ as the ordinary geometric reflection with respect to the nonisotropic vector α_i ; i.e.,

$$r_i\varphi = \varphi - \frac{2\{\varphi, \alpha_i\}}{\{\alpha_i, \alpha_i\}}\alpha_i$$

for all $\varphi \in \mathfrak{h}_1{}^*$. This gives the action of W on $\mathfrak{h}_1{}^*$. Note that in practice, the formula $r_i\alpha_j = \alpha_j - \tilde{A}_{ij}\alpha_i$ is easy to use, assuming that we know the integers $\tilde{A}_{ij}$.

We now have the concepts needed to state Macdonald's identities and the Weyl–Kac denominator and character formulas.

Return to the setting of the general GCM Lie algebra I (with symmetrizable Cartan matrix). The dominator formula asserts:

THEOREM 2.6. *We have*

$$\prod_{\varphi\in\Delta_+} (1 - e(-\varphi))^{\dim \mathfrak{l}^\varphi} = \sum_{w\in W} (-1)^{l(w)} e(-\langle\Phi_w\rangle),$$

where the symbol $e(\cdot)$ is a formal exponential. This identity takes place in the formal power series ring $\mathbb{Z}[[e(-\alpha_0),..., e(-\alpha_l)]]$ in the $l+1$ analytically independent variables $e(-\alpha_i)$.

Remark. Recall that $-\langle\Phi_w\rangle = w\rho - \rho$.

Weyl's classical denominator formula is the special case of this theorem when $\mathfrak{l}$ is finite-dimensional semisimple. (In this case, the power series involved are polynomials; i.e., they terminate.) Macdonald's identities constitute the special case of Theorem 2.6 when $\mathfrak{l}$ is affine, or rather a little more generally, when $\mathfrak{l}$ is "Euclidean."

We shall now rewrite the denominator formula as in [12b, Sect. 13].

Notation. Write $u_i = e(-\alpha_i)$ for all $i = 0,..., l$.

Each identity in Theorem 2.6 may be written as an equality between two formal power series in $u_0 ,..., u_l$. It is convenient to introduce the following:

Notation. For every integral linear combination $\varphi = \sum_{i=0}^{l} c_i\alpha_i$ of the α_i, let $\xi_j(\varphi) = c_j$ ($j = 0,..., l$).

Theorem 2.6 can clearly be reformulated as follows:

COROLLARY 2.7. *In* $\mathbb{Z}[[u_0 ,..., u_l]]$,

$$\prod_{\varphi\in\Delta_+}\left(1-\prod_{i=0}^{l}u_i^{\xi_i(\varphi)}\right)^{\dim\mathfrak{l}^\varphi} = \sum_{w\in W}(-1)^{l(w)}\prod_{i=0}^{l}u_i^{\xi_i(\langle\Phi_n\rangle)}.$$

DEFINITION. Let $(s_0,...,s_l)$ be a sequence of positive integers. Let q be an indeterminate. The homomorphism of power series rings

$$\mathbb{Z}[[u_0 ,..., u_l]] \to Z[[q]]$$

which sends u_i to q^{s_i} for all $i = 0,..., l$ is called the *q-specialization of type* $(s_0 ,..., s_l)$.

We clearly have:

The q-specialization of type $(s_0 ,..., s_l)$ of Corollary 2.7 asserts:

COROLLARY 2.8.

$$\prod_{\varphi\in\Delta_+}(1-q^{\sum_{i=0}^{l}s_i\xi_i(\varphi)})^{\dim\mathfrak{l}^\varphi} = \sum_{w\in W}(-1)^{l(w)}q^{\sum_{i=0}^{l}s_i\xi_i(\langle\Phi_w\rangle)}.$$

We next give the character formula:

Let V be a standard module for $\mathfrak{l}^e$. Recall the weight spaces $V_\mu \subset V$. The weights μ of V lie in $(\mathfrak{h}^e)^*$.

DEFINITION. The *character* $\chi(V)$ is the generating function of the weight multiplicities of V, that is,

$$\chi(V) = \sum_{\mu\in(\mathfrak{h}^e)^*}(\dim V_\mu)\, e(\mu).$$

(Recall that $\dim V_\mu$ is always finite.)

Here is the Weyl–Kac character formula (Weyl's character formula is the special case in which $\mathfrak{l}$ is finite-dimensional semisimple):

THEOREM 2.9. *Let λ be a dominant integral linear form (i.e., $\lambda \in P \subset (\hat{\mathfrak{h}}^e)^*$), and let V be the standard (irreducible) module with highest weight λ. Then*

$$\chi(V) = \frac{\sum_{w \in W} (-1)^{l(w)}\, e(w(\lambda + \rho) - \rho)}{\sum_{w \in W} (-1)^{l(w)}\, e(w\rho - \rho)} .$$

Remark. Note that the denominator is the expression treated in Theorem 2.6. In order to reveal the character formula as an equality of formal power series, we divide both sides by $e(\lambda)$:

COROLLARY 2.10. *In the notation of Theorem* 2.9,

$$\frac{\chi(V)}{e(\lambda)} = \frac{\sum_{w \in W} (-1)^{l(w)}\, e(w(\lambda + \rho) - (\lambda + \rho))}{\sum_{w \in W} (-1)^{l(w)}\, e(w\rho - \rho)} .$$

Both the numerator and denominator on the right-hand side are formal power series in $u_0 ,..., u_l$, as is the left-hand side.

Remark. The last assertion for the numerator follows from the fact that $w(\lambda + \rho) - (\lambda + \rho)$ is a nonnegative integral linear combination of $-\alpha_0 ,..., -\alpha_l$; this fact follows from [9, Proposition 6.1], because $\lambda + \rho \in P$.

The analog of Corollary 2.7 is:

COROLLARY 2.11. *In $\mathbb{Z}[[u_0 ,..., u_l]]$, the right-hand side in Corollary* 2.10 *is*

$$\frac{\sum_{w \in W} (-1)^{l(w)} \prod_{i=0}^{l} u_i^{\xi_i(\lambda+\rho-w(\lambda+\rho))}}{\sum_{w \in W} (-1)^{l(w)} \prod_{i=0}^{l} u_i^{\xi_i(\rho-w\rho)}} .$$

Hence:

COROLLARY 2.12. *The q-specialization of type $(s_0 ,..., s_l)$ of the right-hand side in Corollary* 2.11 *is*

$$\frac{\sum_{w \in W} (-1)^{l(w)} q^{\sum_{i=0}^{l} s_i \xi_i(\lambda+\rho-w(\lambda+\rho))}}{\sum_{w \in W} (-1)^{l(w)} q^{\sum_{i=0}^{l} s_i \xi_i(\rho-w\rho)}} .$$

We have mentioned above that Macdonald's identities occur in the generality of the "Euclidean Lie algebras." Instead of detailing this generalization of the affine Lie algebras (the reader may consult the exposition in [12b], for example), we shall discuss in Section 3 only the one special case that we shall need.

3. The Two Euclidean Lie Algebras for Which $l = 1$

In this section, we shall concretely describe those concepts from Section 2 that are needed to carry out our arguments for the two "simplest" infinite-dimensional GCM Lie algebras, $A_1^{(1)}$ and $A_2^{(2)}$.

First we shall describe the denominator and character formulas for $A_1^{(1)}$. This will illustrate Section 2 in a concrete setting.

We use the usual basis for $\mathfrak{sl}(2, \mathbb{C})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\mathbb{C}h$ is our Cartan subalgebra, $\mathbb{C}e$ is the root space for the positive root, and $\mathbb{C}f$ is the root space for the negative root. The Cartan matrix is the 1×1 matrix (2).

Recall that, in the notation of Section 2,

$$\mathfrak{sl}(2, \mathbb{C})^{\sim} = \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}].$$

This is just the Lie algebra of 2×2 traceless matrices with finite Laurent series entries. Recall also that $\mathfrak{sl}(2, \mathbb{C})_1^{\sim}$ is the semidirect product of the span of the degree derivation D with $\mathfrak{sl}(2, \mathbb{C})^{\sim}$. (We are of course freely using the notation of Section 2.) The roots of $\mathfrak{sl}(2, \mathbb{C})_1^{\sim}$ with respect to $\mathfrak{h}_1 = \mathbb{C}D \oplus \mathbb{C}h$ are easily described using the basis $\{\alpha_1, \gamma\}$ of $\mathfrak{h}_1{}^*$:

$$\begin{aligned} \Delta &= \{j\gamma + k\alpha_1 \mid j \in \mathbb{Z};\ k = 0, \pm 1;\ j\gamma + k\alpha_1 \neq 0\}, \\ \Delta_{\mathrm{R}} &= \{j\gamma + k\alpha_1 \mid j \in \mathbb{Z},\ k = \pm 1\}, \\ \Delta_{\mathrm{I}} &= \{j\gamma \mid j \in \mathbb{Z}, j \neq 0\}. \end{aligned}$$

Note that $\Delta_0 = \{\pm\alpha_1\}$.

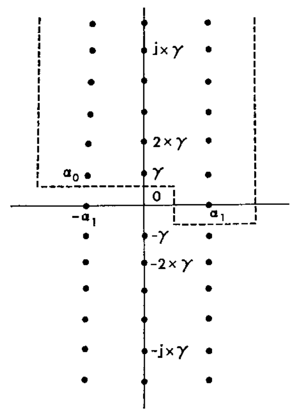
In the present case, since $l = 1$, the imaginary root spaces are one-dimensional. Let $j \in \mathbb{Z}$. Then the root space in $\mathfrak{sl}(2, \mathbb{C})^{\sim}$ for the root $j\gamma + \alpha_1$ is spanned by $e \otimes t^j$; the root space for $j\gamma - \alpha_1$ is spanned by $f \otimes t^j$; and if $j \neq 0$, the root space for $j\gamma$ is spanned by $h \otimes t^j$. We have

$$\Delta_+ = \{j\gamma + k\alpha_1 \mid j \geqslant 0; \text{ and } k = 1 \text{ if } j = 0\}.$$

The simple roots of $\mathfrak{sl}(2, \mathbb{C})_1^{\sim}$ are α_1 and $\alpha_0 = \gamma - \alpha_1$. The Cartan matrix is $\tilde{A} = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Note that $\gamma = \alpha_0 + \alpha_1$.

See Fig. 1 for a useful picture of Δ. The roots are represented by the dots in the plane $\mathfrak{h}_1{}^*$. The positive roots are those in the region enclosed by the dashed boundary. The real roots are the roots in the outer two columns and the imaginary roots are those in the inner column.

FIGURE 1 : ROOT SYSTEM FOR $A_1^{(1)}$



Recall that we have the exact sequence

$$0 \to \mathfrak{c} \to A_1^{(1)} \to \mathfrak{sl}(2, \mathbb{C})^{\sim} \to 0$$

and that this enables us to identify the root system of $A_1^{(1)}$ with that of $\mathfrak{sl}(2, \mathbb{C})^{\sim}$.

Also, the Weyl group W acts naturally on $\mathfrak{h}_1{}^*$ as follows: W is the group generated by the two reflections r_0 and r_1 in $\mathfrak{h}_1{}^*$ given by

$$\begin{aligned} r_0\alpha_0 &= \alpha_0 - \tilde{A}_{00}\alpha_0 = -\alpha_0\,, \\ r_0\alpha_1 &= \alpha_1 - \tilde{A}_{01}\alpha_0 = \alpha_1 + 2\alpha_0\,, \\ r_1\alpha_0 &= \alpha_0 - \tilde{A}_{10}\alpha_1 = \alpha_0 + 2\alpha_1\,, \\ r_1\alpha_1 &= \alpha_1 - \tilde{A}_{11}\alpha_1 = -\alpha_1\,. \end{aligned} \tag{3.1}$$

Recall that the real span of α_0 and α_1 carries a positive semidefinite geometry for which γ is in the radical, and that r_0 and r_1 are the geometric reflections with respect to the nonisotropic vectors α_0 and α_1. Observe that

$$r_0\gamma = \gamma = r_1\gamma,$$

so that W fixes γ. Also,

$$(r_1 r_0)\alpha_0 = -r_1\alpha_0 = -\alpha_0 - 2\alpha_1 = \alpha_0 - 2\gamma.$$

Hence $r_1 r_0$ fixes every multiple of γ, subtracts 2γ from α_0 and adds 2γ to α_1. Thus $r_1 r_0$ is a "vertical shear." The power $(r_1 r_0)^m$ ($m \in \mathbb{Z}$) clearly is the vertical shear which fixes every multiple of γ and adds $2m\gamma$ to α_1. (This holds even if m is negative.)

Concluding Remarks

From Table II (see Appendix), it is apparent that in many cases $A_F^{(k)} = 4A_F^{(k-1)}$ when $\|F\| = k$. The following result explains this phenomenon. By a *star* S_m we mean a tree having m edges and at most one vertex of degree exceeding 1.

FACT 9. *If F is a union of stars and $\|F\| = k$ then*

$$A_F^{(k)} = 4A_F^{(k-1)}.$$

Proof. We first calculate

$$A_{S_m}^{(m-1)} = m, \qquad A_{S_m}^{(m)} = 4m, \qquad \pi(S_m) = m,$$

which implies

$$B_{S_m}^{(m-1)} = 1, \qquad B_{S_m}^{(m)} = 4.$$

The desired result now follows by Fact 8. ∎

Note that by Fact 7, for $\|F\| = k - 1$,

$$A_F^{(k)} < 0 \qquad \text{iff } F \text{ is a tree,}$$
$$A_F^{(k)} = 0 \qquad \text{iff } \ F = 2P_1 .$$

We remark that it is possible to have two different forests F and F' such that $A_F^{(k)} = A_{F'}^{(k)}$ for all k. An example of such a pair is

$$F = S_1 \cup 3S_3 \cup 3S_5 \cup S_{11} ,$$
$$F' = 4S_2 \cup 4S_7 ,$$

for which

$$\|F\| = \|F'\| = 36, \qquad \pi(F) = \pi(F') = 2^{12}3^4,$$
$$A_F^{(35)} = A_{F'}^{(35)} = 2^{11} \cdot 3^3 \cdot 37,$$
$$A_F^{(36)} = A_{F'}^{(36)} = 2^{13} \cdot 3^3 \cdot 37 \qquad \text{(by Fact 9)},$$
$$A_F^{(37)} = A_{F'}^{(37)} = 2^{11} \cdot 3^3 \cdot 31.$$

Since we know $\delta_{n-1}(T) = 0$ for all T, then for $k = n - 1$ the theorem reduces to the simple identity

$$\sum_{\{x,y\}\subseteq T} d(x, y) - \sum_{\{e_1,e_2\}\subseteq \text{edges of } T} d(e_1 , e_2) = \| T \|^2, \tag{43}$$

where $d(e_1 , e_2)$ is defined to be the length of the path joining the edges e_1 and e_2 .

Note that the set Δ_+ of positive roots of $\mathfrak{sl}(2, \mathbb{C})_1^{\sim}$ can also be described as

$$\{-\alpha_1 + j\gamma, j\gamma, \alpha_1 + (j-1)\gamma \mid j \geqslant 1\}, \tag{3.5}$$

or

$$\{j\alpha_0 + (j-1)\alpha_1\,, j\alpha_0 + j\alpha_1\,, (j-1)\alpha_0 + j\alpha_1 \mid j \geqslant 1\}. \tag{3.6}$$

In this case, for $\varphi \in \Delta_+$, $\dim \mathfrak{l}^{\varphi} = 1$. Recall that $W = W_0 \cup r_0 W_0$, $l(w_m) = 2m$, and $l(r_0 w_m) = 2m + 1$.

Substituting expressions (3.3), (3.4), and (3.5) into Theorem 2.6 yields:

COROLLARY 3.7. *The denominator formula for $A_1^{(1)}$ may be written in terms of α_1 and γ as*

$$\prod_{n\geqslant 1} (1 - e(-\gamma)^n)(1 - e(-\gamma)^{n-1}\, e(-\alpha_1))(1 - e(-\gamma)^n\, e(-\alpha_1)^{-1})$$
$$= \sum_{n\in\mathbb{Z}} e(-\alpha_1)^{2n}\, e(-\gamma)^{n(2n-1)} - \sum_{n\in\mathbb{Z}} e(-\alpha_1)^{(2n+1)}\, e(-\gamma)^{(n+1)(2n+1)}. \tag{3.8}$$

Remark. If we set $e(-\alpha_1) = qt$, $e(-\gamma) = q^2$ and simplify then (3.8) gives (1.1) of the Introduction.

Using expressions (3.3), (3.4), and (3.6) with Theorem 2.6, we obtain:

COROLLARY 3.9. *The denominator formula for $A_1^{(1)}$ may be written in terms of α_0 and α_1 as*

$$\prod_{n\geqslant 1} (1 - e(-\alpha_0)^n\, e(-\alpha_1)^n)(1 - e(-\alpha_0)^{n-1}\, e(-\alpha_1)^n)(1 - e(-\alpha_0)^n\, e(-\alpha_1)^{n-1})$$
$$= \sum_{n\in\mathbb{Z}} e(-\alpha_0)^{n(2n-1)}\, e(-\alpha_1)^{n(2n+1)} - \sum_{n\in\mathbb{Z}} e(-\alpha_0)^{(n+1)(2n+1)}\, e(-\alpha_1)^{n(2n+1)}. \tag{3.10}$$

Remark. If we set $e(-\alpha_0) = u$, $e(-\alpha_1) = v$ and let n be $-n$ in the first sum, then (3.10) becomes (1.2) of the Introduction.

Before we discuss the character of every standard irreducible module for $A_1^{(1)}$, recall that $\hat{\mathfrak{h}}^e$ may be assumed three-dimensional (i.e., we need at least one derivation; assume we add *only* this one). Then α_0, α_1, and ρ form a basis of $(\hat{\mathfrak{h}}^e)^*$.

We write concretely the right-hand side of Corollary 2.10. In (3.10) we computed the denominator using $e(-\alpha_0)$ and $e(-\alpha_1)$. All that remains to be done is to compute the numerator. That is, for $\lambda \in P \subset (\hat{\mathfrak{h}}^e)^*$, we study

$$\sum_{w\in W} (-1)^{l(w)}\, e(w(\lambda + \rho) - (\lambda + \rho)). \tag{3.11}$$

Recall that each $\lambda \in P$ is of the form $a\alpha_0 + b\alpha_1 + c\rho$, with $(a\alpha_0 + b\alpha_1 + c\rho)(h_i) \in \mathbb{Z}_+$. Since W fixes γ we can assume that $b = 0$. That is, $\lambda = a\alpha_0 + c\rho$, with $c \pm 2a \in \mathbb{Z}_+$. To carry out our calculation of (3.11) we need:

LEMMA 3.12. *Let W be the (Weyl) group generated by the two reflections r_0 and r_1 in $\mathfrak{h}_1{}^*$ given by* (3.1). *Let $w_m = (r_1 r_0)^m$ $(m \in \mathbb{Z})$, and suppose $\lambda \in P$ is given by $a\alpha_0 + c\rho$. Then*

$$w_m(\lambda + \rho) - (\lambda + \rho) = (2am + (c+1)\, m(2m-1))\,(-\alpha_0) + (2am + (c+1)\, m(2m+1))\,(-\alpha_1), \quad (3.13)$$

$$r_0 w_m(\lambda + \rho) - (\lambda + \rho) = (a(2m+2) + (c+1)(m+1)(2m+1))(-\alpha_0) + (2am + (c+1)\, m(2m+1))\,(-\alpha_1). \quad (3.14)$$

Proof. Note that for each $w \in W$ we have $w\lambda = aw\alpha_0 + c(w\rho - \rho) + c\rho = aw\alpha_0 - c\langle \Phi_w \rangle + c\rho$. Thus, $w(\lambda + \rho) - (\lambda + \rho) = aw\alpha_0 - c\langle \Phi_w \rangle + c\rho + (w\rho - \rho) - \lambda$. But this simplifies to

$$aw\alpha_0 - (c+1)\langle \Phi_w \rangle - a\alpha_0 . \quad (3.15)$$

Now, $w_m\alpha_0 = (1-2m)\alpha_0 - 2m\alpha_1$, and $r_0 w_m \alpha_0 = -(2m+1)\alpha_0 - 2m\alpha_1$. Hence, (3.3) and (3.15) imply that $w_m(\lambda + \rho) - (\lambda + \rho)$ is given by

$$\begin{aligned} &a((1-2m)\alpha_0 - 2m\alpha_1) - (c+1)(m(2m-1)\,\alpha_0 + m(2m+1)\alpha_1) - a\alpha_0 \\ &\quad = (-2am - (c+1)m(2m-1))\alpha_0 + (-2am - (c+1)m(2m+1))\alpha_1 , \end{aligned}$$

which is (3.13).

In the same way (3.4) and (3.15) give (3.14).

Substituting (3.13) and (3.14) into (3.11) we immediately obtain:

COROLLARY 3.16. *Let $\lambda = a\alpha_0 + c\rho$ be a dominant integral linear form and let V be the standard module for $A_1^{(1)}$ with highest weight λ. Then the expression*

$$\chi(V) \sum_{w \in W} (-1)^{l(w)}\, e(w\rho - \rho)/e(\lambda) = \sum_{w \in W} (-1)^{l(w)}\, e(w(\lambda + \rho) - (\lambda + \rho)) \quad (3.17)$$

may be written in terms of $e(-\alpha_0)$ and $e(-\alpha_1)$ as

$$\sum_{n \in \mathbb{Z}} e(-\alpha_0)^{(2an + (c+1)n(2n-1))}\, e(-\alpha_1)^{(2an + (c+1)n(2n+1))} - \sum_{n \in \mathbb{Z}} e(-\alpha_0)^{(a(2n+2) + (c+1)(n+1)(2n+1))}\, e(-\alpha_1)^{(2an + (c+1)n(2n+1))}. \quad (3.18)$$

Remark. The expression in (3.17) is the numerator of the character $\chi(V)$, divided by $e(\lambda)$.

Now that we have set up the necessary material on $A_1^{(1)}$, we turn to the other Euclidean Lie algebra for with $l = 1$, namely, $A_2^{(2)}$. (See, for example, [12b] for an exposition of Euclidean Lie algebras.)

This time, we start with the eight-dimensional rank 2 simple Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. Let θ be the negative transpose map of $\mathfrak{sl}(3, \mathbb{C})$ to itself. Then θ is a

Lie algebra automorphism of order 2. The fixed set of θ is the three-dimensional rank 1 subalgebra $\mathfrak{g}_0 = \mathfrak{so}(3, \mathbb{C})$ consisting of the skew-symmetric matrices. Let $\mathfrak{g}_1$ be the -1-eigenspace of θ in $\mathfrak{sl}(3, \mathbb{C})$, i.e., the five-dimensional space of symmetric traceless 3×3 matrices. Then

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \qquad \text{and} \qquad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0. \tag{3.19}$$

Choose any Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, and let $\pm\alpha_1$ be the roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$, with α_1 the positive root. Then the weights (with respect to $\mathfrak{h}_0$) of $\mathfrak{g}_0$ acting on $\mathfrak{g}_1$ are 0, $\pm\alpha_1$, $\pm 2\alpha_1$, each of multiplicity one.

Now

$$\mathfrak{sl}(3, \mathbb{C})^{\sim} = \mathfrak{sl}(3, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}].$$

Relations (3.19) imply that

$$\mathfrak{a} = \coprod_{j \in \mathbb{Z}} \mathfrak{g}_{j(\mathrm{mod}\, 2)} \otimes t^j$$

is a Lie subalgebra of $\mathfrak{sl}(3, \mathbb{C})^{\sim}$. Let E be the degree derivation of $\mathfrak{sl}(3, \mathbb{C})^{\sim}$, so that E acts as scalar multiplication by j on $\mathfrak{sl}(3, \mathbb{C}) \otimes t^j$. Then E preserves $\mathfrak{a}$ and acts as a derivation on $\mathfrak{a}$. Define $\mathfrak{a}_1$ to be the corresponding semidirect product Lie algebra $\mathbb{C}E \oplus \mathfrak{a}$, and define $\mathfrak{h}_{0,1}$ to be the Abelian subalgebra $\mathbb{C}E \oplus \mathfrak{h}_0$ of $\mathfrak{a}_1$. Then $\mathfrak{a}_1$ has a natural and obvious "root space decomposition" with respect to $\mathfrak{h}_{0,1}$. Let $\gamma \in (\mathfrak{h}_{0,1})^*$ be the element which is 0 in $\mathfrak{h}_0$ and 1 on E. Identifying $\mathfrak{h}_0^*$ with the subspace of $\mathfrak{h}_{0,1}^*$ consisting of the elements vanishing on E, we see that $\mathfrak{h}_{0,1}^* = \mathfrak{h}_0^* \oplus \mathbb{C}\gamma$, and that $\{\alpha_1, \gamma\}$ is a basis of $\mathfrak{h}_{0,1}^*$. The set Δ of roots (defined in the obvious way) of $\mathfrak{a}_1$ with respect to $\mathfrak{h}_{0,1}$ is given by

$$\Delta = \{ j\gamma + k\alpha_1 \mid j \in \mathbb{Z};\ k = 0, \pm 1 \text{ if } j \text{ is even};$$
$$k = 0, \pm 1, \pm 2 \text{ if } j \text{ is odd};\ j\gamma + k\alpha_1 \neq 0\}. \tag{3.20}$$

Define the set Δ_{R} of real roots to be $\{\varphi \in \Delta \mid \varphi \neq 0 \text{ on } \mathfrak{h}_0\}$, and the set Δ_{I} of imaginary roots to be $\{\varphi \in \Delta \mid \varphi = 0 \text{ on } \mathfrak{h}_0\}$. Then Δ_{I} consists of the elements of Δ which are multiples of γ, and Δ_{R} consists of all other elements of Δ. Note that all the root spaces (real or imaginary) are one-dimensional. Let $\Delta_+ \subset \Delta$ consist of α_1 and the roots which are positive on E. Then $\Delta = \Delta_+ \cup (-\Delta_+)$. Define the simple roots to be those positive roots which cannot be written as a sum of two positive roots. Then the simple roots are α_1 and $\alpha_0 = \gamma - 2\alpha_1$. The Cartan matrix of $\mathfrak{a}$ (or of $\mathfrak{a}_1$) is defined to be the 2×2 matrix B with

$$B_{ij} = \frac{2(\alpha_i \mid \mathfrak{h}_0, \alpha_j \mid \mathfrak{h}_0)}{(\alpha_i \mid \mathfrak{h}_0, \alpha_i \mid \mathfrak{h}_0)}.$$

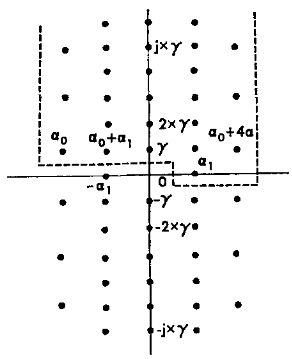
Then

$$B = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}. \tag{3.21}$$

We have $\gamma = \alpha_0 + 2\alpha_1$.

In Fig. 2, we indicate the roots by dots in the plane $\mathfrak{h}_{0,1}^*$. The positive roots are the roots in the region enclosed by the dashed boundary. The real roots are the roots in the outer four columns, and the imaginary roots are those in the middle column.

FIGURE 2 : ROOT SYSTEM FOR $A_2^{(2)}$



Let $A_2^{(2)}$ be the GCM Lie algebra $\mathfrak{l}(B)$. Then the center of $A_2^{(2)}$ is a one-dimensional subalgebra $\mathfrak{c}$, and there is an exact sequence

$$0 \to \mathfrak{c} \to A_2^{(2)} \to \mathfrak{a} \to 0$$

by means of which we can identify the root system of $A_2^{(2)}$ with that of $\mathfrak{a}$ (described above). Note the analogy between this situation and that of $A_1^{(1)}$. The reason for the notation $A_2^{(2)}$ is that $\mathfrak{a}$ was defined using an automorphism of order 2 (the superscript in $A_2^{(2)}$) of the Lie algebra A_2 (Cartan's notation for $\mathfrak{sl}(3, \mathbb{C})$). Incidentally, in this (Kac's) notational scheme for Euclidean Lie algebras, $\mathfrak{sl}(2, \mathbb{C})^\wedge$ is also denoted $A_1^{(1)}$, in that it comes analogously from the automorphism of order 1 of $A_1 = \mathfrak{sl}(2, \mathbb{C})$.

Let W be the Weyl group of $A_2^{(2)}$. Then W acts naturally on $\mathfrak{h}_{0,1}^*$ as follows: W is the group generated by the two reflections r_0 and r_1 in $\mathfrak{h}_{0,1}^*$ given by

$$\begin{aligned} r_0\alpha_0 &= \alpha_0 - B_{00}\alpha_0 = -\alpha_0, \\ r_0\alpha_1 &= \alpha_1 - B_{01}\alpha_0 = \alpha_1 + \alpha_0, \\ r_1\alpha_0 &= \alpha_0 - B_{10}\alpha_1 = \alpha_0 + 4\alpha_1, \\ r_1\alpha_1 &= \alpha_1 - B_{11}\alpha_1 = -\alpha_1. \end{aligned} \tag{3.22}$$

Just as in the case of $A_1^{(1)}$, the real span of α_0 and α_1 in $\mathfrak{h}_{0,1}^*$ has a natural positive semidefinite geometry (by restricting functionals to $\mathfrak{h}_0$ and then using the nonsingular form on $\mathfrak{h}_0{}^*$ obtained by restricting the Killing form of $\mathfrak{sl}(3, \mathbb{C})$ to $\mathfrak{h}_0$). The imaginary root γ is in the radical of this semidefinite form. Also, r_0 and r_1 are the geometric reflections with respect to the nonisotropic vectors α_0 and α_1. Observe that $r_0\gamma = \gamma = r_1\gamma$, so that W fixes γ. Also, $(r_1r_0)\alpha_0 = -r_1\alpha_0 = -\alpha_0 - 4\alpha_1 = \alpha_0 - 2\gamma$, and $(r_1r_0)(\alpha_0 + \alpha_1) = r_1(-\alpha_0 + \alpha_1 + \alpha_0) = \alpha_0 + \alpha_1 - (\alpha_0 + 2\alpha_1) = \alpha_0 + \alpha_1 - \gamma$. Hence r_1r_0 fixes every multiple of γ, subtracts 2γ from α_0, adds γ to α_1, subtracts γ from $\alpha_0 + \alpha_1$, and adds 2γ to $\alpha_0 + 4\alpha_1$. Thus r_1r_0 is a "vertical shear." The power $(r_1r_0)^m$ $(m \in \mathbb{Z})$ clearly is the vertical shear which fixes every multiple of γ, adds $m\gamma$ to α_1, and adds $2m\gamma$ to $\alpha_0 + 4\alpha_1$. (This holds even if m is negative.)

Notation. Let W_0 be the subgroup of W consisting of the integral powers of r_1r_0. W_0 is a normal subgroup of index 2 in W, and W is the disjoint union $W = W_0 \cup r_0W_0$. The elements of W_0 have even length while those of r_0W_0 have odd length.

We now write the denominator formula for $A_2^{(2)}$ in concrete form first using α_1 and γ, and *then* α_0 and α_1. To do this, we need:

LEMMA 3.23. *Let W be the (Weyl) group generated by the two reflections r_0 and r_1 in $\mathfrak{h}_{0,1}^*$ given by* (3.22). *Let $w_m = (r_1r_0)^m$ $(m \in \mathbb{Z})$. Then*

$$\begin{aligned}\langle \Phi_{w_m} \rangle &= m(3m-1)/2\alpha_0 + m(3m+2)\alpha_1 \\ &= 3m\alpha_1 + m(3m-1)/2\gamma, \end{aligned} \tag{3.24}$$

$$\begin{aligned}\langle \Phi_{r_0w_m} \rangle &= (3m^2+5m+2)/2\alpha_0 + (3m^2+2m)\alpha_1 \\ &= -(3m+2)\alpha_1 + (3m^2+5m+2)/2\gamma. \end{aligned} \tag{3.25}$$

Proof. Recall that $w_m\gamma = \alpha_0 + 2\alpha_1$, $w_m\alpha_0 = \alpha_0 - 2m\gamma$, $w_m(\alpha_0 + \alpha_1) = (\alpha_0 + \alpha_1) - m\gamma$, $w_m\alpha_1 = \alpha_1 + m\gamma$, and $w_m(\alpha_0 + 4\alpha_1) = (\alpha_0 + 4\alpha_1) + 2m\gamma$. By definition, $\Phi_{w_m} = \Delta_+ \cap w_m\Delta_-$. If $m \geqslant 0$, then

$$\Phi_{w_m} = \{\alpha_1 + i\gamma, (\alpha_0 + 4\alpha_1) + 2i\gamma \mid 0 \leqslant i \leqslant m-1\}, \tag{3.26}$$

since $w_m(\alpha_1 - i\gamma) = \alpha_1 + (m-i)\gamma$, and $w_m(\alpha_0 + 4\alpha_1 - 2i\gamma) = \alpha_0 + 4\alpha_1 + 2(m-i)\gamma$. Hence, $\langle \Phi_{w_m} \rangle = (m\alpha_1 + m(m-1)/2\gamma) + (m(\alpha_0 + 4\alpha_1) + m(m-1)\gamma) = 3m\alpha_1 + m(3m-1)/2\gamma = m(3m-1)/2\alpha_0 + m(3m+2)\alpha_1$. If $m < 0$, then

$$\Phi_{w_m} = \{(\alpha_0 + \alpha_1) + i\gamma, \alpha_0 + 2i\gamma \mid 0 \leqslant i \leqslant -m-1\}, \tag{3.27}$$

since $w_m(\alpha_0 + \alpha_1 - i\gamma) = \alpha_0 + \alpha_1 + (-m-i)\gamma$, and $w_m(\alpha_0 - 2i\gamma) = \alpha_0 + 2(-m-i)\gamma$.

Hence,

$$\begin{aligned}\langle \Phi_{w_m} \rangle &= (-m(\alpha_0 + \alpha_1) + m(m+1)/2\gamma) + (-m\alpha_0 + m(m+1)\gamma) \\ &= 3m\alpha_1 + m(3m-1)/2\gamma = m(3m-1)/2\alpha_0 + m(3m+2)\alpha_1 .\end{aligned}$$

This establishes (3.24).

From Lemma 2.1 and (3.24) we immediately have

$$\langle \Phi_{r_0 w_m} \rangle = r_0(3m\alpha_1 + m(3m-1)/2\gamma) + \langle \Phi_{r_0} \rangle.$$

Now $\langle \Phi_{r_0} \rangle = \alpha_0 = \gamma - 2\alpha_1$. This gives

$$\begin{aligned}\langle \Phi_{r_0 w_m} \rangle &= 3m(\alpha_0 + \alpha_1) + m(3m-1)/2\gamma + \alpha_0 \\ &= -(3m+2)\alpha_1 + (3m^2 + 5m + 2)/2\gamma \\ &= (3m^2 + 5m + 2)/2\alpha_0 + (3m^2 + 2m)\alpha_1 ,\end{aligned}$$

which is (3.25). Q.E.D.

Note that the set Δ_+ of positive roots of $\mathfrak{a}_1$ can also be described as

$$\{-2\alpha_1 + (2j-1)\gamma, -\alpha_1 + j\gamma, j\gamma, \alpha_1 + (j-1)\gamma, 2\alpha_1 + (2j-1)\gamma \mid j \geqslant 1\} \quad (3.28)$$

or

$$\begin{aligned}\{(2j-1)\,\alpha_0 + (4j-4)\alpha_1 , j\alpha_0 + (2j-1)\alpha_1 , j\alpha_0 + 2j\alpha_1 , \\ (j-1)\,\alpha_0 + (2j-1)\alpha_1 , (2j-1)\,\alpha_0 + 4j\alpha_1 \mid j \geqslant 1\}. \quad (3.29)\end{aligned}$$

In this case, for $\varphi \in \Delta_+$, $\dim \mathfrak{l}^{\varphi} = 1$. Just as in $A_1^{(1)}$, substituting expressions (3.24), (3.25), and (3.28) into Theorem 2.6 yields:

COROLLARY 3.30. *The denominator formula for $A_2^{(2)}$ may be written in terms of α_1 and γ as*

$$\begin{aligned}&\prod_{n \geqslant 1} (1 - e(-\gamma)^n)(1 - e(-\gamma)^{n-1} e(-\alpha_1))(1 - e(-\gamma)^n e(-\alpha_1)^{-1}) \\ &\qquad \times (1 - e(-\gamma)^{(2n-1)} e(-\alpha_1)^2)(1 - e(-\gamma)^{(2n-1)} e(-\alpha_1)^{-2}) \\ &= \sum_{n \in \mathbb{Z}} e(-\alpha_1)^{3n} e(-\gamma)^{n(3n-1)/2} - \sum_{n \in \mathbb{Z}} e(-\alpha_1)^{-(3n+2)} e(-\gamma)^{(n+1)(3n+2)/2}. \quad (3.31)\end{aligned}$$

Remark. If we set $e(-\alpha_1) = t^{-1}$, $e(-\gamma) = q$, replace n by $-n$ in the first sum, and n by $-(n+1)$ in the second, then (3.31) gives (1.5) of the Introduction.

Using expressions (3.24), (3.25), and (3.29) with Theorem 2.6 we obtain:

COROLLARY 3.32. *The denominator formula for $A_2^{(2)}$ may be written in terms of α_0 and α_1 as*

$$\prod_{n\geqslant 1} (1 - e(-\alpha_0)^{(2n-1} e(-\alpha_1)^{(4n-4)})(1 - e(-\alpha_0)^n e(-\alpha_1)^{(2n-1)})$$
$$\times (1 - e(-\alpha_0)^n e(-\alpha_1)^{2n})(1 - e(-\alpha_0)^{(n-1)} e(-\alpha_1)^{(2n-1)})$$
$$\times (1 - e(-\alpha_0)^{(2n-1)} e(-\alpha_1)^{4n})$$
$$= \sum_{n\in\mathbb{Z}} e(-\alpha_0)^{n(3n-1)/2} e(-\alpha_1)^{n(3n+2)} - \sum_{n\in\mathbb{Z}} e(-\alpha_0)^{(n+1)(3n+2)/2} e(-\alpha_1)^{n(3n+2)}. \quad (3.33)$$

Remark. If we set $e(-\alpha_0) = u$, $e(-\alpha_1) = v$, replace n by $-n$ in the first sum, and n by $-(n+1)$ in the second, then (3.33) gives (1.6) of the Introduction.

Before we discuss the character of every standard irreducible module for $A_2^{(2)}$, we define $\hat{\mathfrak{h}}_0$ and $\hat{\mathfrak{h}}_0{}^e$ to be the objects denotes $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^e$, respectively, in Section 2. Note that $\hat{\mathfrak{h}}_0{}^e$ may be assumed three-dimensional (i.e., we need at least one derivation; assume we add *only* this one). Then α_0, α_1, and ρ form a basis of $(\hat{\mathfrak{h}}_0{}^e)^*$.

As before, we write concretely the right-hand side of Corollary 2.10. In (3.33) we computed the denominator using $e(-\alpha_0)$ and $e(-\alpha_1)$. All that remains to be done is to compute the numerator. That is, for $\lambda \in P \subset (\hat{\mathfrak{h}}_0{}^e)^*$, we study (3.11).

Note that each $\lambda \in P$ is of the form $a\alpha_0 + b\alpha_1 + c\rho$, with $(a\alpha_0 + b\alpha_1 + c\rho)(h_i) \in \mathbb{Z}_+$. Since W fixes γ we can assume that $b = 0$. That is, $\lambda = a\alpha_0 + c\rho$, with $c + 2a$, $c - 4a \in \mathbb{Z}_+$. To carry out our calculation of (3.11) we need:

LEMMA 3.34. *Let W be the (Weyl) group generated by the two reflections r_0 and r_1 in $\mathfrak{h}_{0,1}^*$ given by (3.22). Let $w_m = (r_1 r_0)^m$ ($m \in \mathbb{Z}$), and suppose $\lambda \in P$ is given by $a\alpha_0 + c\rho$. Then,*

$$w_m(\lambda + \rho) - (\lambda + \rho) = (2am + (c+1)\, m(3m-1)/2)(-\alpha_0) + (4am + (c+1)\, m(3m+2))(-\alpha_1), \quad (3.35)$$

$$r_0 w_m(\lambda + \rho) - (\lambda + \rho) = ((2m+2)\, a + (c+1)(3m^2 + 5m + 2)/2)(-\alpha_0) + (4ma + (c+1)(3m^2 + 2m))(-\alpha_1). \quad (3.36)$$

Proof. Recall from (3.15) that for $\lambda = a\alpha_0 + c\rho$, $w(\lambda + \rho) - (\lambda + \rho) = aw\alpha_0 - (c+1)\langle\Phi_w\rangle - \alpha_0$. Now, $w_m\alpha_0 = (1-2m)\alpha_0 - 4m\alpha_1$, and $r_0 w_m \alpha_0 = -(2m+1)\alpha_0 - 4m\alpha_1$. Hence, (3.24) implies that $w_m(\lambda + \rho) - (\lambda + \rho)$ is given by $a((1-2m)\alpha_0 - 4m\alpha_1) - (c+1)(m(3m-1)/2\alpha_0 + m(3m+2)\alpha_1) - a\alpha_0 = (-2am - (c+1)\, m(3m-1)/2)\alpha_0 + (-4am - (c+1)\, m(3m+2))\alpha_1$, which is (3.35).

In the same way (3.25) and (3.15) give (3.36). Q.E.D.

Substituting (3.35) and (3.36) into (3.11) we immediately obtain:

COROLLARY 3.37. *Let $\lambda = a\alpha_0 + c\rho$ be a dominant integral linear form and V be the standard module for $A_2^{(2)}$ with highest weight λ. Then the expression*

$$\chi(V) \sum_{w \in W} (-1)^{l(w)} e(w\rho - \rho)/e(\lambda) = \sum_{w \in W} (-1)^{l(w)} e(w(\lambda + \rho) - (\lambda + \rho)) \tag{3.38}$$

may be written in terms of $e(-\alpha_0)$ and $(-\alpha_1)$ as

$$\sum_{n \in \mathbb{Z}} e(-\alpha_0)^{(2an+(c+1)n(3n-1)/2} e(-\alpha_1)^{(4an+(c+1)n(3n+2))}$$
$$- \sum_{n \in \mathbb{Z}} e(-\alpha_0)^{((2n+2)a+(c+1)(3n^2+5n+2)/2)} e(-\alpha_1)^{(4an+(c+1)(3n^2+2n))}. \tag{3.39}$$

We now have the necessary material for $A_2^{(2)}$. The techniques used to write down concretely the denominator formulas for $A_1^{(1)}$ and $A_2^{(2)}$ will be applied in Section 4. Corollaries 3.9, 3.16, 3.32, and 3.37 will be essential to our proof of the "numerator formula" for $A_1^{(1)}$ and $A_2^{(2)}$.

4. THE MULTIVARIABLE VECTOR PARTITION THEOREM FOR $A_l^{(1)}$

In this section we state and prove an abstract vector partition theorem which was described in the Introduction. For the important special cases $A_l^{(1)}$ we describe explicitly the allowable parts (including multiplicity) that can occur in any vector partition, and also give a concrete algorithm for constructing the sets Φ_w, $w \in W$, the Weyl group of $A_l^{(1)}$. To this end, we study the structure of the group W. We close this section by writing out concretely the unspecialized denominator formula for $A_2^{(1)}$. We are grateful to R. Wilson for help in the writing of this section.

Note that the exponents of the variables $e(-\alpha_0)$ and $e(-\alpha_1)$ in the monomials on the sum side of Corollary 3.9 are quadratic in the variable of summation. This fact is a direct consequence of our computation of $\langle \Phi_w \rangle$ in Lemma 3.2. The key step in the proof of this lemma is to express the set Φ_{w_m} as $\{\alpha_1 + i\gamma \mid 0 \leqslant i \leqslant 2m - 1\}$. When the roots of Φ_{w_m} are added together, the coefficient of α_1 is linear in m while the coefficient of $\gamma = \alpha_0 + \alpha_1$ is quadratic in m. Setting $u = qt$ and $v = q/t$ in (3.10) gives (1.1), where the exponent of q is quadratic. Although this is the simplest example, the situation we have just described is typical. That is, the exponents of the variables e(-simple roots) in the monomials on the sum side of the denominator formula given in Theorem 2.6 (in the case of affine or Euclidean GCM Lie algebras) are quadratic in the "variables of summation." We will discuss this remark later.

For now, just keep in mind that the fundamental reason for the quadratic exponents is the fact that consecutive integral multiples of a fixed imaginary root γ are added together in the computation of $\langle \Phi_w \rangle$. This reasoning involving imaginary roots "explains" why certain combinatorial identities always have quadratic exponents for the variable q.

Before specializing to $A_l^{(1)}$, we work in the full generality of GCM Lie algebras (see Section 2).

LEMMA 4.1. *Consider the denominator formula (Corollary 2.7)*

$$\prod_{\varphi\in\Delta_+}\left(1-\prod_{i=0}^{l} u_i^{\xi_i(\varphi)}\right)^{\dim \mathfrak{l}^\varphi} = \sum_{w\in W}(-1)^{l(w)}\prod_{i=0}^{l} u_i^{\xi_i(\langle\Phi_w\rangle)}, \tag{4.2}$$

in $\mathbb{Z}[[u_0,..., u_l]]$, *with* $u_i = e(-\alpha_i)$. *Then the monomials* $\prod_{i=0}^{l} u_i^{\xi_i(\langle\Phi_w\rangle)}$ *as* w *ranges over the Weyl group* W *are all different. That is, the coefficient of* $\prod_{i=0}^{l} u_i^{\xi_i(\langle\Phi_w\rangle)}$ *is indeed* $(-1)^{l(w)}$.

Proof. It is sufficient to show that if $w_1, w_2 \in W$, and $w_1 \neq w_2$ then $\langle\Phi_{w_1}\rangle \neq \langle\Phi_{w_2}\rangle$. But this follows immediately by taking the contrapositive of Lemma 2.5, which asserts that $\langle\Phi_{w_1}\rangle = \langle\Phi_{w_2}\rangle$ implies that $w_1 = w_2$. Q.E.D.

At this point it is convenient to introduce:

DEFINITION 4.3. Suppose that $\varphi = \sum_{i=0}^{l} c_i\alpha_i$ is a root of $\mathfrak{l}$ such that dim $\mathfrak{l}^\varphi = d$. Then by $\xi^d(\varphi)$ we mean the collection of "colored" $(l+1)$-vectors which is formed by d duplicates of the vector $(c_0, c_1,..., c_l)$, each with a different color. We indicate this by the set $\{\xi(\varphi_j) = (c_0,..., c_l)_j \mid 1 \leqslant j \leqslant d\}$. We say that φ_j is the *copy* of the root φ corresponding to the vector $\xi(\varphi_j)$. If $d = 1$, we suppress the subscript j. If Φ is a collection of roots of $\mathfrak{l}$, possibly including any allowable copies φ_j of φ, then by $\langle\Phi\rangle$ we mean the sum of all the roots in Φ, including multiplicity. For example, the sum of φ_{i_1} and φ_{i_2} is 2φ. $\xi(\Phi)$ is the collection of vectors $\{\xi(\varphi) \mid \varphi \in \Phi\}$.

DEFINITION 4.4 (*vector partitions*). Given $\beta = \sum_{i=0}^{l} c_i\alpha_i$, with $c_i \in \mathbb{Z}$, let $\xi^0(\beta)$ be the vector $(c_0, c_1,..., c_l)$. We say that $\xi(\Phi)$ is a vector partition of $\xi^0(\beta)$ iff: (1) Φ is a collection of positive roots of $\mathfrak{l}$, possibly including any allowable copies φ_j of φ, and (2) the sum of the vectors in $\xi(\Phi)$, including multiplicity, is $\xi^0(\beta)$. For example, if $\varphi_i = (b_0, b_1,..., b_l)_i$, the vector sum of φ_{i_1} and φ_{i_2} is $(2b_0, 2b_1,..., 2b_l)$. So $\xi(\Phi)$ is a vector partition of $\xi^0(\beta)$ iff $\xi^0(\langle\Phi\rangle) = \xi^0(\beta)$.

With these definitions in mind, we state:

LEMMA 4.5. *Let* $w \in W$. *Then there is only one vector partition of* $\xi^0(\langle\Phi_w\rangle)$, *namely,* $\xi(\Phi_w)$. *Moreover, all of the vectors in the vector partition* $\xi(\Phi_w)$ *correspond to real roots.*

Proof. This is an immediate consequence of Lemma 2.2, which asserts that if $\langle\Phi_w\rangle = \langle\Phi\rangle + \beta$, with Φ a finite subset of Δ_+, and $\beta \in \operatorname{span}\Delta$ a finite sum of not necessarily distinct positive imaginary roots, then $\beta = 0$ and $\Phi = \Phi_w$.

Also, Φ_w consists of real roots only. But in view of our definition of vector partitions, this is just a statement of what we wanted to prove. Q.E.D.

We are now ready to state the following abstract vector partition theorem corresponding to the GCM Lie algebra I:

THEOREM 4.6. *The excess of the number of vector partitions of* $(c_0, c_1, \ldots, c_l)$ *into an even number of distinct parts (note: vectors of different colors are regarded as distinct) over those into an odd number of such parts is* $(-1)^{l(w)}$ *or* 0 *according as* $(c_0, c_1, \ldots, c_l)$ *is of the type* $\xi^0(\langle\Phi_w\rangle)$ *or not. Moreover,* $\xi^0(\langle\Phi_w\rangle)$ *has only one vector partition, namely,* $\xi(\langle\Phi_w\rangle)$.

Proof. Imagine the left-hand side of (4.2) expanded out as a formal power series in the u_i, and simply refer to Lemmas 4.1 and 4.5. Q.E.D.

We now pass to the special case $\mathrm{I} = A_l^{(1)}$ $(l \geqslant 1)$. First we shall describe $\Delta_+ \subset \Delta$, in order to exhibit concretely the allowable parts occurring in the vector partitions in Theorem 4.6. To this end, we first describe the roots of $\mathfrak{sl}(n, \mathbb{C})$, (note: set $n = l + 1$), as presented in [10].

Let I be the $\mathbb{Z}$-span of the standard basis vectors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ in $\mathbb{R}^n$. Let E be the l-dimensional subspace of $\mathbb{R}^n$ orthogonal to the vector $\epsilon_1 + \cdots + \epsilon_n$. Let $I' = I \cap E$ and take Δ_0 to be the set of all vectors $\alpha \in I'$ for which $(\alpha, \alpha) = 2$. It is clear that $\Delta_0 = \{\epsilon_i - \epsilon_j \mid i \neq j\}$. The roots of $\mathfrak{sl}(n, \mathbb{C})$ can be identified with the set Δ_0, and the simple roots with the subset $\{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. The vectors α_i are independent and $\epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, if $i < j$. This last relation implies that the positive roots of $\mathfrak{sl}(n, \mathbb{C})$ are of the form

$$\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k-1}, \tag{4.7}$$

with $1 \leqslant k \leqslant l$ and $1 \leqslant i \leqslant l - k + 1$. The negative roots are of the form

$$-(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k-1}). \tag{4.8}$$

The real roots of $A_l^{(1)}$ can be identified with the functionals of the form $j\gamma + \varphi$, where $j \in \mathbb{Z}$, $\varphi \in \Delta_0$, and γ is the smallest positive imaginary root of $A_l^{(1)}$. The imaginary roots of $A_l^{(1)}$ are identified with functionals of the form $j\gamma$, where $j \in \mathbb{Z}, j \neq 0$. Now $\Delta_+ \subset \Delta$ is the union of the positive roots in Δ_0 with the set of roots in Δ which are positive on D. By means of (4.7) and (4.8) we may describe the set Δ_+ of positive roots of $A_l^{(1)}$ concretely as

$$\begin{aligned}\Delta_+ = \{&(s-1)\gamma + \alpha_i + \cdots + \alpha_{i+k-1},\ s\gamma - (\alpha_i + \cdots + \alpha_{i+k-1}),\\ &s\gamma \mid 1 \leqslant k \leqslant l,\ 1 \leqslant i \leqslant l - k + 1,\ s \in Z_+ - \{0\}\}.\end{aligned} \tag{4.9}$$

From (4.2), (4.9), $\gamma = \alpha_0 + \alpha_1 + \cdots + \alpha_l$, and the fact that real roots have multiplicity 1 and imaginary roots multiplicity l, it is not hard to prove:

THEOREM 4.10. *Whenever* $l = A_l^{(1)}$ $(l \geqslant 1)$, *the product side of the denominator formula in* (4.2) *can be written as*:

$$\prod_{s\geqslant 1}\left(1-\prod_{j=0}^{l} u_j{}^s\right)^l \prod_{k=1}^{l}\prod_{i=0}^{n-k}(1-u_0^{s-1}u_1^{s-1}\cdots u_{i-1}^{s-1}u_i{}^s\cdots u_{i+k-1}^{s}u_{i+k}^{s-1}\cdots u_l^{s-1})$$

$$\times\prod_{i=n-k+1}^{l}(1-u_0{}^s\cdots u_{i+k-n-1}^{s}u_{i+k-n}^{s-1}\cdots u_{i-1}^{s-1}u_i{}^s\cdots u_l{}^s). \tag{4.11}$$

Furthermore, if $\mathbf{c} = (c_0, c_1, \ldots, c_l)$ *is an allowable part in any vector partition occurring in Theorem* 4.6 *then* $\mathbf{c}$ *must either be one of the* l *different copies of the* n*-tuple* $(s, s, \ldots, s)$ *or, exactly* k *cyclicly consecutive coordinates* c_i *of* $\mathbf{c}$ *must be* s *and the rest* $s-1$, *where* $1 \leqslant k < n$. *That is, we must have either*

$$\begin{aligned} c_j &= s, &&\text{if } i \leqslant j \leqslant i+k-1,\\ &= s-1, &&\text{otherwise}, \quad \text{for } 0 \leqslant i \leqslant n-k,\end{aligned}$$

or

$$\begin{aligned} c_j &= s-1, &&\text{if } i+k-n \leqslant j \leqslant i-1,\\ &= s, &&\text{otherwise}, \quad \text{for } n-k+1 \leqslant i < n.\end{aligned}$$

Remark. Only for $l = 1$ are colors unnecessary for the description of the allowable parts in any vector partition occurring in Theorem 4.6.

It is now time to study the structure of W, in preparation for constructing the sets Φ_w.

Recall that the simple roots of $A_l^{(1)}$ consist of the simple roots $\alpha_1, \ldots, \alpha_l$ of $\mathfrak{sl}(n, \mathbb{C})$ together with $\alpha_0 = \gamma - \psi$, where γ is the smallest positive imaginary root of $A_l^{(1)}$ and $\psi = \alpha_1 + \cdots + \alpha_l$ is the highest root of $\mathfrak{sl}(n, \mathbb{C})$. The Cartan matrix A of $\mathfrak{sl}(n, \mathbb{C})$ is an l by l matrix that is 2 on the diagonal, -1 on the super- and subdiagonals, and 0 elsewhere. The Cartan matrix $\tilde{A}$ of $A_l^{(1)}$ is obtained from A by adding a zeroth row and column so that the resulting row and column sums are 0. That is, $\tilde{A}_{ij} = A_{ij}$ for $1 \leqslant i, j \leqslant l$, $\tilde{A}_{00} = 2$, $\tilde{A}_{01} = \tilde{A}_{0l} = \tilde{A}_{10} = \tilde{A}_{l0} = -1$, and otherwise $\tilde{A}_{ij} = 0$.

We have by definition that $\alpha_j(h_i) = \tilde{A}_{ij}$ for $0 \leqslant i, j \leqslant l$. Given any root φ for $A_l^{(1)}$ the ith reflection r_i of φ is defined by

$$r_i(\varphi) = \varphi - \varphi(h_i)\alpha_i, \qquad 0 \leqslant i \leqslant l.$$

Every element of W fixes γ.

DEFINITION 4.12. A (*vertical*) *shear* is a linear automorphism T of $R = \operatorname{span} \Delta$ such that for every $\varphi \in R$, $T(\varphi) = \varphi + c(\varphi)\gamma$, where $c \in R^*$ and $c(\gamma) = 0$. That is, T fixes γ and induces the identity on $R/\mathbb{R}\gamma$.

Our first task is to construct a vertical shear from the reflections r_i. To this end we prove:

LEMMA 4.13. *Let $\varphi \in R$, and γ the smallest positive imaginary root for $A_l^{(1)}$. Then we have*

$$r_1 r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2 r_1 r_0 \varphi = \varphi + \varphi(h_1 + \cdots + h_l)\gamma. \tag{4.14}$$

Proof. We first show by induction that

$$r_l r_{l-1} \cdots r_2 r_1 \varphi = \varphi - \sum_{i=1}^{l} \varphi(h_1 + \cdots + h_i)\alpha_i .$$

We have by definition that $r_1\varphi = \varphi - \varphi(h_1)\alpha_1$. For $1 \leqslant m \leqslant l - 1$, suppose that

$$r_m r_{m-1} \cdots r_2 r_1 \varphi = \varphi - \sum_{i=1}^{m} \varphi(h_1 + \cdots + h_i)\alpha_i .$$

Then,

$$\begin{aligned} r_{m+1} r_m \cdots r_2 r_1 \varphi &= \varphi - \varphi(h_{m+1})\alpha_{m+1} \\ &\quad - \sum_{i=1}^{m} \varphi(h_1 + \cdots + h_i)(\alpha_i - \alpha_i(h_{m+1})\alpha_{m+1}). \end{aligned}$$

Now $\alpha_i(h_{m+1}) = \tilde{A}_{m+1,i} = -1$ if $i = m$ and 0 otherwise. Thus,

$$\begin{aligned} r_{m+1} r_m \cdots r_2 r_1 \varphi &= \varphi - \sum_{i=1}^{m} \varphi(h_1 + \cdots + h_i)\alpha_i \\ &\quad - (\varphi(h_1 + \cdots + h_m) + \varphi(h_{m+1}))\alpha_{m+1} . \end{aligned}$$

Now just set $m = l - 1$, and simplify.

Next we show that

$$\begin{aligned} & r_{l-m} r_{l-m+1} \cdots r_l r_{l-1} \cdots r_2 r_1 \varphi \\ &\quad = \varphi - \sum_{i=1}^{l-1-m} \varphi(h_1 + \cdots + h_i)\alpha_i - \varphi(h_1 + \cdots + h_l) \\ &\qquad \times (\alpha_{l-m} + \alpha_{l-m+1} + \cdots + \alpha_l). \end{aligned} \tag{4.15}$$

Note that we have shown (4.15) for $m = 0$ and that we must have $l - 1 \geqslant m \geqslant 0$. We first carry out the inductive step for $2 \leqslant l - m - 1 \leqslant l - 1$ and then, as a special case do $1 = l - m - 1$. We suppose that (4.15) holds and compute

$$\begin{aligned} & r_{l-m-1}(r_{l-m} r_{l-m+1} \cdots r_l r_{l-1} \cdots r_2 r_1 \varphi) \\ &= \varphi - \varphi(h_{l-m-1})\alpha_{l-m-1} - \sum_{i=1}^{l-1-m} \varphi(h_1 + \cdots + h_i)\alpha_i \\ &\quad + \sum_{i=1}^{l-1-m} \varphi(h_1 + \cdots + h_i)\, \alpha_i(h_{l-m-1})\alpha_{l-m-1} \\ &\quad - \varphi(h_1 + \cdots + h_l)(\alpha_{l-m} + \alpha_{l-m+1} + \cdots + \alpha_l) \\ &\quad + \varphi(h_1 + \cdots + h_l)(\alpha_{l-m} + \alpha_{l-m+1} + \cdots + \alpha_l)(h_{l-m-1})\alpha_{l-m-1}. \end{aligned} \tag{4.16}$$

Note that $\sum_{i=1}^{l-1-m} \varphi(h_1 + \cdots + h_i)\alpha_i(h_{l-m-1})\alpha_{l-m-1}$ equals $(-\varphi(h_1 + \cdots + h_{l-m-2}) + 2\varphi(h_1 + \cdots + h_{l-m-1}))\alpha_{l-m-1}$ if $l - m - 1 > 2$ and $2\varphi(h_{l-m-1})\alpha_{l-m-1}$ if $l - m - 1 = 1$. Also, $(\alpha_{l-m} + \alpha_{l-m+1} + \cdots + \alpha_l)\ (h_{l-m-1})\alpha_{l-m-1}$ equals $(-1)\alpha_{l-m-1}$ if $l - m - 1 \geqslant 1$.

Substituting the above relations into (4.16) when $l - m - 1 \geqslant 2$ we obtain

$$\begin{aligned} & r_{l-(m+1)}(r_{l-m} r_{l-m+1} \cdots r_l r_{l-1} \cdots r_2 r_1 \varphi) \\ &= \left\{\varphi - \sum_{i=1}^{l-1-(m+1)} \varphi(h_1 + \cdots + h_i)\alpha_i - \varphi(h_1 + \cdots + h_l)(\alpha_{l-(m+1)} + \cdots + \alpha_l)\right\} \\ &\quad - \varphi(h_1 + \cdots + h_{l-m-1})\alpha_{l-m-1} - \varphi(h_{l-m-1})\alpha_{l-m-1} \\ &\quad - \varphi(h_1 + \cdots + h_{l-m-2})\alpha_{l-m-1} + 2\varphi(h_1 + \cdots + h_{l-m-1})\alpha_{l-m-1}. \end{aligned}$$

But this is just (4.15) with m replaced by $m + 1$.

When $l - m - 1 = 1$, (4.16) becomes

$$\begin{aligned} & r_{l-(m+1)}(r_{l-m} r_{l-m+1} \cdots r_l r_{l-1} \cdots r_2 r_1 \varphi) \\ &= r_1 r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2 r_1 \varphi \\ &= \left\{\varphi - \sum_{i=1}^{l-1-(m+1)} \varphi(h_1 + \cdots + h_i)\alpha_i - \varphi(h_1 + \cdots + h_l)(\alpha_{l-(m+1)} + \cdots + \alpha_l)\right\} \\ &\quad - \varphi(h_1 + \cdots + h_{l-(m+1)}) - \varphi(h_{l-(m+1)})\alpha_{l-(m+1)} + 2\varphi(h_1)\alpha_{l-(m+1)} \\ &= \varphi - \varphi(h_1 + \cdots + h_l)(\alpha_1 + \alpha_2 + \cdots + \alpha_l). \end{aligned}$$

Thus if $\varphi \in R$, we have

$$\begin{aligned} & r_1 r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2 r_1 \varphi \\ &\quad = \varphi - \varphi(h_1 + \cdots + h_l)(\alpha_1 + \cdots + \alpha_l) = \varphi - \varphi(h_1 + \cdots + h_l)\psi. \end{aligned} \tag{4.17}$$

By definition $r_0\varphi = \varphi - \varphi(h_0)\alpha_0$. Hence, by (4.17) we obtain

$$r_1 r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2 r_1 r_0 \varphi$$
$$= \varphi - \varphi(h_1 + \cdots + h_l)\psi - \varphi(h_0)(\alpha_0 - \alpha_0(h_1 + \cdots + h_l)\psi).$$

However, $\alpha_0(h_1 + \cdots + h_l) = \tilde{A}_{10} + \tilde{A}_{20} + \cdots + \tilde{A}_{l0} = -1 + 0 + \cdots + -1 = -2$. Thus we have $\varphi - \varphi(h_1 + \cdots + h_l)\psi - \varphi(h_0)(\alpha_0 + 2\psi)$. Recall that $\gamma = \alpha_0 + \psi$. Substituting γ gives us

$$\varphi - \varphi(h_1 + \cdots + h_l)\psi - \varphi(h_0)(\gamma + \psi) + \varphi(h_1 + \cdots + h_l)\gamma - \varphi(h_1 + \cdots + h_l)\gamma$$
$$= \varphi + \varphi(h_1 + \cdots + h_l)\gamma - \varphi(h_0 + h_1 + \cdots + h_l)(\gamma + \psi) = \varphi + \varphi(h_1 + \cdots + h_l)\gamma,$$

since $\varphi(h_0 + h_1 + \cdots + h_l)$ equals 0. Q.E.D.

Hereafter we define the shear s_0 by the identity $s_0 = r_1 r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2 r_1 r_0$, so that

$$s_0\varphi = r_1 r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2 r_1 r_0 \varphi = \varphi + \varphi(h_1 + \cdots + h_l)\gamma. \tag{4.18}$$

Remark. Let r_ψ be the reflection with respect to ψ in the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$. Then $s_0 = r_\psi r_0$.

It is known (cf. [14]) and we shall show below, that the Weyl group W of $A_l^{(1)}$ is isomorphic to the semidirect product of the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$ (the symmetric group on n letters $\mathscr{S}_n$) with a free Abelian group on l generators. This latter group is known as the group of translations or shears of $A_l^{(1)}$. In order to study W we first describe the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$ as in [10]. Thinking in terms of the expressions $\epsilon_i - \epsilon_j$, notice that the reflection r_i with respect to α_i permutes the subscripts i, $i + 1$ and leaves all other subscripts fixed. Thus, r_i corresponds to the transposition $(i, i + 1)$ in the symmetric group $\mathscr{S}_n$; these transpositions generate $\mathscr{S}_n$, so we may obtain a natural isomorphism of the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$ onto $\mathscr{S}_n$. In fact, if $\sigma \in \mathscr{S}_n$, then σ acts on the root $\epsilon_i - \epsilon_j$ as follows:

$$\sigma(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)}. \tag{4.19}$$

Since $\sigma(\gamma) = \gamma$, (4.19) determines the action of $\mathscr{S}_n$ on the roots φ of $\mathfrak{sl}(n, \mathbb{C})$, and hence on the roots of $A_l^{(1)}$.

Consider the permutation $p_0 = (1\ 2 \cdots n) \in \mathscr{S}_n$. Note that by (4.19),

$$p_0(\epsilon_i - \epsilon_{i+1}) = \epsilon_{i+1} - \epsilon_{i+2}, \quad \text{if } 1 \leqslant i < n - 1,$$
$$= -((\epsilon_1 - \epsilon_2) + (\epsilon_2 - \epsilon_3) + \cdots + (\epsilon_{n-1} - \epsilon_n)), \quad \text{if } i = n - 1.$$

That is,

$$p_0(\alpha_i) = \alpha_{i+1}, \quad \text{if } 1 \leqslant i < l,$$
$$= -(\alpha_1 + \cdots + \alpha_l), \quad \text{if } i = l. \tag{4.20}$$

Iterating (4.20), we immediately have that

$$\begin{aligned} p_0^k(\alpha_i) &= \alpha_{i+k}, && \text{if } 1 \leqslant i \leqslant l-k, \\ &= -(\alpha_1 + \cdots + \alpha_l), && \text{if } i = l-k+1, \\ &= \alpha_{i-(l-k+1)}, && \text{if } (l-k+2) \leqslant i \leqslant l. \end{aligned} \tag{4.21}$$

We get different sets of values provided that $1 \leqslant k \leqslant n$. Note that $p_0^n(\alpha_i) = \alpha_i$, and so $p_0^n = 1$.

It makes sense to conjugate the shear s_0 defined in (4.18) by the permutation p_0. We obtain

$$p_0 s_0 p_0^{-1} \varphi = \varphi + p_0^{-1} \varphi(h_1 + \cdots + h_l)\gamma. \tag{4.22}$$

In exactly the same way as in (4.22), we may write

$$p_0^k s_0 p_0^{-k} \varphi = \varphi + p_0^{-k} \varphi(h_1 + \cdots + h_l)\gamma. \tag{4.23}$$

For all $k = 1,..., l$, we define the shear

$$s_k = p_0^k s_0 p_0^{-k}, \tag{4.24}$$

and we let Λ be the subgroup of W generated by $s_1,..., s_l$. We shall show that W is the semidirect product of $\mathscr{S}_n$ with the normal subgroup Λ, and that Λ is the free Abelian group on the basis $\{s_1,..., s_l\}$. We shall also show how the shears s_k act on the roots of $\mathfrak{sl}(n, \mathbb{C})$. First we prove:

LEMMA 4.25. *W is generated by its subgroups $\mathscr{S}_n$ and Λ.*

Proof. Since W is generated by $r_0,..., r_l$, it suffices to show that r_0 lies in the subgroup generated by $\mathscr{S}_n$ and Λ. But s_0 lies in this subgroup, and hence so does r_0. Q.E.D.

The elements s_k can be conveniently rewritten:

LEMMA 4.26. *For all $i = 0,..., l$ and $\varphi \in R$, we have*

$$s_i \varphi = \varphi - \varphi(h_i)\gamma. \tag{4.27}$$

Moreover, for all $i, j = 0,..., l$,

$$s_i \alpha_j = \alpha_j - \tilde{A}_{ij}\gamma. \tag{4.28}$$

Proof. For $i = 0$, (4.28) follows from (4.18). For $i, j > 0$, (4.28) is a consequence of (4.22) and (4.23), together with (4.20) and (4.21). Finally, to prove (4.28) for $j = 0$, add (4.28) for $j = 0,..., n$ and use the fact that $s_i\gamma = \gamma$. Formula (4.27) follows immediately from (4.28), by linearity. Q.E.D.

LEMMA 4.29. *Λ is a normal Abelian subgroup of W.*

Proof. The commutativity is clear, since shears commute. All we need to show is that $r_k s_i r_k \in \Lambda$ for $i, k = 1,..., l$. But by (4.28), for $j = 0,..., l$,

$$\begin{aligned} r_k s_i r_k \alpha_j &= r_k s_i(\alpha_j - \alpha_j(h_k)\alpha_k) \\ &= r_k\alpha_j - \tilde{A}_{ij}\gamma - \alpha_j(h_k)\, r_k(\alpha_k - \tilde{A}_{ik}\gamma) \\ &= \alpha_j - \alpha_j(h_k)\,\alpha_k + \alpha_j(h_k)\alpha_k - (\tilde{A}_{ij} - \tilde{A}_{ik}\alpha_j(h_k))\gamma \\ &= \alpha_j - \alpha_j(h_i - \tilde{A}_{ik}h_k)\gamma. \end{aligned}$$

Hence

$$r_k s_i r_k \varphi = \varphi - \varphi(h_i - \tilde{A}_{ik}h_k)\gamma$$

for all $\varphi \in R$, and so by (4.27), $r_k s_i r_k$ is the product of s_i with an integral power of s_k. Q.E.D.

We now have:

PROPOSITION 4.30. *W is the semidirect product of $\mathscr{S}_n$ with the normal subgroup Λ, which is precisely the set of shears in W. Moreover, Λ is the free Abelian group generated by $s_1 ,..., s_l$.*

Proof. Since no nontrivial element of $\mathscr{S}_n$ can be a shear, W is the semidirect product of $\mathscr{S}_n$ with Λ, and Λ coincides with the set of shears in W. All that remains to be shown is the freeness of Λ. Suppose then that for some integers $c_1 ,..., c_l$, we have

$$s_1^{c_1} s_2^{c_2} \cdots s_l^{c_l} = 1.$$

Then by (4.27),

$$\varphi\left(\sum_{i=1}^{l} c_i h_i\right) = 0$$

for all $\varphi \in R$, and in particular, for $\varphi = \alpha_1 ,..., \alpha_l$. Thus

$$\sum_{i=1}^{l} c_i A_{ij} = 0$$

for each $j = 1,..., l$. But it is well known that the Cartan matrix A of $\mathfrak{sl}(n, \mathbb{C})$ is nonsingular, and in fact that it has determinant n. Thus each $c_i = 0$. Q.E.D.

We next want to characterize concretely the shears in W. For a shear L and $j = 1,..., l$, we have

$$L\alpha_j = \alpha_j + L_j\gamma, \tag{4.31}$$

where L_j is a scalar. We shall find necessary and sufficient conditions on the L_j in order for L to be in W. From (4.28), we see that the condition for a shear to be a product of powers of $s_1, \ldots, s_l$ is that the l-tuple $(L_1, \ldots, L_l)$ be an integral linear combination of the l rows of the Cartan matrix A of $\mathfrak{sl}(n, \mathbb{C})$. But using a suitable sequence of elementary row operations (in fact, the successive adding of an integral multiple of one row to another row), we can transform A to the matrix

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 2 \\ 0 & 0 & \cdots & 0 & 3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & n-2 \\ 0 & 0 & \cdots & 0 & n \end{pmatrix}.$$

By taking arbitrary integral linear combinations of the rows of B, we conclude:

PROPOSITION 4.32. *The shear L described by* (4.31) *lies in W* (*or equivalently, in Λ*), *if and only if* $L_1, \ldots, L_l \in \mathbb{Z}$ *and n divides* $L_l - \sum_{i=1}^{l-1} iL_i$.

Recall that $\Delta_- = -\Delta$. By Propositions 4.30 and 4.32 and (4.9), we have an algorithm for computing Φ_w for $w \in W$:

PROPOSITION 4.33. *The most general element $w \in W$ may be written uniquely in the form $w = \sigma L$, where $\sigma \in \mathcal{S}_n$ and L is a shear of the type described in Proposition 4.32. The sign $(-1)^{l(w)}$ equals the sign of the permutation σ. The set $\Phi_w = w\Delta_- \cap \Delta_+$ is now constructible by the indicated sequence of steps.*

Remarks. (1) Each element of each set Φ_w is expressed as a sum of the roots γ, $\alpha_1, \ldots, \alpha_l$. By means of the formula $\gamma = \alpha_0 + \cdots + \alpha_l$, we may write each element as a sum of the simple roots $\alpha_0, \ldots, \alpha_l$.

(2) Note that Φ_w is a finite union of finite "columns" either of the form $\{\beta, \beta + \gamma, \beta + 2\gamma, \ldots, \beta + m\gamma\}$, where $\beta \in \Delta_0 \cap \Delta_+$ and $m \in Z_+$, or else of the form $\{\beta + \gamma, \beta + 2\gamma, \ldots, \beta + m\gamma\}$, where $\beta \in \Delta_0 \cap \Delta_-$ and $m \in Z_+ - \{0\}$. These "columns" are the generalizations of the sets such as $\{\alpha_1 + i\gamma \mid 0 \leqslant i \leqslant 2m - 1\}$ which occurred in the case $A_1^{(1)}$ in Lemma 3.2. Thus the "reason" for the quadratic character of the exponents of the variables e(-simple roots) in the monomials in the summation side of the denominator formula (see the discussion below) is similar to the corresponding "reason" in the case $A_1^{(1)}$ (recall the beginning of Section 4).

We shall now describe how results in [12b] can be used to write concrete quadratic expressions for the exponents of the variables u_i in the right-hand side of the denominator formula (4.2). This process works in the generality of all affine, and even Euclidean, GCM Lie algebras. First, one applies Proposition 13.13 of [12b] to the sequence $(s_0, s_1, \ldots, s_l)$ for which a certain $s_i = 1$ and all other components are 0. This gives a formula for $\xi_i(\langle \Phi_w \rangle)$. We next use the fact that W, which may be identified with the "affine Weyl group," is the semidirect product of a finite Weyl group with a lattice of translations. (For $A_l^{(1)}$, these two subgroups are identified with $\mathscr{S}_n$ and the shear group Λ, respectively.) Just as in Proposition 16.8 of [12b] and the discussion preceding this proposition, the right-hand side in (4.2) may be written as a double sum over the finite Weyl group and over the lattice. The exponents of the u_i are quadratic functions on the lattice.

We invite the reader to use this to write down a concrete multivariable vector partition theorem for each Euclidean GCM Lie algebra. Here is the result in the special case $A_2^{(1)}$:

Denominator Formula for $A_2^{(1)}$

$$
\begin{aligned}
&\prod_{s \geq 1} (1 - u_0^s u_1^s u_2^s)^2 (1 - u_0^s u_1^{s-1} u_2^{s-1})(1 - u_0^{s-1} u_1^s u_2^{s-1})(1 - u_0^{s-1} u_1^{s-1} u_2^s) \\
&\qquad \times (1 - u_0^{s-1} u_1^s u_2^s)(1 - u_0^s u_1^{s-1} u_2^s)(1 - u_0^s u_1^s u_2^{s-1}) \\
&= \sum_{\substack{r_j \equiv 0 \\ (\mathrm{mod}\ 3)}} u_0^{\frac{1}{6}(r_1(r_1-2)+r_2^2+r_3(r_3+2))} u_1^{\frac{1}{6}(r_1^2+r_2(r_2+2)+r_3(r_3-2))} u_2^{\frac{1}{6}(r_1(r_1+2)+r_2(r_2-2)+r_3^2)} \\
&\quad - \sum_{\substack{r_1 \equiv 1 \\ r_2 \equiv 0 \\ r_3 \equiv 2 \\ (\mathrm{mod}\ 3)}} u_0^{\frac{1}{6}(r_1(r_1+2)+r_2^2+r_3(r_3-2))} u_1^{\frac{1}{6}(r_1(r_1-2)+r_2(r_2+2)+r_3^2)} u_2^{\frac{1}{6}(r_1^2+r_2(r_2-2)+r_3(r_3+2))} \\
&\quad + \sum_{\substack{r_j \equiv 1 \\ (\mathrm{mod}\ 3)}} u_0^{\frac{1}{6}(r_1(r_1+2)+r_2(r_2-2)+r_3^2)} u_1^{\frac{1}{6}(r_1(r_1-2)+r_2^2+r_3(r_3+2))} u_2^{\frac{1}{6}(r_1^2+r_2(r_2+2)+r_3(r_3-2))} \\
&\quad - \sum_{\substack{r_1 \equiv 0 \\ r_2 \equiv 2 \\ r_3 \equiv 1 \\ (\mathrm{mod}\ 3)}} u_0^{\frac{1}{6}(r_1(r_1-2)+r_2(r_2+2)+r_3^2)} u_1^{\frac{1}{6}(r_1^2+r_2(r_2-2)+r_3(r_3+2))} u_2^{\frac{1}{6}(r_1(r_1+2)+r_2^2+r_3(r_3-2))} \\
&\quad + \sum_{\substack{r_j \equiv 2 \\ (\mathrm{mod}\ 3)}} u_0^{\frac{1}{6}(r_1^2+r_2(r_2+2)+r_3(r_3-2))} u_1^{\frac{1}{6}(r_1(r_1+2)+r_2(r_2-2)+r_3^2)} u_2^{\frac{1}{6}(r_1(r_1-2)+r_2^2+r_3(r_3+2))} \\
&\quad - \sum_{\substack{r_1 \equiv 2 \\ r_2 \equiv 1 \\ r_3 \equiv 0 \\ (\mathrm{mod}\ 3)}} u_0^{\frac{1}{6}(r_1^2+r_2(r_2-2)+r_3(r_3+2))} u_1^{\frac{1}{6}(r_1(r_1+2)+r_2^2+r_3(r_3-2))} u_2^{\frac{1}{6}(r_1(r_1-2)+r_2(r_2+2)+r_3^2)}.
\end{aligned}
\tag{4.34}
$$

Each sum is over $r_1, r_2, r_3 \in \mathbb{Z}$, with $r_1 + r_2 + r_3 = 0$.

5. The Numerator Formula for $A_1^{(1)}$

In this section, we work in the setting of $A_1^{(1)}$, and we use the notation of the first half of Section 3. We are ready to state:

THEOREM 5.1. (numerator formula). *Let λ be a dominant integral linear form and let V be the standard module for $A_1^{(1)}$ with highest weight λ. Then when we set $e(-\alpha_0) = e(-\alpha_1) = q$, the numerator of $\chi(V)/e(\lambda)$ factors into an infinite product:*

$$\chi(V) \sum_{w\in W} (-1)^{l(w)} e(w\rho - \rho)/e(\lambda) \,|_{e(-\alpha_0)=e(-\alpha_1)=q}$$

$$= \prod_{n=1}^{\infty} (1 - q^{(n_0+n_1)n})(1 - q^{(n_0+n_1)n-n_0})(1 - q^{(n_0+n_1)n-n_1})$$

$$= \sum_{w\in W} (-1)^{l(w)} e(w\rho - \rho) \,|_{e(-\alpha_i)=q^{n_i}}, \tag{5.2}$$

where $n_i = (\lambda + \rho)(h_i)$ $(i = 0, 1)$.

Proof. By adding to λ a multiple of γ if necessary, we may assume that $\lambda = a\alpha_0 + c\rho$, where $c \pm 2a \in \mathbb{Z}_+$. Note that $n_0 = c + 2a + 1$ and $n_1 = c - 2a + 1$. Setting $e(-\alpha_0) = e(-\alpha_1) = q$ in Corollary 3.16 we find after some algebraic simplification that

$$\chi(V) \sum_{w\in W} (-1)^{l(w)} e(w\rho - \rho)/e(\lambda) \,|_{e(-\alpha_0)=e(-\alpha_1)=q} \tag{5.3}$$

$$= \sum_{n\in\mathbb{Z}} q^{4(c+1)n^2+4an} - \sum_{n\in\mathbb{Z}} q^{4(c+1)n^2+4(a+c+1)n+2a+c+1} \tag{5.4}$$

Next, observe that if we set $e(-\alpha_0) = q^{m_0}$ and $e(-\alpha_1) = q^{m_0}$ $(m_0, m_1 \in \mathbb{Z}_+ - \{0\})$ in the denominator formula for $A_1^{(1)}$ given in Corollary 3.9 we have

$$\sum_{n\in\mathbb{Z}} q^{2(m_0+m_1)n^2+(m_1-m_0)n} - \sum_{n\in\mathbb{Z}} q^{2(m_0+m_1)n^2+(3m_0+m_1)n+m_0} \tag{5.5}$$

$$= \prod_{n=1}^{\infty} (1 - q^{(m_0+m_1)n})(1 - q^{(m_0+m_1)n-m_0})(1 - q^{(m_0+m_1)n-m_1}). \tag{5.6}$$

Replacing n by $-n$ leaves the first sum in (5.5) unchanged as the summation is over $\mathbb{Z}$. We may then rewrite the expression in (5.5) as

$$\sum_{n\in\mathbb{Z}} q^{2(m_0+m_1)n^2+(m_0-m_1)n} - \sum_{n\in\mathbb{Z}} q^{2(m_0+m_1)n^2+(3m_0+m_1)n+m_0}. \tag{5.7}$$

In order to factor the expression in (5.3) into an infinite product, we determine when (5.4) and (5.7) are equal. These two combinations of sums are equal if and only if the following relations hold:

$$4(c+1) = 2(m_0+m_1), \qquad 4a = m_0 - m_1, \qquad 4(a+c+1) = 3m_0 + m_1,$$

and $2a + c + 1 = m_0$.

But these relations hold if and only if

$$\begin{aligned} m_0 &= c + 2a + 1 \in \mathbb{Z}_+, \\ m_1 &= c - 2a + 1 \in \mathbb{Z}_+. \end{aligned} \tag{5.8}$$

Thus, (5.3) can always be factored into the infinite product given in (5.6) with m_0 and m_1 determined by (5.8). Q.E.D.

As a direct consequence of Theorem 5.1 we may now state:

THEOREM 5.9. *Let λ be a dominant integral linear form and let V be the standard module for $A_1^{(1)}$ with highest weight λ. Then, when we set $e(-\alpha_0) = e(-\alpha_1) = q$ in $\chi(V)/e(\lambda)$, we obtain*

$$\prod_{n=1}^{\infty} (1 - q^{2n-1})\, \chi(V)/e(\lambda) \,|_{e(-\alpha_0)=e(-\alpha_1)=q} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(\lambda+\rho)(h_0) (\mathrm{mod}\, (\lambda+\rho)(h_0+h_1))}}^{\infty} (1-q^n)^{-1}. \tag{5.10}$$

If $(\lambda+\rho)(h_0) = (\lambda+\rho)(h_1)$, we make the obvious modifications; see (5.20).

Proof. First we observe by Corollary 3.9 that

$$\sum_{w \in W} (-1)^{l(w)} e(w\rho - \rho) \,|_{e(-\alpha_0)=e(-\alpha_1)=q} = \prod_{n=1}^{\infty} (1 - q^{2n-1}) \prod_{n=1}^{\infty} (1 - q^n). \tag{5.11}$$

Dividing both sides of (5.2) by (5.11) we immediately have

$$\chi(V)/e(\lambda) \,|_{e(-\alpha_0)=e(-\alpha_1)=q} = \prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1} \prod_{\substack{n=1 \\ n \equiv 0, (\lambda+\rho)(h_0), (\lambda+\rho)(h_1) (\mathrm{mod}\, (\lambda+\rho)(h_0+h_1))}}^{\infty} (1-q^n) \Big/ \prod_{n=1}^{\infty} (1-q^n),$$

from which the result follows by algebraic simplification of the right-hand side. Q.E.D.

We close this section by recovering Theorem 1.11 as well as a similar theorem for the generalizations and analogs of the Rogers–Ramanujan identities due to Gordon, Göllnitz and Gordon, and Andrews.

We first study a generalization of Gordon's generalized Rogers–Ramanujan identities due to Andrews [1b, p. 111]. These identities assert that: For $1 \leqslant i \leqslant k$, $k \geqslant 2$, $|q| < 1$, and $(q)_n = \prod_{l=1}^{n} (1 - q^l)$,

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geqslant 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q)_{n_1}(q)_{n_2} \cdots (q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i \pmod{2k+1}}}^{\infty} (1 - q^n)^{-1}, \tag{5.12}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Note that for $k = 2$ and $1 \leqslant i \leqslant 2$ we obtain from (5.12) the original pair of Rogers–Ramanujan identities.

Now in Theorem 5.9, set $i = (\lambda + \rho)(h_0)$ and $2k + 1 = (\lambda + \rho)(h_0 + h_1)$. This immediately implies that

$$\begin{aligned} \lambda(h_0) &= i - 1, \\ \lambda(h_1) &= 2k - i. \end{aligned} \tag{5.13}$$

Since λ is dominant integral we must have $1 \leqslant i \leqslant 2k$. In order to recover distinct products, we need only consider $1 \leqslant i \leqslant k$. Theorem 5.9 combined with (5.13) immediately gives:

THEOREM 5.14. *Let λ be any dominant integral linear form such that $\lambda(h_0) = i - 1$, $\lambda(h_1) = 2k - i$ with $1 \leqslant i \leqslant k$, and let V be the standard module for $A_1^{(1)}$ with highest weight λ. Then,*

$$\prod_{n=1}^{\infty} (1 - q^{2n-1})\, \chi(V)/e(\lambda)\, |_{e(-\alpha_0)=e(-\alpha_1)=q} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i \pmod{2k+1}}}^{\infty} (1 - q^n)^{-1}. \tag{5.15}$$

Remark. When $k = 1$, (5.15) becomes simply $\prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1} = \prod_{n=1}^{\infty} (1 + q^n)$. That is, $\prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1}$ is itself the principally specialized character for the standard module corresponding to the dominant integral linear form λ such that either $\lambda(h_0) = 0$ and $\lambda(h_1) = 1$, or $\lambda(h_0) = 1$ and $\lambda(h_1) = 0$. We have now established:

THEOREM 5.16. *After multiplication by* (1.12), *the product sides of the generalized Rogers–Ramanujan identities in* (5.12) *become the principally specialized characters for the irreducible modules for* $A_1^{(1)}$ *corresponding to the dominant integral linear forms* λ *such that* $\lambda(h_0) = i - 1$, *and* $\lambda(h_1) = 2k - i$, *where* $2 \leqslant k$ *and* $1 \leqslant i \leqslant k$. *(Concretely,* λ *may be taken to be*

$$(i/2 - k/2 - 1/4)\,\alpha_0 + (k - 1/2)\rho.)$$

Furthermore, expression (1.12) *is itself the principally specialized character for the standard module for* $A_1^{(1)}$ *corresponding to* λ *such that either* $\lambda(h_0) = 0$ *and* $\lambda(h_1) = 1$, *or* $\lambda(h_0) = 1$ *and* $\lambda(h_1) = 0$. *(For example, take* $\lambda = -1/4\,\alpha_0 + 1/2\rho$.)

Taking $\lambda(h_0 + h_1) = 3$ in Theorem 5.16 gives Theorem 1.11.

We finally examine the products which occur in Andrews' extension of the Göllnitz–Gordon analog of the Rogers–Ramanujan identities which appear in [1b, p. 115–116].

When written in the same form as (5.12) the product side of these identities is

$$\prod_{\substack{n=1\\ n\not\equiv 0,\pm(2i-1)\pmod{4k}}}^{\infty} (1 - q^n)^{-1}, \tag{5.17}$$

with $1 \leqslant i \leqslant k$.

For example, when $i = k$ the identity is

$$\prod_{n\not\equiv 2 \pmod 4} (1 - q^n) \sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{(-q;q^2)_{N_1} q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{k-1}}}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0,\pm(2k-1)\pmod{4k}}}^{\infty} (1 - q^n)^{-1},$$

where $(a;q)_n = (1 - a)(1 - aq)\cdots(1 - aq^{n-1})$.

In exactly the same way that we proved Theorem 5.14 we obtain:

THEOREM 5.18. *Let* λ *be any dominant integral linear form such that* $\lambda(h_0) = i - 1$, *and* $\lambda(h_1) = 2k - i - 1$ *with* $1 \leqslant i \leqslant k$, *and let* V *be the standard module for* $A_1^{(1)}$ *with highest weight* λ. *(Concretely,* λ *may be taken to be* $(i/2 - k/2)\alpha_0 + (k - 1)\rho$.) *Then,*

$$\prod_{n=1}^{\infty} (1 - q^{2n-1})\, \chi(V)/e(\lambda)\,|_{e(-\alpha_0)=e(-\alpha_1)=q}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0,\pm i\pmod{2k}}}^{\infty} (1 - q^n)^{-1}, \qquad \textit{if}\quad 1 \leqslant i \leqslant k - 1, \tag{5.19}$$

and

$$= \prod_{\substack{n=1 \\ n\equiv k (\mathrm{mod}\ 2k)}}^{\infty} (1-q^n) \Big/ \prod_{\substack{n=1 \\ n\not\equiv 0,k (\mathrm{mod}\ 2k)}}^{\infty} (1-q^n), \quad \textit{if} \quad i = k. \tag{5.20}$$

By replacing k by $2k$ and i by $2i - 1$ in (5.19) we obtain the products in (5.17). Thus we also have a theorem analogous to Theorem 5.16 for the Göllnitz–Gordon–Andrews identities given in [1b, pp. 115–116]. These generalizations of the Rogers–Ramanujan identities correspond to only some of the standard modules. Recently, identities of Rogers–Ramanujan type which correspond to the rest of the standard modules in (5.19), as well as (5.20), have been obtained in [2]. Thus, there is a known partition identity of Rogers–Ramanujan type corresponding to every dominant integral linear form for $A_1^{(1)}$.

In contrast to the infinite families of identities associated with $A_1^{(1)}$, only 21 of the dominant integral linear forms for $A_2^{(2)}$ correspond as in Theorem 5.16 to known partition identities. We discuss this situation in Section 6.

6. The Numerator Formula for $A_2^{(2)}$

Just as in Section 5, we factor the numerator of a principally specialized character by using the denominator formula, this time for $A_2^{(2)}$. We use the notation of the second half of Section 3. We prove:

Theorem 6.1. (numerator formula). *Let λ be a dominant integral linear form and let V be the standard module for $A_2^{(2)}$ with highest weight λ. Then we have*

$$\chi(V) \sum_{w\in W} (-1)^{l(w)}\, e(w\rho - \rho)/e(\lambda)\, |_{e(-\alpha_0)=e(-\alpha_1)=q}$$

$$= \prod_{n=1}^{\infty} (1 - q^{2(n_1+2n_0)n-(n_1+4n_0)})(1 - q^{2(n_1+2n_0)n-n_1})$$

$$\times (1 - q^{(n_1+2n_0)n})(1 - q^{(n_1+2n_0)n-(n_1+n_0)})$$

$$\times (1 - q^{(n_1+2n_0)n-n_0})$$

$$= \sum_{w\in W} (-1)^{l(w)} e(w\rho - \rho)\, |_{e(-\alpha_i)=q^{n_{1-i}}}, \tag{6.2}$$

where $n_i = (\lambda + \rho)(h_i)$ $(i = 0, 1)$.

Proof. Just as in the proof of Theorem 5.1 assume $\lambda = a\alpha_0 + c\rho$ where $(c + 2a,\ c - 4a \in \mathbb{Z}_+)$. Note that $n_0 = c + 2a + 1$ and $n_1 = c - 4a + 1$. Setting

$e(-\alpha_0) = e(-\alpha_1) = q$ in Corollary 3.37 we find after algebraic simplification that

$$\chi(V) \sum_{w \in W} (-1)^{l(w)} e(w\rho - \rho)/e(\lambda) \,|_{e(-\alpha_0)=e(-\alpha_1)=q} \tag{6.3}$$

$$= \sum_{n \in \mathbb{Z}} q^{\frac{9}{2}(c+1)n^2 + (6a+3c/2+3/2)n}$$

$$- \sum_{n \in \mathbb{Z}} q^{\frac{9}{2}(c+1)n^2 + (6a+9c/2+9/2)n+c+2a+1}. \tag{6.4}$$

Next, observe that if we set $e(-\alpha_0) = q^{m_1}$ and $e(-\alpha_1) = q^{m_0}$ ($m_1, m_0 \in \mathbb{Z}_+ - \{0\}$) in the denominator formula for $A_2^{(2)}$ given in Corollary 3.32 we have

$$\sum_{n \in \mathbb{Z}} q^{3(m_1/2+m_0)n^2 + (2m_0 - m_1/2)n}$$

$$- \sum_{n \in \mathbb{Z}} q^{3(m_1/2+m_0)n^2 + (2m_0+5m_1/2)n + m_1}$$

$$= \prod_{n=1}^{\infty} (1 - q^{2(m_1+2m_0)n - (m_1+4m_0)})(1 - q^{2(m_1+2m_0)n - m_1}) \tag{6.5}$$

$$\times (1 - q^{(m_1+2m_0)n})(1 - q^{(m_1+2m_0)n - (m_1+m_0)})(1 - q^{(m_1+2m_0)n - m_0}).$$

Replacing n by $-(n+1)$ leaves the second sum in (6.5) unchanged as the summation is over $\mathbb{Z}$. We may then rewrite the expression in (6.5) as

$$\sum_{n \in \mathbb{Z}} q^{3(m_1/2+m_0)n^2 + (2m_0 - m_1/2)n} - \sum_{n \in \mathbb{Z}} q^{3(m_1/2+m_0)n^2 + (4m_0+m_1/2)n + m_0}. \tag{6.6}$$

In order to factor the expression in (6.3) into an infinite product, we determine when (6.4) and (6.6) are equal. These two combinations of sums are equal if and only if the following relations hold:

$$9(c+1)/2 = 3(m_1/2 + m_0),$$
$$(6a + 3c/2 + 3/2) = 2m_0 - m_1/2,$$
$$6a + 9c/2 + 9/2 = 4m_0 + m_1/2,$$

and

$$c + 2a + 1 = m_0.$$

But these relations hold if and only if

$$\begin{aligned} m_1 &= c - 4a + 1 \in \mathbb{Z}_+, \\ m_0 &= c + 2a + 1 \in \mathbb{Z}_+. \end{aligned} \tag{6.7}$$

Thus, (6.3) can always be factored into the infinite product given in (6.5) with m_0 and m_1 determined by (6.7). Q.E.D.

In the same way that we proved Theorem 5.9 from the numerator formula for $A_1^{(1)}$, we establish:

THEOREM 6.8. *Let λ be a dominant integral linear form and let V be the standard module for $A_2^{(2)}$ with highest weight λ. Then when we set $e(-\alpha_0) = e(-\alpha_1) = q$ in $\chi(V)/e(\lambda)$, we obtain*

$$\prod_{\substack{n=1 \\ n\equiv\pm1 (\mathrm{mod}\ 6)}}^{\infty} (1-q^n)^{-1} \prod_{\substack{n=1 \\ n\not\equiv 0, (\lambda+\rho)(2h_0+h_1), \pm(\lambda+\rho)(h_1), \\ \pm(\lambda+\rho)(h_0), \pm(\lambda+\rho)(h_0+h_1) \\ (\mathrm{mod} (\lambda+\rho)(4h_0+2h_1))}}^{\infty} (1-q^n)^{-1}. \tag{6.9}$$

If $(\lambda+\rho)(h_0) = (\lambda+\rho)(h_1)$, we make the obvious modification.

Proof. First, note by Corollary 3.32 that

$$\sum_{w\in W} (-1)^{l(w)} e(w\rho-\rho) |_{e(-\alpha_0)=e(-\alpha_1)=q}$$

$$= \prod_{n=1}^{\infty} (1-q^{6n-5})(1-q^{6n-1}) \prod_{n=1}^{\infty} (1-q^n). \tag{6.10}$$

Dividing both sides of (6.2) by (6.10) we immediately have

$$\chi(V)/e(\lambda) |_{e(-\alpha_0)=e(-\alpha_1)=q}$$

$$= \prod_{\substack{n=1 \\ n\equiv\pm1 (\mathrm{mod}\ 6)}}^{\infty} (1-q^n)^{-1} \prod_{\substack{n=1 \\ n\equiv 0, n_1+2n_0, \pm n_1, \pm n_0, \\ \pm(n_1+n_0) (\mathrm{mod}\ 2n_1+4n_0)}}^{\infty} (1-q^n) \Big/ \prod_{n=1}^{\infty} (1-q^n), \tag{6.11}$$

where $n_i = (\lambda+\rho)(h_i)$, $(i = 0, 1)$.

The result in (6.9) follows by the definition of n_i and algebraic simplification of the right-hand side of (6.11). Q.E.D.

The two identities of Rogers and Slater that are quoted in [1a] but not covered by Andrews' theory are

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n}} = \prod_{n=1; n\not\equiv 0,\pm1,\pm8,\pm9,10 (\mathrm{mod}\ 20)}^{\infty} (1-q^n)^{-1}, \tag{6.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n+1}} = \prod_{n=1; n\not\equiv 0,\pm3,\pm4,\pm7,10 (\mathrm{mod}\ 20)}^{\infty} (1-q^n)^{-1}. \tag{6.13}$$

Connor [5] has provided a combinatorial interpretation of (6.11) and (6.13). Note that if we set $n_0 = 1$ and $n_1 = 2$ in (6.12) we obtain

$$\prod_{\substack{n=1 \\ n\equiv\pm1 (\mathrm{mod}\ 6)}}^{\infty} (1-q^n)^{-1}. \tag{6.14}$$

The products in (6.12) and (6.13) are obtained by setting ($n_0 = (\lambda + \rho)(h_0)$, $n_1 = (\lambda + \rho)(h_1)$) to be (1,8) and (3,4), respectively, in

$$\prod_{\substack{n=1;\, n \not\equiv 0, \pm n_1, \pm n_0, \pm(n_1+n_0) \\ n_1+2n_0 \pmod{2n_1+4n_0}}}^{\infty} (1-q^n)^{-1}, \tag{6.15}$$

which is just the product in (6.11) divided by the product in (6.14).

We have now established:

THEOREM 6.16. *After multiplication by* (6.14), *the product sides of the partition identities in* (6.12) *and* (6.13) *become the principally specialized characters for the irreducible modules for* $A_2^{(2)}$ *corresponding to the dominant integral linear forms*

TABLE I

Identities for $A_2^{(2)}$

Equation number	Divisibility conditions on the product side	(n_0, n_1)	$\lambda = a\alpha_0 + c\rho$ (a, c)
62	$\not\equiv 0, \pm 1, \pm 4, \pm 3, 5 \pmod{10}$	(1, 3)	$(-\frac{1}{3}, \frac{2}{3})$
63	$\not\equiv 0, \pm 1, \pm 2, \pm 3, 5 \pmod{10}$	(2, 1)	$(\frac{1}{6}, \frac{2}{3})$
80	$\not\equiv 0, \pm 2, \pm 3, \pm 5, 7 \pmod{14}$	(2, 3)	$(-\frac{1}{6}, \frac{4}{3})$
81	$\not\equiv 0, \pm 1, \pm 5, \pm 6, 7 \pmod{14}$	(1, 5)	$(-\frac{2}{3}, \frac{4}{3})$
82	$\not\equiv 0, \pm 1, \pm 3, \pm 4, 7 \pmod{14}$	(3, 1)	$(\frac{1}{3}, \frac{4}{3})$
83	$\not\equiv 0, \pm 1, \pm 6, \pm 7, 8 \pmod{16}$	(1, 6)	$(-\frac{5}{6}, \frac{5}{3})$
84	$\not\equiv 0, \pm 2, \pm 4, \pm 6, 8 \pmod{16}$	(2, 4)	$(-\frac{1}{3}, \frac{5}{3})$
86	$\not\equiv 0, \pm 2, \pm 3, \pm 5, 8 \pmod{16}$	(3, 2)	$(\frac{1}{6}, \frac{5}{3})$
94	$\not\equiv 0, \pm 3, \pm 4, \pm 7, 10 \pmod{20}$	(3, 4)	$(-\frac{1}{6}, \frac{7}{3})$
96	$\not\equiv 0, \pm 2, \pm 4, \pm 6, 10 \pmod{20}$	(4, 2)	$(\frac{1}{3}, \frac{7}{3})$
98	$\not\equiv 0, \pm 2, \pm 6, \pm 8, 10 \pmod{20}$	(2, 6)	$(-\frac{2}{3}, \frac{7}{3})$
99	$\not\equiv 0, \pm 1, \pm 8, \pm 9, 10 \pmod{20}$	(1, 8)	$(-\frac{7}{6}, \frac{7}{3})$
107[a]	$\not\equiv 0, \pm 3, \pm 6, \pm 9, 12 \pmod{24}$	(3, 6)	$(-\frac{1}{2}, 3)$
108[a]	$\not\equiv 0, \pm 2, \pm 5, \pm 7, 12 \pmod{24}$	(5, 2)	$(\frac{1}{2}, 3)$
117	$\not\equiv 0, \pm 3, \pm 8, \pm 11, 14 \pmod{28}$	(3, 8)	$(-\frac{5}{6}, \frac{11}{3})$
118	$\not\equiv 0, \pm 1, \pm 12, \pm 13, 14 \pmod{28}$	(1, 12)	$(-\frac{11}{6}, \frac{11}{3})$
119	$\not\equiv 0, \pm 4, \pm 5, \pm 9, 14 \pmod{28}$	(5, 4)	$(\frac{1}{6}, \frac{11}{3})$
121	$\not\equiv 0, \pm 2, \pm 12, \pm 14, 16 \pmod{32}$	(2, 12)	$(-\frac{5}{3}, \frac{13}{3})$
123	$\not\equiv 0, \pm 4, \pm 6, \pm 10, 16 \pmod{32}$	(6, 4)	$(\frac{1}{3}, \frac{13}{3})$
124[a]	$\not\equiv 0, \pm 5, \pm 8, \pm 13, 18 \pmod{36}$	(5, 8)	$(-\frac{1}{2}, 5)$
125[a]	$\not\equiv 0, \pm 4, \pm 7, \pm 11, 18 \pmod{36}$	(7, 4)	$(\frac{1}{2}, 5)$

[a] Before comparing with (6.15), replace q by $-q$.

λ_1 and λ_2, respectively, such that $\lambda_1(h_0) = 0$, $\lambda_1(h_1) = 7$, and $\lambda_2(h_0) = 2$, $\lambda_2(h_1) = 3$. (Concretely, λ_1 and λ_2 may be taken to be $-\frac{7}{6}\alpha_0 + \frac{7}{3}\rho$ and $-\frac{1}{6}\alpha_0 + \frac{7}{3}\rho$, respectively.)

Furthermore, expression (6.14) is itself the principally specialized character for the standard module for $A_2^{(2)}$ corresponding to λ such that $\lambda(h_0) = 0$ and $\lambda(h_1) = 1$. (For example, take $\lambda = -\frac{1}{6}\alpha_0 + \frac{1}{3}\rho$.)

Slater [18] has a list of identities among which there are 21 of Rogers–Ramanujan type that are not included in Andrews' theory. All 21 (including (6.12) and (6.13)) are "explained" by a theorem exactly like Theorem 6.16. For convenience, we just include a table (see Table I) which gives:

(1) the equation number in [18] for the identity,

(2) whether or not we replace q by $-q$ before comparing the identity with (6.15),

(3) the pair $(n_0 = (\lambda + \rho)(h_0),\ n_1 = (\lambda + \rho)(h_1))$ of integers which when substituted in (6.15) give the product side of the identity, and

(4) a particular dominant integral linear form $\lambda = a\alpha_0 + c\rho$ which corresponds to the identity as in Theorem 6.16.

We strongly suspect that methods similar to the generalized hypergeometric series and difference equation techniques of Andrews can be used to extend the 21 identities from Slater's list in [18] to an infinite family of identities of Rogers–Ramanujan type such as those in (5.12).

Clearly, one should look for an identity of Rogers–Ramanujan type for each standard module for each Euclidean GCM Lie algebra.

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