Linear Oscillatory Hydromagnetic Flow Between Two Coaxial Cylinders

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Hydromagnetic flow between two coaxial circular cylinders is discussed when the inner cylinder oscillates axially under a radial magnetic field. Exact solution is given for the case of a perfectly conducting fluid. Expressions for velocity, induced magnetic field, current density, electric field, viscous drag and energy transfer are derived and expressed in polar forms so as to facilitate the study of magnitude and phase variations. Current sheets are found to exist on the two boundaries.

1. INTRODUCTION

Magnetohydrodynamic flow due to an infinite cylinder oscillating axially has been studied by Arora and Gupta [1]. This work is extended here to the flow of a perfectly conducting fluid between two coaxial circular cylinders, when the inner cylinder axially oscillates under a radially applied magnetic field. The corresponding problem of parallel planes, with one of the planes oscillating in its own plane, has been discussed by Arora and Gupta [2] for the case when the dissipative effect of viscosity is negligible in comparison to that of the finite conductivity. The present work forms a generalization of the study of oscillatory viscous flow between coaxial cylinders by Khamrui [3] and Arora and Gupta [4] to the case of perfectly conducting fluids. Unlike Khamrui [3], there is no radial-injection flow present. The problem approximates to that of the single oscillating cylinder, when the distance separating the two cylinders is made infinite.

Expressions for velocity, induced magnetic field, electric field and current density are obtained. Both the electric field and the current density are found to exist at the oscillating cylinder, whereas, on the fixed cylinder, only the current density has a finite value. Also, the current sheets are observed on the boundaries. Viscous drag experienced by the cylinders and the transfer of energy from the oscillating cylinder to the fluid have been calculated.
Consider the harmonic oscillatory motion of an incompressible, perfectly conducting fluid confined between two coaxial, infinitely long circular cylinders of radii $R_1$ and $R_2$ ($R_2 > R_1$). The axis is taken along the $z$-direction of a cylindrical system $(r, \phi, z)$. A radial magnetic field, $k/r$, is impressed across the fluid. The constant $k$ determines the strength of the applied field. Let the axial velocity of the inner cylinder be $u = u_0 \cos \omega t$, which we write as $u = u_0 \exp(i\omega t)$ in order to obtain the solution [1]. $u_0$ and $\omega$ are the velocity amplitude and the angular frequency of vibration, respectively. The outer cylinder is held stationary. We wish to study the resulting motion of the fluid.

2. Flow Equations

The equations, in rationalized M.K.S. units, governing the flow of an incompressible, viscous and electrically conducting fluid [5] may be taken as

$$\text{div } \mathbf{v} = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{\rho} \nabla p + \frac{1}{\mu \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{v},$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{B}$ the magnetic induction, $\mu$ the permeability, $\rho$ the matter density, $\nu$ the kinematic viscosity, $\lambda$ ($= 1/\mu \sigma$) the magnetic diffusivity, $\sigma$ the electrical conductivity and $p$ the hydrostatic pressure. In the above equations, displacement current and free charge density have been neglected.

The solution of Eqs. (2) and (3) provides the expressions for velocity and induced magnetic field. The equations to determine induced current density $\mathbf{j}$ and electric field $\mathbf{E}$ are

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B},$$

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \mathbf{v} \times \mathbf{B},$$

Eqs. (1)–(3), when expressed in cylindrical coordinates [6], allow a solution

$$v_r = 0, \quad v_\phi = 0, \quad v_z = v(r, t).$$

$$B_r = k/r, \quad B_\phi = 0, \quad B_z = b(r, t),$$

where $v$ and $b$ are functions of $r$ and $t$. 

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for the present problem corresponding to a perfectly conducting fluid \((\lambda = 0)\), provided that

\[
\frac{\partial \rho}{\partial r} + \frac{1}{\mu} b \frac{\partial b}{\partial r} = 0, \tag{7}
\]

\[
\left[ \frac{\partial}{\partial t} - \nu \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \right] v = \frac{k}{\mu \rho r} \frac{\partial b}{\partial r}, \tag{8}
\]

\[
\frac{\partial b}{\partial t} = \frac{k}{r} \frac{\partial v}{\partial r}. \tag{9}
\]

Equation (7) gives the pressure distribution in the radial direction, once the induced magnetic field, \(b\), is obtained from Eqs. (8) and (9). Elimination of \(b\) or \(v\) from Eqs. (8) and (9) leads to

\[
\left[ \frac{\partial^2}{\partial t^2} - \frac{\nu}{3\nu} \frac{\partial^2}{\partial r^2} - \nu \frac{\partial^3}{\partial r \partial r^2} + \frac{k^2}{\mu \rho r^3} \frac{\partial}{\partial r} - \frac{k^2}{\mu \rho r^2} \frac{\partial^2}{\partial r^2} \right] \left( \frac{\partial}{\partial t} \right) (v) = 0. \tag{10}
\]

3. Solution and Basic Results

The general solution of differential equation (10) for the velocity field has been given by Arora and Gupta [1] as

\[
v(r, t) = e^{i\omega t} \left[ C_1 I_0(r') + C_2 K_0(r') \right] \tag{11}
\]

where

\[
r' = (\omega / \nu)^{1/2} (MR_1^2 + ir^2)^{1/2}, \quad M = (k^2 / \mu \rho \omega R_1^2). \tag{12}
\]

\(I_n\) and \(K_n\) are \(n\)th order modified Bessel functions of first and second kinds, respectively; \(C_1\) and \(C_2\) being two arbitrary constants. \(M\) is a dimensionless parameter, characteristic of the flow.

Inserting this value of \(v\) in Eq. (9) and integrating the resulting equation with respect to time, we obtain

\[
b(r, t) = \frac{k}{\nu r} e^{i\omega t} \left[ C_1 I_1(r') - C_2 K_1(r') \right] \tag{13}
\]

for the induced magnetic field. Here, we have taken the constant of integration to be zero, since a constant, independent of time, shall render \(b\) aperiodic. Moreover, with the substitution of Eqs. (11) and (13), Eq. (8) is identically
satisfied. Using Eqs. (4) and (13), we get for the induced current density $j(0, i, 0)$,

$$j(r, t) = \frac{-1}{\mu} \frac{\partial b}{\partial r} = -\frac{i\omega k r}{\mu \nu^2 r^3} e^{i \omega t} \left\{ C_1 [r' I_0 (r') - 2 I_1 (r')] + C_2 [r' K_0 (r') + 2 K_1 (r')] \right\}.$$

From Eqs. (5) and (11), the induced electric field $E(0, E, 0)$, for the case $\lambda = 0$, can be written as

$$E(r, t) = -(k/r) v = -(k/r) e^{i \omega t} \left\{ C_1 I_0 (r') + C_2 K_0 (r') \right\}.$$

In order to evaluate $C_1$ and $C_2$, we subject the velocity expression (11) to the following boundary conditions:

$$v(r = R_1) = u_0 \exp i \omega t, \quad v(r = R_2) = 0,$$

i.e., in contact with the oscillating cylinder, the fluid velocity is same as that of the cylinder and as we move towards the outer cylinder, the velocity approaches zero. Writing

$$r' = \xi e^{i \beta}$$

with

$$\xi = (\omega / \nu)^{1/2} (M^2 R_1^4 + r^4)^{1/4}, \quad \beta = \frac{1}{2} \tan^{-1}(r^2 / MR_1^2),$$

the constants are found to be

$$C_1 = u_0 A K_0 (\xi e^{i \beta}), \quad C_2 = -u_0 A I_0 (\xi e^{i \beta}),$$

where

$$A = 1 / \left[ K_0 (\xi e^{i \beta}) I_0 (\xi e^{i \beta}) - I_0 (\xi e^{i \beta}) K_0 (\xi e^{i \beta}) \right].$$

$\xi_1$, $\beta_1$ and $\xi_2$, $\beta_2$ are the values of $\xi$, $\beta$ at the inner and outer cylinders, respectively. Equations (11) and (13)–(15) now reduce to

$$v(r, t) = u_0 A e^{i \omega t} \left[ K_0 (\xi_2 e^{i \beta_2}) I_0 (\xi e^{i \beta}) - I_0 (\xi_2 e^{i \beta_2}) K_0 (\xi e^{i \beta}) \right],$$

$$b(r, t) = -u_0 \frac{k A}{\nu \xi} e^{i (\omega t - \beta)} \left[ K_0 (\xi_2 e^{i \beta_2}) I_0 (\xi e^{i \beta}) + I_0 (\xi_2 e^{i \beta_2}) K_0 (\xi e^{i \beta}) \right],$$

$$j(r, t) = -u_0 \frac{i \omega k A r}{\mu \nu^2 \xi} e^{i (\omega t - 2 \beta)} \left[ K_0 (\xi_2 e^{i \beta_2}) \left[ \xi e^{i \beta} I_0 (\xi e^{i \beta}) - 2 I_1 (\xi e^{i \beta}) \right] - I_0 (\xi_2 e^{i \beta_2}) \left[ \xi e^{i \beta} K_0 (\xi e^{i \beta}) + 2 K_1 (\xi e^{i \beta}) \right] \right],$$

$$E(r, t) = -u_0 \frac{k A}{r} e^{i \omega t} \left[ K_0 (\xi_2 e^{i \beta_2}) I_0 (\xi e^{i \beta}) - I_0 (\xi_2 e^{i \beta_2}) K_0 (\xi e^{i \beta}) \right].$$
To express Eqs. (21)–(24) in a simple form, we write

\[ I_n(x e^{i\gamma}) = P_n(x, y) e^{i\alpha_n(x, y)}, \]  

\[ K_n(x e^{i\gamma}) = R_n(x, y) e^{i\theta_n(x, y)}, \]  

where

\[ P_n(x, y) = \left[ U_n^2(x, y) + V_n^2(x, y) \right]^{1/2}, \quad \alpha_n(x, y) = \tan^{-1} \frac{V_n(x, y)}{U_n(x, y)}, \]  

\[ R_n(x, y) = \left[ E_n^2(x, y) + F_n^2(x, y) \right]^{1/2}, \quad \theta_n(x, y) = \tan^{-1} \frac{F_n(x, y)}{E_n(x, y)}, \]  

\[ U_n(x, y) = \sum_{m=0}^{\infty} \frac{\left[ \frac{1}{2} x \right]^{n+2m}}{m!(n+m)!} \cos(n+2m)y, \]  

\[ V_n(x, y) = \sum_{m=0}^{\infty} \frac{\left[ \frac{1}{2} x \right]^{n+2m}}{m!(n+m)!} \sin(n+2m)y, \]  

\[ E_n(x, y) = (-1)^{n+1} \left( y - \log 2 + \log x \right) U_n(x, y) + \frac{(-1)^n y V_n(x, y)}{2} \]  

\[ + \frac{1}{2} \sum_{m=0}^{n-1} (-1)^m \frac{(n-m-1)!}{m!} \left( \frac{2}{x} \right)^{n-2m} \cos(n-2m)y, \]  

\[ F_n(x, y) = (-1)^{n+1} \left( y - \log 2 + \log x \right) V_n(x, y) + \frac{(-1)^n y U_n(x, y)}{2} \]  

\[ + \frac{1}{2} \sum_{m=0}^{n-1} (-1)^{m+1} \frac{(n-m-1)!}{m!} \left( \frac{2}{x} \right)^{n-2m} \sin(n-2m)y, \]  

\[ E_n(x, y) = (-1)^{n+1} \left( y - \log 2 + \log x \right) U_n(x, y) + \frac{(-1)^n y V_n(x, y)}{2} \]  

\[ + \frac{1}{2} \sum_{m=0}^{n-1} (-1)^m \frac{(n-m-1)!}{m!} \left( \frac{2}{x} \right)^{n-2m} \cos(n-2m)y, \]  

Here, \( n \) is the order of various functions, \( \gamma \) the Euler's constant and \( x, y \) are the general variables. We, further, write

\[ [K_0(\zeta e^{i\beta_2}) I_n(x e^{i\nu}) - \frac{(-1)^n}{2} J_0(\zeta e^{i\beta_2}) K_n(x e^{i\nu})] \]  

\[ = X_n(x, y) \exp[i[D_n(x, y) + \alpha_n(x, y) + \beta_0(\zeta_2 , \beta_2)]], \]
where the functions $X_n(x, y)$ and $D_n(x, y)$ are obtained from the following set of equations:

$$
R_n(\zeta_2, \beta_2) P_n(x, y) - (-1)^n P_n(\zeta_2, \beta_2) R_n(x, y) \cos Q_n(x, y) = X_n(x, y) \cos D_n(x, y);
(-1)^n P_n(\zeta_2, \beta_2) R_n(x, y) \sin Q_n(x, y) = X_n(x, y) \sin D_n(x, y). \tag{34}
$$

In Eq. (34), $Q_n(x, y)$ is given by

$$
Q_n(x, y) = \alpha_n(x, y) - \theta_n(x, y) - \alpha_0(\zeta_2, \beta_2) + \theta_0(\zeta_2, \beta_2). \tag{35}
$$

Now, using the polar form (33), Eqs. (21–24) become

$$
\begin{align}
\varphi(r, t) &= u_0 \frac{X_0(\xi_1, \beta)}{X_0(\xi_2, \beta)} \\
&\times \exp[i(\omega t + D_0(\xi, \beta) + \alpha_0(\xi, \beta) - D_0(\zeta_1, \beta_1) - \alpha_0(\zeta_1, \beta_1))], \tag{36}
\end{align}
$$

$$
\begin{align}
b(r, t) &= u_0 R_1(\mu_0 M)^{1/2} (M^2 R_1^4 + r^4)^{-1/4} \frac{X_1(\xi, \beta)}{X_0(\xi_1, \beta_1)} \\
&\times \exp[i(\omega t + D_1(\xi, \beta) + \alpha_1(\xi, \beta) - D_0(\zeta_1, \beta_1) - \alpha_0(\zeta_1, \beta_1) - \beta)] \tag{37},
\end{align}
$$

$$
\begin{align}
j(r, t) &= u_0 R_1(\mu_0 M)^{1/2} r R_1 (M^2 R_1^4 + r^4)^{-3/4} \frac{G'(\xi, \beta)}{X_0(\xi_1, \beta_1)} \\
&\times \exp \left[ i \left( \omega t + D_1(\xi, \beta) + \alpha_1(\xi, \beta) - D_0(\zeta_1, \beta_1) - \alpha_0(\zeta_1, \beta_1) - \beta^2 \frac{3\pi}{2} \right) \right], \tag{38}
\end{align}
$$

$$
\begin{align}
E(r, t) &= u_0 (\mu_0 \omega M)^{1/2} \frac{R_1}{r} \frac{X_0(\xi, \beta)}{X_0(\xi_1, \beta_1)} \\
&\times \exp[i(\omega t + D_0(\xi, \beta) + \alpha_0(\xi, \beta) - D_0(\zeta_1, \beta_1) - \alpha_0(\zeta_1, \beta_1) - \beta)], \tag{39}
\end{align}
$$

where $G'(\xi, \beta)$ and $\xi'(\xi, \beta)$ are determined by

$$
2X_1(\xi, \beta) - \xi X_0(\xi, \beta) \cos \psi(\xi, \beta) = G'(\xi, \beta) \cos \xi'(\xi, \beta), \tag{40}
$$

$$
\begin{align}
\xi X_0(\xi, \beta) \sin \psi(\xi, \beta) &= G'(\xi, \beta) \sin \xi'(\xi, \beta), \\
\psi(\xi, \beta) &= D_0(\xi, \beta) + \alpha_0(\xi, \beta) - D_1(\xi, \beta) - \alpha_1(\xi, \beta) + \beta.
\end{align}
$$

The real parts of Eqs. (36)–(39) represent the distribution of velocity, induced magnetic field, current density and electric field, respectively.
From Eqs. (33), (34) and (40) we observe that

\[ X_0(\xi_1, \beta_1) \neq 0, \quad X_1(\xi_1, \beta_1) \neq 0, \quad G'(\xi_1, \beta_1) \neq 0, \]
\[ X_0(\xi_2, \beta_2) = 0, \quad X_1(\xi_2, \beta_2) = \frac{1}{\xi_2}, \quad G'(\xi_2, \beta_2) = \frac{2}{\xi_2}, \]

\[ [D_1(\xi_2, \beta_2) + \alpha_1(\xi_2, \beta_2) + \theta_0(\xi_2, \beta_2) + \beta_2] = 0, \quad \xi'(\xi_2, \beta_2) = 0. \]

In virtue of these values of various functions at the two boundaries, we shall have, from Eqs. (38) and (39), following values of current density and electric field at the surface of the cylinders:

\[ j(R_1, t) = u_0(\rho M/\mu)^{1/2} \frac{1}{R_1} (M^2 + 1)^{-3/4} \frac{G'(\xi_1, \beta_1)}{X_0(\xi_1, \beta_1)} \times \cos \left[ \omega t \right] D_1(\xi_1, \beta_1) + \alpha(\xi_1, \beta_1) D_0(\xi_1, \beta_1) \]
\[ = \alpha_0(\xi_1, \beta_1) - \xi'(\xi_1, \beta_1) - 3\beta_1 - \frac{3\pi}{2}, \]

\[ j(R_2, t) = u_0 \frac{k}{\mu \omega} R_2 (M^2 R_1^4 + R_2^4)^{-1} \frac{1}{X_0(\xi_1, \beta_1)} \times \cos \left[ \omega t - \theta_0(\xi_2, \beta_2) - D_0(\xi_1, \beta_1) - \alpha_0(\xi_1, \beta_1) - 4\beta_2 - \frac{3\pi}{2} \right], \]

\[ E(R_1, t) = u_0 \frac{k}{R_1} \cos(\omega t - \pi), \]

\[ E(R_2, t) = 0. \]

Hence, at the inner cylinder both the electric field and the current density have nonzero values; whereas, at the outer stationary cylinder the electric field is zero and the current density is finite. From Eq. (44), we find that the surface electric field is directly proportional to the value of applied magnetic field at the oscillating cylinder. Comparing Eqs. (36) and (39), we see that a constant phase difference \( \pi \) exists everywhere between the velocity and the induced electric field.

Equation (37) reveals that the induced magnetic field has finite values at \( r = R_1 \) and \( r = R_2 \) and so there appear current sheets at both the boundaries, irrespective of the nature of the cylinders. Their densities, \( j_s(0, j_s, 0) \), per unit distance, measured tangentially normal to the current [7], are given by

\[ j_s(R_1, t) = -\frac{1}{\mu} b(R_1, t) = u_0(\rho M/\mu)^{1/2} (M^2 + 1)^{-1/4} \frac{X_1(\xi_1, \beta_1)}{X_0(\xi_1, \beta_1)} \times \cos[\omega t + D_1(\xi_1, \beta_1) + \alpha_0(\xi_1, \beta_1) - D_0(\xi_1, \beta_1)] \]
\[ = -\alpha_0(\xi_1 \beta_1) - \beta_1, \]

\[ j_s(R_2, t) = -\frac{1}{\mu} b(R_2, t) = u_0 \frac{k}{\mu \omega} (M^2 R_1^4 + R_2^4)^{-1/2} \frac{1}{X_0(\xi_1, \beta_1)} \times \cos[\omega t - \theta_0(\xi_2, \beta_2) - D_0(\xi_1, \beta_1) - \alpha_0(\xi_1, \beta_1) - 2\beta_2]. \]
4. Viscous Drag and Energy Transfer

With the help of Eqs. (21) and (33), the tangential viscous stress \([8]\) at any fluid level \(r\) can be written as

\[
\sigma_{\tau r}(r, t) = \rho v \frac{\partial v}{\partial r} = u_0 \rho \langle \omega v \rangle^{1/2} r (M^2 R_1^4 + r^4)^{-1/4} \frac{X_1(\zeta_1, \beta_1)}{X_0(\zeta_1, \beta_1)}
\]

\[
\times \cos \left[ \omega t + D_1(\zeta_1, \beta_1) + \alpha_1(\zeta_1, \beta_1) - D_0(\zeta_1, \beta_1) - \alpha_0(\zeta_1, \beta_1) \right] - \beta + \frac{\pi}{2} \right].
\]

Thus, the viscous drag on the oscillating cylinder is

\[
\sigma_{\tau r}(R_1, t) = u_0 \rho \langle \omega v \rangle^{1/2} (M^2 + 1)^{-1/4} \frac{X_1(\zeta_1, \beta_1)}{X_0(\zeta_1, \beta_1)}
\]

\[
\times \cos \left[ \omega t + D_1(\zeta_1, \beta_1) + \alpha_1(\zeta_1, \beta_1) - D_0(\zeta_1, \beta_1) \right] - \alpha_0(\zeta_1, \beta_1) - \beta_1 + \frac{\pi}{2} \right],
\]

while that on the fixed cylinder is

\[
- \sigma_{\tau r}(R_2, t) = u_0 \rho \nu R_2 (M^2 R_1^4 + R_2^4)^{-1/2} \frac{1}{X_0(\zeta_1, \beta_1)}
\]

\[
\times \cos \left[ \omega t - \theta_2(\zeta_2, \beta_2) - D_0(\zeta_1, \beta_1) - \alpha_0(\zeta_1, \beta_1) \right] - 2\beta_2 - \frac{\pi}{2} \right].
\]

The net energy transfer from the oscillating cylinder to the fluid is

\[
P = - (\sigma_{\tau r} v)(R_1, t) = \frac{1}{2} u_0 \rho \langle \omega v \rangle^{1/2} (M^2 + 1)^{-1/4} \frac{X_1(\zeta_1, \beta_1)}{X_0(\zeta_1, \beta_1)}
\]

\[
\times \cos \left[ D_1(\zeta_1, \beta_1) + \alpha_1(\zeta_1, \beta_1) - D_0(\zeta_1, \beta_1) \right] - \alpha_0(\zeta_1, \beta_1) - \beta_1 - \frac{\pi}{2} \right].
\]

5. General Remarks

The results of this problem reduce to those of the case of a single cylinder \([1]\), when the radius of the outer cylinder is made infinite \((R_2 \rightarrow \infty)\). Under the limit \(k \rightarrow 0\), we arrive at the corresponding results for the viscous hydrodynamic flow \([4]\).
REFERENCES


