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The Asymptotic Number of Labeled Graphs with Given Degree Sequences

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Asymptotics are obtained for the number of $n \times n$ symmetric non-negative integer matrices subject to the following constraints: (i) each row sum is specified and bounded, (ii) the entries are bounded, and (iii) a specified "sparse" set of entries must be zero. The result can be interpreted in terms of incidence matrices for labeled graphs.

1. STATEMENT OF RESULTS

Let $\mathscr{M}(n, z)$ be the set of all $n \times n$ symmetric (0, 1) matrices with at most z zeroes in each row. Let $\mathbf{r}$ be an n long vector over $[0, d] = \{0, 1, \ldots, d\}$. For $M \in \mathscr{M}(n, z)$, let $G(M, \mathbf{r}, t)$ be the number of $n \times n$ symmetric matrices (g_{ij}) over $[0, t]$ such that

(i) $g_{ij} = 0$ whenever $m_{ij} = 0$,

(ii) $\sum_j g_{ij} = r_i$.

By "$A(\omega) \sim B(\omega)$ uniformly for $\omega \in \Omega$ as $f(\omega) \to \infty$" we mean

$$\lim_{k\to\infty} \sup_{\substack{\omega\in\Omega \\ f(\omega)=k}} \left| \frac{A(\omega)}{B(\omega)} - 1 \right| = 0,$$

where the supremum over the empty set is 0 and 0/0 = 1 by convention.

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We prove

THEOREM 1. *For given d, t, and z*

$$G(M, \mathbf{r}, t) \sim T(f, \delta)\, e^{\epsilon a - b} / \Pi r_i\,!$$

uniformly for $(M, \mathbf{r}) \in \bigcup_{n=1}^{\infty} (\mathcal{M}(n, z) \times [0, d]^n)$ *as* $f \to \infty$ *where*

$$f = \sum_i r_i ,$$

$$\epsilon = -1 \text{ if } t = 1 \textit{ and } +1 \text{ if } t > 1,$$

$$a = \Big(\sum_i \binom{r_i}{2}\Big/f\Big)^2 + \sum_{m_{ii}=1} \binom{r_i}{2}\Big/f,$$

$$b = \Big(\sum_{\substack{m_{ij}=0\\ i<j}} r_i r_j + \sum_i \binom{r_i}{2}\Big)\Big/f,$$

$$\delta = \sum_{m_{ii}=0} r_i ,$$

and

$T(f, \delta)$ is the number of involutions on $[1, f]$ such that no element in some specified set of size δ is fixed.

We also prove the following:

THEOREM 2. *For $T(f, \delta)$ defined as above,*

$$T(f, \delta) \sim (f/e)^{f/2} \exp(-k/2f + k^2/4f^2) \times (\exp(k/f^{1/2}) + (-1)^f \exp(-k/f^{1/2}))/2^{1/2}$$

uniformly as $f \to \infty$ *where* $k = f - \delta$.

Remarks. Since $f \leqslant dn$, we have $n \to \infty$ as $f \to \infty$. Since z is fixed, the *forced* zeroes in the matrices being counted become very sparse as $f \to \infty$; however, most entries will be zero since row sums are bounded and $n \to \infty$. Two graphical interpretations are of particular interest: Let I be the $n \times n$ identity matrix and J the $n \times n$ matrix of ones. Then $G(J - I, \mathbf{r}, 1)$ is the number of loopless labeled graphs with degree sequence $\mathbf{r}$ and $G(J, \mathbf{r}, 1)$ counts those graphs when loops are allowed. N. G. de Bruijn (unpublished) has obtained the asymptotic formula for $G(J - I, \mathbf{r}, 1)$ with all r_i's equal.

A result similar to Theorem 1 without the symmetry condition was proved in [2]. We adapt the technique there to the symmetric case. The situation is more complicated because we are enumerating certain classes of involutions rather than certain classes of permutations.

2. Proof of Theorem 2

We have

$$T(f, \delta) = \sum_i \binom{k}{i} c_{f-i} \tag{1}$$

where c_m is the number of ways to partition an m-set into blocks all of size 2 and so

$$c_m = \begin{cases} 0 \text{ if } m \text{ odd} \\ \dfrac{m!}{2^{m/2}(m/2)!} \sim 2^{1/2}(m/e)^{m/2} \text{ if } m \text{ even.} \end{cases}$$

It follows that

$$c_{f-i} \sim 2^{1/2}(f/e)^{f/2} f^{-i/2} e^{i^2/4f}$$

uniformly provided $f - i$ is even and $i^3 = O(f^{2-\sigma})$ for some $\sigma > 0$. Define $\epsilon = 1/10$.

Case (i). $k \geqslant f^{1/2+\epsilon}$. Using the idea in [1, Sect. 3] we find that the maximum term occurs near $i = k/f^{1/2}$ and that those terms with $| i - k/f^{1/2} |^2/(k/f^{1/2})$ large do not contribute significantly to the sum. Summing over $| i - k/f^{1/2} | \leqslant f^{\epsilon/7}(k/f^{1/2})^{1/2}$ we obtain the estimate

$$\frac{1}{2} \sum \binom{k}{i} 2^{1/2} \left(\frac{f}{e}\right)^{f/2} f^{-i/2} e^{i^2/4f},$$

where the factor of 1/2 is due to the fact that $c_m = 0$ when m is odd. In this range $i^2/4f = k^2/4f^2 + o(1)$. Thus

$$T(f, \delta) \sim (f/e)^{f/2} e^{k^2/4f^2} (1 + f^{-1/2})^k/2^{1/2}.$$

The estimate is uniform.

Case (ii). $k \leqslant f^{1/2+\epsilon}$. The terms with $i > f^{1/4-\epsilon} = I$ now turn out to be negligible and the remaining sum is as given in the theorem because

$$\sum_{i<I} \binom{k}{i} c_{f-i} \sim 2^{1/2}(f/e)^{f/2} \sum \frac{(k/f^{1/2})^i}{i!},$$

where the sum is over $i < I$ and $f - i$ even. When f is even this is asymptotic to $\cosh(k/f^{1/2})$ and when f is odd it is asymptotic to $\sinh(k/f^{1/2})$. ∎

The most important (for us) aspects of Theorem 2 are given below in (2) and (3). They follow easily from the theorem. Below and *throughout the paper* we make the following convention:

$$T(C, \beta) = T(C, \max(0, \beta)).$$

Recall that $A = \omega(B)$ means $B = o(A)$.

COROLLARY. *We have*

$$T(f-j, \delta-j) \sim f^{-j/2} T(f, \delta) \text{ uniformly as } f \to \infty \tag{2a}$$

provided $j^2 = o(f)$ *uniformly and either* j *is even or* $k = \omega(f^{1/2})$ *uniformly*;

$$T(f-j, \delta-j) \leqslant (K/f)^{j/2} \cdot T(f, \delta) \tag{2b}$$

when j *is even or* $k \geqslant f^{1/2}/2$. *Also,*

$$T(f, \delta \pm e(f)) \sim T(f, \delta) \text{ uniformly as } f \to \infty \tag{3}$$

provided $e(f) = o(f^{1/2})$ *uniformly and* $k = \omega(e(f))$ *uniformly.*

We shall need the obvious

$$T(r, s) \leqslant T(r, s') \quad \text{for} \quad s \leqslant s' \tag{4}$$

from time to time.

3. A LEMMA ON INVOLUTIONS

Let a set $\mathcal{O}$ of objects and a finite set $\mathcal{F}$ of properties be given. For each $X \subseteq \mathcal{F}$, let $N_{\geqslant}(X)$ be the number of objects in $\mathcal{O}$ having at least the properties in X and let $N_{=}(X)$ be the number which, in addition, have none of the properties in $\mathcal{F} - X$. Define

$$P_u = \sum_{|X|=u} N_{=}(X) \qquad S_u = \sum_{|X|=u} N_{\geqslant}(X).$$

LEMMA 1. *Let* Δ *be a subset of* $[1, f]$ *and* $\mathcal{P}$ *be a collection of unordered pairs* $\{i, j\}$ *with* $i \neq j$ *and* $1 \leqslant i, j \leqslant f$. *If* $\operatorname{Inv}(\mathcal{P}, \Delta)$ *denotes the number of involutions* σ *on* $[1, f]$ *such that* $\sigma(i) \neq j$ *whenever* $\{i, j\} \in \mathcal{P}$ *and* $\sigma(i) \neq i$ *whenever* $i \in \Delta$, *then*

$$\operatorname{Inv}(\mathcal{P}, \Delta) \sim T(f, \delta) \cdot e^{-\lambda}$$

uniformly as $f \to \infty$ for all Δ and all $\mathscr{P}$ satisfying

$$|\{i: \{i,j\} \in \mathscr{P}\}| \leqslant C$$

for some C and all j and f, where $\delta = |\Delta|$ and $\lambda = |\mathscr{P}|/f$.

Proof. Let $\mathcal{O}$ be all involutions σ on $[1,f]$ such that no element of Δ is fixed. Let $\mathscr{F} = \mathscr{P}$ where $\{i,j\}$ is the property "$\sigma(i) = j$." We want to compute P_0. Suppose $X \subseteq \mathscr{F}$ and $|X| = u$. Then $N_{\geqslant}(X) = T(f-2u, \delta - l(X))$ where

(i) $l(x) = |\{i \in \Delta: \{i,j\} \in X \text{ for some } j\}|$

(ii) X is independent; i.e., each $i \in [1,f]$ appears in at most one pair $\{i,j\} \in X$.

If X is not independent, $N_{\geqslant}(X) = 0$. Note that $S_0 = T(f, \delta)$.

First suppose that $|\mathscr{P}| \geqslant f^{1/4}$. Using (3) with $e(f) = f^{1/9}$ and (4) we have

$$N_{\geqslant}(X) \sim T(f-2u, \delta - 2u)$$

uniformly for $u = o(f^{1/9})$ and $\delta \leqslant f - f^{1/8}$. When $u = o(f^{1/9})$ almost all subsets of size u are independent (see [1, p. 493]). Hence

$$S_u \sim \binom{|\mathscr{P}|}{u} T(f-2u, \delta-2u) \tag{5}$$

uniformly for $u = o(f^{1/9})$ and $\delta \leqslant f - f^{1/8}$.

Now suppose $\delta > f - f^{1/8}$. We claim (5) still holds. The idea is that for almost all subsets $X \subseteq \mathscr{P}$, X consists entirely of pairs made from Δ. Indeed, the number of subsets X with $|X| = u$ and *not* made from Δ is bounded above by $f^{1/8}C\binom{|\mathscr{P}|-1}{u-1}$, where the first factor selects $i \notin \Delta$, the second selects j with $\{i,j\} \in \mathscr{P}$, and the third completes X. This product is $o(f^{-1/72}\binom{|\mathscr{P}|}{u})$. By (4), $T(f-2u, \delta-2u) \geqslant T(f-2u, \delta - l(x))$, which proves (5) for all δ. By (2) and (5)

$$u!S_u \sim \lambda^u T(f, \delta).$$

By [1, (4.3)] the lemma follows for $|\mathscr{P}| \geqslant f^{1/4}$.

Now suppose that $|\mathscr{P}| \leqslant f^{1/4}$. Then $S_1 \leqslant |\mathscr{P}|\, T(f-2, \delta-2) = O(f^{-1/4}T(f, \delta))$. Since $S_0 \geqslant P_0 \geqslant S_0 - S_1$, we are done. ∎

4. Outline of the Proof

Suppose that d, z, and t are fixed, $M \in \mathscr{M}(n, z)$ and $\mathbf{r} \in [0, d]^n$. Consider a set F of cardinality f and an ordered partition

$$F = R_1 \cup R_2 \cup \cdots \cup R_n$$

where $|R_i| = r_i$. Define the set $\mathcal{O}$ of objects to be all involutions σ on F such that

(a) $\sigma(R_i) \cap R_j = \varnothing$ whenever $m_{ij} = 0$,
(b) if α and $\sigma(\alpha)$ are in the same R_i, then $\sigma(\alpha) = \alpha$.

By the lemma of the previous section,

$$S_0 \sim T(f, \delta)e^{-b} \qquad \text{uniformly as } f \to \infty. \tag{6}$$

Let Φ be a map from involutions σ to $n \times n$ symmetric matrices G defined by $g_{ij} = |\sigma(R_i) \cap R_j|$. The set $\Phi(\mathcal{O})$ is precisely those matrices counted by $G(m, \mathbf{r}, d)$ and for $G \in \Phi(\mathcal{O})$,

$$|\Phi^{-1}(G)| = \prod r_i! \Big/ \prod_{i \leqslant j} g_{ij}!. \tag{7}$$

We now define a set $\mathcal{Q}$ of properties for $\mathcal{O}$. These have the form "$|\sigma(R_i) \cap R_j| = k$" where $d \geqslant k \geqslant 2$ and $m_{ij} = 1$. We can always take $i \leqslant j$. We abbreviate the property as "$(i, j) = k$." We will consider three classes of $X \subseteq \mathcal{Q}$:

(I) All k's $= 2$, no m appears in more than one (i, j);
(II) All k's $= 2$, some m appears in more than one (i, j);
(III) Some $k \geqslant 3$.

Let the contribution of the three classes to P_u and S_u be denoted by P'_u, P''_u, P'''_u, S'_u, S''_u, and S'''_u respectively. We shall show that class I provides the only significant contribution and that large u is unimportant. This is summarized in

LEMMA 2. *Suppose that for some C*

$$\begin{aligned} u!\,S'_u &= S_0 a^u(1 + o(1)) + o(S_0 C^u) \\ u!\,S''_u &= o(S_0 C^u) \\ u!\,S'''_u &= o(S_0 C^u) \end{aligned} \tag{8}$$

uniformly as $f \to \infty$ for $u \leqslant \Lambda(f)$, where $\Lambda(f) \to \infty$. Suppose further that

$$u!\,S_u \leqslant S_0 K^u \tag{9}$$

for all $u > 0$ and some K. Then Theorem 1 is valid.

Proof. Let $t(f)$ be a function such that $t(f) \to 0$ and the little-ohs in (8) are bounded by $S_0 C^u t(f)$. Since $S_0 C^u t(f) \leqslant S_0(t(f))^{1/2}$ for $u \leqslant \min(\Lambda, -1/2 \log t/|\log C|)$, we may assume $C = 1$ in (8) provided Λ is replaced with the above minimum and t is replaced by $\tau = t^{1/2}$.

Suppose first that $2 \log a \geqslant -(-\log \tau)^{1/2}$ and that (8) holds. For $u \leqslant \min(\Lambda, (-\log \tau)^{1/2})$ we have uniformly $u!\, S'_u \sim S_0 a^u$, $u!\, S''_u = o(S_0 a^u)$, and $u!\, S'''_u = o(S_0 a^u)$. By [1, (4.3)]

$$P'_k \sim P_k \sim S_0 e^{-a} a^k / k! \tag{10}$$

uniformly for $k \leqslant \sigma(f)$ where σ is some function such that $\sigma \to \infty$.

For $t = 1$, Theorem 1 then follows by (6) and (10) with $k = 0$. Suppose $t > 1$. By (7) as in [2, pp. 222–223]

$$\prod r_i\, !\, G(M, \mathbf{r}, t) \geqslant \sum_{k \leqslant \sigma} P'_k 2^k \sim S_0 e^a$$

uniformly by (10). Also,

$$\begin{aligned}\prod r_i\, !\, G(M, \mathbf{r}, t) &\leqslant \sum_{k \leqslant \sigma} P'_k 2^k + \sum_{k \leqslant \sigma} (S''_k + S'''_k)(d!)^k + \sum_{k > \sigma} S_k (d!)^k \\ &= S_0 e^a (1 + o(1)) + o(S_0 e^{d!}) + O(S_0 (eKd!/\sigma)^\sigma)\end{aligned}$$

uniformly as $f \to \infty$ by (8) and (9). This proves that Theorem 1 is valid in the case $2 \log a \geqslant -(-\log \tau)^{1/2}$.

Now suppose that $2 \log a \leqslant -(-\log \tau)^{1/2}$. By (8), $S_0 \sim S'_0$ and $u!\, S_u = o(S_0)$ for $1 \leqslant u \leqslant \Lambda(f)$. As in [1, (4.3)] one easily has $P_0 \sim S_0$ uniformly. As in the previous paragraph,

$$\prod r_i\, !\, G(M, \mathbf{r}, t) \geqslant P_0 \sim S_0, \text{ and}$$

$$\begin{aligned}\prod r_i\, !\, G(M, \mathbf{r}, t) &\leqslant P_0 + \sum_{k=1}^{\Lambda} S_k (d!)^k + \sum_{k > \Lambda} S_k (d!)^k \\ &= S_0 (1 + o(1)) + o(S_0 e^{d!}) + O(S_0 (eKd!/\Lambda)^\Lambda) \\ &\sim S_0 . \quad \blacksquare\end{aligned}$$

5. Details of the Proof

First we reduce to the case in which no r_i is zero. Let $\mathbf{r}^*$ denote $\mathbf{r}$ with the zeroes deleted and M^* denote M with the corresponding rows and columns deleted. Since $G(M, \mathbf{r}, t) = G(M^*, \mathbf{r}^*, t)$, $f = f^*$, $a = a^*$, $b = b^*$, and $\delta = \delta^*$, the theorem is unchanged. Hence we may assume $r_i \neq 0$. Thus $nd \geqslant f \geqslant n$.

If $f = \delta$ is odd, then $G = T = 0$. Hence we may assume $T(f, \delta) \neq 0$.

We begin with (8) and (9) for class III. Let X be a set of properties of class III, containing w properties of the form $(i_p, j_p) = k_p$ and $u - w$ of the form $(l_q, l_q) = m_q$. Let $\Sigma k_p = 2w + A$ and $\Sigma m_q = 2(u - w) + B$. Then

$w(d-2) \geqslant A \geqslant 0$, $(u-w)(d-2) \geqslant B \geqslant 0$, and $A+B \geqslant 1$. Suppose σ is counted in $N_{\geqslant}(X)$. We can define σ to satisfy $(l,l)=m$ in at most

$$\binom{r_l}{m} \leqslant \binom{r_l}{2} d^{m-2} \leqslant \binom{r_l}{2} d^t$$

ways and to satisfy $(i,j)=k$ in at most

$$\binom{r_i}{k}\binom{r_j}{k} k! \leqslant 2 \binom{r_i}{2}\binom{r_j}{2} d^{2t}$$

ways. The part of σ not specified by X can be chosen in at most $T(f-L, \delta^*)$ ways where $L = 2u + 2w + 2A + B$ and $\delta - \delta^* \leqslant L$. Note that if $u = w$, then $B = 0$ and so L is even. By (4) and the above

$$N_{\geqslant}(X) \leqslant d^{2tu} \prod_q \binom{r_{l_q}}{2} \prod_p 2 \binom{r_{i_p}}{2}\binom{r_{j_p}}{2} T(f-L, \delta-L). \tag{11}$$

Now fix u. Summing over all X with $u = w$ we obtain

$$u! \sum_{w=u} N_{\geqslant}(X) \leqslant d^{2tu} \left(\left(\sum_i \binom{r_i}{2}\right)^2\right)^u T(f,\delta)/(K/f)^{2u+1}$$

by (2b). Since $\sum_i \binom{r_i}{2}/f \leqslant (d-1)/2$, we have

$$u! \sum_{w=u} N_{\geqslant}(X) = C^u T(f,\delta)\, O(1/f) \tag{12}$$

for some C. Summing (11) over all X with $u > w$ we obtain

$$u! \sum_{w<u} N_{\geqslant}(X) \leqslant \sum_{w<u} d^{2tu} \binom{u}{w} \left(\sum_{m_{ii}=1} \binom{r_i}{2}\right)^{u-w} \left(\sum_i \binom{r_i}{2}\right)^{2w} T_w$$

where

$$T_w = \max T(f-L, \delta-L) \text{ where } L \text{ ranges over } (u+w)\,t \geqslant L > 2u+2w$$
$$\leqslant \max_{l>0} T(f-l, \delta-l)(K/f)^{u+w} \text{ by (2b).}$$

Hence

$$\begin{aligned} u! \sum_{w<u} N_{\geqslant}(X) &\leqslant R \sum_{w<u} \binom{u}{w} D^{u-w} N^w \\ &= R((D+N)^u - N^u) \end{aligned} \tag{13}$$

where $R = (d^tK)^{2u} \max_{l>0} T(f-l, \delta - l)$

$$D = \sum_{m_{ii}=1} \binom{r_i}{2}\Big/f$$

$$N = \Big(\sum_i \binom{r_i}{2}\Big/f\Big)^2.$$

If $f - \delta \geqslant f^{2/3}$, we can apply (2b) and combine the result with (12) to obtain $u!\,S'''_u = o(C^uT(f, \delta))$ uniformly for some C and all u. If $k = f - \delta < f^{2/3}$, we have

$$D \leqslant k\, d/2f = O(k/f)$$

and $(x+h)^u - x^u \leqslant (x+h)^{u-1}\, uh$ for $h \geqslant 0$.

Using these in (13):

$$u! \sum_{w<u} N_{\geqslant}(X) = O(C^ukf^{-1} \max_{l>0} T(f-l, \delta - l)). \tag{14}$$

The maximum occurs at either $l = 1$ or $l = 2$. If it occurs at $l = 2$, or if $k \geqslant f^{1/2}/2$ then (2b) applies. When $k < f^{1/2}/2$ it is easily shown from Theorem 2 that $T(f-1, \delta-1) < 2(1 + o(1))\, T(f, \delta)/k$. Combining this with (12) and (14) we obtain $u!\,S'''_u = o(C^uT(f, \delta))$ again.

The argument for class II is similar. Define X, u, and w as before. We then have equation (11) with $L = 2u + 2w$, since A and B are zero for X in class II. Summing (11) over all X of class II, using (2b)

$$\begin{aligned} u! \sum N_{\geqslant}(X) &\leqslant \sum_{w>0} \binom{u}{w} \Big(\sum_i \binom{r_i}{2}\Big)^{2w-1} \Big(\sum_{m_{ii}=1} \binom{r_i}{2}\Big)^{u-w} \\ &\quad \cdot (u + w - 1)\binom{d}{2}\, T(f, \delta) \cdot (K/f)^{u+w} \cdot d^{2tu} \\ &\leqslant C^u \cdot T(f, \delta) \cdot O(u/f). \end{aligned} \tag{15}$$

This establishes (8) for class II.

We now estimate class I. Let X be class I with w properties "$(i_p, j_p) = 2$" and $u - w$ properties "$(l_q, l_q) = 2$". The number of σ counted in $N_{\geqslant}(X)$ can be factored as $I(X)\, J(X)$ where

$$I(X) = \prod_p 2 \binom{r_{i_p}}{2}\binom{r_{i_p}}{2} \prod_q \binom{r_{l_q}}{2}$$

and $J(X)$ is the number of ways to complete the definition of σ on the remaining $f - 2u - 2w$ elements of F in such a way that the properties of X are preserved. Thus $J(X)$ is the number of involutions on a subset F'

of F containing $f - 2u - 2w$ elements and, in the notation of Lemma 1, $\mathscr{P}$ is the set of all $\{\alpha, \beta\}$ such that $\alpha, \beta \in F'$, $\alpha < \beta$, and

(i) $\alpha \in R_i$, $\beta \in R_j$, "$(i, j) = 2$" $\in X$; or

(ii) $\alpha \in r_i$, $\beta \in R_j$, $m_{ij} = 0$; or

(iii) $\alpha, \beta \in R_i$.

Also Δ consists of all $\alpha \in F'$ such that

(i) $\alpha \in R_i$ and $m_{ii} = 0$; or

(ii) $\alpha \in R_i$ and "$(i, i) = 2$" $\in X$.

By a careful consideration of definitions:

$$bf + wd^2 \geqslant |\mathscr{P}| \geqslant bf - 4\,du(z+1),$$
$$\delta + d(u - w) \geqslant |\Delta| \geqslant \delta - 4w,$$

where b and δ are as in Theorem 1. The collection of all $\mathscr{P}$ arising above satisfy Lemma 2. Hence

$$J(X) \sim T(f - 2u - 2w, \delta + r(x))\, e^{-b}$$

where $|r(X)| \leqslant 4ud$, uniformly for $u = o(f)$. By (2)

$$J(X) \sim T(f, \delta + r(X) + 2u + 2w) f^{-u-w} e^{-b} \tag{16}$$

uniformly. Notice that $r(X) + 2u + 2w \geqslant 2(u - w) \geqslant 0$ and so by (4) the right side of (16) is bounded by $T(f, \delta) f^{-u-w} e^{-b}$.

We now distinguish cases. Define

$$P = |\{i: r_i > 1\}|,$$
$$Q = |\{i: r_i > 1 \text{ and } m_{ii} = 1\}|.$$

The cases are

(A) $P < f^{7/8}$

(B) $P > f^{7/8}$, $Q > f^{2/3}$

(C) $P > f^{7/8}$, $Q \leqslant f^{2/3}$.

Consider case A. We have

$$a < (Pd^2/2f)^2 + Qd^2/2f < (Pd^2/f)^2 < d^4/f^{1/4}.$$

Proceeding as for classes II and III.

$$u!\,S'_u = O\left(\sum_w \binom{u}{w} N^w D^{u-w} T(f,\delta)\right)$$

$$= O(a^u T(f,\delta)) = o(S_0) \text{ for } u > 0.$$

Since $a = O(f^{-1/4})$, this establishes (8).

Consider case B. We have

$$f - \delta = \sum_{m_{ii}=1} r_i \geqslant 2Q > f^{2/3}.$$

By (16), (3), and (6)

$$J(X) \sim T(f,\delta) f^{u-w} e^{-b} \sim S_0 f^{-u-w} \tag{17}$$

uniformly provided u does not grow too rapidly. Summing over all X of class I and cardinality u:

$$S'_u \sim S_0 \cdot \sum \left(\prod_p 2\binom{r_{i_p}}{2}\binom{r_{j_p}}{2}\Big/f^2\right)\prod_q \binom{r_{l_q}}{2}\Big/f$$

where the sum ranges over all i_p, j_p, l_q such that there are no repeats, $m_{i_p,j_p} = 1$, $m_{l_q,l_q} = 1$, and the number of p's plus q's equals u. An upper bound for this sum is

$$\sum_w \frac{1}{w!}\left(\sum_i \binom{r_i}{2}\Big/f\right)^{2w} \frac{1}{(u-w)!}\left(\sum_{m_{ii}=1}\binom{r_i}{2}\Big/f\right)^{u-w} = a^u/u!. \tag{18}$$

A lower bound for this sum is

$$\sum_w \frac{1}{w!\,(u-w)!}\left\{N^w D^{u-w} - \binom{u+w}{2} n(z+1)(d^2/2f)^2 N^w D^{u-w-2}\right\}$$

$$> \sum_w \frac{N^w D^{u-w}}{w!\,(u-w)!}\left(1 - u^2 n(z+1)\,d^4/2f^2D^2\right)$$

where

$$N = \left(\sum_i \binom{r_i}{2}\Big/f\right)^2,$$

$$D = \sum_{m_{ii}=1}\binom{r_i}{2}\Big/f \geqslant Q/f > f^{-1/3}.$$

Thus

$$u^2 n(z+1)\, d^4/2f^2D^2 = O(u^2 f^{-1/3}).$$

Hence $S'_u \sim a^u S_0$ uniformly provided u does not grow too rapidly.

Now consider case C. First consider those X with $u - w > 0$. We have by (16) and the idea leading to (18)

$$\begin{aligned} u! \sum_{w<u} N_{\geqslant}(X) &\leqslant CT(f, \delta)\, e^{-b} \sum_{w<u} \binom{u}{w} N^w D^{u-w} \\ &= CT(f, \delta)\, e^{-b}((N+D)^u - N^u) \\ &\leqslant CT(f, \delta)\, e^{-b} \frac{uD}{N} (N+D)^u \\ &= O(a^u S_0 uD/N) \text{ by (6).} \end{aligned}$$

Since $N \geqslant (P/f)^2 \geqslant f^{-1/4}$ and $D \leqslant Qd^2/f \leqslant d^2 f^{-1/3}$ we see that when $u = o(f^{1/12})$ this is $o(a^u S_0)$. Hence

$$u!\, S'_u = \sum N_{\geqslant}(X) + o(a^u S_0) \tag{19}$$

uniformly as $f \to \infty$ and u slowly growing, where the sum is over those X of class I, cardinality u and no "$(i, i) = 2$." We now estimate this sum. The procedure is like case B but the estimates are a bit easier. Using this, (19), and $N \sim a$, we obtain (8).

We now prove (9) for class I. We return to $J(X)$ and note that

$$\begin{aligned} J(X) &\leqslant T(f - 2u - 2w, |\Delta|) \\ &\leqslant T(f - 2u - 2w, \delta - 4w) \\ &\leqslant T(f - 2u - 2w, \delta - 2u - 2w) \end{aligned}$$

by (4). Using (2b)

$$J(X) \leqslant (K/f)^{u+w}\, T(f, \delta).$$

Using this and the idea leading to (18):

$$\sum N_{\geqslant}(X) \leqslant (aK^2)^u\, T(f, \delta)/u! \leqslant (K')^u S_0 .$$

This proves (9) for class I.

References

1. E. A. Bender, Asymptotic methods in enumeration, *SIAM Rev.* **16** (1974), 485–515.
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