Kalman–Bucy filtering equations of forward and backward stochastic systems and applications to recursive optimal control problems

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Received 13 April 2006
Available online 3 January 2008
Submitted by J. Glaz

Abstract

This paper is concerned with Kalman–Bucy filtering problems of a forward and backward stochastic system which is a Hamiltonian system arising from a stochastic optimal control problem. There are two main contributions worthy pointing out. One is that we obtain the Kalman–Bucy filtering equation of a forward and backward stochastic system and study a kind of stability of the aforementioned filtering equation. The other is that we develop a backward separation technique, which is different to Wonham’s separation theorem, to study a partially observed recursive optimal control problem. This new technique can also cover some more general situation such as a partially observed linear quadratic non-zero sum differential game problem is solved by it. We also give a simple formula to estimate the information value which is the difference of the optimal cost functionals between the partial and the full observable information cases.

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Keywords: Backward stochastic differential equation; Feynman–Kac formula; Kalman–Bucy filtering; Linear quadratic non-zero sum differential game; Recursive optimal control; Stability

1. Introduction

To solve partially observed stochastic optimal control problems consists of two components. One is estimation, the other is control. The estimation part is related to filtering problems. The most successful result of filtering theory was obtained for linear systems by Kalman [5] and Kalman and Bucy [6] in 1960 and 1961, respectively. In the case of linear systems, partially observed optimal control problems can be partly treated by a separation theorem originally obtained by Wonham [17] in 1968. This theorem allows us to first compute filtering of states, and then to solve fully

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\textsuperscript{*} This work is partially supported by the Natural Science Foundation of PR China (10671112) and Shandong Province (Z2006A01), the National Basic Research Program of PR China (973 Program, No. 2007CB814904) and the New Century Excellent Young Teachers Program of Education Ministry, PR China.

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observed optimal control problems driven by the filtering states. A systematic introduction of linear filtering theory and its application to optimal control can be found in the books of Liptser and Shiryaev [9] and Bensoussan [2].

However we note that the signal processes in the above filtering problems are the solutions of forward stochastic differential equations (SDEs in short). Nonlinear backward stochastic differential equations (BSDEs in short) have been independently introduced by Pardoux and Peng [12] and Duffie and Epstein [3]. For a BSDE coupled with a forward SDE, Peng [13] gave a probabilistic interpretation for a large kind of the second order quasi-linear partial differential equation (PDE in short). This result generalized the well-known Feynman–Kac formula to a non-linear case. El Karoui et al. [7] gave some important properties of BSDEs and their applications to optimal control and financial mathematics.

Peng [14] derived a general maximum principle for a fully observed forward stochastic control system. It is well known that an optimal control can be represented by an adjoint process which is the solution of a BSDE. Then the optimal state equation and the adjoint equation consist of a Hamiltonian system which is a forward and backward stochastic differential equation (FBSDE in short). The study of coupled FBSDEs started in last early 90s. In his PhD dissertation, Antonelli [1] obtained the first result on the solvability of an FBSDE over a small time duration. Ma et al. [10] provided explicit relations among the forward and the backward components of the adapted solution via a quasi-linear PDE, but they required the non-degeneracy held for the forward diffusion and the non-randomness held for the coefficients. Hu and Peng [4], Peng and Wu [15] got the existence and uniqueness result of an FBSDE with the arbitrarily fixed large time duration under a monotonicity condition on the coefficients, which is restrictive in a different way. We refer the reader to the book of Ma and Yong [11] for a systematic introduction of FBSDEs.

In [8] and [16], Li and Tang derived some general maximum principles for partially observed forward stochastic control systems, which covered most of the results of references therein. To get an observable maximum principle, they used backward stochastic PDEs to characterize the corresponding Hamiltonian system. In fact, it is a natural request to characterize the Hamiltonian system by filtering for FBSDEs. However, there exists few work dealing with this topic. In our paper, we will study filtering problems of a forward and backward stochastic system arising from an optimal control problem. And then the theoretical result is applied to a partially observed recursive optimal control problem in Section 4. To our best knowledge, these kinds of results have not been found in existing works.

In the coming section, we present the Kalman–Bucy filtering equation corresponding to the aforementioned forward and backward stochastic system. In Section 3, we study a kind of stability of the filtering equation obtained in the above section. We also give an example of a forward and backward stochastic system which has a stable explicit observable solution.

Duffie and Epstein [3] presented a concept of stochastic differential recursive utility which is an extension of the standard additive utility with the instantaneous utility depending not only on an instantaneous consumption rate \( c(\cdot) \) but also on the future utility. As has been noted by El Karoui et al. [7], the (stochastic differential) recursive utility process can be regarded as the solution of a special BSDE. From BSDEs’ point of view, El Karoui et al. [7] gave the formulation of recursive utilities and their properties. Using solutions of BSDEs to describe cost functionals of control systems, we get recursive optimal control problems. In Section 4, we study a partially observed recursive optimal control problem. Using a new technique which is different to Wonham’s separation theorem, we obtain a unique optimal control which is a linear feedback of the state filtering estimation. We notice that this new technique can cover some more general situation. For example, it can be used to solve a partially observed linear quadratic non-zero sum differential game problem, which is more general than the aforementioned recursive problem.

From the financial mathematics point of view, Yang and Ma [18] gave a definition of information value. In the last section, our task is to establish a formula, which shows the importance of more observable information to controllers. How to estimate the information value of the recursive optimal control problem is also studied in this section.

2. Kalman–Bucy filtering equations

In this section, we first introduce a forward and backward stochastic system arising from a classical optimal control problem, and then derive the Kalman–Bucy filtering equation for this kind of system.

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \) be a filtered complete probability space equipped with a natural filtration \( \mathcal{F}_t = \sigma(\xi, W_1(s), W_2(s); 0 \leq s \leq t), \mathcal{F} = \mathcal{F}_T, \) where \((W_1(\cdot), W_2(\cdot))\) is a 2-dimensional standard Brownian motion defined on the space, and \( T > 0 \) is a fixed real number. \( \xi \) is a Gaussian random variable, independent of \((W_1(\cdot), W_2(\cdot))\), with the mean \( m_0 \) and the variance \( n_0 \geq 0 \).
Throughout the paper, for the sake of convenience, we only consider 1-dimensional stochastic system. For the multi-dimensional case, we can get similar results by the same method.

Suppose that we have a control system, whose evolution is described by the following equation:

\[
\begin{align*}
\begin{cases}
    dX(t) = \left( A(t)X(t) + B(t)v(t) \right)dt + C_1(t)dW_1(t) + C_2(t)dW_2(t), \\
    X(0) = \xi,
\end{cases}
\end{align*}
\]

where \( v(\cdot) \) is defined by

\[
U_{ad} = \left\{ v(\cdot) \mid v(t) \text{ is an } \mathcal{F}_t\text{-adapted process valued in } \mathbb{R} \text{ and satisfies } \mathbb{E} \int_0^T v^4(t)dt < +\infty \right\}.
\]

Every element in \( U_{ad} \) is called an admissible control.

We suppose that \( X(\cdot) \) has an effect on the wealth of a controller, however the controller cannot influence the system, and acts to protect his advantages by \( v(\cdot) \in U_{ad} \). The payoff corresponding to \( v(\cdot) \in U_{ad} \) is recursive, which means that the cost functional is given by

\[
J(v(\cdot)) = Y(0) = \mathbb{E}Y(0),
\]

where \( (Y(\cdot), Z_1(\cdot), Z_2(\cdot)) \) is a solution of the BSDE

\[
\begin{align*}
\begin{cases}
    -dY(t) = \left( a(t)X^2(t) + b(t)Y(t) + f_1(t)Z_1(t) + f_2(t)Z_2(t) + c(t)v^2(t) \right)dt \\
    -Z_1(t)dW_1(t) - Z_2(t)dW_2(t),
\end{cases}
\end{align*}
\]

\( Y(T) = X^2(T). \)

We need the following hypothesis:

(H1) \( a(\cdot) \geq 0, c(\cdot) \geq \varepsilon > 0, A(\cdot), B(\cdot), C_1(\cdot), C_2(\cdot), f_1(\cdot) \) and \( f_2(\cdot) \) are uniformly bounded deterministic functions with respect to \( t \in [0, T] \).

Since the drift term in (3) contains \( (Z_1(\cdot), Z_2(\cdot)) \), it brings us some trouble to express the cost functional (2). To simplify it, we define a probability measure \( Q \) on the space \( (\Omega, \mathcal{F}) \) by

\[
\frac{dQ}{dP} = \exp \left\{ \int_0^T f_1(t)dW_1(t) + \int_0^T f_2(t)dW_2(t) - \frac{1}{2} \int_0^T \left( f_1^2(t) + f_2^2(t) \right)dt \right\}.
\]

From (H1), according to Girsanov’s theorem, it follows that \( (U(\cdot), V(\cdot)) \) defined by

\[
\begin{align*}
U(t) = W_1(t) - \int_0^t f_1(s)ds \quad \text{and} \quad V(t) = W_2(t) - \int_0^t f_2(s)ds
\end{align*}
\]

is a 2-dimensional standard Brownian motion defined on the space \( (\Omega, \mathcal{F}, (\mathcal{F}_t), Q) \). It is easy to prove that \( (U(\cdot), V(\cdot)) \) and \( \xi \) remain mutually independent and \( \xi \) keeps the same probability law as before on \( (\Omega, \mathcal{F}, (\mathcal{F}_t), Q) \).

Then we can rewrite (1) and (3) as follows:

\[
\begin{align*}
\begin{cases}
    dX(t) = \left( A(t)X(t) + B(t)v(t) + C_1(t)f_1(t) + C_2(t)f_2(t) \right)dt + C_1(t)dU(t) + C_2(t)dV(t), \\
    X(0) = \xi, \\
    -dY(t) = \left( a(t)X^2(t) + b(t)Y(t) + c(t)v^2(t) \right)dt - Z_1(t)dU(t) - Z_2(t)dV(t),
\end{cases}
\end{align*}
\]

\( Y(T) = X^2(T). \)

By the definition of \( U_{ad} \), we know that if \( v(\cdot) \in U_{ad} \) then \( \mathbb{E}_Q \int_0^T v^4(t)dt < +\infty \). In this case, \( \mathbb{E}_Q X^4(\cdot) < +\infty \), i.e., \( \mathbb{E}_Q Y^2(T) < +\infty \). So there exists a unique solution for (4) and (5), respectively. Therefore the corresponding cost functional is rewritten as
Remark 2.1. The linear combinations of $(y(t), z_1(t), z_2(t))$ can be considered in the drift term of the observation equation (9). For this case, we can still deal with it by same techniques, so we only consider the observation equation as above.

Our filtering problem is to find explicit expressions for the best estimation (in the sense of square error) with respect to the observations $Z(\cdot)$ up to time $t$, denoted by $(\hat{x}(t), \hat{y}(t), \hat{z}_1(t), \hat{z}_2(t))$, for the state $(x(t), y(t), z_1(t), z_2(t))$, i.e., we want to find the explicit expressions for

\[
\hat{x}(t) = \mathbb{E}_Q[x(t) \mid Z_t], \quad \hat{y}(t) = \mathbb{E}_Q[y(t) \mid Z_t], \quad \hat{z}_1(t) = \mathbb{E}_Q[z_1(t) \mid Z_t], \quad \hat{z}_2(t) = \mathbb{E}_Q[z_2(t) \mid Z_t]
\]

and their square error estimation. Here $Z_t = \sigma \{ Z(s); 0 \leq s \leq t \}$. Where $\mathbb{E}_Q$ denotes the mathematical expectation on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$.

Minimizing (6) subject to $v(\cdot) \in \mathcal{U}_{ad}$ and (4) formulates a fully observed optimal control problem. For simplicity, we denote this problem by Problem (FO). Any $u(\cdot) \in \mathcal{U}_{ad}$ satisfying

\[
J(u(\cdot)) = \min_{v(\cdot) \in \mathcal{U}_{ad}} J(v(\cdot))
\]

called an optimal control. The corresponding state trajectory and the cost functional are called an optimal state trajectory and an optimal cost functional denoted by $x(\cdot)$ and $J(u(\cdot))$, respectively.

Since the drift term in (4) contains the deterministic function $C_1(\cdot) f_1(\cdot) + C_2(\cdot) f_2(\cdot)$, the classical technique of completing squares cannot be used directly to solve Problem (FO). However, Peng’s maximum principle (see Peng [14]) is still an alternative tool. From the maximum principle, it is easy to check that

\[
u(t) = -\frac{1}{2} B(t) c^{-1}(t) e^{-\int_0^t b(s) ds} y(t), \quad \text{a.e., a.s.}
\]

is an optimal control of Problem (FO). Here the adjoint process $y(\cdot)$ satisfies the following Hamiltonian system which is an FBSDE

\[
\begin{cases}
\frac{dx(t)}{dt} = \left( A(t)x(t) - \frac{1}{2} B^2(t)c^{-1}(t)e^{-\int_0^t b(s) ds} y(t) + C_1(t) f_1(t) + C_2(t) f_2(t) \right) dt \\
\quad + C_1(t) dU(t) + C_2(t) dV(t), \\
-dy(t) = (2a(t)e^{\int_0^t b(s) ds} x(t) + A(t)y(t)) dt - z_1(t) dU(t) - z_2(t) dV(t), \\
x(0) = \xi, \quad y(T) = 2e^{\int_0^T b(s) ds} x(T).
\end{cases}
\]

From a result in Peng and Wu [15], we know that (8) admits a unique solution $(x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot))$ and the FBSDE has a practical background in optimal control.

In the following, we will discuss the filtering problem for the forward and backward stochastic system (8). For simplicity, we keep same notations as before. Suppose that the state variable $(x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot))$ cannot be observed directly, however we can observe a noisy process $Z(\cdot)$ related to $x(\cdot)$, whose dynamic is described by the equation

\[
\begin{cases}
\frac{dZ(t)}{dt} = \left( D(t)x(t) + F(t)Z(t) \right) dt + H(t) dW_2(t), \\
Z(0) = 0,
\end{cases}
\]

in other way,

\[
\begin{cases}
\frac{dZ(t)}{dt} = \left( D(t)x(t) + F(t)Z(t) + f_2(t) H(t) \right) dt + H(t) dV(t), \\
Z(0) = 0.
\end{cases}
\]

We introduce the following hypothesis:

\textbf{(H2)} $D(\cdot), F(\cdot), |H(\cdot)| \geq \varepsilon > 0$ and $H^{-1}(\cdot)$ are uniformly bounded deterministic functions with respect to $t$.

Obviously, if (H2) holds, then there exists a unique solution for (9) as well as (10).
Our method is first to look for the relations of \((x(\cdot), y(\cdot), z(\cdot))\) then to compute \((\hat{x}(\cdot), \hat{y}(\cdot), \hat{z}_1(\cdot), \hat{z}_2(\cdot))\) by classical filtering theory for forward SDEs.

From the general non-linear Feynman–Kac formula (see Peng [13]), if we set \(y(t) = u(t, x(t))\), then \(z_1(t)\) and \(z_2(t)\) can be written as
\[
\begin{align*}
    z_1(t) &= C_1(t) \frac{\partial}{\partial x} u(t, x(t)), \\
    z_2(t) &= C_2(t) \frac{\partial}{\partial x} u(t, x(t)),
\end{align*}
\]
where \(u(t, x)\) is a classical solution of the following PDE:
\[
\begin{align*}
    \frac{\partial}{\partial t} u(t, x) + Lu(t, x) + 2a(t)e^{\int_0^t b(s)ds} x + A(t)u(t, x) &= 0, \\
    u(T, x) &= 2e^{T} b(s)ds,
\end{align*}
\]
Here
\[
L u(t, x) = \frac{1}{2} \left( C_1^2(t) + C_2^2(t) \right) \frac{\partial^2}{\partial x^2} u(t, x) + \left( A(t) x - \frac{1}{2} B^2(t) c^{-1}(t) e^{-\int_0^t b(s)ds} u(t, x) + C_1(t) f_1(t) + C_2(t) f_2(t) \right) \frac{\partial}{\partial x} u(t, x).
\]
By the terminal condition of (13), we set \(\bar{u}(t, x) = \Pi(t) x + \pi(t)\), where \(\Pi(\cdot)\) and \(\pi(\cdot)\) satisfy respectively
\[
\begin{align*}
    \bar{\Pi}(t) + 2A(t)\Pi(t) - \frac{1}{2} B^2(t) c^{-1}(t) e^{-\int_0^t b(s)ds} \Pi^2(t) + 2a(t) e^{\int_0^t b(s)ds} &= 0, \\
    \bar{\Pi}(T) &= 2e^{T} b(s)ds,
\end{align*}
\] and
\[
\begin{align*}
    \bar{\pi}(t) + \left( A(t) - \frac{1}{2} B^2(t) c^{-1}(t) e^{-\int_0^t b(s)ds} \Pi(t) \right) \pi(t) + \left( C_1(t) f_1(t) + C_2(t) f_2(t) \right) \Pi(t) &= 0, \\
    \bar{\pi}(T) &= 0.
\end{align*}
\]
From the classical Riccati differential equation theory, we know that there exists a unique solution for (14) and (15), respectively.

By (12) and (14), we get
\[
y(t) = \Pi(t) x(t) + \pi(t), \quad z_1(t) = C_1(t) \Pi(t), \quad z_2(t) = C_2(t) \Pi(t),
\]
where \(x(\cdot)\) satisfies
\[
\begin{align*}
    dx(t) &= \left[ \left( A(t) - \frac{1}{2} B^2(t) c^{-1}(t) \Pi(t) e^{-\int_0^t b(s)ds} \right) x(t) + C_1(t) f_1(t) + C_2(t) f_2(t) \\
    &\quad - \frac{1}{2} B^2(t) c^{-1}(t) \pi(t) e^{-\int_0^t b(s)ds} \right] dt + C_1(t) dU(t) + C_2(t) dV(t),
\end{align*}
\]
Here \(\Pi(\cdot)\) and \(\pi(\cdot)\) come from (14) and (15), respectively.

From (10), (16) and (17), it is easy to see that \(x(\cdot)\) is Gaussian, then \(Z(\cdot)\) is Gaussian, so is \((x(\cdot), y(\cdot), z(\cdot))\) valued in \(\mathbb{R}^3\). Therefore there exists a recursive filtering formula for \((\hat{x}(\cdot), \hat{y}(\cdot), \hat{z}_1(\cdot), \hat{z}_2(\cdot))\). In fact, here we applied the mutual independence of \(\xi\) and \((U(\cdot), V(\cdot))\).

Obviously,
\[
\hat{z}_1(t) = C_1(t) \Pi(t), \quad \hat{z}_2(t) = C_2(t) \Pi(t).
\]
Then we only need to compute \(\hat{x}(t)\) and \(\hat{y}(t)\) defined by (11). Let \(P(t) = \mathbb{E}_Q(x(t) - \hat{x}(t))^2\) be the square error of the estimation \(\hat{x}(t)\). From the fact that \((x(t) - \hat{x}(t)) \perp Z_t\) and \(x(t) - \hat{x}(t)\) is Gaussian, we know that \(x(t) - \hat{x}(t)\) is independent of \(Z_t\). So
\[
P(t) = \mathbb{E}_Q(x(t) - \hat{x}(t))^2 = \mathbb{E}_Q[(x(t) - \hat{x}(t))^2 | Z_t].
\]
Thanks to Theorem 8.1 in Liptser and Shiryaev [9], we obtain the following equations for \( \hat{x}(t) \) and \( P(t) \):

\[
\begin{aligned}
\frac{d\hat{x}(t)}{dt} &= \left[ \left( A(t) - \frac{1}{2} B^2(t)c^{-1}(t)\pi(t)e^{-\int_0^t b(s)ds} \right) \right] \hat{x}(t) + C_1(t)f_1(t) + C_2(t)f_2(t) \\
&\quad - \frac{1}{2} B^2(t)c^{-1}(t)\pi(t)e^{-\int_0^t b(s)ds} dt + \left( C_2(t) + D(t)H^{-1}(t)P(t) \right) d\tilde{W}(t), \\
\hat{x}(0) &= m_0, \\
\dot{P}(t) &= -2 \left( A(t) - \frac{1}{2} B^2(t)c^{-1}(t)\pi(t)e^{-\int_0^t b(s)ds} \right) P(t) + \left( C_2(t) + D(t)H^{-1}(t)P(t) \right)^2 \\
&\quad - C_1^2(t) - C_2^2(t) = 0, \\
P(0) &= n_0,
\end{aligned}
\]

where the process

\[
\tilde{W}(t) = \int_0^t H^{-1}(t)\left( dZ(t) - D(t)\dot{x}(s) - F(t)Z(t) - f_2(t)H(t) \right) dt = V(t) + \int_0^t D(s)H^{-1}(s)(x(s) - \hat{x}(s)) ds
\]

is an observable 1-dimensional standard Brownian motion defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), Q)\), which is the so-called innovation process.

Taking conditional expectations on both sides of (16), we get

\[
\hat{y}(t) = \Pi(t)\hat{x}(t) + \pi(t),
\]

where \( \hat{y}(\cdot) \) is the solution of (20).

So we have

**Theorem 2.2.** Let (H1) and (H2) hold. Then the filtering estimation \((\hat{x}(\cdot), \hat{y}(\cdot), \hat{z}_1(\cdot), \hat{z}_2(\cdot))\) of the state \((x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot))\), which is the solution of (8), are given by (20), (23) and (18).

This result will be used to study a partially observed recursive optimal control problem in Section 4.

### 3. Stability of Kalman–Bucy filtering equations

In this section, we will study a kind of stability of the filtering equations (20) and (23) with respect to their initial values and prove that (20) and (23) are stable under our framework. Furthermore, we also give a worked-out example of a forward and backward stochastic system which has a stable explicit observable filtering solution.

We first give

**Definition 3.1.** For any \( 0 \leq t \leq T \), assume that \( \hat{x}_1(0) \) and \( \hat{x}_2(0) \) are two initial values, and that \( \hat{x}_1(t) \) and \( \hat{x}_2(t) \) are the corresponding filtering estimation values. The filtering equation (20) is called stable, if for any \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that when \( \mathbb{E}_Q(\hat{x}_1(0) - \hat{x}_2(0))^2 < \delta \), we always have

\[
\mathbb{E}_Q \sup_{0 \leq t \leq T} (\hat{x}_1(t) - \hat{x}_2(t))^2 < \varepsilon.
\]

In practice, we hope when the initial filtering estimation value changes little, there is also little difference of the filtering estimation at any time, i.e., we can get a stable filtering estimation. Otherwise, the filtering result has little practical sense. For our filtering equation (20), we can give a more general result, the continuous dependence of solutions with respect to parameters, which implies our desired stable result.

Let us now consider the following equation depending on a parameter \( \delta \in \mathbb{R} \):

\[
\begin{aligned}
\frac{d\hat{x}^\delta(t)}{dt} &= \left( g^\delta(t)\hat{x}^\delta(t) + h^\delta(t)D^\delta(t)\hat{x}^\delta(t) + r^\delta(t) \right) dt + h^\delta(t)H^\delta(t) dV(t), \\
\hat{x}^\delta(0) &= m_0^\delta,
\end{aligned}
\]
where
\[ h^\delta(t) = (H^\delta(t))^{-2}(\mathcal{C}_2^\delta(t)H^\delta(t) + D^\delta(t)P^\delta(t)), \]
\[ g^\delta(t) = A^\delta(t) - \frac{1}{2}(B^\delta(t))^{-2}(c^\delta(t))^{-1}\mathcal{P}^\delta(t)e^{-\int_0^t b^\delta(s)ds} \]
\[ r^\delta(t) = C_1^\delta f^\delta(t) + C_2^\delta f^\delta(t) - \frac{1}{2}(B^\delta(t))^{-2}(c^\delta(t))^{-1}\mathcal{P}^\delta(t)e^{-\int_0^t b^\delta(s)ds}, \]
and \( \mathcal{P}^\delta(\cdot), \pi^\delta(\cdot), x^\delta(\cdot) \) and \( P^\delta(\cdot) \) are solutions of (14), (15), (17) and (21) respectively where all coefficients depend on the parameter \( \delta \). In fact, (24) can be obtained from (20) and (22).

Obviously, if all variables of (24) do not depend on the parameter \( \delta \), Eq. (24) can be regarded as the filtering equation (20) corresponding to the system (8) and (10).

We assume the following hypothesis:

(H3) \( a^\delta(\cdot), b^\delta(\cdot), c^\delta(\cdot), f^\delta(\cdot), f_2^\delta(\cdot), A^\delta(\cdot), B^\delta(\cdot), C_1^\delta(\cdot), C_2^\delta(\cdot), D^\delta(\cdot), H^\delta(\cdot) \) and \( [H^\delta(\cdot)]^{-1} \) are continuous with respect to \( \delta \) and uniformly bounded with respect to \( t \) and \( \delta \).

Then we have

**Theorem 3.2.** Let (H3) hold. Then the solution \( \hat{x}^\delta(\cdot) \) of (24) is continuous about the parameter \( \delta \in \mathbb{R} \).

**Proof.** For notational convenience, we set
\[
\begin{align*}
\bar{x}(t) &= \hat{x}^\delta_1(t) - \hat{x}^\delta_2(t), \quad \tilde{x}(t) = x^\delta_1(t) - x^\delta_2(t), \\
\bar{g}(t) &= g^\delta_1(t) - g^\delta_2(t), \quad \tilde{r}(t) = r^\delta_1(t) - r^\delta_2(t), \\
(\bar{h}D)(t) &= h^\delta_1(t)D^\delta_1(t) - h^\delta_2(t)D^\delta_2(t), \quad (\bar{h}H)(t) = h^\delta_1(t)H^\delta_1(t) - h^\delta_2(t)H^\delta_2(t).
\end{align*}
\]
So, we have
\[
\bar{x}(t) = \int_0^t [\bar{g}(s)\bar{x}(s) + \tilde{g}(s)\tilde{x}(s) + h^\delta_1(s)D^\delta_1(s)\bar{x}(s) + (\bar{h}H)(s)x^\delta_1(s) + \tilde{r}(s)]ds
\]
\[
+ \int_0^t (\bar{h}H)(s)dV(s) + \bar{x}(0).
\]
Hölder’s inequality implies that
\[
\begin{align*}
\tilde{x}^2(t) &\leq 7 \int_0^t (g^\delta(s))^2ds + 7 \int_0^t \tilde{x}^2(s)ds + 7 \int_0^t (\tilde{x}^\delta_2(s))^2ds + 7 \int_0^t (h^\delta_1(s)D^\delta_1(s))^2ds \int_0^t \tilde{x}^2(s)ds \\
&\quad + 7 \int_0^t (\bar{h}D)^2(s)ds \int_0^t (\tilde{x}^\delta_2(s))^2ds + 7T \int_0^t \tilde{r}^2(s)ds + 7 \int_0^t (\bar{h}H)(s)dV(s) + 7\tilde{x}^2(0),
\end{align*}
\]
and
\[
\sup_{0 \leq t \leq T} \tilde{x}^2(t) = 7C_0 \int_0^T \tilde{x}^2(s)ds + 7 \int_0^T \tilde{g}^2(s)ds \int_0^T (\tilde{x}^\delta_2(s))^2ds + 7 \int_0^T (h^\delta_1(s)D^\delta_1(s))^2ds \int_0^T \tilde{x}^2(s)ds \\
&\quad + 7 \int_0^T (\bar{h}D)^2(s)ds \int_0^T (\tilde{x}^\delta_2(s))^2ds + 7T \int_0^T \tilde{r}^2(s)ds + 7 \sup_{0 \leq t \leq T} \left( \int_0^t (\bar{h}H)(s)dV(s) \right)^2 + 7\tilde{x}^2(0).
\]
From B–D–G inequality, we have
\[
\mathbb{E}_Q \sup_{0 \leq t \leq T} \tilde{x}^2(t) \leq 7C_0 \mathbb{E}_Q \int_0^T \sup_{0 \leq s \leq t} \tilde{x}^2(s) \, dt + \psi_T^0(\delta_1, \delta_2),
\]
where
\[
\psi_T^0(\delta_1, \delta_2) = 7 \int_0^T \tilde{g}_2(s) \, ds \mathbb{E}_Q \int_0^T (\tilde{x}^{\delta_2}(s))^2 \, ds + 7C_0 \mathbb{E}_Q \int_0^T \sup_{0 \leq r \leq s} \tilde{x}^2(r) \, ds + 7 \int_0^T \bar{r}(s)^2 \, ds + 7 \mathbb{E}_Q \bar{x}^2(0).
\]

By (H1) and (H2), we know that \(\psi_T^0(\delta_1, \delta_2)\) converges to 0 in \(L^2_\mathcal{P}(T)\) as \(\delta_1 \rightarrow \delta_2\). Here we have already applied the continuous dependence property of the solution of SDE on the parameter \(\delta\). From Gronwall’s inequality and (25), we get
\[
\mathbb{E}_Q \sup_{0 \leq t \leq T} \tilde{x}^2(t) \leq \psi_T^0(\delta_1, \delta_2) e^{7C_0 T}.
\]

The proof is completed. \(\square\)

It is easy to check that the stability of the filtering equation (20) is a particular case of the continuous dependence of the solution on the parameter. Therefore we have

**Corollary 3.3.** Let (H3) hold. Then the filtering equation (20) is stable in the sense of Definition 3.1.

**Example 3.4.** We set \(B(\cdot) = C_2(\cdot) \equiv 0\) and all the other coefficients in (8) and (10) be non-zero constants.

By Theorem 2.2, we get the corresponding Riccati equation
\[
\begin{cases}
\dot{P}(t) - 2AP(t) + D^2 H^{-2} P^2(t) - C_1^2 = 0, \\
P(0) = 0,
\end{cases}
\]
which has a solution
\[
P(t) = \frac{\lambda_1 - \lambda_2 \lambda_3 \exp\left\{ \frac{(\lambda_2 - \lambda_1)D^2}{H^2} \right\}}{1 - \lambda_3 \exp\left\{ \frac{(\lambda_2 - \lambda_1)D^2}{H^2} \right\}},
\]
where
\[
\lambda_1 = D^2 H(AH - \sqrt{A^2 H^2 + C_1^2 D^2}), \quad \lambda_2 = D^2 H(AH + \sqrt{A^2 H^2 + C_1^2 D^2}), \quad \lambda_3 = \frac{n_0^2 - \lambda_1}{n_0^2 - \lambda_2}.
\]

The filtering equation
\[
\begin{cases}
d\tilde{x}(t) = (A\tilde{x}(t) + C_1 f_1) \, dt + DH^{-1} P(t) \, d\tilde{W}(t), \\
\tilde{x}(0) = m_0
\end{cases}
\]
has a solution
\[
\tilde{x}(t) = m_0 + C_1 f_1 A^{-1}(e^{At} - 1) + DH^{-1} \int_0^t P(s) e^{A(t-s)} \, d\tilde{W}(s).
\]
Therefore
\[ \hat{y}(t) = \Pi(t) \hat{x}(t) + \pi(t), \quad \hat{z}(t) = C_1 \Pi(t), \]
where
\[ \Pi(t) = \left( \frac{a}{A} + 2 \right) e^{2A(T-t)} \frac{a}{A}, \]
\[ \pi(t) = C_1 f_1 A^{-1} \left[ \left( \frac{a}{A} + 2 \right) \left( e^{2A(T-t)} - e^{A(T-t)} \right) - \frac{a}{A} \left( e^{A(T-t)} - 1 \right) \right]. \]

Obviously, the solution \((\hat{x}(\cdot), \hat{y}(\cdot), \hat{z}(\cdot))\) of our filtering problem is stable in the sense of Definition 3.1.

4. A partially observed recursive optimal control problem

The objective of this section is to study a partially observed recursive optimal control problem, which has a close connection with results in Section 2. We use the same notations as those in Section 2.

Let us consider the following state and observation equations:
\begin{align}
\dot{X}_1(t) &= (A(t)X_1(t) + C_1(t) f_1(t) + C_2(t) f_2(t)) \, dt + C_1(t) \, dU(t) + C_2(t) \, dV(t), \\
\dot{Z}_1(t) &= (D(t)X_1(t) + F(t)Z_1(t) + f_2(t)H(t)) \, dt + H(t) \, dV(t), \\
X_1(0) &= \xi, \quad \hat{Z}_1(0) = 0,
\end{align}

\begin{align}
\dot{X}_2(t) &= A(t)X_2(t) + B(t)v(t), \quad X_2(0) = 0, \\
\dot{Z}_2(t) &= D(t)X_2(t) + F(t)\hat{Z}_2(t), \quad \hat{Z}_2(0) = 0,
\end{align}

where \(v(\cdot) \in \mathcal{U}_{ad}\) and all coefficients satisfy (H1) and (H2). For any \(v(\cdot) \in \mathcal{U}_{ad}\), it is easy to check that \(X_1(\cdot) + X_2(\cdot)\) and \(\hat{Z}_1(\cdot) + \hat{Z}_2(\cdot)\) are the unique solution of (4) and (10), respectively, i.e., \(X(\cdot) = X_1(\cdot) + X_2(\cdot), Z(\cdot) = \hat{Z}_1(\cdot) + \hat{Z}_2(\cdot)\).

Set \(\hat{Z}_t = \sigma(\hat{Z}_1(s); \ 0 \leq s \leq t)\). We present the following:

**Definition 4.1.** A control variable \(v(\cdot)\) is called admissible, if \(v(t)\) is an \(\mathbb{R}\)-valued stochastic process adapted to \(Z_t\) and \(\hat{Z}_t\) and satisfying \(E \int_0^T v^2(t) \, dt < +\infty\). The set of admissible controls is denoted by \(\mathcal{U}_{ad}\).

**Remark 4.2.** From Definition 4.1, we claim that if \(v(\cdot) \in \mathcal{U}_{ad}\) then \(Z_t = \hat{Z}_t, \ 0 \leq t \leq T\). In fact, it is clear that \(Z_t \supset \hat{Z}_t, \ 0 \leq t \leq T\). On the other hand, if \(v(\cdot) \in \mathcal{U}_{ad}\), from (27) we know that \(X_2(t)\) is \(\hat{Z}_t\)-adapted, so is \(\hat{Z}_2(t)\). Thus \(Z(t) = \hat{Z}_1(t) + \hat{Z}_2(t)\) is \(\hat{Z}_t\)-adapted. That is to say, \(Z_t \subseteq \hat{Z}_t, \ 0 \leq t \leq T\). Definition 4.1 implies us to determine the control by the observable process. But the observable process does not depend on the control. Otherwise, there is an immediate difficulty when the observable process depends on the control. It is the main reason that the state and the observation equations are decoupled.

It follows from Definition 4.1 and Remark 4.2 that
\[ \dot{X}(t) = E_0 [X(t) | Z_t] + X_2(t) = \dot{\hat{X}}_1(t) + \dot{X}_2(t). \]

Since (26) is similar to (10) and (17), from Theorem 2.2, we easily get the following result.

**Proposition 4.3.** For any \(v(\cdot) \in \mathcal{U}_{ad}\), let (H1) and (H2) hold. Then the state variable \(X(\cdot)\), which is the solution of (4), has a filtering estimation
\begin{align}
\dot{X}(t) &= (A(t)X(t) + B(t)v(t) + C_1(t) f_1(t) + C_2(t) f_2(t)) \, dt \\
&\quad + (C_2(t) + D(t)H^{-1}(t)\Delta(t)) \, d\tilde{U}(t), \\
\dot{X}(0) &= m_0,
\end{align}

where \(\tilde{U}(\cdot)\) is a \(\mathcal{U}_{ad}\)-controlled \(\mathbb{R}\)-valued stochastic process with \(\hat{Z}_t\)-adapted.
where the observable 1-dimensional standard Brownian motion $\tilde{U}(\cdot)$ is defined as

$$\tilde{U}(t) = V(t) + \int_0^t D(s)H^{-1}(s)(X(s) - \hat{X}(s)) \, ds$$

and $\Delta(\cdot) = \mathbb{E}_Q(X(\cdot) - \hat{X}(\cdot))^2$ satisfies

$$\begin{cases}
\dot{\Delta}(t) - 2A(t)\Delta(t) + (C_2(t) + D(t)H^{-1}(t)\Delta(t))^2 - C_1(t) - C_2(t) = 0, \\
\Delta(0) = n_0.
\end{cases}$$

Remark 4.4. Obviously, the solution $\Delta(\cdot)$ of (29) does not depend on the admissible control $v(\cdot) \in \tilde{U}_{ad}$. This is very important to solve the following Problem (PO).

Our problem is to seek a suitable $v(\cdot) \in \tilde{U}_{ad}$ to minimize the cost functional $J(v(\cdot))$ defined by (6) subject to (4) and (10). If an admissible control $\bar{u}(\cdot) \in \tilde{U}_{ad}$ satisfies

$$J(\bar{u}(\cdot)) = \min_{v(\cdot) \in \tilde{U}_{ad}} J(v(\cdot)),$$

then $\bar{u}(\cdot)$ is called an optimal control, and the corresponding state trajectory determined by (4) is denoted by $\hat{x}(\cdot)$.

For simplicity, we denote the above problem by Problem (PO). This is a partially observed recursive optimal control problem. A classical solving method is to combine Wonham’s separation theorem with a direct construction method introduced in Bensoussan [2]. However, under our framework, we will introduce a new technique to solve it in three steps. In details, we first regard Problem (PO) as Problem (FO) for a moment and seek its optimal solution, next we conjecture a candidate optimal control $\bar{u}(\cdot)$ of Problem (PO). To get an explicit observable optimal control, we apply the filtering estimation of BSDEs to characterize the adjoint process $\hat{y}(\cdot)$. This is different to Li and Tang [8], [16], in which backward stochastic PDEs was used to describe adjoint processes. At the last step, we verify that $\bar{u}(\cdot)$ defined by (36) is indeed an optimal control. In contrast with Wonham’s separation theorem, our method can be regarded as a backward separation technique. Follow our new technique, it is much more convenient, direct and valid to solve Problem (PO) than using the method introduced in Bensoussan [2]. It needs to point out that this idea is inspired by Li and Tang [8] and Tang [16], in which some theoretical results of maximum principles were derived, however they did not illustrate how to use their theoretical results to get an explicit observable optimal control of a partially observed optimal control problem. Moreover, to apply Girsanov’s theorem, which is necessary to obtain a maximum principle, Li and Tang need a crucial assumption, i.e., the drift term in their observation equation is uniformly bounded with respect to the state $x(\cdot)$ and the control $v(\cdot)$. Although in our setting, it still does not contain the control $v(\cdot)$, but linear with respect to $(X(\cdot), Z(\cdot))$, which partly generalizes the results of Li and Tang. This is another main difference to theirs.

Step 1. Optimal solution of Problem (FO).

Recalling the optimal control defined by (7), we claim that it is also unique. In fact, let $u_1(\cdot)$ and $u_2(\cdot)$ be optimal, and the corresponding trajectories be $X_1(\cdot)$ and $X_2(\cdot)$. Since (4) is a linear system, $\frac{X_1(\cdot) + X_2(\cdot)}{2}$ and $\frac{X_1(\cdot) - X_2(\cdot)}{2}$ are the trajectories under the controls $\frac{u_1(\cdot) + u_2(\cdot)}{2}$ and $\frac{u_1(\cdot) - u_2(\cdot)}{2}$. Set $J(u_1(\cdot)) = J(u_2(\cdot)) = \alpha$, where $\alpha$ is a constant. From Parallelogram law, it follows that

$$2\alpha = J(u_1(\cdot)) + J(u_2(\cdot)) = 2J\left(\frac{u_1(\cdot) + u_2(\cdot)}{2}\right) + 2J\left(\frac{u_1(\cdot) - u_2(\cdot)}{2}\right) = 2J\left(\frac{u_1(\cdot) + u_2(\cdot)}{2}\right)$$

$$+ \frac{1}{2}\mathbb{E}_Q \left\{ \int_0^T e^{\int_0^r b(s) \, ds} \left[ a(t)(X_1(t) - X_2(t))^2 + c(t)(u_1(t) - u_2(t))^2 \right] \, dt + e^{\int_0^T b(t) \, dt} (X_1(T) - X_2(T))^2 \right\}$$

$$\geq 2\alpha + \frac{1}{2}\mathbb{E}_Q \int_0^T e^{\int_0^r b(s) \, ds} c(t)(u_1(t) - u_2(t))^2 \, dt.$$
which implies
\[ E_Q \int_0^T e^{\int_0^t b(s) ds} c(t)(u_1(t) - u_2(t))^2 dt \leq 0. \]
Thus \( u_1(\cdot) \equiv u_2(\cdot) \), i.e., Problem (FO) exists a unique optimal control.

**Step 2.** Conjecture.

Obviously, \( \mathcal{U}_{ad} \subseteq \mathcal{U}_{ad} \), i.e., the optimal control \( \hat{u}(\cdot) \) of Problem (PO) is an element of \( \mathcal{U}_{ad} \). For Problem (PO), we cannot fully observe the state variable \( X(\cdot) \), and we also cannot observe the adjoint process \( \hat{y}(\cdot) \), but we can observe the noisy process \( Z(\cdot) \) related to \( X(\cdot) \). Our intuition is to replace \( \hat{y}(\cdot) \) by its filtering estimation \( \tilde{y}(\cdot) \). Introduce an observable control variable
\[ \hat{u}(t) = -\frac{1}{2} B(t) c^{-1}(t) e^{-\int_0^t b(s) ds} \tilde{y}(t), \]  
(30)
where \( (\hat{x}(\cdot), \tilde{y}(\cdot), \hat{z}_1(\cdot), \hat{z}_2(\cdot)) \) satisfies the FBSDE
\[
\begin{align*}
  d\hat{x}(t) &= \left( A(t) \hat{x}(t) - \frac{1}{2} B^2(t) c^{-1}(t) e^{-\int_0^t b(s) ds} \tilde{y}(t) + C_1(t) f_1(t) + C_2(t) f_2(t) \right) dt \\
  &\quad + C_1(t) dU(t) + C_2(t) dV(t), \\
  -d\tilde{y}(t) &= \left( 2a(t) e^{\int_0^t b(s) ds} \hat{x}(t) + A(t) \tilde{y}(t) \right) dt - \hat{z}_1(t) dU(t) - \hat{z}_2(t) dV(t), \\
  \hat{x}(0) &= \xi, \quad \tilde{y}(T) = 2e^{\int_0^T b(s) ds} \tilde{x}(T). 
\end{align*}
\]  
(31)
The above equation (31) is similar to (8), except that the drift term of the forward SDE in (31) contains the observable process \( \hat{y}(\cdot) \). For mathematically rigorous, we assume \( E_Q \int_0^T \tilde{y}^4(t) dt < +\infty \). Then the forward SDE in (31) admits a unique solution. So is the BSDE there. That is to say, for a given suitable \( \hat{y}(\cdot) \), there exists a unique solution to the FBSDE (31). On the other hand, the following formulas (32) and (34) show that the aforementioned assumption about \( \hat{y}(\cdot) \) is indeed reasonable.

Since \( E_Q \int_0^T \tilde{y}^4(t) dt < +\infty \), it is clear that \( \hat{u}(\cdot) \) defined by (30) is admissible. From step one, we conjecture that \( \hat{u}(\cdot) \) is a candidate optimal control of Problem (PO). To prove the conjecture is true in step 3, next we will give a more explicit form of \( \hat{u}(\cdot) \) by computing \( \hat{\pi}(\cdot), \tilde{y}(\cdot) \). Although in (31), the drift term of the forward SDE contains \( \tilde{y}(\cdot) \). Fortunately, \( \tilde{y}(\cdot) \) is observable. So it does not bring difficulty for us to compute \( \hat{\pi}(\cdot), \tilde{y}(\cdot) \). From Proposition 4.3, we easily derive that
\[
\begin{align*}
  d\hat{x}(t) &= \left( A(t) \hat{x}(t) - \frac{1}{2} B^2(t) c^{-1}(t) e^{-\int_0^t b(s) ds} \tilde{y}(t) + C_1(t) f_1(t) + C_2(t) f_2(t) \right) dt \\
  &\quad + (C_2(t) + D(t) H^{-1}(t) A(t)) dU(t), \\
  \hat{x}(0) &= m_0. 
\end{align*}
\]  
(32)
Solving (31) by usual techniques for BSDEs, we get
\[
\hat{y}(t) = 2e^{\int_0^T b(s) ds} \hat{x}(T) + \int_T^t a(s) e^{\int_0^s b(r) dr} + \int_0^s A(r) dr E_Q\left[ \hat{\pi}(s) | Z_t \right] ds. 
\]  
(33)
We now claim that
\[
\hat{y}(t) = \Pi(t) \hat{\pi}(t) + \pi(t), 
\]  
(34)
where \( \Pi(\cdot), \pi(\cdot) \) and \( \hat{\pi}(\cdot) \) are the solutions of (14), (15) and (32). In fact, if we let \( \Psi_t \) be the fundamental solution of
\[
\Psi_t = \begin{pmatrix}
A(t) - A(t) & 0 \\
-\Lambda(t) & A(t)
\end{pmatrix} \Psi_t
\]
combining (31) with (32), then we have
\[
\left( \begin{array}{c}
\hat{x}(s) \\
\bar{x}(s)
\end{array} \right) = \Psi(s, t) \left( \begin{array}{c}
\hat{x}(t) \\
\bar{x}(t)
\end{array} \right) + \int_{t}^{s} \Psi(s, r) \left( \begin{array}{c}
1 \\
1
\end{array} \right) \lambda(r) \, dr
\]
\[
+ \int_{t}^{s} \Psi(s, r) \left( \begin{array}{cc}
C_2(r) + D(r)H^{-1}(r)\Delta(r) & 0 \\
0 & C_1(r)
\end{array} \right) d \left( \begin{array}{c}
\bar{U}(r) \\
U(r)
\end{array} \right),
\]
where
\[
\Lambda(t) = \frac{1}{2} B_2(t) c^{-1}(t) \Pi(t) e^{-\int_{0}^{t} b(s) ds},
\]
\[\lambda(t) = C_1(t) f_1(t) + C_2(t) f_2(t) - \frac{1}{2} B_2(t) c^{-1}(t) \pi(t) e^{-\int_{0}^{t} b(\mu) d\mu}.
\]

It is easy to check that
\[
\mathbb{E}_Q \left[ \bar{x}(s) \mid Z_t \right] = \left( \begin{array}{c}
0 \\
1
\end{array} \right) \Psi(s, t) \left( \begin{array}{c}
\hat{x}(t) \\
\bar{x}(t)
\end{array} \right) + \int_{t}^{s} \left( \begin{array}{c}
0 \\
1
\end{array} \right) \Psi(s, r) \left( \begin{array}{c}
1 \\
1
\end{array} \right) \lambda(r) \, dr
\]
\[
= e^{\int_{T}^{t} (A(r) - \Lambda(r)) \, dr} \hat{x}(t) + \int_{t}^{s} e^{\int_{T}^{r} (A(\mu) - \Lambda(\mu)) \, d\mu} \lambda(r) \, dr.
\]

Substituting (35) into (33), we have
\[
\hat{y}(t) = \bar{\Pi}(t) \bar{x}(t) + \bar{\pi}(t)
\]
with
\[
\bar{\Pi}(t) = 2e^{\int_{0}^{t} b(s) ds} + \int_{i}^{T} e^{\int_{i}^{s} (2A(s) - \Lambda(s)) ds} \, ds + 2 \int_{i}^{T} a(s) e^{\int_{i}^{s} b(r) dr} + \int_{i}^{T} (2A(s) - \Lambda(s)) \, ds \, ds,
\]
\[
\bar{\pi}(t) = 2e^{\int_{0}^{t} b(s) ds} + \int_{i}^{T} A(s) ds + \int_{i}^{T} e^{\int_{i}^{T} (A(\mu) - \Lambda(\mu)) \, d\mu} \lambda(r) \, dr
\]
\[
+ 2 \int_{i}^{T} a(s) e^{\int_{i}^{s} b(r) dr} + \int_{i}^{T} A(s) ds + \int_{i}^{T} e^{\int_{i}^{T} (A(\mu) - \Lambda(\mu)) \, d\mu} \lambda(r) \, dr \, ds.
\]

From the existence and uniqueness of solutions to (14) and (15), it is easy to verify that \(\bar{\Pi}(\cdot)\) and \(\bar{\pi}(\cdot)\) satisfy (14) and (15), i.e., \(\bar{\Pi}(\cdot) \equiv \bar{\Pi}(\cdot), \bar{\pi}(\cdot) \equiv \bar{\pi}(\cdot).\) That is to say, the claim (34) is true. On the other hand, combining (32) and (34), we verify \(\mathbb{E}_Q \int_{0}^{T} \hat{x}^4(t) \, dt < +\infty.\) So does \(\bar{y}(\cdot),\) i.e., the aforementioned assumption about \(\hat{y}(\cdot)\) is reasonable. Thus the solvability of (31) is furthermore confirmed.

**Step 3.** Proof of optimization.

In this step, we will prove that
\[
\tilde{u}(t) = -\frac{1}{2} B(t) c^{-1}(t) e^{-\int_{0}^{t} b(s) ds} \left( \Pi(t) \hat{x}(t) + \pi(t) \right)
\]
is a unique optimal control of Problem (PO), where \(\hat{x}(\cdot)\) satisfies (32) with \(\hat{y}(\cdot)\) displaced by (34).
Since \(\hat{X}(\cdot) \perp (X(\cdot) - \hat{X}(\cdot)),\) the cost functional (6) can be rewritten as
\[
J(v(\cdot)) = J(v(\cdot)) + \int_{0}^{T} e^{\int_{0}^{t} b(s) ds} a(t) \Delta(t) \, dt + e^{\int_{0}^{T} b(s) ds} \Delta(T)
\]
with
\[ J(v) = \mathbb{E}_Q \left[ \int_0^T e^\int_0^t b(s)ds \left( ta(t)\hat{X}(t) + c(t)v^2(t) \right) dt + e^{\int_0^T b(s)ds} \hat{X}(T) \right], \]  
(38)

where \( \hat{X}(\cdot) \) and \( \Delta(\cdot) \) satisfy (28) and (29), respectively. For any \( v(\cdot) \in \bar{U}_{ad} \), we easily derive that
\[ J(v) - J(\bar{u}(\cdot)) = \mathbb{E}_Q \left\{ \int_0^T e^\int_0^t b(s)ds \left[ a(t)(\hat{X}(t) - \hat{\bar{x}}(t))^2 + c(t)(v(t) - \hat{\bar{u}}(t))^2 \right] dt 
+ e^{\int_0^T b(s)ds} (\hat{X}(T) - \hat{\bar{x}}(T))^2 \right\} + \Theta \]  
(39)

with
\[ \Theta = 2\mathbb{E}_Q \left\{ \int_0^T e^\int_0^t b(s)ds \left[ a(t)\hat{\bar{x}}(t)(\hat{X}(t) - \hat{\bar{x}}(t)) + c(t)\hat{\bar{u}}(t)(v(t) - \hat{\bar{u}}(t)) \right] dt 
+ e^{\int_0^T b(s)ds} \hat{\bar{x}}(T)(\hat{X}(T) - \hat{\bar{x}}(T)) \right\}. \]  
(40)

Since all the terms depending on \( \Delta(\cdot) \) have disappeared and the first term at the right-hand side of (39) is non-negative, we know that
\[ J(v(\cdot)) - J(\bar{u}(\cdot)) \geq \Theta. \]  
(41)

We claim that \( \Theta \equiv 0 \), which implies that \( \bar{u}(\cdot) \) defined by (36) is optimal. In fact, noting (14), (15), (28), (32) and (34), it follows from Itô’s formula that
\[ -d\hat{y}(t) = (A(t)\hat{y}(t) + 2a(t)e^\int_0^t b(s)ds \hat{\bar{x}}(t)) dt - \Pi(t)(C_2(t) + D(t)H^{-1}(t)\Delta(t)) d\hat{U}(t), \]
\[ \mathbb{E}_Q[\hat{y}(T)(\hat{X}(T) - \hat{\bar{x}}(T))] = \mathbb{E}_Q \left[ \int_0^T (\hat{X}(t) - \hat{\bar{x}}(t)) d\hat{y}(t) \right] 
+ \mathbb{E}_Q \left[ \int_0^T [A(t)(\hat{X}(t) - \hat{\bar{x}}(t)) + B(t)(v(t) - \hat{\bar{u}}(t))] \hat{y}(t) dt \right]. \]

Noting (34), (36) and \( \hat{y}(T) = 2e^\int_0^T b(s)ds \hat{x}(T) \), substituting the above two formulas into (40), we get
\[ \Theta = \mathbb{E}_Q \int_0^T (2e^\int_0^t b(s)ds c(t)\hat{\bar{u}}(t) + B(t)\hat{y}(t)) \{v(t) - \hat{\bar{u}}(t)\} dt \equiv 0. \]

Thus (41) implies that \( \bar{u}(\cdot) \) defined by (36) is an optimal control. Applying Parallelogram law similar to step one, we can also prove the uniqueness of \( \bar{u}(\cdot) \).

Now we only need to compute \( J(\bar{u}(\cdot)) \). Substituting \( \bar{u}(\cdot) \) into (38), we get
\[ J(\bar{u}(\cdot)) = \Sigma + \frac{1}{4} \int_0^T B^2(t)c^{-1}(t)\pi^2(t)e^{-\int_0^t b(s)ds} dt, \]  
(42)

where
The equations are as follows:

Example 4.8. For simplicity, let us only consider the case of two players. The 1-dimensional state and observation game problem.

Theorem 4.5. Let (H1) and (H2) hold. Then the unique optimal control and the cost functional of Problem (PO) are given by (36) and (43), respectively.

Remark 4.6. If we set \( b(\cdot) = f_1(\cdot) = f_2(\cdot) = 0 \), our Theorem 4.5 reduces to the classical results obtained in Liptser and Shiryayev [9] and Bensoussan [2], in which Wonham’s separation theorem is used to get an explicit observable optimal control.

Remark 4.7. We find that the filtering estimation \( (\hat{x}(\cdot), \hat{y}(\cdot)) \) of \( (\hat{x}(\cdot), \hat{y}(\cdot)) \), which is the solution of (31), play an important role in looking for an optimal control \( \hat{u}(\cdot) \) of Problem (PO). Although in (31), the drift term of the forward SDE contains \( \hat{y}(\cdot) \). Fortunately, \( \hat{y}(\cdot) \) is observable. So it does not bring any difficulty for us to compute \( (\hat{x}(\cdot), \hat{y}(\cdot)) \). This is also a motivation for us to study the filtering problems of a Hamiltonian system which is an FBSDE in Section 2.

In fact, the above backward separation technique can cover a general situation than Problem (PO). For example, we can formulate the following partially observed linear quadratic non-zero sum differential game problem. Following the backward technique and applying Theorem 2.2, we obtain an explicit observable Nash equilibrium point of the game problem.

Example 4.8. For simplicity, let us only consider the case of two players. The 1-dimensional state and observation equations are as follows:

\[
\begin{align*}
\frac{dx(t)}{dt} &= (Ax(t) + B_1v_1(t) + B_2v_2(t))dt + C_1dW_1(t) + C_2dW_2(t), \\
x(0) &= \xi, \\
\frac{dZ(t)}{dt} &= D(t)x(t)dt + F(t)dW_2(t), \\
Z(0) &= 0.
\end{align*}
\]

Here and below, for convenience, we let \( M_i \geq 0, N_i \geq 0, Q_i \geq 0, B_i, C_i (i = 1, 2) \) and \( A \) be constants, \( D(\cdot) \) and \( F(\cdot) \) be bounded deterministic in \([0, T] \), \( F^{-1}(\cdot) \) be also bounded. \( \xi \) is an \( \mathcal{F}_0 \)-measurable Gaussian random variable, independent of \((W_1(\cdot), W_2(\cdot))\), with the mean \( m_0 \) and the variance \( n_0 \geq 0 \).
The cost functionals of two players are denoted by
\[
J_1(v_1(\cdot), v_2(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( M_1 x^2(t) + N_1 v_1^2(t) \right) dt + Q_1 x^2(T) \right].
\]

Our problem is to find a pair of \((u_1(\cdot), u_2(\cdot))\) such that
\[
J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot) \in \tilde{U}_{ad}} J_1(v_1(\cdot), u_2(\cdot)), \quad J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in \tilde{U}_{ad}} J_2(u_1(\cdot), v_2(\cdot)),
\]
where
\[
\tilde{U}_{ad} = \left\{ v_1(\cdot) \mid v_1(t) \text{ is an } \mathbb{R}\text{-valued stochastic process adapted to } Z_t \text{ and } \tilde{Z}_t \text{ and satisfies } \mathbb{E} \int_0^T v_1^2(t) dt < +\infty \right\}.
\]

Then \((u_1(\cdot), u_2(\cdot))\) is called a Nash equilibrium point of the game problem (46) subject to (44) and (45). Since the drift term of the state equation (44) contains two admissible controls corresponding to two players, it is more general than Problem (PO). Fortunately, both of them are observable. And the state and observation equations (44) and (45) are similar to (4) and (10). Thus it does not bring any trouble to estimate the state \(x(\cdot)\) by filtering.

The solving method is similar to Problem (PO), so we omit some of the similar proofs and only give key results here. Introduce the following Riccati differential equations:
\[
\begin{align*}
\dot{\Sigma}_1(t) + 2A \Sigma_1(t) - N_2^{-1} B_2^2 \Sigma_1(t) \Sigma_2(t) - N_1^{-1} B_1^2 \Sigma_1^2(t) + M_1 = 0, \\
\Sigma_1(T) = Q_1, \\
\end{align*}
\]
(47)
\[
\begin{align*}
\dot{\Sigma}_2(t) + 2A \Sigma_2(t) - N_1^{-1} B_1^2 \Sigma_1(t) \Sigma_2(t) - N_2^{-1} B_2^2 \Sigma_2^2(t) + M_2 = 0, \\
\Sigma_2(T) = Q_2.
\end{align*}
\]
(48)

The above two equations are coupled together. To prove that there exist solutions to them, we need an additional assumption \(N_1^{-1} B_1^2 = N_2^{-1} B_2^2\). Introduce the following equations:
\[
\begin{align*}
\dot{\Sigma}(t) + 2A \Sigma(t) - N_1^{-1} B_1^2 \Sigma^2(t) + M_1 + M_2 = 0, \quad \Sigma(T) = Q_1 + Q_2, \\
\dot{\tilde{\Sigma}}_1(t) + (2A - N_2^{-1} B_2^2 \Sigma(t)) \tilde{\Sigma}_1(t) + M_1 = 0, \quad \tilde{\Sigma}_1(T) = Q_1, \\
\dot{\tilde{\Sigma}}_2(t) + (2A - N_1^{-1} B_1^2 \Sigma(t)) \tilde{\Sigma}_2(t) + M_2 = 0, \quad \tilde{\Sigma}_2(T) = Q_2.
\end{align*}
\]
(49)
(50)
(51)

It is clear that (49) admits a unique solution. Thus (50) and (51) also exist a unique solution, respectively. Let \(\tilde{\Sigma}(\cdot) = \tilde{\Sigma}_1(\cdot) + \tilde{\Sigma}_2(\cdot)\). We can verify that \(\tilde{\Sigma}(\cdot)\) satisfies (49), i.e., \(\dot{\tilde{\Sigma}}(\cdot) = \Sigma(\cdot)\). Substituting \(\Sigma(\cdot) = \tilde{\Sigma}_1(\cdot) + \tilde{\Sigma}_2(\cdot)\) into (50) and (51), we easily know that (47) and (48) exist a unique solution, respectively.

Following the backward technique and applying Theorem 2.2, we get an explicit observable Nash equilibrium point
\[
\hat{u}_i(t) = -N_i^{-1} B_i \Sigma_i(t) \hat{x}(t),
\]
(52)
where \(\hat{x}(\cdot)\) is the solution of
\[
\begin{align*}
d\hat{x}(t) &= \left( A - N_1^{-1} B_1^2 \Sigma_1(t) - N_2^{-1} B_2^2 \Sigma_2(t) \right) \hat{x}(t) dt + \left( C_2 + D(t) F^{-1}(t) \Delta(t) \right) d\tilde{W}(t), \\
\hat{x}(0) &= m_0.
\end{align*}
\]
(53)

The square error \(\Delta(\cdot) = \mathbb{E}(x(\cdot) - \hat{x}(\cdot))^2\) and the innovation process \(\tilde{W}(\cdot)\) satisfy respectively
\[
\begin{align*}
\dot{\Delta}(t) - 2A \Delta(t) + \left( C_2 + D(t) F^{-1}(t) \Delta(t) \right)^2 - C_2^2(t) - C_2^2(t) = 0, \\
\Delta(0) &= n_0,
\end{align*}
\]
\[
\tilde{W}(t) = W_2(t) + \int_0^t D(s) F^{-1}(s) (x(s) - \hat{x}(s)) ds.
\]
5. Computing the information value

In this section, our task is to compute the difference of the optimal recursive cost functionals of Problem (PO) and Problem (FO)

\[ ΔJ = J(\tilde{u}(\cdot)) - J(u(\cdot)), \]

which is called the information value.

Substituting \( u(\cdot) \) given in (7) into the cost functional (6), by usual techniques, we have

\[
J(u(\cdot)) = \int_0^T \left[ \frac{1}{2} (C_2^2(t) + D(t)H^{-2}(t)) \Delta(t) (D(t)\Delta(t) + 2C_2(t)H(t)) - C_1^2(t) \Pi(t) \right] dt \\
+ \frac{1}{2} (m_0^2 + n_0) \Pi(0) + m_0 \pi(0).
\]

From (43) and (54) we easily get the following information value formula

\[
ΔJ = \frac{1}{2} \int_0^T \left[ C_2^2(t) + D(t)H^{-2}(t) \Delta(t) (D(t)\Delta(t) + 2C_2(t)H(t)) - C_1^2(t) \Pi(t) \right] dt \\
+ \int_0^T a(t) \Delta(t)e_{0}^{\int_0^t b(s)ds} dt + Δ(T)e_{0}^{\int_0^T b(s)ds} - \frac{1}{2} n_0 \Pi(0).
\]

Therefore we have

**Theorem 5.1.** Let (H1) and (H2) hold. Then the information value \( ΔJ \) defined by (54) can be written as (55).

**Remark 5.2.** The information value formula shows the following fact: to minimize the cost functional, it is very important for controllers to collect more information. \( ΔJ \) does not depend on \( m_0 \), which is the mean of the Gaussian random variable \( \xi \), and increases in \( \Delta(\cdot) \), the square error of the filtering estimation. Obviously, these results coincide with our intuition.

**Remark 5.3.** We note that there are no state and control variables in the diffusion coefficients of the FBSDE (8) and the control system (1). For that case, we cannot get the explicit filtering estimation for this kind of forward and backward stochastic system, and the optimal control for the partially observed recursive optimal control problem. To our best knowledge, it is still an open problem. For partially observed forward and backward stochastic systems, there exists few theoretical result up to now. We hope that we could furthermore develop this kind of theory and find more applications in our future work.

**Acknowledgments**

The authors would like to thank the anonymous referee for the careful reading and helpful comments and suggestions that led to an improved version of this paper. The authors also thank Dr. Mingyu Xu for many helpful advices.

**References**