Normal Forms for Periodic Systems

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INTRODUCTION

The theory of normal forms for autonomous differential equations based on the Lie bracket has proved to be an important technique in determining the behaviour of solutions of nonlinear differential equations near a stationary solution. It has been widely used in the bifurcation theory for autonomous ordinary differential equations (see, e.g., [1, 3, 6, 9, 11]). In its simplest form the theory applies to an equation of the form

$$y' = Ay + Q_2(y) + Q_3(y) + \cdots + Q_j(y) + \cdots,$$

(0.1)
in which $A$ is an $n \times n$ matrix and $Q_j(y)$ is a homogeneous polynomial of degree $j$ (in each component) in $y \in \mathbb{R}^n$. One seeks to make a near identity transformation of the form

$$y = u + D_2(u) + \cdots + D_k(u),$$

(0.2)
in which the $D_j(u)$, homogeneous of degree $j$ in $u$, $2 \leq j \leq k$, are selected in such a way that resulting equation for $u$

$$u' = Au + \tilde{Q}_2(u) + \cdots + \tilde{Q}_k(u) + \cdots,$$

(0.3)
has as simple a form (normal form) as possible through terms of order $k$. The theory amounts to a systematic way of choosing $D_j(u)$, $2 \leq j \leq k$, in (0.2) so as to eliminate as many terms of order less than or equal to $k$ in $u$ as possible from (0.1). Ultimate success would be $Q_j(u) \equiv 0$, $2 \leq j \leq k$, and this can sometimes be accomplished depending on the spectrum of $A$. Suffice it to say here that the normal form (0.3) is obtained by solving linear equations in the vector spaces of homogeneous polynomials of degree $j$, $2 \leq j \leq k$. For more complete description of the theory we refer the reader to one of the sources [1, 4, 11].

It appears not to be well known that this theory has been extended to periodic systems and that significant applications can be made of this
theory. The author rediscovered the generalization of normal forms to periodic systems in recent work involving a perturbed Hill's equation [10]. The method is briefly described in a recent translation of a book of V. I. Arnold [12]. Our purpose in this paper is to flesh out the description in [12] and to systematically study the qualitative properties of normal forms. We show how the theory can be modified to approximate integral manifolds of solutions and how the theory can be useful in bifurcation problems. Much of this work is patterned after similar results in the recent text of S. N. Chow and J. K. Hale [11] for autonomous systems. We do not, however, follow the mathematical approach to normal forms described in [11], preferring the conceptually simpler view of successive changes of variable characteristic of the method of averaging.

Briefly, we consider the periodic system

\[ y' = A(t) y + Q_2(t, y) + Q_3(t, y) + \cdots + Q_k(t, y) + \cdots, \quad (0.4) \]

in which the \( n \times n \) matrix and the homogeneous polynomials of degree \( j \) are \( T \)-periodic in \( t \). The assumption of periodicity in the higher order terms in (0.4) is not absolutely necessary. One can sometimes get away with almost periodic or bounded (in \( t \)) coefficients in the \( Q_j(t, y) \) but not generally because of the failure of the Fredholm alternative for periodic linear differential equations with almost periodic inhomogeneities. Since \( A(t) \) is periodic in \( t \), the Floquet theory implies that we can make a Floquet change of variables

\[ y = P(t) \tilde{y}, \]

where \( P(t) \) is \( T \)-periodic in \( t \) and nonsingular so that the resulting equation for \( \tilde{y} \) has the same form as (0.4) except that \( A(t) \) is now a constant matrix the eigenvalues of which are the Floquet exponents of the linear system

\[ z' = A(t) z \quad (0.5) \]

Indeed, for a fundamental matrix solution \( \Phi(t) \) of (0.5), \( \Phi(t) = P(t) e^{At} \), where \( A \) is the above mentioned constant matrix. We assume this Floquet change of variables has been performed so that \( A \) is constant in (0.4).

We will show that a near identity change of variables of the form

\[ y = u + D_2(t, u) + \cdots + D_k(t, u) \quad (0.6) \]

can be determined in such a way that the resulting differential equation for \( u \) has generally a simpler (normal form) through terms of order \( k \) in \( u \)

\[ u' = Au + \bar{Q}_2(t, u) + \cdots + \bar{Q}_k(t, u) + \cdots. \quad (0.7) \]
In (0.6) the $D_j(t, u)$ are homogeneous polynomials of degree $j$ in $u$ with $T$-periodic coefficients. The theory will consist of a systematic way of finding $\bar{Q}_j(t, u)$ and choosing $D_j(t, u)$ which will involve solving constant coefficient linear inhomogeneous ordinary differential equations with $T$-periodic inhomogeneities in the vector space of homogeneous polynomial functions of degree $j$, $2 \leq j \leq k$.

It is well known that one can study (0.4) by considering its Poincaré map (time $T$-map) which maps an initial condition to the point on its trajectory a time $T$ later. One can then apply the theory of normal forms for mappings to the Poincaré map. This is a well developed theory: a particularly readable description of this method can be found in [7]. The major drawback to both the theory of normal forms for mappings applied to the Poincaré map and the normal form (0.7) which we develop in this work is the difficulty of computing the coefficients, whether of the mapping or the $\bar{Q}_k(t, u)$ in (0.7). We feel, however, that it is simpler to deal with the differential equation (0.4) than with its Poincaré map, the nonlinear part of which is very difficult to compute.

In the following section we introduce some notation and conventions which we will use throughout the paper. Our main results are contained in Section 1 and are patterned after the corresponding results for autonomous equations described in [1]. Our presentation, as mentioned earlier, is from a different point of view than in [1]. In Section 2 we apply our results to a well-understood model problem, namely, the bifurcation of an invariant cylinder of solutions from a periodic solution of a nonautonomous periodic planar system as two Floquet exponents exit the unit circle in a non-resonant fashion. We also consider two resonant cases. This example illustrates the usefulness of the results to bifurcation problems. The author, in previous work [10], has applied the normal form theory to a perturbed Hill's equation.

**Preliminaries**

In the following sections we will require some notation and modes of representation for homogeneous polynomials of a fixed degree. We develop the necessary material here. Denote by $V(k, m, n)$ the finite dimensional vector space of functions $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ each component of which is a homogeneous polynomial of degree $k$ in $x \in \mathbb{R}^m$. Let $e_1, e_2, \ldots, e_n$ be the usual basis vectors in $\mathbb{R}^n$. We choose an ordered basis for the vector space $V(k, m, n)$ as follows

$$B = \{ x_1^k e_1, x_1^{k-1} x_2 e_1, \ldots, x_m^k e_1, x_m^k e_2, \ldots, x_m^k e_n \}.$$
More precisely, let \( q = (q_1, q_2, \ldots, q_m) \) be a multi-index, \( q_i > 0 \) a positive integer, \( |q| = \sum q_i \). We use \( (\cdot, \cdot) \) for the ordinary inner product on \( \mathbb{R}^m \). Then \( B \) consists of the elements \( x^i e_j, \ |q| = k, \ 1 \leq j \leq n, \) where \( x^q \equiv x_1^{q_1} x_2^{q_2} \cdots x_m^{q_m} \). The element \( x^i e_j \) precedes \( x^q e_i \) in the ordering if \( j < i \) or if \( j = i \) and the first nonzero difference \( q_1 - q_i, q_2 - q_2, \ldots, q_m - q_m \) is positive. The basis \( B \) for \( V(k, m, n) \) induces an inner product on \( V(k, m, n) \) by taking the elements of \( B \) to be an orthonormal basis for \( V(k, m, n) \). We use \( \langle \cdot, \cdot \rangle \) to denote this inner product on \( V(k, m, n) \).

It will be convenient to have the following notation concerning \( T \)-periodic functions with values in some normed vector space, \( V \). Let \( P_T(V) \) denote the space of continuous, \( T \)-periodic, \( V \)-vector valued functions where \( V \) is a fixed normed vector space. If \( V \) is \( \mathbb{R} \) we have the usual \( T \)-periodic continuous scalar functions. The mean value of such a function will be denoted by \( [ \cdot ]_T \),

\[
[f]_T = \frac{1}{T} \int_0^T f(s) \, ds.
\]

I. MAIN RESULTS

We will be concerned with the periodic system of differential equations

\[
y' = Ay + Q_2(t, y) + Q_3(t, y) + \cdots, \quad (1.1)
\]

in which \( A \) is an \( n \times n \) constant matrix and \( Q_j(t, y) \in P_T(V(j, n, n)), \ j \geq 2 \). This rather fancy notation will soon be justified. Typically only the first few terms of (1.1) will interest us so that we need not assume the right-hand side of (1.1) is analytic.

Our aim is to understand the behavior of solutions of (1.1) near \( y = 0 \) by making a \( T \)-periodic time-dependent near-identity transformation of variables such that the resulting equations have a much simpler "normal form." More precisely, we introduce the change of coordinates

\[
y = v + D_2(t, v) + \cdots + D_{k-1}(t, v), \quad D_j \in P_T(V(j, n, n)) \quad (1.2)
\]

and seek to determine, \( D_j, \ 2 \leq j \leq k - 1, \) so that the resulting differential equation for \( v \) has as simple a form as possible through order \( (k - 1) \) terms,

\[
v' = Av + \bar{Q}_2(t, v) + \cdots + \bar{Q}_{k-1}(t, v) + \bar{Q}_k(t, v) + \bar{Q}_{k+1}(t, v) \cdots, \quad (1.3)
\]

where \( \bar{Q}_j(t, v) \in P_T(V(j, n, n)) \) depend on certain choices of projections \( P_j, \ 2 \leq j \leq k - 1, \) on the spaces \( P_T(V(j, n, n)) \). The truncated version of (1.3)

\[
v' = Av + \bar{Q}_2(t, v) + \cdots + \bar{Q}_{k-1}(t, v) \quad (1.4)
\]
(which has still to be properly defined) will be called the normal form of
(1.1) through order \((k - 1)\) based on the projections \(P_j \in P_T(V(j, n, n))\). We
make all this precise by showing how one obtains the normal form of (1.1)
through order \(k\) from (1.3). Introduce the change of variables

\[ v = u + D_k(t, u), \quad D_k \in P_T(V(k, n, n)) \]  

in (1.3). A straightforward computation gives the differential equation for \(u\),

\[ u' = Au + \tilde{Q}_2(t, u) + \cdots + \tilde{Q}_{k-1}(t, u) \]

\[ + \left[ \tilde{Q}_k(t, u) - \frac{\partial}{\partial t} D_k(t, u) + AD_k - \left( \frac{\partial D_k}{\partial u} \right)(Au) \right] + O(|u|^{k+1}). \]  

Note that the transformation (1.5) did not effect the normal form of order
\((k - 1)\) but it certainly did effect the terms of order \(k\) and the higher order
terms which are not explicitly presented. Our task now is to select \(D_k\) so
that the term in brackets has a simple form. For this we need some
additional notation. Define

\[ M_k : V(k, n, n) \to V(k, n, n) \quad \text{by} \quad (M_k f)(u) = \left( \frac{\partial f}{\partial u} \right)(Au) - Af(u). \]  

It is easily seen that \(M_k\) is a linear operator on \(V(k, n, n)\). Now consider the
differential equation for \(f(t) \in V(k, n, n)\),

\[ f' + M_k f - \phi, \phi \in P_T(V(k, n, n)). \]  

Equation (1.8) can be solved for \(f \in P_T(V(k, n, n))\) if and only if

\[ \mathcal{A}_{\phi}(\phi) = [\langle \phi(t), g(t) \rangle_T = 0 \]  

for every solution \(g \in P_T(V(k, n, n))\) of the adjoint equation

\[ g' - M^*_k g = 0, \]  

where \(M^*_k\) is the transpose of \(M_k\) with respect to the basis \(B\) and inner
product \(\langle \cdot, \cdot \rangle\). Let \(\{g_1(t), g_2(t), \ldots, g_l(t)\}\) be a linearly independent set of
solutions of (1.10) which span the \(T\)-periodic solutions of (1.10). Then it
follows that the range of the operator \((d/dt + M_k)\) on \(P_T(V(k, n, n))\) is

\[ R_k = \{ \phi \in P_T(V(k, n, n)) : \mathcal{A}_{g_i}(\phi) = 0, \quad 1 \leq i \leq l \}. \]

Let \(S_k\) be a (finite dimensional) complement to \(R_k\)

\[ P_T(V(k, n, n)) = R_k \oplus S_k\]
and $P_k$ be the projection onto $S_k$ along $R_k$. Finally, let $N_k$ denote the null space of $((d/dt) + M_k)$ on $P_T(V(k, n, n))$ and let $0_k$ denote a choice of a complement to $N_k$ in $P_T(V(k, n, n))$

$$P_T(V(k, n, n)) = N_k \oplus 0_k.$$ 

Then

$$\left( \frac{d}{dt} + M_k \right): 0_k \to R_k$$

is an isomorphism and we denote by $K_k$ its inverse.

Now consider the order $k$ terms in brackets in (1.6). This term may be rewritten in our notations as

$$- CP_k + M_kD_k - (I - P/c) \bar{Q}_k - Pd.$$  \hspace{1cm} (1.11)

Since $(I - P_k) \bar{Q}_k \in R_k$ we can select $D_k \in P_T(V(k, n, n))$, $D_k = \bar{Q}_k(I - P/c) \bar{Q}_k$, \hspace{1cm} (1.12)

so that the term in parentheses in (1.11) vanishes. The remainder we define to be $\bar{Q}_k(t, u)$,

$$\bar{Q}_k(t, u) = P_k \bar{Q}_k.$$ \hspace{1cm} (1.13)

Thus we obtain the normal form of (1.1) through terms of order $k$,

$$u' = Au + \bar{Q}_2(t, u) + \cdots + \bar{Q}_k(t, u) + O(|u|^{k+1}).$$ \hspace{1cm} (1.14)

It is important to note that this normal form depends on the matrix $A$ and the choice of complements to the spaces $R_j$ and $N_j$, $2 \leq j \leq k$.

Now, to perform the computations outlined above we need to know the ranges $R_k$ and null spaces $N_k$ of $((d/dt) + M_k)$ on $P_T(V(k, n, n))$. For this, one must know the spectrum of $M_k$, $\sigma(M_k)$.

**Proposition 1.1** (see, e.g., [1]). If $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ then

$$\sigma(M_k) = \{(q, \lambda) - \lambda_j: |q| = k, 1 \leq j \leq n\},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$.

It follows that (1.10) has solutions in $P_T(V(k, n, n))$ if and only if

$$(H1) \quad (q, \lambda) - \lambda_j = 2m\pi i/T,$$

for some $q$, $|q| = k$, nonnegative integer $m$, and $j$, $1 \leq j \leq n$. The above observation proves
THEOREM 1.2. If

\((H1)_k \quad (q, \lambda) - \lambda_j \neq 2\pi mi/T, \quad 2 \leq |q| \leq k, \quad m \geq 0, \quad 1 \leq j \leq n, \) then every normal form for (1) through order \(k\) is \(u' = Au\).

If \((H1)_k\) holds for all \(k > 2\), then there is a formal change of coordinates such that (1) becomes

\[ u' = Au. \quad (1.15) \]

Theorem 1.2 follows from the fact that \(R_j = P_T(V(j, n, n))\) and \(P_j = 0, 1 \leq j \leq k\). Since the infinite series describing the change of coordinates effecting (1.15) may not converge, we term this a formal change of coordinates. Hereafter we will be concerned with obtaining qualitative information about normal forms for (1.1) when \((H1)_k\) fails for some \(k\).

THEOREM 1.3. Suppose that \((H1)\) fails to hold for some set of \(k\), but fails only for \(m = 0\). More precisely, suppose

\((H2)_k \quad (q, \lambda) - \lambda_j \neq 2\pi mi/T, \quad 2 \leq |q| \leq k, \quad m > 1, \quad 1 \leq j \leq n. \)

Then some normal form for (1.1) through order \(k\) is autonomous,

\[ u' = Au + \mathcal{Q}_2(u) + \cdots + \mathcal{Q}_k(u). \quad (1.16) \]

If \((H2)_k\) holds for \(k = \infty\) then a formal change of variables can be found reducing (1.1) to an autonomous equation.

Proof. \((H2)_k\) is equivalent to the statement that Eq. (1.10) can have only \(t\)-independent solutions in \(P_T(V(j, n, n))\), \(2 \leq j \leq k\). If we take \(S_j = \text{span}\{g_1, \ldots, g_j\}\), where \(\{g_1, \ldots, g_j\}\) is an orthonormal set spanning the solutions of (1.10) in \(P_T(V(j, n, n))\) and define

\[ (P_j \phi)(t) = \sum_{i=1}^{k} \frac{A_{g_i}(\phi)}{A_{g_i}(g_i)} g_i, \quad \phi \in P_T(V(j, n, n)), \]

then the theorem follows (see (1.13)).

Theorem 1.3 can be very useful in applications since in the new coordinates the system may be regarded as a periodic perturbation of the autonomous equation (1.16), i.e., we have

\[ u' = Au + \mathcal{Q}_2(u) + \cdots + \mathcal{Q}_k(u) + \mathcal{Q}_{k+1}(u) + \cdots. \quad (1.17) \]

Standard theorems from ordinary differential equations (see, e.g., [5]) assert that hyperbolic steady states of (1.16) near \(u = 0\) correspond to \(T\)-periodic solutions of (1.16) while hyperbolic periodic orbits near \(u = 0\) of
(1.17) correspond to invariant cylinders of solutions of (1.16) in $\mathbb{R}^n \times \mathbb{R}$ which may be viewed as tori on the identification $(x, t) \sim (x, t + T)$.

Below we give an example illustrating some of the computations involved in computing a normal form.

**Example 1.** Consider the $T$-periodic system

$$
y'' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y' + Q_z(t, y) + Q_s(t, y) + \cdots. \tag{1.18}
$$

We compute a normal form through order 2 for (1.18). We write

$$Q_2(t, y) = q_1(t) y_1^2 e_1 + q_2(t) y_1' y_2 e_1 + q_3(t) y_2^2 e_1 + q_4(t) y_1^2 e_2$$

$$+ q_5(t) y_1 y_2 e_2 + q_6(t) y_2^2 e_2 = (q_1, \ldots, q_6).$$

The operator $M_2: V(2, 2, 2) \to V(2, 2, 2)$ is given by

$$\left( M_2 f \right) = \frac{\partial f}{\partial y} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f(y),$$

or, in terms of the basis above,

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $\sigma(A) = \{0\}$, $\sigma(M_2) = 0$. The null space of $M_2^*$ is easily computed; it is spanned by $v_1 = \text{col}(0, 0, 0, 1, 0, 0)$ and $v_2 = \text{col}(2, 0, 0, 0, 1, 0)$. Note Theorem 1.3 applies to this example. We set $p_1 = v_1$ and $p_2 = \text{col}(0, 0, 0, 1, 0, 0)$ so $\langle p_i, v_j \rangle = \delta_{ij}$ and define the projection $P$ on $P_1(V(2, 2, 2))$ onto $S = \text{span} \{p_1, p_2\}$ by

$$(Pf)(t) = \left[ \langle f(t), q_1 \rangle \right]_T p_1 + \left[ \langle f(t), q_2 \rangle \right]_T p_2.$$

A simple calculation shows that

$$(PQ_2) = \alpha y_1^2 e_2 + \beta y_1 y_2 e_2,$$

where

$$\alpha = \left[ q_4(t) \right]_T,$$

$$\beta = \left[ 2q_1(t) + q_5(t) \right]_T.$$
The normal form of order 2 for (1.18) is then given by

\[ u_1' = u_2, \]
\[ u_2' = \alpha u_1^2 + \beta u_1 u_2. \]  

(1.19)

The reader may check that if we had used the projection in the proof of Theorem 1.3 our normal form would have contained an additional term. In [10] the author obtained a two parameter unfolding of the normal form (1.19) in a study of the dynamics of a perturbed Hill's equation near an eigenvalue corresponding to \( \pi \)-periodic solutions.

We return to the study of qualitative properties of normal forms. Consider the case that \( A = \text{diag}[B, C] \), where \( B \) is \( n_1 \times n_1 \) and \( C \) is \( n_2 \times n_2 \), \( n_1 + n_2 = n \). Let the eigenvalues of \( B \) be \( \{\mu_1, \ldots, \mu_{n_1}\} \) and those of \( C \) be \( \{v_1, \ldots, v_{n_2}\} \). System (1.1) becomes

\[ y_1' = By_1 + Q_2(t, y_1, y_2) + \cdots + Q_{k_1}(t, y_1, y_2) + \cdots, \]
\[ y_2' = Cy_2 + Q_2(t, y_1, y_2) + \cdots + Q_{k_2}(t, y_1, y_2). \]  

(1.20)

**Theorem 1.4.** If

\[ (H3)_k \quad (r, \mu) - v_j \neq 2\pi mi/T, \quad 2 \leq |r| \leq k, \quad m > 0, \quad 1 \leq j \leq n_2, \]

then a normal form for (1.20) of order \( k \) has the form

\[ u_1' = Bu_1 + Q_{21}(t, u_1, u_2) + \cdots + Q_{k_1}(t, u_1, u_2). \]  

(1.21a)

\[ u_2' = Cu_2 + Q_{22}(t, u_1, u_2) + \cdots + Q_{k_2}(t, u_1, u_2), \]  

(1.21b)

in which \( Q_{j}(t, u_1, u_2) = 0 \), \( 2 \leq j \leq k \), i.e., \( u_2 = 0 \) is an invariant manifold for (1.21). In addition, if we set \( u_2 = 0 \) in (1.21a), the resulting equation is in a normal form.

If \( u_2 \) is set to zero in (1.21) the resulting equation

\[ u_1' = Bu_1 + Q_{21}(t, u_1, 0) + \cdots + Q_{k_1}(t, u_1, 0) \]  

(1.22)

is in a normal form. This equation gives the \( k \)th-order approximation to the dynamics on the invariant manifold \( u_2 = 0 \).

The proof of Theorem 1.4 (to follow) proceeds simply by decomposing the various equations (1.8), (1.10) into components and solving each component separately making sure to pick the various projections to be compatible with this decomposition. As a by product of using the framework described earlier and used to obtain Theorems 1.2 and 1.3, we have perhaps wasted effort on putting all of (1.20) in normal form (1.21) whereas we may only be really concerned with making sure \( u_2 = 0 \) is an invariant manifold.
for (1.21) and (1.22) is in normal form. The effort expended in putting all of (1.20) in normal form has another unfortunate side effect, namely, in requiring us to make the change of variables

\[ y_1 = u_1 + D_{21}(t, u_1, u_2) + \cdots + D_{k1}(t, u_1, u_2), \]
\[ y_2 = u_2 + D_{22}(t, u_1, u_2) + \cdots + D_{k2}(t, u_1, u_2). \]

As a corollary to the proof of Theorem 1.4 it will follow that a change of variables of the form

\[ y = u + D_2(t, u_1) + \cdots + D_k(t, u_1), \quad D_j \in P_1(V(j, n_1, n)) \quad (1.23) \]

whose higher order terms depend only on \( u_1 \) can be found so that the resulting equation for \( u \) has the form (1.21) with \( \tilde{Q}_2(t, u_1, 0) = 0 \) and such that (1.22) is in normal form. However, the transformed equations (1.21) will not be in normal form in the sense of our earlier definition.

The approximation of the invariant manifold to order \( k \) in the original variables \( y \) can be obtained directly from (1.23),

\[ y_2 = D_{22}(t, u_1(y_1)) + \cdots + D_{k2}(t, u_1(y_1)), \]

where \( u_1(y_1) \) is the inverse transformation of

\[ y_1 = u_1 + D_{21}(t, u_1) + \cdots + D_{k1}(t, u_1). \]

**Proof of Theorem 1.4.** We decompose \( V(k, n, n) \) in the natural way \( V(k, n, n) = V(k, n, n_1) \oplus V(k, n, n_2) \), writing \( D_k = D_{k1} + D_{k2} \). The term in brackets in (1.6) becomes

\[
\begin{bmatrix}
\tilde{Q}_{k1} - \frac{\partial D_{k1}}{\partial t} + BD_{k1} - \frac{\partial D_{k1}}{\partial u_1} (B u_1) - \frac{\partial D_{k1}}{\partial u_2} (C u_2) \\
\tilde{Q}_{k2} - \frac{\partial D_{k2}}{\partial t} + CD_{k2} - \frac{\partial D_{k2}}{\partial u_1} (B u_1) - \frac{\partial D_{k2}}{\partial u_2} (C u_2)
\end{bmatrix}.
\]

(1.24)

This is to say that the operator \( M_k = M^1_k \oplus M^2_k \) is reduced by the decomposition of \( V(k, n, n) \),

\[ M^i_k : V(k, n, n_i) \to V(k, n, n_i), \quad i = 1, 2 \]

\[ (M^i_k f_i)(u) = \frac{\partial f_i}{\partial u_1} (B u_1) + \frac{\partial f_i}{\partial u_2} (C u_2) - B f_i, \quad (1.25) \]

\[ (M^2_k f_2)(u) = \frac{\partial f_2}{\partial u_1} (B u_1) + \frac{\partial f_2}{\partial u_2} (C u_2) - C f_2. \]
Clearly, Eq. (1.8) decomposes as well:

\[ f_i' + M_{k_i}^1 f_i' = \phi_1, \quad \phi = \phi_1 + \phi_2, \]
\[ f_1' + M_{k_1}^2 f_1' = \phi_2, \quad f = f_1 + f_2, \]  

(1.26)
in which \( \phi_i \in \mathcal{P}_r(V(k, n_i, n_i)), \ i = 1, 2, \) and \( \mathcal{P}_r(V(k, n, n_i)) = \mathcal{P}_r(V(k, n_i, n_i)) \oplus \mathcal{P}_r(V(k, n, n_2)). \) We need to make two more decompositions,

\[ V(k, n, n_i) = V(k, n_i, n_i) \oplus V_0(k, n, n_i), \]
\[ V(k, n, n_2) = V(k, n_1, n_2) \oplus V_0(k, n, n_2). \]
defined by \( f(u_1, u_2) = f(u_1, 0) + (f(u_1, u_2) - f(u_1, 0)) \) for \( f \in V(k, n, n_i). \) It is easily seen from (1.25) that these decompositions reduce \( M_k^1 \) and \( M_k^2 \) as follows:

\[ M_k^1 = M_{k_1}^{11} \oplus M_{k_1}^{12}, \]
\[ (M_{k_1}^{11} f_{11})(u_1) = \frac{\partial f_{11}}{\partial u_1} (Bu_{11}) - Bf_{11}, \quad f_1 = f_{11} + f_{12}, \]
\[ (M_{k_1}^{12} f_{12})(u_1, u_2) = \frac{\partial f_{12}}{\partial u_1} (Bu_{12}) + \frac{\partial f_{12}}{\partial u_2} (Cu_{2}) - Bf_{12}. \]

Using the fact that \( f_{12}(u_1, 0) = 0, \) the reader may easily verify that \( (M_{k_1}^{12} f_{12}) \in V_0(k, n_1, n_1). \) Similarly for \( M_k^2 \)

\[ M_k^2 = M_{k_2}^{21} \oplus M_{k_2}^{22}, \]
\[ (M_{k_2}^{21} f_{21})(u_1) = \frac{\partial f_{21}}{\partial u_1} (Bu_{21}) - Cf_{21}, \]
\[ (M_{k_2}^{22} f_{22})(u_1, u_2) = \frac{\partial f_{22}}{\partial u_1} (Bu_{22}) + \frac{\partial f_{22}}{\partial u_2} (Cu_{2}) - Cf_{22}. \]

Thus (1.26) further decomposes into

\[ f_{11}' + M_{k_1}^{11} f_{11}' = \phi_{11}, \quad \phi_1 = \phi_{11} + \phi_{12}, \]
\[ f_{12}' + M_{k_1}^{12} f_{12}' = \phi_{12}, \quad f_1 = f_{11} + f_{12}, \]
\[ f_{21}' + M_{k_2}^{21} f_{21}' = \phi_{21}, \quad \phi_2 = \phi_{21} + \phi_{22}, \]
\[ f_{22}' + M_{k_2}^{22} f_{22}' = \phi_{22}, \quad f_2 = f_{21} + f_{22}. \]  

(1.27)

Now \( (H3)_k \) is precisely the hypothesis that \( f_{21}' + M_{k_2}^{21} f_{21} = 0 \) has no non-trivial solutions in \( \mathcal{P}_r(V(j, n_i, n_i)), \) \( 2 \leq j \leq k \) (see [1] where it is shown that \( \sigma(M_{j_i}^{21}) = \{(r, \mu) - \nu_i : |r| = j, 1 \leq i \leq n_1\} \)). Hence the third equation in
(1.27) can be solved for all \( \phi_{2i} \in P_T(V(k, n_1, n_2)) \) for a unique \( f_{2i} \in P_T(V(k, n_1, n_2)) \). Choose a complementary subspace to \( R_i((d/dt) + M_i^j) \) for \( ij = 11, 12, 22 \) and corresponding projections \( P_{k1}^i, P_{k2}^i, P_{k2}^{2i} \) and let \( P_k = P_{k1}^i \oplus P_{k2}^i \oplus 0 \oplus P_{k2}^{2i} \). Proceed similarly with each of the null spaces of \( (d/dt) + M_i^j \), defining complements \( 0_i^j \) and \( K_k = K_k^{1i} \oplus K_k^{2i} \oplus K_k^{12} \oplus K_k^{22} \). The term in brackets in (1.24) can be written as

\[
-(D_{k11}' + M_{k1}^i D_{k11} - \tilde{Q}_{k11}) + (D_{k12}' + M_{k1}^i D_{k12} - \tilde{Q}_{k12})
\left[
(D_{k21}' + M_{k1}^i D_{k21} - \tilde{Q}_{k21}) + (D_{k22}' + M_{k1}^i D_{k22} - \tilde{Q}_{k22})
\right]
\]

The proof is completed by the observation that the first term in the second component above vanishes.

**Corollary 1.5.** \((H3)_k\) holds for all \( k > 2 \) if any one of the following holds

(i) \( \text{Re} \sigma(B) < 0, \text{Re} \sigma(C) > 0 \),

(ii) \( \text{Re} \sigma(B) > 0, \text{Re} \sigma(C) \leq 0 \),

(iii) \( \text{Re} \sigma(B) = 0, \text{Re} \sigma(C) > 0, \) or \( \text{Re} \sigma(C) < 0 \).

In case (i), the invariant manifold \( u_2 = 0 \) is the stable manifold, in case (ii), \( u_2 = 0 \) is the unstable manifold, and in case (iii) \( u_2 = 0 \) is the center manifold. In each case (1.22) gives an approximation valid to order \( k \) of the dynamics on the manifold.

2. **An Application**

The following problem is now a rather standard one in the differential equations literature (see, e.g., [8]). Consider the differential equation

\[
x' = X(x, t, \eta) = X(x, t + T, \eta)
\]

for \( x \in \mathbb{R}^2 \) depending on the parameter \( \eta \). We assume (2.1) has a \( T \)-periodic solution \( x = p(t, \eta) \) for \( \eta \) near \( \eta_0 \) for some real number \( \eta_0 \). Assume that the variational equation about \( x = p(t, \eta) \),

\[
w' = A_\eta(t) w, \quad A_\eta(t) \equiv \frac{\partial X}{\partial x}(p(t, \eta), t, \eta)
\]

has two Floquet multipliers satisfying

\[
\lambda(\eta_0) = e^{\pm i\omega T}, \quad 0 < \omega T < \pi,
\]

\[
\left. \frac{d}{d\eta} \right|_{\eta = \eta_0} |\lambda(\eta)| > 0.
\]
That is, the periodic solution $p(t, \eta)$, stable for $\eta < \eta_0$, loses stability as $\eta$ increases through $\eta_0$. We introduce the perturbation variables

$$x = p(t, \eta) + z,$$

$$\eta = \eta_0 + \varepsilon.$$

The differential equation governing the perturbation $z$ can be expanded in the small quantities $(z, \varepsilon)$ as follows:

$$z' = A_0(t) z + \varepsilon A_1(t) z + Q_{21}(t, z) + \varepsilon^2 A_2(t) z$$

$$+ \varepsilon Q_{22}(t, z) + Q_3(t, z) + O(|(z, \varepsilon)|^4)$$

(2.4)

in which

$$A_0(t) = A_0(t) + \varepsilon A_1(t) + \varepsilon^2 A_2(t) + O(\varepsilon^3),$$

$$\frac{1}{2!} \frac{\partial^2 X}{\partial x^2} (p(t, \eta), t, \eta)[z]^2 = Q_{21}(t, z) + \varepsilon Q_{22}(t, z),$$

$$\frac{1}{3!} \frac{\partial^3 X}{\partial x^3} (p(t, \eta_0), t, \eta_0)[z]^3 = Q_3(t, z).$$

By Floquet theory, there is a real $T$-periodic nonsingular matrix $P(t)$ such that

$$\Phi(t) = P(t) e^{At}, \quad A = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a fundamental matrix solution of (2.2) for $\eta = \eta_0$. Letting

$$z = P(t) y$$

in (2.4) we obtain the differential equation for $y$

$$y' = Ay + \varepsilon \tilde{A}_1(t) y + \varepsilon^2 \tilde{A}_2(t) y + \tilde{Q}_{21}(t, y) + \varepsilon \tilde{Q}_{22}(t, y)$$

$$+ \tilde{Q}_3(t, y) + O(|(y, \varepsilon)|^4)$$

(2.5)

in which

$$\tilde{A}_i(t) = P^{-1}(t) A_i(t) P(t), \quad i = 1, 2,$$

$$\tilde{Q}_{2i}(t, y) = P^{-1}(t) Q_{2i}(t, P(t) y),$$

$$\tilde{Q}_3(t, y) = P^{-1}(t) Q_3(t, P(t) y),$$

If to (2.5) we add the trivial equation $\varepsilon' = 0$ the result is a system to which
our normal form theory applies directly. We will show that there is a near identity change of variables of the form

\[ y = u + \varepsilon L_1(t) u + \varepsilon^2 L_2(t) u + D_{21}(t, u) + \varepsilon D_{22}(t, u) + D_3(t, u) \tag{2.6} \]

in which the matrices \( L_i(t) \) are \( T \)-periodic, \( D_{21} \in P_T(V(2, 2, 2)), \ i = 1, 2, \]
\( D_3 \in P_T(V(3, 2, 2)) \), such that in the new variable (2.5) becomes

\[
\begin{align*}
    u_1' &= (\omega + \varepsilon_2) u_2 + \varepsilon_1 u_1 + \alpha u_1 r^2 + \beta u_2 r^2 + O(\|u, \varepsilon\|^4), \\
    u_2' &= -(\omega + \varepsilon_2) u_1 + \varepsilon_1 u_2 + \alpha u_2 r^2 - \beta u_1 r^2 + O(\|u, \varepsilon\|^4),
\end{align*} \tag{2.7}
\]

where

\[
\begin{align*}
    r &= u_1^2 + u_2^2, \quad \gamma_1 = \frac{1}{2} \left[ \text{tr} \tilde{A}_1 \right]_T, \\
    \varepsilon_1 &= \varepsilon \gamma_1 + \varepsilon^2 \gamma_2, \quad \zeta_1 = -\frac{1}{2} \left[ \text{tr} J \tilde{A}_1 \right]_T, \\
    \varepsilon_2 &= \varepsilon^2 \gamma_1 + \varepsilon^3 \zeta_2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\end{align*}
\]

provided the nonresonance condition

\[
\frac{\omega T}{2\pi} \neq \frac{1}{4}, \frac{1}{3} \tag{2.8}
\]

is satisfied. Expressions for \( \alpha \) and \( \beta \) will be given. The terms in (2.7) represented by \( O(\|u, \varepsilon\|^4) \) are \( T \)-periodic in \( t \). If the nonsingular matrix \( P(t) = (p_1(t), p_2(t)) \) then the approximate relation between (2.1) and (2.7) is given by

\[
x(t) = p(t, \eta) + u_1(t) p_1(t) + u_2(t) p_2(t). \tag{2.9}
\]

The quantities \( \gamma_1 \) and \( \zeta_1 \) can be expressed in terms of the Floquet multipliers as

\[
\begin{align*}
    \gamma_1 &= \frac{1}{T} \left. \frac{d}{d\eta} \right|_{\eta = \eta_0} |\lambda(\eta)| > 0, \\
    \zeta_1 &= \frac{1}{T} \left. \frac{d}{d\eta} \arg \lambda(\eta) \right|_{\eta = \eta_0}.
\end{align*} \tag{2.10}
\]

Returning to Eq. (2.7) we note that in polar coordinates it has the simple form

\[
\begin{align*}
    r' &= r(\varepsilon_1 + \alpha r^2), \\
    \theta' &= -(\omega + \varepsilon_2 + \beta r^2), \tag{2.11}
\end{align*}
\]
in which the higher order terms have been neglected. We assume $\alpha \neq 0$, that is, we assume the stability of the periodic solution $p(\tau, \eta_0)$ is determined by the cubic in (2.7). If $\alpha < 0$ so that $p(\tau, \eta_0)$ is stable then (2.11) undergoes a supercritical, stable, Hopf bifurcation while if $\alpha > 0$ so $p(\tau, \eta_0)$ is unstable the Hopf bifurcation is subcritical and unstable.

Well-known result from the theory of ordinary differential equations (see, e.g., [5, Chap. 8]) imply that the Hopf bifurcation in (2.11) corresponds to the bifurcation of an invariant manifold of solutions of (2.7), a cylinder in $(\tau, t)$-space about the periodic solution $p(\tau, \eta_0 + \varepsilon)$. Since (2.7) is periodic in $\tau$, the invariant manifold is also periodic in $\tau$ and can be viewed as a torus on the identification $(\tau, t + T) \sim (\tau, t)$. The second of equations (2.11) can be used to estimate the rotation number of the flow on the torus in the usual way.

We outlined the steps involved in obtaining (2.7). First, we obtain the normal form through quadratic terms in $(\varepsilon, \tau)$ by the change of variables

$$y = u + \varepsilon L_1(\tau) u + D_{21}(\tau, u).$$

The resulting equation for $u$ is given by

$$u' = Au + \varepsilon [\tilde{A}_1 - L_1 + AL_1 - L_1 A] u$$

$$+ \left[ \tilde{Q}_{21} - D_{21} + AD_{21} - \frac{\partial D_{21}}{\partial u} Au \right]$$

$$+ \varepsilon^2 \tilde{A}_2 u + \varepsilon \tilde{Q}_{22}(\tau, u) + \tilde{Q}_3(t, u) \tag{2.12}$$

in which

$$\tilde{A}_2 u = \tilde{A}_2 u + \tilde{A}_1 L_1 u - L_1 (\tilde{A}_1 - L_1 + AL_1) u - \frac{\partial D_{21}}{\partial u} (\tilde{A}_2 + \tilde{A}_1 L_1) u,$$

$$\tilde{Q}_{22} = \tilde{Q}_2 + \tilde{A}_1 D_{21} + \frac{\partial \tilde{Q}_{21}}{\partial u} L_1 u - L_1 (\tilde{Q}_{21} - D_{21} + AD_{21})$$

$$- \frac{\partial D_{21}}{\partial u} (\tilde{A}_1 - L_1 + AL_1) u,$$

$$\tilde{Q}_3 = \tilde{Q}_3 + \frac{\partial \tilde{Q}_{21}}{\partial u} D_{21} - \frac{\partial D_{21}}{\partial u} (\tilde{Q}_{21} - D_{21} + AD_{21}).$$

It is easy to check that the range of the operator $L' + (L B - BL)$ on the $T$-periodic, $2 \times 2$-matrices $L$ is the set of $T$-periodic matrices $\phi$ satisfying $[\text{tr } \phi]_T = [\text{tr } J\phi]_T = 0$. If we choose a projection $P$ by

$$P\phi = \frac{[\text{tr } \phi]_T J}{2} - \frac{[\text{tr } J\phi]_T J}{2},$$
then $P$ projects onto a complementary subspace to the range. Hence we can choose $L_1$ in the first bracket in (2.12) so that the term in that bracket becomes $PA_1$. This justifies the lowest order in $\varepsilon$ terms in (2.7).

The quadratic terms in the second bracket in (2.12) can be completely eliminated by an appropriate choice of $D_{21}$ since the spectrum of the operator $M_2$: $V(2, 2, 2) \rightarrow V(2, 2, 2)$ is \{ $\pm i\omega$, $\pm 3i\omega$ \} and $n \frac{2\pi}{\omega} \neq T$, $n \frac{2\pi}{3\omega} \neq T$ for any positive integer $n$ by (2.8).

Now that $L_1$ and $D_{21}$ are known, the higher order terms in (2.12) are known and can be simplified somewhat. For example,

$$Q_3 = Q_3 + \frac{\partial Q_{21}}{\partial u} D_{21} - \frac{\partial D_{21}}{\partial u} \left( \frac{\partial D_{21}}{\partial u} Au \right).$$

The cubic terms in (2.12) can now be put in normal form. Let

$$u = v + \varepsilon^2 L_2 v + \varepsilon D_{22}(t, v) + D_3(t, v).$$

It is a straightforward exercise to see that in the resulting equation for $v$, whose quadratic terms in $(\varepsilon, v)$ are the same as those in (2.7), the term $\varepsilon^2 A_2 v$ can be treated exactly as was $\varepsilon A_1 u$ and the term $\varepsilon Q_{22}$ can be eliminated exactly as was $Q_{21}$. Hence we need only treat the cubic terms in $v$, $Q_3(t, v)$. The appropriate term in the equation for $v$ will be

$$Q_3 - D_3 - \frac{\partial D_3}{\partial u} Au - AD_3.$$

The spectrum of the map $M_3$ on $V(3, 2, 2)$ is \{ $0$, $\pm 2i\omega$, $\pm 4i\omega$ \}. The range of the operator $D_3 + M_3 D_3$ on the $T$-periodic $V(3, 2, 2)$-valued maps is easily verified to be those functions $f = (q_1(t), q_2(t), ..., q_8(t))$ satisfying

$$\sum_{i=1}^{8} \langle q_i(t), n_i \rangle = 0,$$

where $n_1 = (0, -1, 0, -3, 3, 0, 1, 0)$ and $n_2 = (3, 0, 1, 0, 0, 1, 0, 3)$. A projection onto a complementary subspace is given by

$$P_q = \frac{1}{8} \left[ \langle q(t), n_1 \rangle \right]_T m_1 + \frac{1}{8} \left[ \langle q(t), n_2 \rangle \right]_T m_2,$$

where $m_2 = (1, 0, 1, 0, 0, 1, 0, 1)$ and $m_1 = (0, 1, 0, 1, -1, 0, -1, 0)$ span the null space of $M_3$. Hence $D_3$ can be chosen so (2.13) becomes $P\tilde{Q}_3$ and this gives the pure cubic terms in $u$ in (2.7). Explicitly

$$\alpha = -\frac{1}{8} \left[ \langle \tilde{Q}_3, n_2 \rangle \right]_T,$$

$$\beta = \frac{1}{8} \left[ \langle \tilde{Q}_3, n_1 \rangle \right]_T.$$
The computation of $\alpha$ and $\beta$ is made difficult because $\tilde{Q}_3$ depends on $D_{21}$ which is the unique $T$-periodic solution of

$$D_{21}' + M_2 D_{21} = \tilde{Q}_{21},$$

where $M_2: V(2, 2, 2) \rightarrow V(2, 2, 2)$ is given in the usual basis by

$$M_2 = \begin{bmatrix}
0 & -1 & 0 & -1 & 0 & 0 \\
2 & 0 & -2 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 2 & 0 & -2 \\
0 & 0 & 1 & 0 & 1 & 0 
\end{bmatrix}.$$ More generally, if

$$\frac{\omega T}{2\pi} \in \left\{ \frac{l}{2k+1} \right\}_{l=0}^{t} \sum_{l=0}^{t} \frac{l}{2k+2} : 1 \leq l \leq t \leq n \right\},$$

then a near identity transformation of variables containing terms through order $2n + 1$ in $(u, \varepsilon)$ can be found such that the differential equation for $u$ is given by

$$u_1' = (\omega + \varepsilon_1) u_2 + \varepsilon_1 u_1 + u_1 f_1(r^2, \varepsilon) + u_2 f_2(r^2, \varepsilon) + O(||u, \varepsilon||)^{2n + 2},$$

$$u_2' = - (\omega + \varepsilon_2) u_1 + \varepsilon_1 u_2 + u_2 f_1(r^2, \varepsilon) - u_1 f_2(r^2, \varepsilon) + O(||u, \varepsilon||)^{2n + 2},$$

in which

$$\varepsilon_1 = \varepsilon_{r_1} + \varepsilon_{r_2}^2 + \cdots + \varepsilon_{r_{2n}}^2,$$

$$\varepsilon_2 = \varepsilon_{r_1}^2 + \varepsilon_{r_2}^2 + \cdots + \varepsilon_{r_{2n}}^2,$$

and $f_i(r^2, \varepsilon)$ is a polynomial of degree $2n$ in $(r, \varepsilon)$ the first few terms of which are given in (2.7).

The above assertion follows from two facts. First, $\sigma(M_{2n}) = \{ \pm (2n + 1) i\omega, \pm (2n - 1) i\omega, \cdots, \pm i\omega \}$ so by (2.14) all terms which are even order in $u$ and of order less than $2n + 1$ in $(u, \varepsilon)$ can be eliminated just as $\tilde{Q}_{21}, \tilde{Q}_{22}$ were eliminated from (2.5). Second, $\sigma(M_{2n + 1}) = \{ \pm (2n + 2) i\omega, \pm 2ni\omega, \cdots, \pm 2i\omega, 0 \}$ so the odd order terms in $u$ cannot be eliminated. The nonresonance condition (2.14) is a way of saying that the only nontrivial periodic solutions of (1.10) are constant solutions and they exist only if $0 \in \sigma(M_{k})$, i.e., for $k$ odd. It is not difficult to see that
In polar coordinates

\[ f_1 = r^{2n+1} f_1(\cos \theta, \sin \theta) = r^{2n+1} f_1(\theta). \]

The null space of $M_{2n+1}$ is easily computed by solving

\[ \frac{\partial f_1}{\partial \theta} - f_2 = 0, \]
\[ \frac{\partial f_2}{\partial \theta} + f_1 = 0, \]

hence

\[ f_1(\theta) = A \cos \theta + B \sin \theta, \]
\[ f_2(\theta) = A \sin \theta - B \cos \theta, \]

or

\[ f = Ar^{2n} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + Br^{2n} \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}. \]

Since $V(2n + 1, 2, 2) = \text{Range } M_{2n+1} \oplus \text{Ker}(M_{2n+1})$ the form of the nonlinear terms in (2.15) follows:

2.1. $\omega T/2\pi = \frac{1}{3}$

We briefly discuss the resonance $\omega T/2\pi = \frac{1}{3}$ in this section. The only terms of (2.12) which are affected by the resonance are the quadratic terms since $\sigma(M_2) = \{ \pm \omega_0, \pm 3\omega_0 \}$. The result is that quadratic terms in $y$ can not be eliminated from (2.5). Because $\pm 3\omega_0$ are simple eigenvalues of $M_2$, (1.10) with $M = M_2$, has two linearly independent $T$-periodic solutions which we have computed:

\[ (\cos 3\omega t, -\sin 3\omega t, -\cos 3\omega t, -\cos 3\omega t, \sin 3\omega t), \]
\[ (-\sin 3\omega t, -\cos 3\omega t, \sin 3\omega t, -\cos 3\omega t, \sin 3\omega t, \cos 3\omega t). \]

The range of the operator $(d/dt) + M_2$ on $P_T(V((2, 2, 2))$ are those elements of $P_T(V(2, 2, 2))$ orthogonal to these elements. For a two-dimen-
sional subspace complementary to the range we take the span of the two vectors,

\[(\cos 3\omega t, -2 \sin 3\omega t, -\cos 3\omega t, -\sin 3\omega t, -2 \cos 3\omega t, \sin 3\omega t)\]

and

\[(-\sin 3\omega t, -2 \cos 3\omega t, \sin 3\omega t, -\cos 3\omega t, -2 \sin 3\omega t, \cos 3\omega t).\]

The resulting normal form then becomes

\[u'_1 = (\omega + \varepsilon_2) u_2 + \varepsilon_1 u_1 + (\delta \cos 3\omega t - \gamma \sin 3\omega t) u_1^2 - 2(\delta \sin 3\omega t + \gamma \cos 3\omega t) u_1 u_2 - (\delta \cos 3\omega t - \gamma \sin 3\omega t) u_2^2,\]

\[u'_2 = - (\omega + \varepsilon_2) u_1 + \varepsilon_2 u_2 - (\delta \sin 3\omega t + \gamma \cos 3\omega t) u_1^2 - 2(\delta \cos 3\omega t + \gamma \sin 3\omega t) u_1 u_2 + (\delta \sin 3\omega t + \gamma \cos 3\omega t) u_2^2,\]

where we have ignored the higher order terms beginning with the cubic terms in (2.7). If we write \(u = u_1 + iu_2\) then the differential equation for \(u\) is given by

\[u' = [\varepsilon_1 - i(\omega + \varepsilon_2)] u + Ae^{-(3\omega t + \phi)}u^2 + (\alpha - i\beta)|u|^2u + O(|u, \varepsilon|^4),\]

where \(A = \sqrt{\delta^2 + \gamma^2}\), \(\tan \phi = \delta/\gamma\) and we have included the cubic terms exactly as in (2.7) since they are unaffected by the resonance. Recall that the terms \(O(|(u, \varepsilon)|^4)\) are \(T\)-periodic in \(t\). Substituting

\[u = ve^{-i\omega t}\]

into the above equation, we obtain an equation for \(v\) in which the \(O(|(v, \varepsilon)|^4)\) terms are now \(3T\)-periodic:

\[v' = \varepsilon z_1 v + z_2 v^2 + z_3 |v|^2v + O(|(v, \varepsilon)|^4)\]

in which

\[\varepsilon z_1 = \varepsilon_1 - \varepsilon_2,\]

\[z_2 = Ae^{-i\phi},\]

\[z_3 = \alpha - i\beta,\]

the steady states of (2.1.1), ignoring the order four terms, \(v = \varepsilon w\), are solutions of

\[0 = z_1 w + z_2 w^2 + \varepsilon z_3 |w|^2w.\]
Setting $\varepsilon = 0$ in (2.1.2), we find nontrivial $w$

$$w = \rho e^{i\psi_k}, \quad \rho = \left| \frac{z_1}{z_2} \right|, \quad \psi_k = -\left( \frac{\arg(-z_1/z_2)}{3} + \frac{2k\pi}{3} \right),$$

$$k = 0, 1, 2,$$

where $0 \leq \arg(-z_1/z_2) < 2\pi$, provided of course that $z_2 \neq 0$. The Implicit Function Theorem implies these three solutions can be continued to solutions of (2.1.2) for small $\varepsilon$.

Thus there is ($z_2 \neq 0$) a transcritical bifurcation of 3$T$-periodic solutions given by

$$u_0(t) = \varepsilon e^{i\omega t - \psi_0} + O(\varepsilon^2),$$

$$u_1(t) = u_0(t + 1),$$

$$u_2(t) = u_0(t + 2).$$

2.2. $\omega T/2\pi = \frac{1}{4},$

In this case, the cubic terms of the normal form are affected. The quadratic terms can be eliminated exactly as in the nonresonant case. Since $\sigma(M_3) = \{0, \pm 2i\omega, \pm 4i\omega\}$ on $V(3, 2, 2)$, the adjoint equation (1.10) has, in addition to the constant solutions associated with the zero eigenvalue, some time dependent $T$-periodic solutions associated with the eigenvalues $\pm 4i\omega$. These are

$$(-\sin 4\omega t, -\cos 4\omega t, \sin 4\omega t, \cos 4\omega t, -\cos 4\omega t, \sin 4\omega t, -\sin 4\omega t)$$

and

$$(\cos 4\omega t, -\sin 4\omega t, -\cos 4\omega t, \sin 4\omega t, -\sin 4\omega t, -\cos 4\omega t, \sin 4\omega t, \cos 4\omega t).$$

For a complementary subspace to the range of $(d/dt) + M_3$ we take the span of the vectors

$$(-\sin 4\omega t, -3 \cos 4\omega t, 3 \sin 4\omega t, \cos 4\omega t, -\cos 4\omega t, 3 \sin 4\omega t, 3 \cos 4\omega t, -\sin 4\omega t),$$

$$(\cos 4\omega t, -3 \sin 4\omega t, -3 \cos 4\omega t, \sin 4\omega t, -\sin 4\omega t, -3 \cos 4\omega t, 3 \sin 4\omega t, \cos 4\omega t),$$

$$(1, 0, 1, 0, 0, 1, 0, 1), \quad (0, 1, 0, 1, -1, 0, -1, 0),$$

the last two of which are the same as in the nonresonant case.
Proceeding exactly as in (2.1.1), we can write a differential equation for $u = u_1 + iu_2$ as

$$u' = \left[ e_1 - i(\omega + e_2) \right] u + z_2 e^{-4\epsilon u} u^3 + z_3 |u|^2 u + O(|u, \epsilon|)^4).$$

Letting $u = ve^{-\epsilon t}$ we obtain the differential equation for $v$ as

$$v' = \epsilon z_1 v + z_2 v^3 + z_3 |v|^2 v + O(|v, \epsilon|)^4), \quad (2.2.1)$$

where $\epsilon z_1 = e_1 - i\epsilon_2$ and the $O(|v, \epsilon|)^4$ terms are $4T$-periodic in $t$. Ignoring these higher order terms, the steady states of (2.2.1) can be obtained by setting

$$\epsilon = q \delta^2, \quad q = \pm 1$$

$$v = \delta w.$$  

Then the steady state equation becomes

$$qz_1 w + z_2 \bar{w}^3 + z_3 |w|^2 w = 0.$$  

If we multiply through by $\bar{w}$ and rearrange we find

$$\bar{w}^4 = -qz_1 |w|^2 + z_3 |w|^4 \quad (2.2.2)$$

provided $z_2 \neq 0$, which we assume to be the case. Writing $w = \rho e^{i\theta}$ and dividing (2.2.2) through by $\rho^4$ it is easily seen that (2.2.2) has solutions if and only if

$$(|z_2|^2 - |z_3|^2) \rho^4 - 2qz_1\rho^3(z_1, z_3) - |z_1|^2 = 0,$$

where $(z_1, z_3) = \text{Re} z_1 \text{ Re} z_3 + \text{Im} z_1 \text{ Im} z_3$. There are three cases depending on the discriminant

$$A = (z_1, z_3)^2 + |z_1|^2 (|z_2|^2 - |z_3|^2).$$
Case I. $\Delta > 0$ and $|z_3| > |z_2|$.

In this case there are two positive roots, $\rho^2$, if the sign of $q_1$ is such that $q_1(z_1, z_3) > 0$. Hence there is a sub- or super-critical bifurcation of two families of $4T$-periodic solutions as depicted in the schematic bifurcation diagram, Fig. 1, the direction of bifurcation depending on the sign of $q_1$.

Case II. $|z_2| > |z_3|$.

In this case, for each choice of $q = \pm 1$, there is one positive root $\rho^2$. The bifurcation diagram is two sided as depicted in Fig. 2. Each branch corresponds to a $4T$-periodic solution.

Case III. $\Delta < 0$.

In this case there are no nontrivial $4T$-periodic solutions.

REFERENCES

