Gradient estimates in Orlicz space for nonlinear elliptic equations

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Abstract

In this paper we generalize gradient estimates in $L^p$ space to Orlicz space for weak solutions of elliptic equations of $p$-Laplacian type with small BMO coefficients in $\delta$-Reifenberg flat domains. Our results improve the known results for such equations using a harmonic analysis-free technique.

Keywords: Elliptic PDE of $p$-Laplacian type; Reifenberg flat domain; BMO space; Orlicz space

1. Introduction

Let us assume $1 < p < \infty$ is fixed. We then consider the following nonlinear elliptic boundary value problem of $p$-Laplacian type:

$$\text{div} \left( (A \nabla u \cdot \nabla u)^{\frac{p-2}{2}} A \nabla u \right) = \text{div} (|f|^{p-2} f) \quad \text{in } \Omega, \quad (1.1)$$

$$u = 0 \quad \text{on } \partial \Omega, \quad (1.2)$$

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where $\Omega$ is an open bounded domain in $\mathbb{R}^n$ with nonsmooth boundary $\partial \Omega$, $f = (f^1, \ldots, f^n) \in L^p(\Omega)$ is a given vector field with $|f|^p$ belonging to the general Orlicz space, and $A = \{a_{ij}(x)\}_{n \times n}$ is a symmetric matrix with discontinuous coefficients satisfying the uniform ellipticity condition; namely,

$$A^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq A|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, for almost every $x \in \mathbb{R}^n$ and for some positive constant $\Lambda$.

As usual, the solutions of (1.1)–(1.2) are taken in a weak sense. We now state the definition of weak solutions.

**Definition 1.1.** Function $u \in W^{1,p}_0(\Omega)$ is a weak solution of (1.1)–(1.2) if for any $\varphi \in W^{1,p}_0(\Omega)$, we have

$$\int_{\Omega} (A \nabla u \cdot \nabla u)^{\frac{p-2}{2}} A \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} |f|^{p-2} f \cdot \nabla \varphi \, dx.$$ 

According to classical theory, the problem (1.1)–(1.2) has a unique weak solution $u \in W^{1,p}_0(\Omega)$ with the estimate

$$\int_{\Omega} |\nabla u|^p \, dx \leq C \int_{\Omega} |f|^p \, dx \quad (1.3)$$

if $f \in L^p(\Omega)$, $A$ is uniformly elliptic and $\Omega$ is bounded. Our interest is to study how the regularity of $f$ is reflected to the solutions in the setting of Orlicz spaces under minimal assumptions on the matrix of coefficient $A$ and the geometry of the domain $\Omega$.

There have been a wide research activities on the study on $W^{1,q}$ regularity for (1.1)–(1.2); that is, obtaining local/global $L^q$ estimates for the gradients of weak solutions of (1.1)–(1.2) with given $f \in L^q(\Omega)$. The techniques for $W^{1,q}$ regularity results have been mainly based on the maximal function method. When $A$ is the identity matrix, DiBenedetto and Manfredi, and Iwaniec obtained $W^{1,q}$ regularity results (see [10,11], respectively). Their results were extended by Kinnunen and Zhou in [16,17] to the case $A \in \text{VMO}$ and $\partial \Omega \in C^{1,\alpha}$. In the recent paper [6] Byun, Wang and Zhou obtained the global $W^{1,q}$ regularity when $A$ is $(\delta, R)$-vanishing (see Definition 1.2) and $\Omega$ is $(\delta, R)$-Reifenberg flat (see Definition 1.3).

The purpose of this paper is to extend the results in [6] in the setting of the general Orlicz spaces. In particular, we are interested in the estimate like

$$\int_{\Omega} \phi(|\nabla u|^p) \, dx \leq C \int_{\Omega} \phi(|f|^p) \, dx, \quad (1.4)$$

where $C$ is a constant independent from $u$ and $f$. Indeed, if $\phi(x) = |x|^{q/p}$ with $q > p$, (1.4) is reduced to the estimate obtained in [6] (see Remark 1.11). Especially when $p = 2$, Eq. (1.1) is a linear elliptic one. In this case Jia, Li and Wang [12,13] have obtained the similar estimates in Orlicz spaces. Their approach is mainly based on the maximal functions. For this case, the authors [26] have also obtained estimate (1.4) under the assumptions that $A$ is $(\delta, R)$-vanishing and
$\Omega$ is $(\delta, R)$-Lipschitz. We should point out that a $(\delta, R)$-Lipschitz domain is $(\delta, R)$-Reifenberg flat if its Lipschitz constant is small enough (see [23]).

Recently E. Acerbi and G. Mingione [1] obtained local $W^{1,q}$ estimates for the degenerate parabolic $p$-Laplacian systems where they used the stopping time argument and large-M-inequality, avoiding the use of the maximal function operator. Here modifying the techniques introduced in [1] and applying the method of approximation used in [6] (see Lemmas 3.1, 3.2), we will show that the estimates (1.4) still hold under the same assumptions as in [6] (see Theorem 1.12).

We assume that the coefficients of $A = \{a_{ij}\}$ are in the BMO space and their semi-norms are small enough. More precisely, we use the following definition.

**Definition 1.2 (Small BMO condition).** We say that the matrix $A$ of coefficients is $(\delta, R)$-vanishing if

$$\sup_{0 < r \leq R} \sup_{x \in \mathbb{R}^n} \int_{B_r(x)} |A(y) - \bar{A}_{B_r(x)}| \, dy \leq \delta,$$

where

$$\bar{A}_{B_r(x)} = \int_{B_r(x)} A(y) \, dy.$$

Recently integrability of the gradient of solutions for elliptic/parabolic problems with discontinuous coefficients of VMO/BMO type have been extensively studied by many authors (see [4–6,16,17,19]). We would like to point out that if a function satisfies the VMO condition, then it satisfies the small BMO condition which we treat in this paper.

The domain considered in this paper is a $(\delta, R)$-Reifenberg domain.

**Definition 1.3.** We say that a domain $\Omega$ is $(\delta, R)$-Reifenberg flat if for every $x \in \partial \Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, \ldots, y_n\}$, which depends on $r$ and $x$, so that $x = 0$ in this coordinate system and

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

A Reifenberg flat set was introduced by Reifenberg in the noteworthy paper [22] where he showed that it is locally a topological disk if $\delta$ is sufficiently small. A good example of Reifenberg flat domains is a flat version of the well-known Van Koch snowflake when the angle of the spike with respect to the horizontal is sufficiently small (see [23]). They arise naturally in many areas such as applied mathematics, harmonic analysis and geometric measure theory. They look like coast lines, zigzag functions, atomic clusters. For a further discussion of Reifenberg flat domains we refer to [4,6,9,14,15,23].

**Remark 1.4.** Throughout this paper we mean $\delta$ to be a small positive constant. As one can see in Definitions 1.2 and 1.3, $\delta$ is invariant under a scaling and we will determine it in the proof of Theorem 1.12. We also remark that $R$ can be any positive number by a scaling.
The objective of this paper is to discuss the regularity of nonlinear elliptic equations of \( p \)-Laplacian type in the general Orlicz space. Orlicz spaces have been studied as a generalization of \( L^p \) spaces since it was introduced by Orlicz [20] (see [3,8,12,18,25]). The theory of Orlicz spaces plays a crucial role in a very wide spectrum (see [21]). Here for reader’s convenience, we will give some definitions and preliminary lemmas on the general Orlicz spaces. We denote by \( \Phi \) the function class that consists of all functions \( \phi : [0, +\infty) \to [0, +\infty) \) which are increasing and convex.

**Definition 1.5.** A function \( \phi \in \Phi \) is said to be a Young function if

\[
\lim_{t \to 0^+} \frac{\phi(t)}{t} = \lim_{t \to +\infty} \frac{t}{\phi(t)} = 0.
\]

**Definition 1.6.** A Young function \( \phi \) is said to satisfy the global \( \Delta_2 \) condition, denoted by \( \phi \in \Delta_2 \), if there exists a positive constant \( K \) such that for every \( t > 0 \),

\[
\phi(2t) \leq K \phi(t).
\]

Moreover, a Young function \( \phi \) is said to satisfy the global \( \nabla_2 \) condition, denoted by \( \phi \in \nabla_2 \), if there exists a number \( a > 1 \) such that for every \( t > 0 \),

\[
\phi(t) \leq \frac{\phi(at)}{2a}.
\]

**Lemma 1.7.** Let \( \phi \) be a Young function. Then \( \phi \in \Delta_2 \cap \nabla_2 \) if and only if there exist constants \( A_2 \geq A_1 > 0 \) and \( \alpha_1 \geq \alpha_2 > 1 \) such that for any \( 0 < s \leq t \)

\[
A_1 \left( \frac{s}{t} \right)^{\alpha_1} \leq \frac{\phi(s)}{\phi(t)} \leq A_2 \left( \frac{s}{t} \right)^{\alpha_2}.
\]

(1.5)

Moreover the condition (1.5) implies that for \( 0 < \theta_1 \leq 1 \leq \theta_2 < \infty \),

\[
\phi(\theta_1 t) \leq A_2 \theta_1^{\alpha_2} \phi(t) \quad \text{and} \quad \phi(\theta_2 t) \leq A_1^{-1} \theta_2^{\alpha_1} \phi(t).
\]

(1.6)

**Remark 1.8.** The simplest examples for functions \( \phi(t) \) satisfying the \( \Delta_2 \cap \nabla_2 \) condition are power functions \( \phi(t) = t^q \) with \( q > 1 \). Another kind of examples is the type \( \phi(t) = t^q (1 + |\log t|) \) with \( q > 1 \). Therefore, we remark that the global \( \Delta_2 \cap \nabla_2 \) condition makes the function grow moderately. Examples such as \( t \log(1 + t) \) are ruled out by \( \nabla_2 \), and those such as \( \exp(t^2) \) are ruled out by \( \Delta_2 \).

**Definition 1.9.** Let \( \phi \) be a Young function. Then the Orlicz class \( K^\phi(\Omega) \) is the set of all measurable functions \( g : \Omega \to \mathbb{R} \) satisfying

\[
\int_{\Omega} \phi(|g|) \, dx < \infty.
\]

The Orlicz space \( L^\phi(\Omega) \) is the linear hull of \( K^\phi(\Omega) \); that is, the smallest linear space (under pointwise addition and scalar multiplication) containing \( K^\phi(\Omega) \).
Lemma 1.10. (See [2].) Let $\phi$ be a Young function satisfying $\phi \in \Delta_2 \cap \nabla_2$. Then

1. $K_\phi(\Omega) = L_\phi(\Omega)$.
2. $C_0^\infty(\Omega)$ is dense in $L_\phi(\Omega)$.
3. $L^{\alpha_1}(\Omega) \subset L_\phi(\Omega) \subset L^{\alpha_2}(\Omega) \subset L^1(\Omega)$ with $\alpha_1 \geq \alpha_2 > 1$ as in Lemma 1.7.

Remark 1.11. We remark that the Orlicz spaces generalize $L^q$ spaces in the sense that if we take $\phi(t) = t^q$, $t \geq 0$, then $\phi \in \Phi$ is a Young function with $\phi \in \Delta_2 \cap \nabla_2$, so for this special case,

$$L_\phi(\Omega) = L^q(\Omega).$$

Now we are set to state the main result.

Theorem 1.12. Given a Young function $\phi \in \Delta_2 \cap \nabla_2$, there exists a small $\delta = \delta(n, p, \phi, \Lambda) > 0$ such that if $A$ is uniformly elliptic and $(\delta, R)$-vanishing, $\Omega$ is $(\delta, R)$-Reifenberg flat and $|f|^p \in L_\phi(\Omega)$, then the unique weak solution $u$ of the problem (1.1)–(1.2) satisfies

$$|\nabla u|^p \in L_\phi(\Omega)$$

with the estimate (1.4).

Remark 1.13. We remark that the global $\Delta_2 \cap \nabla_2$ condition is necessary and sufficient for such kinds of estimates even for the simplest linear equation. Actually, the authors [24] have proved that if $u$ is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} \phi(|D^2 u|) \, dx \leq C \int_{\mathbb{R}^n} \phi(|f|) \, dx$$

holds if and only if $\phi \in \Delta_2 \cap \nabla_2$.

This paper will be organized as follows. In Section 2, we give some notations and preliminary lemmas. We finish the proof of Theorem 1.12 in Section 3.

2. Preliminary materials

2.1. Geometric notation

1. A typical point in $\mathbb{R}^n$ is $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$.
2. $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$ and $\mathbb{R}_-^n = \{x \in \mathbb{R}^n : x_n < 0\}$.
3. $B_r = \{y \in \mathbb{R}^n : |y| < r\}$ is an open ball in $\mathbb{R}^n$ with center 0 and radius $r > 0$, $B_r(x) = B_r + x$, $B_r^+ = B_r \cap \mathbb{R}_+^n$.

2.2. Preliminary lemmas

In this subsection we give two lemmas which are very important to obtain the main result, Theorem 1.12. The two lemmas are much influenced by Steps 2 and 4 in [1]. To start with, let
let $u$ be the weak solution of the problem (1.1)–(1.2) and let $\Omega$ be a $(\delta, R)$-Reifenberg flat domain in $\mathbb{R}^n$. Then we write

$$\lambda_0 = \int_\Omega |\nabla u|^p \, dx + \frac{1}{\delta^p} \int_\Omega |f|^p \, dx$$

(2.1)

and

$$E(\lambda) = \{ x \in \Omega : |\nabla u|^p > \lambda \}$$

(2.2)

for any $\lambda > 0$. Moreover, for any $x \in \Omega$ and $\rho > 0$, we write

$$J[B_\rho(x)] = \int_{B_\rho(x) \cap \Omega} |\nabla u|^p \, dy + \frac{1}{\delta^p} \int_{B_\rho(x) \cap \Omega} |f|^p \, dy.$$  

(2.3)

Since $|\nabla u|$ is bounded in $\Omega \setminus E(\lambda)$ for a fixed $\lambda > 0$, we focus our attention on the level set $E(\lambda)$. Without loss of generality we assume that there exists a positive constant $K$ such that

$$|\Omega| \leq |B_{KR}|.$$  

(2.4)

Now we will cover a major portion of $E(\lambda)$ by a family of countably many disjoint balls.

**Lemma 2.1.** Given $\lambda \geq (\frac{20K}{1-\delta})^n \lambda_0$, there exists a family of disjoint balls $\{B^0_i\}_{i \geq 1} = \{B_{\rho_i}(x_i)\}_{i \geq 1}$ with $x_i \in E(\lambda)$ and $\rho_i \in (0, \frac{R}{10}]$ such that

$$J[B^0_1] = \lambda,$$

(2.5)

and

$$E(\lambda) \subset Z \cup \left( \bigcup_{i \geq 1} B^1_i \right),$$

(2.6)

with $Z$ having zero Lebesgue measure, where $B^1_i = B_{5\rho_i(x_i)}$ and for any $\rho_i < \rho \leq R$,

$$J[B_{\rho}(x_i)] < \lambda.$$  

(2.7)

**Proof.**

1. We first assert

$$\sup_{0 < \rho \leq R} \sup_{x \in \Omega} \frac{|B_{\rho}(x)|}{|B_{\rho}(x) \cap \Omega|} \leq \left( \frac{2}{1 - \delta} \right)^n.$$  

(2.8)

To do this, fix any $x \in \Omega$ and $\rho \in (0, R]$. If $B_{\rho}(x) \subset \Omega$, then the assertion (2.8) is obvious, and so suppose $B_{\rho}(x) \not\subset \Omega$. Then since $\Omega$ is $(\delta, R)$-Reifenberg flat, we assume that

$$B_{\rho}(x) \cap \{y_n > \delta \rho\} \subset B_{\rho}(x) \cap \Omega \subset B_{\rho}(x) \cap \{y_n > -\delta \rho\}.$$
in some appropriate coordinate system. Therefore it follows from the geometry that

\[
\frac{|B_\rho(x)|}{|B_\rho(x) \cap \Omega|} \leq \frac{|B_\rho(x)|}{|B_\rho(x) \cap \{y_n > \rho \delta\}|} \leq \left( \frac{2}{1 - \delta} \right)^n.
\]

2. Next, we claim

\[
\sup_{x \in \Omega} \sup_{\frac{R}{10} \leq \rho \leq R} J[B_\rho(x)] \leq \left( \frac{20K}{1-\delta} \right)^n \lambda_0.
\]  

(2.9)

To prove this, fix any \( x \in \Omega \) and \( \rho \in [\frac{R}{10}, R] \). Then it follows from (2.4) and (2.8) that

\[
\int_{B_\rho(x) \cap \Omega} |\nabla u|^p \, dy = \frac{1}{|B_\rho(x) \cap \Omega|} \int_{B_\rho(x) \cap \Omega} |\nabla u|^p \, dy \\
\leq \frac{|B_\rho(x)|}{|B_\rho(x) \cap \Omega|} \frac{|\Omega|}{|B_\rho(x)|} \int_{\Omega} |\nabla u|^p \, dy \\
\leq \left( \frac{2}{1 - \delta} \right)^n \frac{|B_{KR}|}{|B_\rho|} \int_{\Omega} |\nabla u|^p \, dy \\
\leq \left( \frac{20K}{1 - \delta} \right)^n \int_{\Omega} |\nabla u|^p \, dy.
\]

Similarly, we have

\[
\int_{B_\rho(x) \cap \Omega} |\Phi|^p \, dy \leq \left( \frac{20K}{1 - \delta} \right)^n \int_{\Omega} |\Phi|^p \, dy.
\]

Consequently it follows from the two inequalities above and (2.1) that

\[
J[B_\rho(x)] \leq \left( \frac{20K}{1 - \delta} \right)^n \lambda_0,
\]

which implies (2.9) holds.

3. Let \( \lambda \geq \left( \frac{20K}{1 - \delta} \right)^n \lambda_0 \). Now for a.e. \( x \in E(\lambda) \), a version of Lebesgue’s differentiation theorem implies that

\[
\lim_{\rho \to 0} J[B_\rho(x)] > \lambda.
\]

Thus from (2.9) one can select a radius \( \rho_x \in (0, R/10] \) such that

\[
\rho_x = \max \left\{ \rho \in \left( 0, \frac{R}{10} \right]: J[B_\rho(x)] = \lambda \right\}.
\]
Then we observe

\[ J[B_{\rho_x}(x)] = \lambda \]

and for each \( \rho_x < \rho \leq R \),

\[ J[B_{\rho}(x)] < \lambda \]

since \( J[B_{\rho}(x)] \) is a continuous function of \( \rho \). It follows from the argument above that for a.e. \( x \in E(\lambda) \) there exists a ball \( B_{\rho_x}(x) \) constructed as above. Therefore, applying Vitali’s covering lemma, we can find a family of disjoint balls \( \{B^0_i\} = \{B_{\rho_i}(x_i)\} \) so that the results of the lemma hold. This completes the proof. \( \square \)

Next, we obtain the following estimates of \( \{B^0_i\} \).

**Lemma 2.2.** Under the same hypothesis and results as in Lemma 2.1, we have

\[ |B^0_i \cap \Omega| \leq \frac{2}{\lambda} \left( \int_{B^0_i \cap \{x \in \Omega: |\nabla u|^{p/4} > \frac{1}{2}\}} |\nabla u|^p \, dx + \frac{1}{\delta^p} \int_{B^0_i \cap \{x \in \Omega: |f|^p > \delta^p \frac{1}{4}\}} |f|^p \, dx \right). \]

**Proof.** In light of Lemma 2.1, we have

\[ J[B^0_i] = \int_{B^0_i \cap \Omega} |\nabla u|^p \, dx + \frac{1}{\delta^p} \int_{B^0_i \cap \Omega} |f|^p \, dx = \lambda, \]

which implies that

\[ \lambda \cdot |B^0_i \cap \Omega| = \int_{B^0_i \cap \Omega} |\nabla u|^p \, dx + \frac{1}{\delta^p} \int_{B^0_i \cap \Omega} |f|^p \, dx. \]

Now we split the two integrals above as follows:

\[ \lambda \cdot |B^0_i \cap \Omega| \leq \int_{B^0_i \cap \{x \in \Omega: |\nabla u|^{p/4} > \frac{1}{2}\}} |\nabla u|^p \, dx + \frac{\lambda}{4} |B^0_i \cap \Omega| + \int_{B^0_i \cap \{x \in \Omega: |f|^p > \delta^p \frac{1}{4}\}} |f|^p \, dx + \frac{\lambda}{4} |B^0_i \cap \Omega|, \]

and therefore the conclusion follows. \( \square \)
3. Proof of the main result

In Section 3.1, we use an approximation argument to show that the proof of Theorem 1.12 can be reduced to proving an a priori estimate (1.4) with the assumption that $|\nabla u|^p \in L^\phi(\Omega)$, where $\phi$ is a Young function with $\Delta_2 \cap \nabla_2$. In Section 3.2, we finish the proof of Theorem 1.12.

3.1. Approximation

We first recall that the given bounded, open domain $\Omega$ is $(\delta, R)$ is Reifenberg flat. Then for each small $\epsilon > 0$, we write

$$\Omega_\epsilon = \{ x \in \Omega : d(x, \partial \Omega) > \epsilon \},$$

where $d$ is the standard distance function defined by

$$d(x, y) = |x - y| \quad (x, y \in \mathbb{R}^n)$$

and

$$d(x, \partial \Omega) = \inf \{ d(x, y) : y \in \partial \Omega \} \quad (x \in \Omega).$$

In the recent paper [7], the authors showed that an $\epsilon$ inner neighborhood of the $(\delta, R)$-Reifenberg flat domain is a Lipschitz domain with the $(\delta, R)$-Reifenberg flat property for $\delta$ small; that is, $\Omega_\epsilon$ is a Lipschitz domain with the uniform $(\delta, R)$-Reifenberg flat (see [7, Lemma 4.2]). Then according to a standard approximation of a Lipschitz domain by smooth domains, one can construct a further approximation of $\Omega_\epsilon$ for any fixed small $\epsilon > 0$ by smooth domains $\Omega^\eta_\epsilon \subset \Omega_\epsilon$ with the uniform $(\delta, R)$-Reifenberg flat property for a properly chosen $\eta = \eta(\epsilon) > 0$.

We next use a standard diagonal argument to extract a subsequence of smooth domains $\Omega^k$ with the uniform $(\delta, R)$-Reifenberg flat property such that

$$\Omega^k \subset \Omega \quad \text{and} \quad d_H(\partial \Omega^k, \partial \Omega) \to 0 \quad \text{as} \ k \to \infty,$$

(3.1)

where the Hausdorff distance $d_H$ is defined as follows:

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}.$$ 

Now let $\{A_k\}$ be a sequence of smooth functions with the uniform ellipticity and the uniform $(\delta, R)$-vanishing property converging to $A$ in $L^t$ for any $1 < t < \infty$, and $\{f_k\}_{k=1}^\infty$ be a sequence of smooth functions in $C_0^\infty(\Omega; \mathbb{R}^n)$ satisfying

$$f_k \rightarrow f \quad \text{in} \ L^p(\Omega; \mathbb{R}^n), \quad |f_k|^p \rightarrow |f|^p \quad \text{in} \ L^\phi(\Omega)$$

(3.2)

for a given Young function $\phi \in \Delta_2 \cap \nabla_2$. And

$$\int_\Omega |f_k|^p \, dx \leq C \int_\Omega |f|^p \, dx, \quad \int_\Omega \phi(|f_k|^p) \, dx \leq C \int_\Omega \phi(|f|^p) \, dx.$$ 

(3.3)
According to the standard theory for nonlinear uniformly elliptic equations of \( p \)-Laplacian type with the corresponding smooth data on smooth domains, the following Dirichlet problems

\[
\begin{aligned}
&\text{div}\left((A_k \nabla u_k \cdot \nabla u_k)^{\frac{p-2}{2}} A_k \nabla u_k\right) = \text{div}\left(|f_k|^p - f_k\right) & \text{in } \Omega^k, \\
u_k = 0 & \text{on } \partial \Omega^k
\end{aligned}
\] (3.4)

have unique weak solutions \( u_k \in W^{1,p}_0(\Omega^k) \) with the regularity \( u_k \in C^{1,\alpha}(\bar{\Omega}^k) \) for some \( \alpha = \alpha(n, p, k) \in (0, 1) \) with \( u_k = 0 \) on \( \partial \Omega^k \) in the classical sense.

Of course, these smooth solutions satisfy

\[
|\nabla u_k|^p \in L^\phi(\Omega^k). \tag{3.5}
\]

Then with the condition (3.5), our estimates in the next subsection show that these solutions have the uniform gradient estimates in Orlicz space with respect to the above approximation; that is,

\[
\int_{\Omega^k} \phi(|\nabla u_k|^p) \, dx \leq C \int_{\Omega^k} \phi(|f_k|^p) \, dx,
\]

where the constant \( C \) is independent of \( k \in \mathbb{N} \). We first extend \( u_k \) from \( \Omega^k \) to \( \Omega \) by the zero extension and denote by \( \tilde{u}_k \). Then \( \tilde{u}_k \in W^{1,p}_0(\Omega) \) and the inequality above and (3.3) imply that

\[
\int_{\Omega} \phi(|\nabla \tilde{u}_k|^p) \, dx \leq C \int_{\Omega} \phi(|f_k|^p) \, dx \leq C \int_{\Omega} \phi(|f|^p) \, dx. \tag{3.6}
\]

Moreover, from (1.3) and (3.3) we have

\[
\int_{\Omega} |\nabla \tilde{u}_k|^p \, dx \leq C \int_{\Omega} |f_k|^p \, dx \leq C \int_{\Omega} |f|^p \, dx. \tag{3.7}
\]

Therefore there exist a subsequence of \( \{\tilde{u}_k\} \) (still denoted by \( \{\tilde{u}_k\} \)) and a function \( v \in W^{1,p}_0(\Omega) \) such that

\[
\begin{aligned}
\tilde{u}_k & \to v \quad \text{strongly in } L^p(\Omega), \\
\nabla \tilde{u}_k & \rightharpoonup \nabla v \quad \text{weakly in } L^p(\Omega).
\end{aligned}
\] (3.8)

We claim that

\[
\nabla \tilde{u}_k \to \nabla v \quad \text{strongly in } L^p_{\text{loc}}(\Omega). \tag{3.9}
\]

The proof of the above claim will be given later.

It follows from (3.9) and the particular selection of \( A_k, f_k \) that function \( v \in W^{1,p}_0(\Omega) \) is also a weak solution of the problem (1.1)–(1.2). According to the uniqueness of the weak solution for the problem (1.1)–(1.2), we know that \( v = u \) and (3.9) reads as

\[
\nabla \tilde{u}_k \to \nabla u \quad \text{strongly in } L^p_{\text{loc}}(\Omega). \tag{3.10}
\]
Next we use a standard diagonal argument to extract a subsequence of \( \{\tilde{u}_k\} \) (still denoted by \( \{\tilde{u}_k\} \)) such that
\[
\nabla \tilde{u}_k \to \nabla u \quad \text{a.e. in } \Omega.
\] (3.11)

Applying Fatou’s lemma in the left-hand side of (3.6), we finally obtain the a priori estimates (1.4); that is,
\[
\int_{\Omega} \phi(|\nabla u|^p) \, dx \leq C \int_{\Omega} \phi(|f|^p) \, dx.
\]

We now prove the claim (3.9). We only consider the case \( p \geq 2 \). The other case that \( 1 < p < 2 \) can be handled in the same way (see [6,16]). To confirm this, choose a cut-off function \( \zeta \in C_0^\infty(\Omega) \) satisfying
\[
0 \leq \zeta \leq 1, \quad \text{supp} \, \zeta \subset \Omega^2 \quad \text{and} \quad \zeta = 1 \quad \text{on } \Omega^1.
\]

Then function \( \varphi = \zeta^p(\tilde{u}_l - \tilde{u}_n) \) with \( l, n \geq 2 \) is a qualified test function for (3.4) when \( k = l \) or \( k = n \). Thus we have
\[
\int_{\Omega} (A_l \nabla \tilde{u}_l \cdot \nabla \tilde{u}_l)^{p-2} A_l \nabla \tilde{u}_l \cdot \nabla [\zeta^p(\tilde{u}_l - \tilde{u}_n)] \, dx = \int_{\Omega} |f_l|^{p-2} f_l \cdot \nabla \zeta \, dx
\]
and
\[
\int_{\Omega} (A_n \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n)^{p-2} A_n \nabla \tilde{u}_n \cdot \nabla [\zeta^p(\tilde{u}_l - \tilde{u}_n)] \, dx = \int_{\Omega} |f_n|^{p-2} f_n \cdot \nabla \zeta \, dx.
\]

After simple computations we can write the resulting expression as
\[
I_1 = I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\]
where
\[
I_1 = \int_{\Omega} \zeta^p \left[ (A_l \nabla \tilde{u}_l \cdot \nabla \tilde{u}_l)^{p-2} A_l \nabla \tilde{u}_l - (A_l \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n)^{p-2} A_l \nabla \tilde{u}_n \right] \cdot \nabla (\tilde{u}_l - \tilde{u}_n) \, dx,
\]
\[
I_2 = -p \int_{\Omega} \zeta^{p-1}(\tilde{u}_l - \tilde{u}_n)(A_l \nabla \tilde{u}_l \cdot \nabla \tilde{u}_l)^{p-2} A_l \nabla \tilde{u}_l \cdot \nabla \zeta \, dx,
\]
\[
I_3 = p \int_{\Omega} \zeta^{p-1}(\tilde{u}_l - \tilde{u}_n)(A_l \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n)^{p-2} A_l \nabla \tilde{u}_n \cdot \nabla \zeta \, dx,
\]
\[
I_4 = \int_{\Omega} p \zeta^{p-1}(\tilde{u}_l - \tilde{u}_n) \left[ |f_l|^{p-2} f_l - |f_n|^{p-2} f_n \right] \cdot \nabla \zeta \, dx.
\]
\[ I_5 = \int_\Omega \xi^p \left[ |f_l|^{p-2} f_l - |f_n|^{p-2} f_n \right] \cdot \nabla (\bar{u}_l - \bar{u}_n) \, dx, \]

\[ I_6 = -\int_\Omega \xi^p \nabla (\bar{u}_l - \bar{u}_n) \cdot \left[ (A_l \nabla \bar{u}_n \cdot \nabla \bar{u}_n)^{\frac{p-2}{2}} A_l \nabla \bar{u}_n - (A_n \nabla \bar{u}_n \cdot \nabla \bar{u}_n)^{\frac{p-2}{2}} A_n \nabla \bar{u}_n \right] \, dx, \]

\[ I_7 = -p \int_\Omega \xi^{p-1} (\bar{u}_l - \bar{u}_n) \nabla \xi \cdot \left[ (A_l \nabla \bar{u}_n \cdot \nabla \bar{u}_n)^{\frac{p-2}{2}} A_l \nabla \bar{u}_n - (A_n \nabla \bar{u}_n \cdot \nabla \bar{u}_n)^{\frac{p-2}{2}} A_n \nabla \bar{u}_n \right] \, dx. \]

**Estimate of \( I_1 \).** Since \( A_l \) is uniformly elliptic, the vector valued function \( a(\xi, x) = (A_l(x) \times \xi : \xi)^{\frac{p-2}{2}} A_l(x) \xi \) is strictly monotonic; that is,

\[
\left[ (A_l(x) \xi : \xi)^{\frac{p-2}{2}} A_l(x) \xi - (A_l(x) \eta : \eta)^{\frac{p-2}{2}} A_l(x) \eta \right] \cdot (\xi - \eta) \geq c_0 |\xi - \eta|^p
\]

for all \( \xi, \eta \in \mathbb{R}^n \) and for some positive constant \( c_0 \). From this inequality, we have

\[
I_1 \geq c_0 \int_\Omega \xi^p |\nabla (\bar{u}_l - \bar{u}_n)|^p \, dx. \quad (3.12)
\]

**Estimate of \( I_2 \).** Since \( A_l \) is uniformly bounded, we have

\[
I_2 \leq C(p) \int_\Omega \left( \xi |\nabla \bar{u}_l| \right)^{p-1} |(\bar{u}_l - \bar{u}_n) \nabla \xi| \, dx.
\]

Then it follows from Young’s inequality with \( \epsilon \) that

\[
I_2 \leq \epsilon \int_\Omega \xi^p |\nabla \bar{u}_l| \, dx + C(\epsilon, p) \int_\Omega |\bar{u}_l - \bar{u}_n|^p \, dx. \quad (3.13)
\]

**Estimate of \( I_3 \).** Similarly to the estimate of \( I_2 \), we have

\[
I_3 \leq \epsilon \int_\Omega \xi^p |\nabla \bar{u}_n| \, dx + C(\epsilon, p) \int_\Omega |\bar{u}_l - \bar{u}_n|^p \, dx. \quad (3.14)
\]

**Estimate of \( I_4 \).** Using Young’s inequality with \( \epsilon \), we have

\[
I_4 \leq \epsilon \int_\Omega \xi^p \left( |f_l|^p + |f_n|^p \right) \, dx + C(\epsilon, p) \int_\Omega |\bar{u}_l - \bar{u}_n|^p \, dx. \quad (3.15)
\]

**Estimate of \( I_5 \).** From the following inequality

\[
| |\xi|^{p-2} \xi - |\eta|^{p-2} \eta | \leq c(p) (|\xi| + |\eta|)^{p-2} |\xi - \eta|
\]
for all $\xi, \eta \in \mathbb{R}^n$ and for some positive constant $c(p)$, we have

$$I_5 \leq c(p) \int_{\Omega} \xi^p\left[\left(|f_l| + |f_n|\right)^{p-2}|f_l - f_n|\left|\nabla (\tilde{u}_l - \tilde{u}_n)\right|\right] dx.$$ 

Then using Young’s inequality with $\epsilon$ and Hölder’s inequality, we have

$$I_5 \leq \epsilon \int_{\Omega} \xi^p\left|\nabla (\tilde{u}_l - \tilde{u}_n)\right|^p dx + C(p, \epsilon)\left(\int_{\Omega} |f_l|^{p} dx\right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |f_l - f_n|^{p} dx\right)^{\frac{1}{p-1}}.$$

Thus, we get

$$I_5 \leq \epsilon \int_{\Omega} \xi^p\left|\nabla (\tilde{u}_l - \tilde{u}_n)\right|^p dx + C(p, \epsilon)\left(\int_{\Omega} |f_l|^{p} dx\right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |f_l - f_n|^{p} dx\right)^{\frac{1}{p-1}}. \quad (3.16)$$

**Estimate of $I_6$.** Using the following elementary inequality

$$\left|(A_l \xi \cdot \xi)\right|^{p-2} A_l \xi - \left|(A_n \xi \cdot \xi)\right|^{p-2} A_n \xi \leq C(p)|A_l - A_n||\xi|^{p-1}$$

for all $\xi \in \mathbb{R}^n$, we have

$$I_6 \leq C(p) \int_{\Omega} \xi^p\left[|A_l - A_n||\nabla \tilde{u}_n|^{p-1}\left|\nabla (\tilde{u}_l - \tilde{u}_n)\right|\right] dx.$$ 

Now we recall Lemmas 1.10 and 1.7 to observe that for each $v \in L^\phi(\Omega)$ there exist $A_2 > 0$ and $\alpha_2 > 1$ such that

$$\int_{\Omega} |v|^{\alpha_2} dx \leq \int_{\{x \in \Omega: |v| \leq 1\}} |v|^{\alpha_2} dx + \int_{\{x \in \Omega: |v| \geq 1\}} |v|^{\alpha_2} dx \leq |\Omega| + \frac{A_2}{\phi(1)} \int_{\Omega} \phi(|v|) dx.$$ 

But then since $|\nabla \tilde{u}_n|^p \in L^\phi(\Omega)$ in view of (3.6), we find

$$\int_{\Omega} |\nabla \tilde{u}_n|^{p\alpha_2} dx \leq |\Omega| + \frac{A_2}{\phi(1)} \int_{\Omega} \phi(|\nabla \tilde{u}_n|^p) dx < +\infty.$$ 

Thus if we use Young’s inequality with $\epsilon$, we see that
\[ I_6 \leq \epsilon \int \Omega \zeta^p \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx \]

\[ + C(\epsilon, p) \left( \int \Omega |A_l - A_n|^\frac{p\alpha_2}{(p-1)(\alpha_2-1)} \, dx \right)^{1-\frac{1}{\alpha_2}} \left( \int \Omega \zeta^p \left| \nabla \bar{u}_n \right|^p \, dx \right)^{\frac{1}{\alpha_2}} \]

\[ \leq \epsilon \int \Omega \zeta^p \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx + C(\epsilon, p) \left( \int \Omega |A_l - A_n|^\frac{p\alpha_2}{(p-1)(\alpha_2-1)} \, dx \right)^{1-\frac{1}{\alpha_2}}. \]  

Thus

\[ I_6 \leq \epsilon \int \Omega \zeta^p \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx + C(\epsilon, p) \left( \int \Omega |A_l - A_n|^\frac{p\alpha_2}{(p-1)(\alpha_2-1)} \, dx \right)^{1-\frac{1}{\alpha_2}}. \]  

(3.17)

**Estimate of** \( I_7 \). Similarly to the estimate of \( I_4 \), we have

\[ I_7 \leq \epsilon \int \Omega \zeta^p \left| \nabla \bar{u}_n \right|^p \, dx + C(\epsilon, p) \int \Omega |\bar{u}_l - \bar{u}_n|^p \, dx. \]  

(3.18)

Now we finally combine all the estimates (3.12)–(3.18), to obtain that for every \( \epsilon > 0 \),

\[ \int \Omega \zeta^p \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx \]

\[ \leq C(\epsilon, p) \left\{ \int \Omega |\bar{u}_l - \bar{u}_n|^p \, dx + \left( \int \Omega |f_l - f_n|^p \, dx \right)^{\frac{1}{p-1}} + \left( \int \Omega |A_l - A_n|^\frac{p\alpha_2}{(p-1)(\alpha_2-1)} \, dx \right)^{1-\frac{1}{\alpha_2}} \right\} \]

\[ + \epsilon \int \Omega \zeta^p \left( |\nabla u_l|^p + |\nabla u_n|^p + |f_l|^p + |f_n|^p \right) \, dx. \]

Recalling the strong convergence of \( \{\bar{u}_l\} \) in \( L^p(\Omega) \), the particular selection of \( A_k, f_k \) and the arbitrariness of \( \epsilon > 0 \), we conclude that

\[ \int \Omega^1 \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx \leq \int \Omega \zeta^p \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx \to 0 \quad \text{as} \ l, n \to \infty. \]

Similarly, for every fixed \( k \in \mathbb{N} \) we have

\[ \int \Omega^k \left| \nabla (\bar{u}_l - \bar{u}_n) \right|^p \, dx \to 0 \quad \text{as} \ l, n \to \infty, \]

which implies the claim (3.9).
3.2. Final proof

Without loss of generality, we first assume that $\delta$ is a small constant, say, $0 < \delta \leq \frac{1}{8}$. Then

$$\left( \frac{20K}{1 - \delta} \right)^n \lambda_0 \leq \left( \frac{160K}{7} \right)^n \lambda_0.$$ 

Now we set for simplicity

$$\lambda_1 = \left( \frac{160K}{7} \right)^n \lambda_0,$$  

(3.19)

fix any $\lambda \geq \lambda_1$ and normalize by defining

$$u_\lambda = \frac{u}{\lambda^{\frac{1}{p}}} \quad \text{and} \quad f_\lambda = \frac{f}{\lambda^{\frac{1}{p}}}.$$ 

Then $u_\lambda$ is still the weak solution of (1.1)–(1.2) with $f_\lambda$ replacing $f$. According to Lemma 2.1, we can construct a family of disjoint balls $\{B_i^{(0)}\}_{i \geq 1} = \{B_{\rho_i}(x_i)\}_{i \geq 1}$ with $x_i \in \mathcal{E}(\lambda)$ and $0 < \rho_i \leq \frac{R}{10}$. Moreover, if we set

$$B_i^{2} = B_{10\rho_i}(x_i),$$

then $B_i^{2} \subset B_R(x_i)$. Then it follows from Lemma 2.1 that

$$\int_{\Omega \cap B_i^{2}} |\nabla u_\lambda|^p \ dx = \frac{1}{\lambda} \int_{\Omega \cap B_i^{2}} |\nabla u|^p \ dx \leq 1$$  

(3.20)

and

$$\int_{\Omega \cap B_i^{2}} |f_\lambda|^p \ dx = \frac{1}{\lambda} \int_{\Omega \cap B_i^{2}} |f|^p \ dx \leq \delta^p.$$  

(3.21)

We fix again any $i \geq 1$. Then our argument depends upon whether

1. $B_i^{2} \subset \Omega$,
2. $B_i^{2} \not\subset \Omega$.

We study case (1) in Lemma 3.1 and case (2) in Lemma 3.2.

Let us consider the case $B_i^{2} \subset \Omega$. In this case we assume $x_i = 0$ by a translation invariance. Then we have the following scaling analysis.

**Lemma 3.1.** For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if $u \in W_{0}^{1,p}(\Omega)$ is a weak solution of (1.1)–(1.2) with (3.20) and (3.21), then there exists a weak solution $v_\lambda^i \in W^{1,p}(B_i^{2})$ of

$$\text{div}((\bar{A}_{B_i^{2}} \nabla v_\lambda^i \cdot \nabla v_\lambda^i)^{\frac{p-2}{2}} \bar{A}_{B_i^{2}} \nabla v_\lambda^i) = 0 \quad \text{in } B_i^{2} \subset \Omega$$

\[\]
such that
\[
\int_{B_1^i} \left| \nabla (u_\lambda - v_\lambda^i) \right|^p dx \leq \epsilon^p.
\]

Moreover, there exists a constant $N_0 > 1$ such that
\[
\sup_{B_1^i} \left| \nabla v_\lambda^i \right| \leq N_0.
\]

**Proof.** We define by
\[
\begin{align*}
  u_\lambda^i(x) &= \frac{2}{5 \rho_i} u_\lambda \left( \frac{5 \rho_i}{2} x \right), \\
  f_\lambda^i(x) &= f_\lambda \left( \frac{5 \rho_i}{2} x \right), \\
  A^i(x) &= A \left( \frac{5 \rho_i}{2} x \right), \quad x \in B_4.
\end{align*}
\]

Then $u_\lambda^i \in W^{1,p}(B_4)$ is a weak solution of
\[
\text{div} \left( \left( A^i \nabla u_\lambda^i \cdot \nabla u_\lambda^i \right)^{\frac{p-2}{2}} A^i \nabla u_\lambda^i \right) = \text{div} \left( |f_\lambda^i|^{p-2} f_\lambda^i \right) \quad \text{in } B_4.
\]

Now it follows from (3.20), (3.21) and our small BMO condition (see Definition 1.2) that
\[
\int_{B_4} |A^i - \bar{A}^i_{B_4}| dx \leq \delta, \\
\int_{B_4} \left| \nabla u_\lambda^i \right|^p dx \leq 1 \quad \text{and} \quad \int_{B_4} |f_\lambda^i|^p dx \leq \delta^p.
\]

Thus according to [6, Corollary 3.5], there exists a weak solution $v \in W^{1,p}(B_4)$ of
\[
\text{div} \left( \left( \bar{A}^i_{B_4} \nabla v \cdot \nabla v \right)^{\frac{p-2}{2}} \bar{A}^i_{B_4} \nabla v \right) = 0 \quad \text{in } B_4
\]
such that
\[
\int_{B_2} \left| \nabla (u_\lambda^i - v) \right|^p dx \leq \epsilon^p.
\]

Moreover, there exists a constant $N_0 > 1$ such that
\[
\sup_{B_2} |\nabla v| \leq N_0.
\]
Now we define $v_i^\lambda$ in $B_i^1$ by

$$v_i^\lambda(x) = \frac{5\rho_i \lambda^{1/p}}{2} v_i^\lambda \left( \frac{2}{5\rho_i} x \right).$$

Then changing variables, we recover the conclusion of Lemma 3.1. This finishes the proof. □

Now we extend the estimates in the case $B_i^2 = B_{10\rho_i}(x_i) \subset \Omega$ to the case $B_i^2 \not\subset \Omega$ to study the estimates up to the boundary. We write

$$\Omega_\rho = \Omega \cap B_\rho, \quad \Omega_\rho(y) = \Omega_\rho + y \quad (y \in \Omega, \ \rho > 0).$$

We first assume that $\Omega$ is $(\delta, 3R)$-Reifenberg flat by a scaling. Then since $B_i^2 = B_{10\rho_i}(x_i) \not\subset \Omega$,

there exists an appropriate coordinate system such that

$$y_i = x_i, \quad B_i^0 \subset \Omega_{15\rho_i}, \text{ and } B_{30\rho_i}^+ \subset \Omega_{30\rho_i} \subset B_{30\rho_i} \cap \{y_n > -30\rho_i \delta\}. \quad (3.22)$$

Then we have the following scaling analysis near the boundary.

**Lemma 3.2.** For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if $u \in W_0^{1,p}(\Omega)$ is the weak solution of (1.1)–(1.2) with (3.20)–(3.22), then there exists a weak solution $v_i^\lambda \in W^{1,p}(B_{30\rho_i}^+)$ of

$$\text{div} \left( (\bar{A}_{B_{30\rho_i}^+} \nabla v_i^\lambda \cdot \nabla v_i^\lambda)^{\frac{p-2}{2}} \bar{A}_{B_{30\rho_i}^+} \nabla v_i^\lambda \right) = 0 \quad \text{in } B_{30\rho_i}^+$$

with $v_i^\lambda = 0$ on $B_{30\rho_i}^+ \cap \{y_n = 0\}$ such that

$$\int_{\Omega_{15\rho_i}} |\nabla (u_{\lambda} - \tilde{v}_i^\lambda)|^p \ dx \leq \epsilon^p,$$

where $\tilde{v}_i^\lambda$ is the zero extension of $v_i^\lambda$ from $B_{30\rho_i}^+$ to $\Omega_{30\rho_i}$. Moreover, there exists a constant $N_1 > 1$ such that

$$\sup_{\Omega_{15\rho_i}} |\nabla \tilde{v}_i^\lambda| \leq N_1.$$

**Proof.** We define by

$$
\begin{cases}
  u_i^\lambda(x) = \frac{2}{15\rho_i} u_{\lambda} \left( \frac{15\rho_i}{2} x \right), \\
  f_i^\lambda(x) = f_{\lambda} \left( \frac{15\rho_i}{2} x \right), \\
  A_i^\lambda(x) = A \left( \frac{15\rho_i}{2} x \right) \quad (x \in \Omega_4).
\end{cases}
$$
Then \( u^i_\lambda \in W^{1,p}(\Omega_4) \) is a weak solution of
\[
\text{div}(\left(\left( A^i \nabla u^i_\lambda \cdot \nabla u^i_\lambda \right)^{\frac{p-2}{2}} A^i \nabla u^i_\lambda \right) = \text{div}(|f^i_\lambda|^{p-2} f^i_\lambda) \quad \text{in } \Omega_4
\]
and from (3.20), (3.21) and Definition 1.2 one can readily check that
\[
\int_{\Omega_4} |A^i - \tilde{A}^i_{\Omega_4}|^2 \, dx \leq \delta,
\]
\[
\int_{\Omega_4} |\nabla u^i_\lambda|^p \, dx \leq 1, \quad \text{and} \quad \int_{\Omega_4} |f^i_\lambda|^p \, dx \leq \delta^p.
\]
Thus according to [6, Corollary 4.5], there exists a weak solution \( v \in W^{1,p}(B^+_4) \) of
\[
\text{div}(\left(\left( \bar{A}^i_{B^+_4} \nabla v \cdot \nabla v \right)^{\frac{p-2}{2}} \bar{A}^i_{B^+_4} \nabla v \right) = 0 \quad \text{in } B^+_4
\]
with \( v = 0 \) in \( B_4 \cap \{ x_n = 0 \} \) such that
\[
\int_{\Omega_2} |\nabla (u^i_\lambda - \bar{v})|^p \, dx \leq \epsilon^p,
\]
where \( \bar{v} \) is the zero extension of \( v \) from \( B^+_4 \) to \( \Omega_4 \). Moreover, there exists a constant \( N_1 > 1 \) such that
\[
\sup_{\Omega_2} |\nabla \bar{v}| \leq N_1.
\]
Now we define \( \tilde{v}^i_\lambda \) in \( \Omega_{30\rho_i} \) by
\[
\tilde{v}^i_\lambda(x) = \frac{15 \rho_i \lambda^\frac{1}{p}}{2} \bar{v}\left( \frac{2}{15 \rho_i} x \right).
\]
Then changing variables, we recover the conclusion of Lemma 3.2. This finishes the proof. \( \square \)

Before we finish the proof of the main result, we give the following lemmas.

**Lemma 3.3.** Let \( \phi \in \Phi \) be a Young function with \( \phi \in \Delta_2 \cap \nabla_2 \) and \( g \in L^\phi(\Omega) \). Then
\[
\int_{\Omega} \phi(|g|) \, dx = \int_{0}^{\infty} |\{ x \in \Omega: |g| > \mu \}| \, d[\phi(\mu)].
\]
**Lemma 3.4.** Let $\phi \in \Phi$ be a Young function with $\phi \in \Delta_2 \cap \nabla_2$ and $g \in L^\phi(\Omega)$. Then for any $a, b > 0$ we have

$$ I = \int_0^\infty \frac{1}{\lambda} \left[ \int_{\{x \in \Omega: |g| > a\lambda\}} |g| \, dx \right] d\left[ (\phi(b\lambda)) \right] \leq C \int_\Omega \phi(|g|) \, dx, $$

where $C = C(a, b, \phi)$.

**Proof.** Interchanging the order of integration and using integration by parts in $I$, we have

$$ I = \int_\Omega |g| \left[ \int_0^{\frac{|g|}{a}} \frac{1}{\mu} \, d\phi(b\mu) \right] \, dx $$

$$ \leq \int_\Omega |g| \left\{ \frac{\phi(b|g|)}{|g|} + \int_0^{\frac{|g|}{a}} \frac{\phi(b\mu)}{\mu^2} \, d\mu \right\} \, dx. $$

Then Lemma 1.7 implies

$$ I \leq C \int_\Omega \phi(|g|) \, dx + A_2(ab)^{\alpha_2} \int_\Omega \phi \left( \frac{b|g|}{a} \right) |g|^{1-\alpha_2} \left[ \int_0^{\frac{|g|}{a}} \frac{1}{\mu^2} \, d\mu \right] \, dx $$

$$ \leq C \int_\Omega \phi(|g|) \, dx, $$

which completes the proof. \( \square \)

Now we are set to prove the main result, Theorem 1.12.

**Proof.** We only consider the boundary case $B^2 \not\subset \Omega$. Indeed, the interior case $B^2 \subset \Omega$ can be easily handled with a slight modification in the boundary case. According to Lemma 3.2, we see that for any $\epsilon > 0$, there exists a small $\delta(\epsilon) > 0$ such that

$$ \int_{\Omega_{15\rho_1}} \left| \nabla (u_\lambda - \tilde{v}^{i}_\lambda) \right|^p \, dx \leq \epsilon^p \quad \text{and} \quad \sup_{\Omega_{15\rho_1}} \left| \nabla \tilde{v}^{i}_\lambda \right| \leq N_1, $$

with $\tilde{v}^{i}_\lambda$ and $N_1$ as in Lemma 3.2. Then using the inequalities above and the elementary inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for any $a, b > 0$ and $p \geq 1$, we compute as follows:

$$ |B_{5\rho_1}(y_1) \cap \left\{ x \in \Omega: |\nabla u|^p > (2N_1)^p \lambda \right\}| $$

$$ = \left| B_{5\rho_1}(y_1) \cap \left\{ x \in \Omega: |\nabla u_\lambda|^p > (2N_1)^p \right\} \right| $$
\[
\begin{align*}
&\leq \left| B_{15\rho_i} \cap \{ x \in \Omega : |\nabla (u_\lambda - \tilde{v}_\lambda^i)|^p > N_1^p \} \right| + \left| B_{15\rho_i} \cap \{ x \in \Omega : |\nabla \tilde{v}_\lambda^i|^p > N_1^p \} \right| \\
&= \left| B_{15\rho_i} \cap \{ x \in \Omega : |\nabla (u_\lambda - \tilde{v}_\lambda^i)|^p > N_1^p \} \right| \\
&\leq \frac{1}{N_1^p} \int_{\Omega_{15\rho_i}} |\nabla (u_\lambda - \tilde{v}_\lambda^i)|^p \, dx \\
&\leq \frac{\epsilon^p |\Omega_{15\rho_i}|}{N_1^p} \leq \frac{\epsilon^p |B_{15\rho_i}|}{N_1^p} = \frac{15^n \epsilon^p |B_0^i|}{N_1^p},
\end{align*}
\]
and from (2.8) and Lemma 2.2 we observe
\[
\left| B_{5\rho_i}(y_i) \cap \{ x \in \Omega : |\nabla u|^p > (2N_1)^p \lambda \} \right|
\leq \frac{30^n \epsilon^p}{(1-\delta)^p N_1^p} \left| B_0^i \cap \Omega \right|
\leq \frac{2 \cdot 30^n \epsilon^p}{(1-\delta)^p N_1^p \lambda} \left( \int_{B_i^0(\{ x \in \Omega : |\nabla u|^p > \frac{\lambda}{4} \})} |\nabla u|^p \, dx + \frac{1}{\delta^p} \right) \left( \int_{B_i^0(\{ x \in \Omega : |\nabla f|^p > \delta^p \frac{\lambda}{4} \})} |\nabla f|^p \, dx \right).
\]

Now we recall that for given \( \lambda \geq \lambda_1 \), \( B_i^0 \cap \Omega \) are disjoint and that
\[
\bigcup_{i \geq 1} B_i^1 \cap \Omega \supset E(\lambda) = \{ x \in \Omega : |\nabla u|^p > \lambda \},
\]
which implies that
\[
\left| \{ x \in \Omega : |\nabla u|^p > (2N_1)^p \lambda \} \right|
\leq \sum_{i \geq 1} \left| B_i^1 \cap \{ x \in \Omega : |\nabla u|^p > (2N_1)^p \lambda \} \right|
\leq \frac{2 \cdot 30^n \epsilon^p}{(1-\delta)^p N_1^p \lambda} \left( \int_{\Omega(\{ |\nabla u|^p \geq \frac{\lambda}{4} \})} |\nabla u|^p \, dx + \frac{1}{\delta^p} \right) \left( \int_{\Omega(\{ |\nabla f|^p > \delta^p \frac{\lambda}{4} \})} |\nabla f|^p \, dx \right).
\]

Then using Lemma 3.3, we have
\[
\begin{align*}
\int_{\Omega} \phi(|\nabla u|^p) \, dx &= \int_0^{\infty} \left| \{ x \in \Omega : |\nabla u|^p > (2N_1)^p \lambda \} \right| d\left[ \phi \left( (2N_1)^p \lambda \right) \right] \\
&= \int_0^{\lambda_1} \left| \{ x \in \Omega : |\nabla u|^p > (2N_1)^p \lambda \} \right| d\left[ \phi \left( (2N_1)^p \lambda \right) \right] \\
&\quad + \int_{\lambda_1}^{\infty} \left| \{ x \in \Omega : |\nabla u|^p > (2N_1)^p \lambda \} \right| d\left[ \phi \left( (2N_1)^p \lambda \right) \right] \\
&= J_1 + J_2.
\end{align*}
\]
**Estimate of** \( J_1 \). Using (1.3), (2.1) and (3.19) (our notations for \( \lambda_1 \) and \( \lambda_0 \) in Section 2), we deduce that

\[
\lambda_1 = C \left[ \int_\Omega |f|^p \, dx + \frac{1}{\delta^p} \int_\Omega |f|^p \, dx \right]
\]

\[
\leq C \int_\Omega |f|^p \, dx.
\]

Then it follows from (1.6) and Jensen’s inequality that

\[
J_1 \leq \phi \left( (2N_1)^p \lambda_1 \right) |\Omega| \leq C_1 |\Omega| \phi \left( \int_\Omega |f|^p \, dx \right) \leq C_1 \int_\Omega \phi(|f|^p) \, dx,
\]

where \( C_1 = C_1(n, p, \delta, \phi) \).

**Estimate of** \( J_2 \). From Lemma 3.4 we observe

\[
J_2 \leq \frac{2 \cdot 30^n \epsilon^p}{(1 - \delta)^n N_1^p} \left\{ \int_0^\infty \frac{1}{\lambda} \int_{\{x \in \Omega : |\nabla u|^p > \frac{\lambda}{\delta^p} \}} |\nabla u|^p \, dx \, d\left[ \phi \left( (2N_1)^p \lambda \right) \right] \right. \\
+ \frac{1}{\delta^p} \int_0^\infty \frac{1}{\lambda} \left[ \int_{\{x \in \Omega : |f|^p \leq \frac{\lambda}{\delta^p} \}} |f|^p \, dx \right] d\left[ \phi \left( (2N_2)^p \lambda \right) \right] \right\}
\]

\[
\leq \frac{C_2 \epsilon^p}{(1 - \delta)^n} \int_\Omega \phi(|\nabla u|^p) \, dx + C_3 \int_\Omega \phi(|f|^p) \, dx
\]

\[
\leq \frac{C_2 \epsilon^p}{(1 - \delta)^n} \int_\Omega \phi(|\nabla u|^p) \, dx + C_3 \int_\Omega \phi(|f|^p) \, dx,
\]

where \( C_2 = C_2(n, p, \phi) \) and \( C_3 = C_3(n, p, \phi, \delta, \epsilon) \).

Combining the estimates of \( J_1 \) and \( J_2 \), we obtain

\[
\int_\Omega \phi(|\nabla u|^p) \, dx \leq \frac{C_2 \epsilon^p}{(1 - \delta)^n} \int_\Omega \phi(|\nabla u|^p) \, dx + C_4 \int_\Omega \phi(|f|^p) \, dx,
\]

where \( C_4 = C_4(n, p, \phi, \delta, \epsilon) \). Now we recall that we assume

\[
\int_\Omega \phi(|\nabla u|^p) \, dx < +\infty
\]
via our approximation argument discussed in Section 3.1. Then choosing a suitable \( \epsilon \) such that \( \frac{C_2\epsilon^p}{(1-\delta)^p} < 1 \), thereby determining \( \delta \) with \( 0 < \delta \leq \frac{1}{8} \) as in Lemmas 3.1, 3.2, we finally obtain

\[
\int_{\Omega} \phi(|\nabla u|^p) \, dx \leq C \int_{\Omega} \phi(|f|^p) \, dx.
\]

This completes the proof. \( \square \)

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References