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NP-hardness of shop-scheduling problems with three jobs☆

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Abstract

This paper deals with the problem of scheduling n jobs on m machines in order to minimize the maximum completion time or mean flow time of jobs. We extend the results obtained in Sotskov (1989, 1990, 1991) on the complexity of shop-scheduling problems with $n = 3$. The main result of this paper is an NP-hardness proof for scheduling 3 jobs on 3 machines, whether preemptions of operations are allowed or forbidden.

Keywords: Combinatorial optimization; NP-hard problem; Optimal makespan schedule; Optimal mean flow time schedule; Job-shop; Flow-shop; Open-shop

1. Introduction

Let us consider the following scheduling problem. There is a set $J = \{J_1, \ldots, J_n\}$ of n jobs that are to be processed on a machine set $M = \{M_1, \ldots, M_m\}$. At any time each machine $M_l \in M$ can process at most one job $J_i \in J$, and each job can be processed on at most one machine. Each job $J_i \in J$ consists of a sequence of n_i operations, routes (machine orders) $l^i = (l^i_1, \ldots, l^i_{n_i})$, where $M_{l^i_q} \in M$ and $1 \leqslant q \leqslant n_i$, being given in advance (in flow-shop and job-shop) or may be arbitrary (in open-shop). Every operation $\langle i, q \rangle$ of job $J_i \in J$ on machine $M_{l^i_q}$, $1 \leqslant q \leqslant n_i$, requires given processing time (duration) $t_{iq} \geqslant 0$. If schedule $s = s(t)$ is a nonpreemptive one, it is defined by starting times $\underline{t}_{iq}(s) \geqslant 0$ or by completion times $\bar{t}_{iq}(s) \geqslant 0$ of all the operations $\langle i, q \rangle$. In this case $\bar{t}_{iq}(s) = \underline{t}_{iq}(s) + t_{iq}$, $J_i \in J$, $1 \leqslant q \leqslant n_i$. Let $\bar{t}_i(s)$ mean completion time of job $J_i \in J$ with schedule s, i.e. $\bar{t}_i(s) = \bar{t}_{in_i}(s)$.

Our terminology follows the classification of scheduling problems used in [10, 11]. When $l^i = (1, \ldots, m)$ for all jobs $J_i \in J$, i.e., the routes are identical, we have a flow-shop

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problem, indicated by $n|m|F|\Phi$. When n_i and l^i may vary per job we have a job-shop problem $n|m|J|\Phi$. When the order of the machines in l^i is not fixed for any job $J_i \in J$ we have an open-shop problem $n|m|O|\Phi$. The parameter Φ denotes an optimality criterion of a schedule. If $\Phi = C_{\max}$, the problem is to find a schedule $s^* = s^*(t)$ of n jobs minimizing the maximum (total) completion time:

$$C_{\max}(s^*) = \max\{\bar{t}_i(s^*) \mid J_i \in J\}.$$

If $\Phi = \sum C_i$, the problem is to find a schedule $s^* = s^*(t)$ of n jobs minimizing the mean flow time:

$$\frac{1}{n}\sum C_i(s^*) = \frac{1}{n}\sum_{i=1}^{n} \bar{t}_i(s^*).$$

We shall indicate the preemption allowance by a parameter Pr. For example, $n|m|J, \mathrm{Pr}|C_{\max}$. The condition $t_{iq} > 0$ indicates that processing times are strictly postive.

There are many efficient algorithms and complexity results for the different cases of scheduling problems under the usual assumption $n > m$ (see [5, 8–11, 19]). The purpose of this paper is to improve the results obtained in [14–16] on the study of the complexity of shop-scheduling problems with fixed number of jobs when $n \leqslant m$.

In Section 2 we prove that the problems $3|3|J|C_{\max}$, $3|3|J,\mathrm{Pr}|C_{\max}$, $3|3|J|\sum C_i$ and $3|3|J,\mathrm{Pr}|\sum C_i$ are NP-hard. In Section 3 the same results are obtained for $3|m|F,\mathrm{Pr}|C_{\max}$, $3|m|F,\mathrm{Pr}|\sum C_i$, $3|m|F,\mathrm{Pr}, t_{iq} > 0|C_{\max}$, and $3|m|F,\mathrm{Pr}, t_{iq} > 0|\sum C_i$ problems. The proof of $3|3|J|C_{\max}$ problem NP-hardness is rather complicated, and we tend to present it in detail. The other results presented are based on negligible modifications of the polynomial reduction of the PARTITION problem to the problem $3|3|J|C_{\max}$.

The complexity of $n|m|O|C_{\max}$ and $n|m|O|\sum C_i$ problems is discussed in Section 4. A brief survey of known and new results on the complexity of scheduling a fixed number of jobs is given in Section 5.

2. Job-shop

We shall use the NP-complete PARTITION problem in the following form [6]. Let the ordered set $A = \{1, \ldots, 2a\}$ be given. A strictly positive integer e_i is connected with each element $i \in A$, $\sum_{i \in A} e_i = 2E$. If $A_k \subset A$, then $E_k = \sum_{i \in A_k} e_i$. The question is, does there exist a partition of A into subsets A_1 and A_2 such that $E_1 = E_2$ and set A_1 includes exactly one element from each pair $2i-1, 2i$ where $1 \leqslant i \leqslant a$.

If such subsets A_1 and A_2 exist, we shall say that the PARTITION problem has a solution. Without loss of generality, we shall consider nontrivial PARTITION problems with $e_{2i-1} \neq e_{2i}$ for every i, $1 \leqslant i \leqslant a$. Moreover, we assume that $e_{2i-1} > e_{2i}$ for each pair $2i-1, 2i$, where $1 \leqslant i \leqslant a$.

Theorem 1. *The* $3|3|J|C_{\max}$ *problem is* NP-*hard.*

Proof. We shall reduce polynomially the PARTITION problem to the following decision problem: Does there exist a shedule $s^0 = s^0(t)$ for the $3|3|J|C_{\max}$ problem such that $C_{\max}(s^0) \leqslant y$ for a given integer y.

We construct the following $3|3|J|C_{\max}$ instance. Let H be an integer, $H > 8E$. We set $y = 6aH + E$, $l^1 = ([3,2,3,1,3,3]^a)$, $l^2 = ([1,2,2,1]^a)$ and $l^3 = ([1,2,3,2]^a)$. Here and in what follows, $[\alpha]^k$ indicates the sequence of k repetitions of the expression α, e.g., $[1,2]^2$ means 1,2,1,2. If $k = 0$, then $[\alpha]^k = \emptyset$ for any α. We set the processing times to be equal to the following values:

$$t_{1,6k+1} = \sum_{v=-1}^{k-1} \Delta_{2v+1,2v+2}, \qquad t_{1,6k+2} = H - 2\sum_{v=-1}^{k-1} \Delta_{2v+1,2v+2},$$

$$t_{1,6k+3} = H + \sum_{v=-1}^{k-1} \Delta_{2v+1,2v+2} + e_{2k+2}, \qquad t_{1,6k+4} = \Delta_{2k+1,2k+2},$$

$$t_{1,6k+5} = 2H - \Delta_{2k+1,2k+2}, \qquad t_{1,6k+6} = 2H, \qquad t_{2,4k+1} = H,$$

$$t_{2,4k+2} = 2H, \qquad t_{2,4k+3} = 2H, \qquad t_{2,4k+4} = H, \qquad t_{3,4k+1} = e_{2k+1},$$

$$t_{3,4k+2} = e_{2k+2}, \qquad t_{3,4k+3} = \Delta_{2k+1,2k+2}, \qquad t_{3,4k+4} = H,$$

where $\Delta_{j,j+1} = e_j - e_{j+1}$, $\Delta_{-1,0} = 0$, and $k = 0,\dots,a-1$. It should be noted that $t_{1,1} = \sum_{v=-1}^{-1} \Delta_{2v+1,2v+2} = \Delta_{-1,0} = 0$ and so the route l^1 can be represented as $l^1 = (2,3,1,3,3,[3,2,3,1,3,3]^{a-1})$.

This $3|3|J|C_{\max}$ instance will be further referred to as Instance 1. Let us show that: *a schedule* s^0 *with* $C_{\max}(s^0) \leqslant y$ *exists for Instance* 1 *iff the PARTITION problem has a solution.*

Sufficiency. Since $C_{\max}$ is a regular criterion we may consider only active schedules [5]. If the machine $M_l \in M$ processes operation $\langle i,q\rangle$ before the opertion $\langle i',q'\rangle$, we shall denote it by $\langle i,q\rangle \to \langle i',q'\rangle$, and moreover, if the machine M_l processes operation $\langle i,q\rangle$ directly before $\langle i',q'\rangle$ (i.e., there is no operation $\langle i'',q''\rangle$ being processed after $\langle i,q\rangle$ and before $\langle i',q'\rangle$), we shall denote it by $\langle i,q\rangle \Rightarrow \langle i',q'\rangle$.

Suppose that the PARTITION problem has a solution: $A = A_1 \cup A_2$, $E_1 = E_2$. It is easy to construct a no-wait active schedule s' of jobs J_1 and J_2 with $C_{\max}(s') = 6aH + \sum_{k=1}^{a} e_{2k}$. Such schedule s' for $a = 2$ is shown in Fig. 1. So for any schedule s of jobs J_1, J_2, and J_3 the following lower bound is valid: $C_{\max}(s) \geqslant 6aH + \sum_{k=1}^{a} e_{2k}$. Let us construct the schedule s^0 with the following job completion times $\bar{t}_1(s^0) = 6aH + \sum_{i\in A_2} e_i = 6aH + E_2$, $\bar{t}_2(s^0) = \bar{t}_3(s^0) = 6aH + \sum_{i\in A_1} e_i = 6aH + E_1$ and with the following total completion time $C_{\max}(s^0) = y$.

Recall that set A_1 includes exactly one element from the pair $\{1,2\}$. If $1 \in A_1$ and $2 \in A_2$, then the initial part of the active schedule s^0 formed by operation set

$$N_0 = \{\langle 1,\alpha\rangle \mid \alpha = 1,\dots,6\} \cup \{\langle 2,\beta\rangle \mid \beta = 1,\dots,4\} \cup \{\langle 3,\gamma\rangle \mid \gamma = 1,\dots,4\}$$

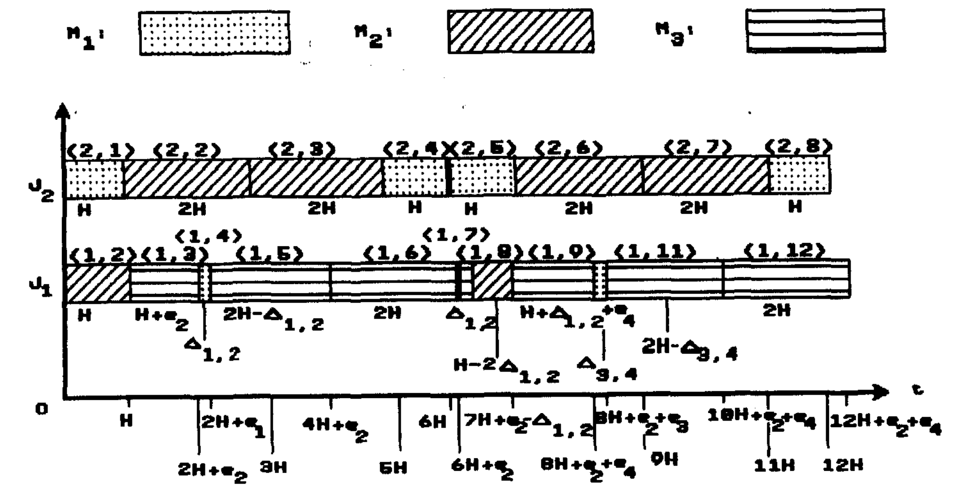

Fig. 1. No-wait schedule of jobs J_1 and J_2 for $a = 2$.

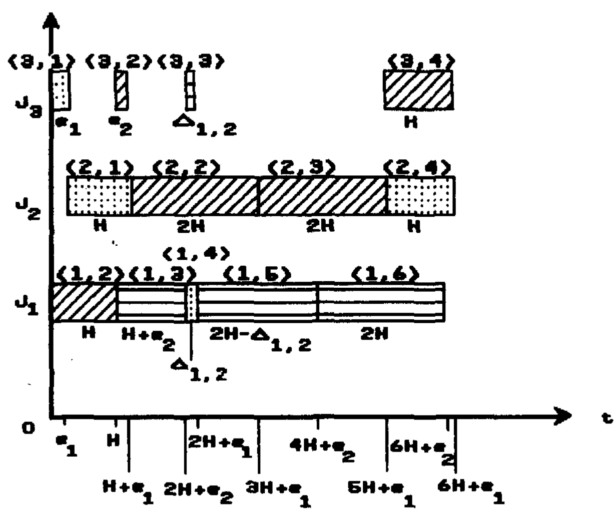

Fig. 2. Initial part of schedule s^0 for $1 \in A_1$ and $2 \in A_2$.

can be uniquely defined by the following conditions:

$$\langle 3,1\rangle \Rightarrow \langle 2,1\rangle, \qquad \langle 1,2\rangle \Rightarrow \langle 3,2\rangle,$$
$$\langle 1,3\rangle \Rightarrow \langle 3,3\rangle, \qquad \langle 2,3\rangle \Rightarrow \langle 3,4\rangle \tag{1}$$

(see Fig. 2). It is easy to make sure that within closed interval $[0, 6H]$ job J_2 is processed with a delay δ_2^0 equal to e_1 and job J_1 is processed without delay: $\delta_1^0 = 0$.

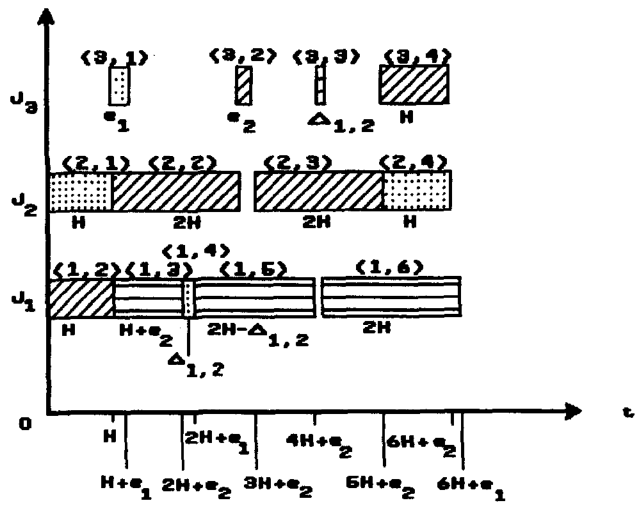

Fig. 3. Initial part of schedule s^0 for $2 \in A_1$ and $1 \in A_2$.

Obviously, the following equalities are true:

$$\bar{t}_{1,6}(s^0) = \sum_{\alpha=1}^{6} t_{1,\alpha} + \delta_1^0 = 6H + e_2,$$

$$\bar{t}_{2,4}(s^0) = \sum_{\beta=1}^{4} t_{2,\beta} + \delta_2^0 = 6H + e_1 \quad \text{and} \quad \bar{t}_{3,4}(s^0) = \bar{t}_{2,4}(s^0)$$

due to the relation $\langle 2,3\rangle \Rightarrow \langle 3,4\rangle$ and equalities $t_{2,4} = t_{3,4} = H$.

If $2 \in A_1$ and $1 \in A_2$, let us build the initial part of the active schedule s^0 under the conditions

$$\langle 2,1\rangle \Rightarrow \langle 3,1\rangle, \qquad \langle 2,2\rangle \Rightarrow \langle 3,2\rangle,$$

$$\langle 1,5\rangle \Rightarrow \langle 3,3\rangle, \qquad \langle 2,3\rangle \Rightarrow \langle 3,4\rangle \tag{2}$$

(see Fig. 3). In this case there is a job J_1 delay δ_1^0 equal to $\Delta_{1,2}$ and a job J_2 delay δ_2^0 equal to e_2. Thus, $\bar{t}_{1,6}(s^0) = \sum_{\alpha=1}^{6} t_{1,\alpha} + \delta_1^0 = 6H + e_2 + \Delta_{1,2} = 6H + e_1$, $\bar{t}_{2,4}(s^0) = \sum_{\beta=1}^{4} t_{2,\beta} + \delta_2^0 = 6H + e_2$ and $\bar{t}_{3,4}(s^0) = \bar{t}_{2,4}(s^0)$ due to $\langle 2,3\rangle \Rightarrow \langle 3,4\rangle$ and $t_{2,4} = t_{3,4} = H$. It will be shown later, that conditions (1) (and conditions (2) also) really specify a part of feasible schedule.

Consider the next part of schedule s^0 formed by operation set $N_1 = \{\langle 1,\alpha\rangle \mid \alpha = 7,\ldots,12\} \cup \{\langle 2,\beta\rangle \mid \beta = 5,\ldots,8\} \cup \{\langle 3,\gamma\rangle \mid \gamma = 5,\ldots,8\}$.

If $3 \in A_1$ and $4 \in A_2$, let machine set M process operations N_1 in a sequence similar to that in Fig. 2, namely,

$$\langle 3,5\rangle \Rightarrow \langle 2,5\rangle, \qquad \langle 1,8\rangle \Rightarrow \langle 3,6\rangle,$$

$$\langle 1,9\rangle \Rightarrow \langle 3,7\rangle, \qquad \langle 2,7\rangle \Rightarrow \langle 3,8\rangle. \tag{3}$$

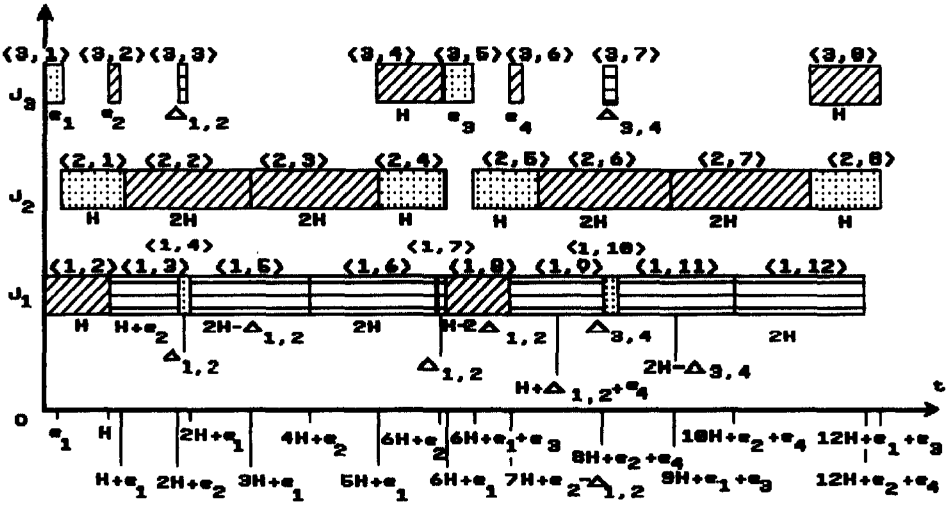

Fig. 4. Part of schedule s^0 for $1 \in A_1$, $3 \in A_1$, $2 \in A_2$ and $4 \in A_2$.

In this case job J_1 is processed without delay ($\delta_1^1 = 0$), and there is a job J_2 delay $\delta_2^1 = e_3$ within closed interval $[6H, 12H]$. Thus,

$$\bar{t}_{1,12}(s^0) = \bar{t}_{1,6}(s^0) + \sum_{\alpha=7}^{12} t_{1,\alpha} + \delta_1^1$$

$$= \begin{cases} (6H + e_2) + (6H + e_4) = 12H + e_2 + e_4 & \text{if } 1 \in A_1, 2 \in A_2, \\ (6H + e_1) + (6H + e_4) = 12H + e_1 + e_4 & \text{if } 2 \in A_1, 1 \in A_2; \end{cases}$$

$$\bar{t}_{2,8}(s^0) = \bar{t}_{2,4}(s^0) + \sum_{\beta=5}^{8} t_{2,\beta} + \delta_2^1$$

$$= \begin{cases} (6H + e_1) + (6H + e_3) = 12H + e_1 + e_3 & \text{if } 1 \in A_1, 2 \in A_2, \\ (6H + e_2) + (6H + e_3) = 12H + e_2 + e_3 & \text{if } 2 \in A_1, 1 \in A_2; \end{cases}$$

and $\bar{t}_{3,4}(s^0) = \bar{t}_{2,4}(s^0)$ due to $\langle 2,7\rangle \Rightarrow \langle 3,8\rangle$ and equalities $t_{2,8} = t_{3,8} = H$. The parts of schedule s^0 for operation sets N_0 and N_1 are represented in Fig. 4 (for $1 \in A_1, 2 \in A_2$) and in Fig. 5 (for $2 \in A_1, 1 \in A_2$).

If $4 \in A_1$ and $3 \in A_2$, then machine set M processes operations N_1 according with the order like in Fig. 3:

$$\langle 2,5\rangle \Rightarrow \langle 3,5\rangle, \qquad \langle 2,6\rangle \Rightarrow \langle 3,6\rangle,$$

$$\langle 1,11\rangle \Rightarrow \langle 3,7\rangle, \qquad \langle 2,7\rangle \Rightarrow \langle 3,8\rangle; \tag{4}$$

and there is a job J_1 delay $\delta_1^1 = \Delta_{3,4}$ and a job J_2 delay $\delta_2^1 = e_4$ within closed interval

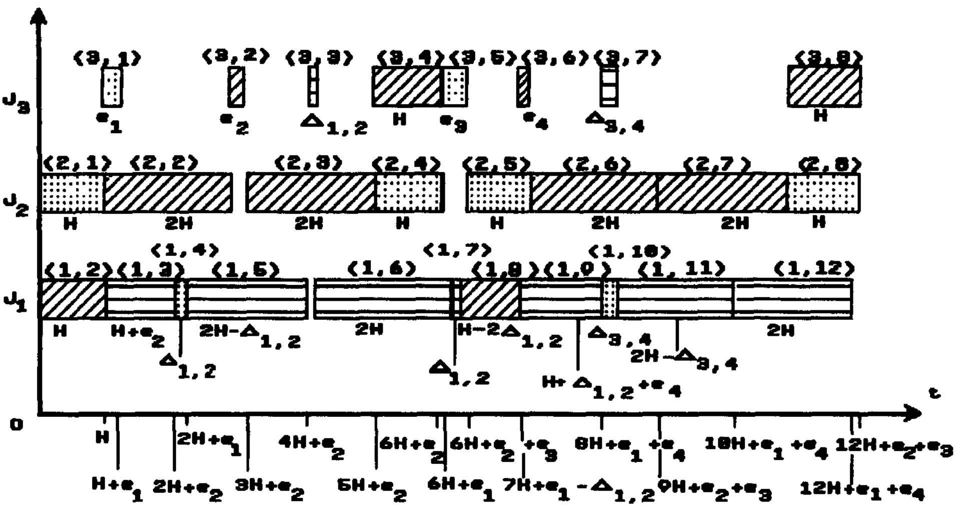

Fig. 5. Part of schedule s^0 for $2 \in A_1$, $3 \in A_1$, $1 \in A_2$ and $4 \in A_2$.

$[6H, 12H]$. Thus, we have

$$\bar{t}_{1,12}(s^0) = \bar{t}_{1,6}(s^0) + \sum_{\alpha=7}^{12} t_{1,\alpha} + \delta_1^1$$

$$= \begin{cases} (6H + e_2) + (6H + e_4) + \Delta_{3,4} = 12H + e_2 + e_3 & \text{if } 1 \in A_1, 2 \in A_2, \\ (6H + e_1) + (6H + e_4) + \Delta_{3,4} = 12H + e_1 + e_3 & \text{if } 2 \in A_1, 1 \in A_2; \end{cases}$$

$$\bar{t}_{2,8}(s^0) = \bar{t}_{2,4}(s^0) + \sum_{\beta=5}^{8} t_{2,\beta} + \delta_2^1$$

$$= \begin{cases} (6H + e_1) + (6H + e_4) = 12H + e_1 + e_4 & \text{if } 1 \in A_1, 2 \in A_2, \\ (6H + e_2) + (6H + e_4) = 12H + e_2 + e_4 & \text{if } 2 \in A_1, 1 \in A_2; \end{cases}$$

and $\bar{t}_{3,8}(s^0) = \bar{t}_{2,8}(s^0)$ due to $\langle 2,7 \rangle \Rightarrow \langle 3,8 \rangle$ and $t_{2,8} = t_{3,8} = H$. The corresponding parts of schedule s^0 for operation sets N_0 and N_1 are represented in Figs. 6 and 7.

Let us show that *each set of conditions* (1), (2), (3), *or* (4) *uniquely specifies a part of an active schedule*, i.e., no machine processes two or three operations simultaneously if only conditions (i) hold, $i \in \{1,2,3,4\}$.

Machine M_1 processing operation sets N_k, $k \in \{0,1\}$. The precedence constraint $\langle 2,4k+1 \rangle \Rightarrow \langle 3,4k+1 \rangle$ or $\langle 3,4k+1 \rangle \Rightarrow \langle 2,4k+1 \rangle$ with $k \in \{0,1\}$ is given. This implies that the operations $\langle 2,4k+1 \rangle$ and $\langle 3,4k+1 \rangle$ never conflict.

We now show that operations $\langle 2,1 \rangle$, $\langle 3,1 \rangle$, and $\langle 1,4 \rangle$ never conflict, as well operations $\langle 2,5 \rangle$, $\langle 3,5 \rangle$, and $\langle 1,10 \rangle$ never conflict.

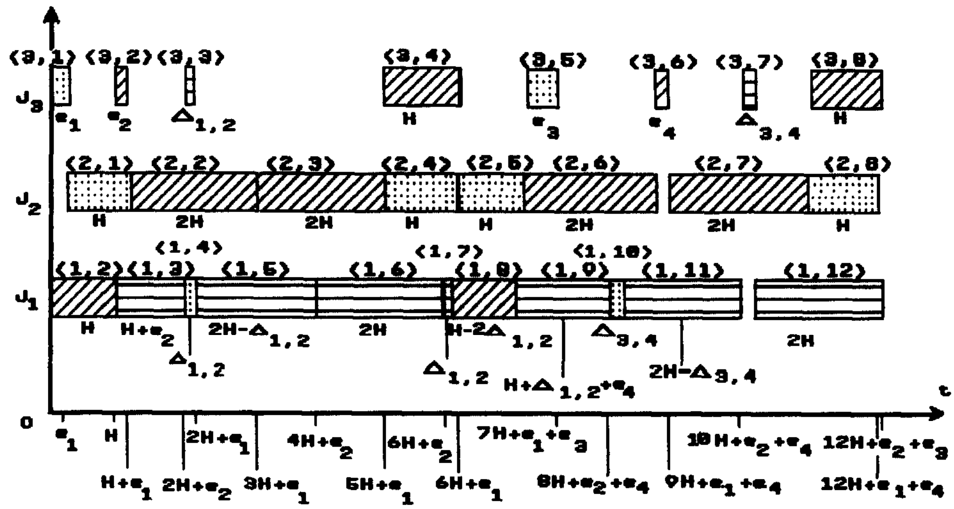

Fig. 6. Part of schedule s^0 for $1 \in A_1$, $4 \in A_1$, $2 \in A_2$ and $3 \in A_2$.

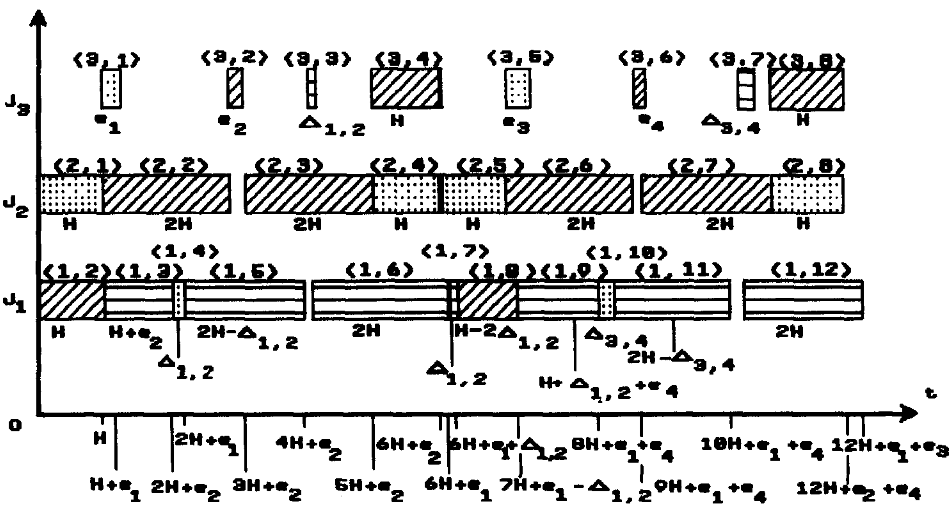

Fig. 7. Part of schedule s^0 for $2 \in A_1$, $4 \in A_1$, $1 \in A_2$ and $3 \in A_2$.

The inequality $\max\{\bar{t}_{2,1}(s^0), \bar{t}_{3,1}(s^0)\} \leqslant \underline{t}_{1,4}(s^0)$ is true since

$$\underline{t}_{1,4}(s^0) = t_{1,1} + t_{1,2} + t_{1,3} = 2H + e_2,$$

$$\max\{\bar{t}_{2,1}(s^0), \bar{t}_{3,1}(s^0)\} = t_{2,1} + t_{3,1} = H + e_1,$$

and so

$$\underline{t}_{1,4}(s^0) - \max\{\bar{t}_{2,1}(s^0), \bar{t}_{3,1}(s^0)\} = (2H + e_2) - (H + e_1)$$
$$= H + e_2 - e_1 > 8E + e_2 - e_1 > 0.$$

Similarly, the inequality $\max\{\bar{t}_{2,5}(s^0), \bar{t}_{3,5}(s^0)\} \leqslant \underline{t}_{1,10}(s^0)$ is true since

$$\begin{aligned}\underline{t}_{1,10}(s^0) &= \bar{t}_{1,6}(s_0) + (t_{1,7} + t_{1,8} + t_{1,9})\\ &\geqslant (6H + \min\{e_1, e_2\}) + (2H + e_4)\\ &= 8H + e_2 + e_4,\end{aligned}$$

$$\begin{aligned}\max\{\bar{t}_{2,5}(s^0), \bar{t}_{3,5}(s^0)\} &= \bar{t}_{2,4}(s^0) + (t_{2,5} + t_{3,5})\\ &\leqslant (6H + \max\{e_1, e_2\}) + (H + e_3)\\ &= 7H + e_1 + e_3,\end{aligned}$$

and so

$$\begin{aligned}\underline{t}_{1,10}(s^0) - \max\{\bar{t}_{2,5}(s^0), \bar{t}_{3,5}(s^0)\} &\geqslant H + e_2 + e_4 - e_1 - e_3\\ &> 8E + e_2 + e_4 - e_1 - e_3 > 0.\end{aligned}$$

As far as $\bar{t}_{3,4}(s^0) = \bar{t}_{2,4}(s^0)$ the relation $\langle 2,4\rangle \Rightarrow \langle 3,5\rangle$ is true in the case of order (3) or relation $\langle 2,4\rangle \Rightarrow \langle 2,5\rangle$ is true in the case of order (4).

The inequality $\bar{t}_{1,4}(s^0) \leqslant \underline{t}_{2,4}(s^0)$ is true since

$$\bar{t}_{1,4}(s^0) = t_{1,1} + t_{1,2} + t_{1,3} + t_{1,4} = 2H + e_2 + \Delta_{1,2} = 2H + e_1,$$

$$\underline{t}_{2,4}(s^0) = t_{2,1} + t_{2,2} + t_{2,3} + \delta_2^0 \geqslant 5H + e_2$$

and

$$\begin{aligned}\underline{t}_{2,4}(s^0) - \bar{t}_{1,4}(s^0) &\geqslant (5H + e_2) - (2H + e_1)\\ &= 3H + e_2 - e_1\\ &> 24E + e_2 - e_1 > 0.\end{aligned}$$

The inequality $\bar{t}_{1,10}(s^0) \leqslant \underline{t}_{2,8}(s^0)$ is true since

$$\begin{aligned}\bar{t}_{1,10}(s^0) &= \bar{t}_{1,6}(s^0) + (t_{1,7} + t_{1,8} + t_{1,9} + t_{1,10})\\ &\leqslant (6H + \max\{e_1, e_2\}) + (2H + e_4 + \Delta_{3,4})\\ &= 8H + e_1 + e_3,\end{aligned}$$

$$\begin{aligned}\underline{t}_{2,8}(s^0) &= \bar{t}_{2,4}(s^0) + (t_{2,5} + t_{2,6} + t_{2,7} + \delta_2^1)\\ &\geqslant (6H + \min\{e_1, e_2\}) + (5H + \min\{e_3, e_4\}) = 11H + e_2 + e_4\end{aligned}$$

and

$$\begin{aligned}\underline{t}_{2,8}(s^0) - \bar{t}_{1,10}(s^0) &\geqslant (11H + e_2 + e_4) - (8H + e_1 + e_3)\\ &= 3H + e_2 + e_4 - e_1 - e_3 > 24E + e_2 + e_4 - e_1 - e_3 > 0.\end{aligned}$$

Machine M_2 processing operation sets N_k, $k \in \{0, 1\}$. The precedence constraint $\langle 2, 4k+3\rangle \Rightarrow \langle 3, 4k+4\rangle$ with $k \in \{0,1\}$ is given for orders (1), (2), (3), and (4).

The inequality $\bar{t}_{3,4}(s^0) \leqslant \underline{t}_{1,8}(s^0)$ is true since

$$\bar{t}_{3,4}(s^0) = \bar{t}_{2,4}(s^0) \leqslant 6H + \max\{e_1, e_2\} = 6H + e_1$$

and

$$\underline{t}_{1,8}(s^0) = \bar{t}_{1,6}(s^0) + t_{1,7} \geqslant (6H + \min\{e_1, e_2\}) + \Delta_{1,2}$$
$$= 6H + e_2 + \Delta_{1,2} = 6H + e_1.$$

If conditions (1) take place for $k = 0$, then the precedence constraint $\langle 1,2\rangle \Rightarrow \langle 3,2\rangle$ is given and the inequality $\bar{t}_{3,2}(s^0) \leqslant \underline{t}_{2,2}(s^0)$ is true since

$$\bar{t}_{3,2}(s^0) = \bar{t}_{1,2}(s^0) + t_{3,2} = H + e_2$$

and

$$\underline{t}_{2,2}(s^0) = t_{3,1} + t_{2,1} = H + e_1.$$

Similarly if the conditions (3) take place for $k = 1$, then the precedence constraint $\langle 1,8\rangle \Rightarrow \langle 3,6\rangle$ is given and the inequality $\bar{t}_{3,6}(s^0) \leqslant \underline{t}_{2,6}(s^0)$ is true since

$$\begin{aligned}
\bar{t}_{3,6}(s^0) &= \bar{t}_{1,8}(s^0) + t_{3,6} \\
&= \bar{t}_{1,6}(s^0) + (t_{1,7} + t_{1,8}) + t_{3,6} \\
&\leqslant (6H + \max\{e_1, e_2\}) + (H - \Delta_{1,2}) + e_4 \\
&= 7H + e_1 - \Delta_{1,2} + e_4 \\
&= 7H + e_2 + e_4, \\
\underline{t}_{2,6}(s^0) &= \bar{t}_{2,4}(s^0) + (t_{3,5} + t_{2,5}) \\
&\geqslant (6H + \min\{e_1, e_2\}) + (e_3 + H) \\
&= 7H + e_2 + e_3, \\
\underline{t}_{2,6}(s^0) - \bar{t}_{3,6}(s^0) &\geqslant (7H + e_2 + e_3) - (7H + e_2 + e_4) \\
&= e_3 - e_4 > 0.
\end{aligned}$$

If the conditions (2) take place for $k = 0$, then the inequality $\bar{t}_{1,2}(s^0) \leqslant \underline{t}_{2,2}(s^0)$ is true since $\bar{t}_{1,2}(s^0) = H = \underline{t}_{2,2}(s^0)$, the precedence constraint $\langle 2,2\rangle \Rightarrow \langle 3,2\rangle$ is given and the constraint $\langle 3,2\rangle \Rightarrow \langle 2,3\rangle$ follows from $\langle 2,2\rangle \Rightarrow \langle 3,2\rangle$, $\langle 2,2\rangle \rightarrow \langle 2,3\rangle$, $\langle 2,3\rangle \Rightarrow \langle 3,4\rangle$, $\langle 3,2\rangle \rightarrow \langle 3,4\rangle$, and it is guaranteed by $\delta_2^0 = t_{3,2} = e_2$.

Similarly, if the conditions (4) take place for $k = 1$, then the inequality $\bar{t}_{1,8}(s^0) \leqslant \underline{t}_{2,6}(s^0)$ is true since

$$\begin{aligned}
\bar{t}_{1,8}(s^0) &= \bar{t}_{1,6}(s^0) + (t_{1,7} + t_{1,8}) \\
&\leqslant (6H + \max\{e_1, e_2\}) + (H - \Delta_{1,2}) \\
&= 7H + e_1 - \Delta_{1,2} = 7H + e_2
\end{aligned}$$

and

$$\underline{t}_{2,6}(s^0) = \bar{t}_{2,4}(s^0) + t_{2,5} \geqslant (6H + \min\{e_1, e_2\}) + H = 7H + e_2,$$

the precedence constraint $\langle 2,6\rangle \Rightarrow \langle 3,6\rangle$ is given and the constraint $\langle 3,6\rangle \Rightarrow \langle 2,7\rangle$ follows from $\langle 2,6\rangle \Rightarrow \langle 3,6\rangle$, $\langle 2,6\rangle \to \langle 2,7\rangle$, $\langle 2,7\rangle \Rightarrow \langle 3,8\rangle$, $\langle 3,6\rangle \to \langle 3,8\rangle$, and it is guaranteed by $\delta_2^1 = t_{3,6} = e_4$.

Machine M_3 processing operation sets N_k, $k \in \{0,1\}$. If the conditions (1) take place for $k=0$ or conditions (3) take place for $k=1$, then the precedence constraint $\langle 1,6k+3\rangle \Rightarrow \langle 3,4k+3\rangle$ is given and the inequality $\bar{t}_{3,4k+3}(s^0) \leqslant \underline{t}_{1,6k+5}(s^0)$ is true since $\bar{t}_{3,4k+3}(s^0) = \bar{t}_{1,6k+3}(s^0) + t_{3,4k+3} = \bar{t}_{1,6k+3}(s^0) + t_{1,6k+4} = \underline{t}_{1,6k+5}(s^0)$.

If conditions (2) take place for $k=0$ or conditions (4) take place for $k=1$, then the precedence constraint $\langle 1,6k+5\rangle \Rightarrow \langle 3,4k+3\rangle$ is given and the constraint $\langle 3,4k+3\rangle \Rightarrow \langle 1,6k+6\rangle$ is fulfilled due to the delay $\delta_1^k = \Delta_{2k+1,2k+2} = t_{3,4k+3}$.

We can continue similarly for sets N_2, N_3, etc. up to set N_{a-1}: If $(2k+1) \in A_1$, $(2k+2) \in A_2$, $k \in \{2,\dots,a-1\}$, then machine set M processes the operation set $N_k = \{\langle 1,6k+\alpha\rangle \mid \alpha = 1,\dots,6\} \cup \{\langle 2,4k+\beta\rangle \mid \beta = 1,\dots,4\} \cup \{\langle 3,4k+\gamma\rangle \mid \gamma = 1,\dots,4\}$ similar to the order shown in Fig. 2, namely,

$$\langle 3,4k+1\rangle \Rightarrow \langle 2,4k+1\rangle, \qquad \langle 1,6k+2\rangle \Rightarrow \langle 3,4k+2\rangle,$$
$$\langle 1,6k+3\rangle \Rightarrow \langle 3,4k+3\rangle, \qquad \langle 2,4k+3\rangle \Rightarrow \langle 3,4k+4\rangle. \tag{5}$$

In this case there is no job J_1 delay ($\delta_1^k = 0$) and there is a job J_2 delay $\delta_2^k = e_{2k+1}$ within closed interval $[6kH, 6kH+6H]$, and the following equalities are true:

$$\bar{t}_{1,6k+6}(s^0) = \bar{t}_{1,6k}(s^0) + \sum_{\alpha=1}^{6} t_{1,6k+\alpha} + \delta_1^k = \bar{t}_{1,6k}(s^0) + 6H + e_{2k+2},$$

$$\bar{t}_{2,4k+4}(s^0) = \bar{t}_{2,4k}(s^0) + \sum_{\beta=1}^{4} t_{2,4k+\beta} + \delta_2^k = \bar{t}_{2,4k}(s^0) + 6H + e_{2k+1},$$

and $\bar{t}_{3,4k+4}(s^0) = \bar{t}_{2,4k+4}(s^0)$ due to the relations $\langle 2,4k+3\rangle \Rightarrow \langle 3,4k+4\rangle$ and $t_{2,4k+4} = t_{3,4k+4} = H$.

Otherwise (i.e., if $(2k+2) \in A_1$ and $(2k+1) \in A_2$) the order

$$\langle 2,4k+1\rangle \Rightarrow \langle 3,4k+1\rangle, \qquad \langle 2,4k+2\rangle \Rightarrow \langle 3,4k+2\rangle,$$
$$\langle 1,6k+5\rangle \Rightarrow \langle 3,4k+3\rangle, \qquad \langle 2,4k+3\rangle \Rightarrow \langle 3,4k+4\rangle \tag{6}$$

is used for sequencing operations N_k (see Fig. 3). In this case jobs J_1 and J_2 are processed with delays $\delta_1^k = \Delta_{2k+1,2k+2}$ and $\delta_2^k = e_{2k+2}$, respectively, within closed interval $[6kH, 6kH+6H]$, and the following equalities are true:

$$\begin{aligned}\bar{t}_{1,6k+6}(s^0) &= \bar{t}_{1,6k}(s^0) + \sum_{\alpha=1}^{6} t_{1,6k+\alpha} + \delta_1^k \\ &= \bar{t}_{1,6k}(s^0) + (6H + e_{2k+2}) + \Delta_{2k+1,2k+2} \\ &= \bar{t}_{1,6k}(s^0) + 6H + e_{2k+1},\end{aligned}$$

$$\bar{t}_{2,4k+4}(s^0) = \bar{t}_{2,4k}(s^0) + \sum_{\beta=1}^{4} t_{2,4k+\beta} + \delta_2^k = \bar{t}_{2,4k}(s^0) + 6H + e_{2k+2},$$

and $\bar{t}_{3,4k+4}(s^0) = \bar{t}_{2,4k+4}(s^0)$ due to the relations $\langle 2, 4k+3\rangle \Rightarrow \langle 3, 4k+4\rangle$ and $t_{2,4k+4} = t_{3,4k+4} = H$.

As in the case of $k \in \{0,1\}$ we show that the conditions (5) or (6) also uniquely specify a part of an active schedule for $k = 2, \ldots, a-1$.

Machine M_1 processing. The precedence constraint $\langle 2,4k+1\rangle \Rightarrow \langle 3,4k+1\rangle$ or $\langle 3,4k+1\rangle \Rightarrow \langle 2,4k+1\rangle$ is given. This implies that the operations $\langle 2,4k+1\rangle$ and $\langle 3,4k+1\rangle$ never conflict.

We now show that these operations cannot conflict with operation $\langle 1,6k+4\rangle$. Indeed the inequality $\max\{\bar{t}_{2,4k+1}(s^0), \bar{t}_{3,4k+1}(s^0)\} \leqslant \underline{t}_{1,6k+4}(s^0)$ is true since

$$\begin{aligned}\underline{t}_{1,6k+4}(s^0) &= \bar{t}_{1,6k}(s^0) + (t_{1,6k+1} + t_{1,6k+2} + t_{1,6k+3})\\ &\geqslant (6kH + \min\{e_1, e_2\} + \min\{e_3, e_4\} + \cdots + \min\{e_{2k-1}, e_{2k}\})\\ &\quad + (2H + e_{2k+2})\\ &= (6k+2)H + e_2 + e_4 + \cdots + e_{2k} + e_{2k+2},\end{aligned}$$

and

$$\begin{aligned}&\max\{\bar{t}_{2,4k+1}(s^0), \bar{t}_{3,4k+1}(s^0)\} = \bar{t}_{2,4k}(s^0) + (t_{2,4k+1} + t_{3,4k+1})\\ &\quad\leqslant (6kH + \max\{e_1, e_2\} + \max\{e_3, e_4\} + \cdots\\ &\qquad\qquad + \max\{e_{2k-1}, e_{2k}\}) + (H + e_{2k+1})\\ &\quad= (6k+1)H + e_1 + e_3 + \cdots + e_{2k-1} + e_{2k+1},\end{aligned}$$

and so

$$\begin{aligned}&\underline{t}_{1,6k+4}(s^0) - \max\{\bar{t}_{2,4k+1}(s^0), \bar{t}_{3,4k+1}(s^0)\}\\ &\quad\geqslant H + (e_2 + e_4 + \cdots + e_{2k} + e_{2k+2}) - (e_1 + e_3 + \cdots + e_{2k-1} + e_{2k+1})\\ &\quad> 8E + (e_2 + e_4 + \cdots + e_{2k} + e_{2k+2}) - (e_1 + e_3 + \cdots + e_{2k-1} + e_{2k+1})\\ &\quad> 0.\end{aligned}$$

As far as $\bar{t}_{3,4k}(s^0) = \bar{t}_{2,4k}(s^0)$, the relation $\langle 2,4k\rangle \Rightarrow \langle 3,4k+1\rangle$ is true in the case of order (5) or relation $\langle 2,4k\rangle \Rightarrow \langle 2,4k+1\rangle$ is true in the case of order (6).

The inequality $\bar{t}_{1,6k+4}(s^0) \leqslant \underline{t}_{2,4k+4}(s^0)$ is true since

$$\begin{aligned}\bar{t}_{1,6k+4}(s^0) &= \bar{t}_{1,6k}(s^0) + (t_{1,6k+1} + t_{1,6k+2} + t_{1,6k+3} + t_{1,6k+4})\\ &\leqslant (6kH + \max\{e_1, e_2\} + \max\{e_3, e_4\} + \cdots + \max\{e_{2k-1}, e_{2k}\})\\ &\quad + (2H + e_{2k+2} + \Delta_{2k+1,2k+2})\\ &= (6k+2)H + (e_1 + e_3 + \cdots + e_{2k-1} + e_{2k+1}),\\ \underline{t}_{2,4k+4}(s^0) &= \bar{t}_{2,4k}(s^0) + (t_{2,4k+1} + t_{2,4k+2} + t_{2,4k+3} + \delta_2^k)\\ &\geqslant (6kH + \min\{e_1, e_2\} + \min\{e_3, e_4\} + \cdots + \min\{e_{2k-1}, e_{2k}\})\\ &\quad + (5H + \min\{e_{2k+1}, e_{2k+2}\})\\ &= (6k+5)H + (e_2 + e_4 + \cdots + e_{2k} + e_{2k+2})\end{aligned}$$

and

$$\underline{t}_{2,4k+4}(s^0) - \bar{t}_{1,6k+4}(s^0) \geqslant 3H + (e_2 + e_4 + \cdots + e_{2k} + e_{2k+2}) - (e_1 + e_3 + \cdots + e_{2k-1} + e_{2k+1}) > 0.$$

Machine M_2 processing. The precedence constraint $\langle 2, 4k+3\rangle \Rightarrow \langle 3, 4k+4\rangle$ is given for both orders (5) and (6).

The inequality $\bar{t}_{3,4k}(s^0) \leqslant \underline{t}_{1,6k+2}(s^0)$ is true since

$$\bar{t}_{3,4k}(s^0) = \bar{t}_{2,4k}(s^0) \leqslant 6kH + \max\{e_1, e_2\} + \max\{e_3, e_4\} + \cdots + \max\{e_{2k-1}, e_{2k}\} = 6kH + e_1 + e_3 + \cdots + e_{2k-1}$$

and

$$\begin{aligned}\underline{t}_{1,6k+2}(s^0) &= \bar{t}_{1,6k}(s^0) + t_{1,6k+1}\\ &\geqslant (6kH + \min\{e_1, e_2\} + \min\{e_3, e_4\} + \cdots + \min\{e_{2k-1}, e_{2k}\})\\ &\quad + (\Delta_{1,2} + \Delta_{3,4} + \cdots + \Delta_{2k-1,2k})\\ &= (6kH + e_2 + e_4 + \cdots + e_{2k}) + (\Delta_{1,2} + \Delta_{3,4} + \cdots + \Delta_{2k-1,2k})\\ &= 6kH + e_1 + e_3 + \cdots + e_{2k-1}.\end{aligned}$$

If conditions (5) take place, then precedence constraint $\langle 1, 6k+2\rangle \Rightarrow \langle 3, 4k+2\rangle$ is given and the inequality $\bar{t}_{3,4k+2}(s^0) \leqslant \underline{t}_{2,4k+2}(s^0)$ is true since

$$\begin{aligned}\bar{t}_{3,4k+2}(s^0) &= \bar{t}_{1,6k+2}(s^0) + t_{3,4k+2}\\ &= \bar{t}_{1,6k}(s^0) + (t_{1,6k+1} + t_{1,6k+2}) + t_{3,4k+2}\\ &\leqslant (6kH + \max\{e_1, e_2\} + \max\{e_3, e_4\} + \cdots + \max\{e_{2k-1}, e_{2k}\})\\ &\quad + (t_{1,6k+1} + t_{1,6k+2}) + t_{3,4k+2}\\ &= 6kH + (e_1 + e_3 + \cdots + e_{2k-1})\\ &\quad + (H - \Delta_{1,2} - \Delta_{3,4} - \cdots - \Delta_{2k-1,2k}) + e_{2k+2}\\ &= (6k+1)H + (e_2 + e_4 + \cdots + e_{2k}) + e_{2k+2},\end{aligned}$$

$$\begin{aligned}\underline{t}_{2,4k+2}(s^0) &= \bar{t}_{2,4k}(s^0) + (t_{3,4k+1} + t_{2,4k+1})\\ &\geqslant (6kH + \min\{e_1, e_2\} + \min\{e_3, e_4\} + \cdots + \min\{e_{2k-1}, e_{2k}\})\\ &\quad + (e_{2k+1} + H)\\ &= (6k+1)H + (e_2 + e_4 + \cdots + e_{2k}) + e_{2k+1},\end{aligned}$$

$$\underline{t}_{2,4k+2}(s^0) - \bar{t}_{3,4k+2}(s^0) \geqslant e_{2k+1} - e_{2k+2} > 0.$$

If conditions (6) take place, then the inequality $\bar{t}_{1,6k+2}(s^0) \leqslant \underline{t}_{2,4k+2}(s^0)$ is true since

$$\begin{aligned}\bar{t}_{1,6k+2}(s^0) &= \bar{t}_{1,6k}(s^0) + (t_{1,6k+1} + t_{1,6k+2})\\ &\leqslant (6kH + \max\{e_1, e_2\} + \max\{e_3, e_4\} + \cdots + \max\{e_{2k-1}, e_{2k}\})\\ &\quad + (t_{1,6k+1} + t_{1,6k+2})\\ &= 6kH + (e_1 + e_3 + \cdots + e_{2k-1})\\ &\quad + (H - \Delta_{1,2} - \Delta_{3,4} - \cdots - \Delta_{2k-1,2k})\\ &= (6k+1)H + (e_2 + e_4 + \cdots + e_{2k})\end{aligned}$$

and

$$\begin{aligned}\underline{t}_{2,4k+2}(s^0) &= \bar{t}_{2,4k}(s^0) + t_{2,4k+1}\\ &\geqslant (6kH + \min\{e_1, e_2\} + \min\{e_3, e_4\} + \cdots + \min\{e_{2k-1}, e_{2k}\})\\ &\quad + t_{2,4k+1}\\ &= (6k+1)H + (e_2 + e_4 + \cdots + e_{2k}),\end{aligned}$$

the constraint $\langle 2, 4k+2\rangle \Rightarrow \langle 3, 4k+2\rangle$ is given and the constraint $\langle 3, 4k+2\rangle \Rightarrow \langle 2, 4k+3\rangle$ follows from $\langle 2, 4k+2\rangle \Rightarrow \langle 3, 4k+2\rangle$, $\langle 2, 4k+2\rangle \rightarrow \langle 2, 4k+3\rangle$, $\langle 2, 4k+3\rangle \Rightarrow \langle 3, 4k+4\rangle$, $\langle 3, 4k+2\rangle \rightarrow \langle 3, 4k+4\rangle$, and it is guaranteed by $\delta_2^k = t_{3,4k+2} = e_{2k+2}$.

At last, the *consideration of machine* M_3 is the same as in the case $k \in \{0, 1\}$.

Thus, we can construct the schedule s^0 consisting of a (here a is not article it means number) fragments, each kth fragment, $k \in \{0, \ldots, a-1\}$, being associated with processing operations N_k on machine set M and satisfying equalities:

$$\begin{aligned}&\bar{t}_{1,6k+6}(s^0) = \bar{t}_{1,6k}(s^0) + 6H + e_{2k+2},\\ &\bar{t}_{2,4k+4}(s^0) = \bar{t}_{2,4k}(s^0) + 6H + e_{2k+1},\\ &\bar{t}_{3,4k+4}(s^0) = \bar{t}_{2,4k+4}(s^0) \quad \text{if } (2k+1) \in A_1 \text{ and } (2k+2) \in A_2,\end{aligned}$$

or equalities

$$\begin{aligned}&\bar{t}_{1,6k+6}(s^0) = \bar{t}_{1,6k}(s^0) + 6H + e_{2k+1},\\ &\bar{t}_{2,4k+4}(s^0) = \bar{t}_{2,4k}(s^0) + 6H + e_{2k+2},\\ &\bar{t}_{3,4k+4}(s^0) = \bar{t}_{2,4k+4}(s^0) \quad \text{if } (2k+2) \in A_1 \text{ and } (2k+1) \in A_2.\end{aligned}$$

These recurrence equalities imply that

$$\bar{t}_1(s^0) = \bar{t}_{1,6a}(s^0) = 6aH + \sum_{i \in A_2} e_i, \qquad \bar{t}_2(s^0) = \bar{t}_{2,4a}(s^0) = 6aH + \sum_{i \in A_1} e_i$$

and

$$\bar{t}_3(s^0) = \bar{t}_2(s^0).$$

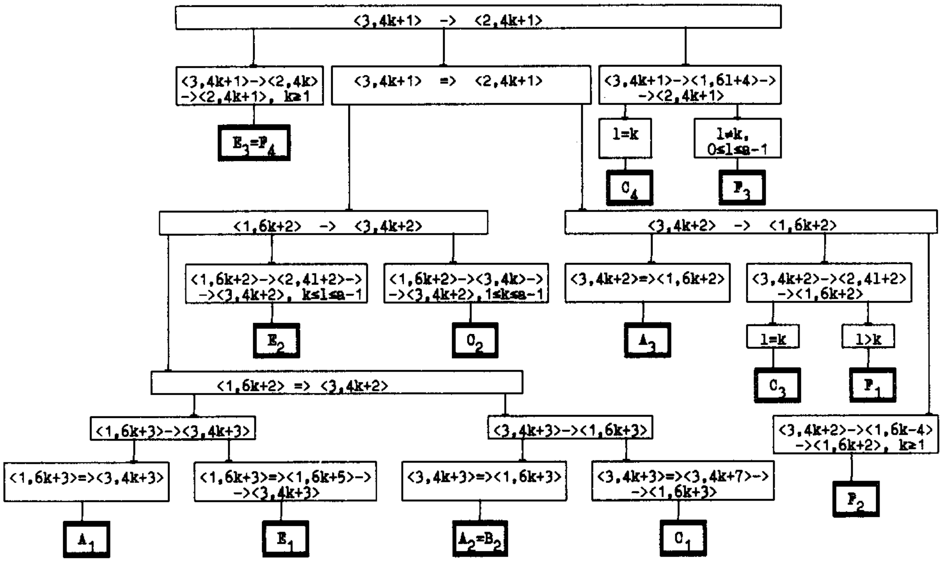

Fig. 8. Overlook tree of schedules satisfying the conditions $\langle 2,4k+3\rangle \to \langle 3,4k+4\rangle$ and $\langle 3,4k+1\rangle \to \langle 2,4k+1\rangle$.

Since $\sum_{i\in A_1} e_i = \sum_{i\in A_2} e_i = E$ we have obtained the schedule s^0 with $C_{\max}(s^0) = 6aH + E = y$.

Necessity. On the other hand suppose that there exists active schedule s^0 with $C_{\max}(s^0) \leqslant y$. Denote the set of such schedules by S:

$$S = \{s \mid C_{\max}(s) \leqslant y\}.$$

Let us prove that *there exists a schedule* $s^* \in S$ *with the orders of operations* N_k, $k = 0, \ldots, a-1$, *corresponding to orders* (5) *or* (6). To do this we show that a schedule $s \in S$ with any other possible orders of operations N_k, $k = 0, \ldots, a-1$, is not better (with respect to $C_{\max}(s)$) than the schedule s^*.

To look over all active schedules for Instance 1 we shall consider each machine M_1, M_2, and M_3 processing different sequences of operations N_k, $k = 0, \ldots, a-1$, which satisfy routes l^1, l^2, and l^3. Let a part of schedule, corresponding to operations N_k, be called a fragment. The way of considering possible sequences of operations N_k (in other words, considering possible fragments) for a fixed $k = 0, \ldots, a-1$ is represented by the overlook tree, shown in Figs. 8 and 9. This tree is constructed under assumption that $\langle 2,4k+3\rangle \to \langle 3,4k+4\rangle$, because otherwise (i.e., if $\langle 3,4k+4\rangle \to \langle 2,4k+3\rangle$) either job J_2 has a delay $\delta_2^k = H$ or job J_1 has a delay $\delta_1^k > H$ and therefore $C_{\max}(s) \geqslant a \cdot 6H + H > y$.

One-half of the tree, represented in Fig. 8, enumerates all schedules satisfying condition $\langle 3,4k+1\rangle \to \langle 2,4k+1\rangle$, and half of the tree, represented in Fig. 9, enumerates all schedules satisfying the inverse condition $\langle 2,4k+1\rangle \to \langle 3,4k+1\rangle$.

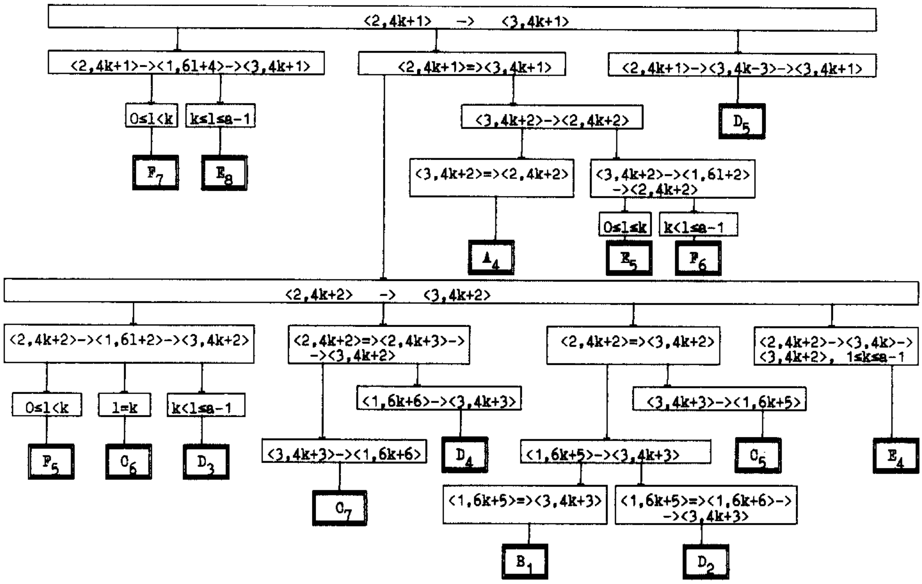

Fig. 9. Overlook tree of schedules satisfying the conditions $\langle 2,4k+3\rangle \to \langle 3,4k+4\rangle$ and $\langle 2,4k+1\rangle \to \langle 3,4k+1\rangle$.

Rectangles drawn by bold lines contain letter A, B, C, D or F (with subscripts) indicating the type of a fragment of an active schedule s (A_i-fragment, B_i-fragment, and so on), the letter E indicates non-active schedule. Integer l varies from 0 up to $a-1$.

We consider these fragment types in order to prove that *either A_1-fragment or B_1-fragment dominates a fragment of any other type.* In particular, we shall show that if a schedule contains some A_i-fragment, then it is not better (with respect to $C_{\max}$) than a schedule containing A_1-fragment; if a schedule contains some B_i-fragment, then it is not better than a schedule containing B_1-fragment; if a schedule contains C_i-, D_i- or F_i-fragment, then it does not belong to class S (since for C_i-fragment there is either job J_1 or job J_2 delay greater than $H/2$, since for D_i-fragment there exists operation $\langle 3,4j+4\rangle$, $0 \leqslant j \leqslant a-1$, such that $\underline{t}_{3,4j+4}(s) > 6jH + 6H$ holds, and since for F_i-fragment there exists either operation $\langle 1,6k\rangle$ with $\bar{t}_{1,6k}(s) > 6kH + E$, or operation $\langle 2,4k\rangle$ with $\bar{t}_{2,4k}(s) > 6kH + E$).

Beforehand, let us note that for any given value k, $0 \leqslant k \leqslant a-1$, the corresponding fragment of schedule s starts with processing operations $\langle 1,\ 6k+1\rangle$ and $\langle 2,4k+1\rangle$. If $k \geqslant 1$, then it is reasonable to suggest that the following inequalities are true

$$6kH + (e_2 + e_4 + \cdots + e_{2k}) \leqslant \bar{t}_{1,6k}(s) \leqslant 6kH + E, \tag{7}$$

$$6kH \leqslant \bar{t}_{2,4}(s) \leqslant 6kH + E. \tag{8}$$

The lower bounds for these values had been estimated earlier (see Fig. 1). If at least one of the upper bounds is exceeded, we get a schedule s with $C_{\max}(s) > 6aH + E$. Such fragments, satisfying inequality $\bar{t}_{1,6k}(s) > 6kH + E$ or inequality $\bar{t}_{2,4k}(s) >$

$6kH + E$, are denoted in Figs. 8 and 9 by letter F with subscripts. So any schedule containing F_i-fragment does not belong to the set S.

Consider A_i-, B_i-, C_i-, and D_i-fragments assuming that inequalities (7) and (8) are fulfilled. Let parameter A_i, B_i, C_i, or D_i of a schedule s indicate a fragment type, being a part of this schedule. For instance, $s(A_1)$ means a schedule containing at least one A_1-fragment.

A_i-fragments. It is easy to make sure that fragment A_1 (see Fig. 8) corresponds to the order (5) with no delay for job J_1 ($\delta_1^k(s(A_1)) = 0$) and with delay $\delta_2^k(s(A_1)) = e_{2k+1}$ for job J_2. The fragments A_2 (see Fig. 8) and A_4 (see Fig. 9) are worse than fragment A_1 since $\delta_1^k(s(A_j)) \geqslant \delta_1^k(s(A_1))$, $\delta_2^k(s(A_j)) \geqslant \delta_2^k(s(A_1))$, $j \in \{2, 4\}$, and one of these inequalities is strict.

The fragment A_3 (see Fig. 8) is not better than fragment A_1 since $\delta_1^k(s(A_3)) = \max\{\bar{t}_{3,4k+2}(s) - \bar{t}_{1,6k+1}(s), 0\} \geqslant \delta_1^k(s(A_1))$ and $\delta_2^k(s(A_3)) = e_{2k+1} = \delta_2^k(s(A_1))$.

B_i-fragments. The fragment B_1 (see Fig. 9) corresponds to the order (6) with delay $\delta_1^k(s(B_1)) = \Delta_{2k+1,2k+2}$ for job J_1 and with delay $\delta_2^k(s(B_1)) = e_{2k+2}$ for job J_2. The fragment B_2 (see Fig. 8) is worse than B_1 since $\delta_1^k(s(B_2)) = e_{2k+2} + \Delta_{2k+1,2k+2} > \delta_1^k(s(B_1))$ and $\delta_2^k(s(B_2)) = e_{2k+1} > \delta_2^k(s(B_1))$.

C_i-fragments. For any C_i-fragment there is either job J_1 or job J_2 delay greater than $H/2$. In this case $C_{\max}(s) \geqslant 6aH + H/2 > y$. So any schedule containing C_i-fragments does not belong to the set S.

D_i-fragments. For any D_i-fragment there exists an operation $\langle 3, 4j+4\rangle$, $0 \leqslant j \leqslant a-1$, such that $\underline{t}_{3,4j+4}(s(D)) > 6jH + 6H$.

If $j = a-1$, then $C_{\max}(s(D)) \geqslant \bar{t}_{3,4j+4}(s(D)) = \bar{t}_{3,4a}(s(D)) > 6jH + 6H + H = 6aH + H > y$.

If $0 \leqslant j \leqslant a-2$ and relations $\langle 3, 4j+4\rangle \Rightarrow \langle 1, 6j+8\rangle$ hold, then there is a job J_1 delay

$$\begin{aligned}\delta_1^{j+1}(s(D)) &= \bar{t}_{3,4j+4}(s(D)) - \bar{t}_{1,6j+7}(s(D))\\ &\geqslant (6(j+1)H + t_{3,4j+4})\\ &\quad - (6(j+1)H + e_1 + e_3 + \cdots + e_{2j+1} + t_{1,6j+7})\\ &> H - 2E - (\Delta_{1,2} + \Delta_{3,4} + \cdots + \Delta_{2j+1,2j+2})\\ &> H - 2E - 2E > H/2\end{aligned}$$

and so $C_{\max}(s(D)) > y$ (see (Fig. 10(a)).

If $0 \leqslant j \leqslant a-2$ and relations $\langle 1, 6j+8\rangle \Rightarrow \langle 3, 4j+4\rangle \Rightarrow \langle 2, 4j+6\rangle$ hold, then there is a job J_2 delay

$$\begin{aligned}\delta_2^{j+1}(s(D)) &= \bar{t}_{3,4j+4}(s(D)) - \bar{t}_{2,4j+5}(s(D))\\ &= \bar{t}_{1,6j+8}(s(D)) + H - \bar{t}_{2,4j+5}(s(D))\\ &\geqslant (6(j+1)H + e_2 + e_4 + \cdots + e_{2j+2} + H\\ &\quad - \Delta_{1,2} - \Delta_{3,4} - \cdots - \Delta_{2j+1,2j+2}) + H\\ &\quad - (6(j+1)H + e_1 + e_3 + \cdots + e_{2j+1} + H)\\ &\geqslant H - 2E - 2E > H/2\end{aligned}$$

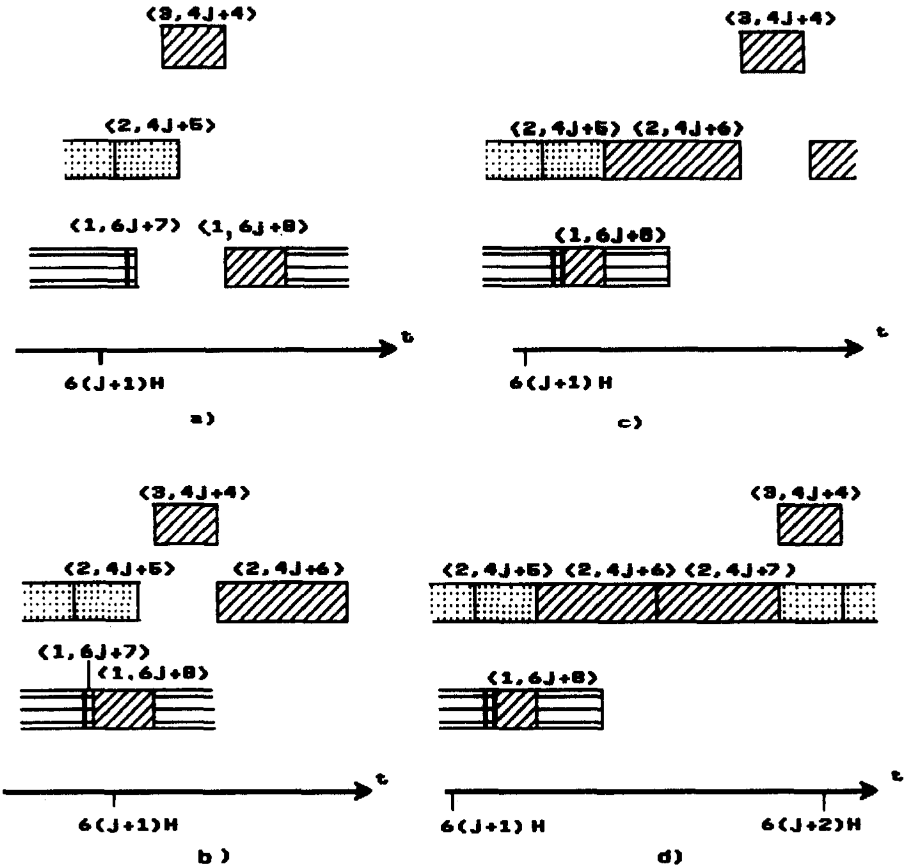

Fig. 10. D_i-fragments of schedules.

and so $C_{\max}(s(D)) > y$ (see (Fig. 10(b)).

If $0 \leqslant j \leqslant a-2$ and relations $\langle 1, 6j+8\rangle \Rightarrow \langle 2, 4j+6\rangle \Rightarrow \langle 3, 4j+4\rangle \Rightarrow \langle 2, 4j+7\rangle$ hold, then there is also a job J_2 delay $\delta_2^{j+1}(s(D)) = H$ within closed interval $[6(j+1)H, 6(j+2)H]$ and so $C_{\max}(s(D)) > y$ (see Fig. 10(c)).

If $0 \leqslant j \leqslant a-2$ and relations $\langle 1, 6j+8\rangle \Rightarrow \langle 2, 4j+6\rangle \Rightarrow \langle 2, 4j+7\rangle \rightarrow \langle 3, 4j+4\rangle$ hold, then $\bar{t}_{3,4j+4}(s(D)) \geqslant \bar{t}_{2,4j+4}(s(D)) \geqslant 6(j+2)H$ (see Fig. 10(d)). So finish part of the schedule $s(D)$ is defined by the set of operations $N_{j+2} \cup N_{j+3} \cup \cdots \cup N_{a-1}$ and by the additional operations $\langle 3, 4j+5\rangle$, $\langle 3, 4j+6\rangle$, $\langle 3, 4j+7\rangle$, and $\langle 3, 4j+8\rangle$, which are to be processed after time $6(j+2)H$. It is easy to make sure that any sequence of these operations gives $C_{\max}(s(D)) > y$.

Thus, we have shown that if a schedule s contains at least one C_i-, D_i-, E_i- or F_i-fragment, it does not belong to S. Moreover, if a schedule s contains only A_i- and/or B_j-fragments, it is not better than a schedule s^* containing only A_1-fragments (i.e., order (5)), and/or B_1-fragments (i.e., order (6)), for all $k = 0, \ldots, a-1$. We conclude that there exists a schedule s^* in the set S such that for any $k = 0, \ldots, a-1$ the operation set N_k is processed according to the order (5) or (6).

It is easy to prove that $C_{\max}(s^*) = y$. In fact, if $C_{\max}(s^*) < y$, then $\bar{t}_1(s^*) < y$, $\bar{t}_2(s^*) < y$ and $\bar{t}_1(s^*) + \bar{t}_2(s^*) < 2y$. However, by construction the schedule s^* we see that $\bar{t}_1(s^*) + \bar{t}_2(s^*) = 6aH + 6aH + 2E = 2y$. Thus, s^* is an optimal schedule for Instance 1.

The schedule s^* defines a solution of the PARTITION problem: If for $k \in \{0, \ldots, a-1\}$ the set of operations N_k is processed according to the order (5) (the order (6), respectively), then $(2k+1) \in A_1$ and $(2k+2) \in A_2$ ($(2k+1) \in A_2$ and $(2k+2) \in A_1$). Either e_{2k+1} causes a job J_2 delay and e_{2k+2} and $\Delta_{2k+1,\,2k+2}$ cause no job J_1 delays, or e_{2k+2} causes a job J_2 delay and $\Delta_{2k+1,\,2k+2}$ causes a job J_1 delay. Since $C_{\max}(s^*) = 6aH + E$ we conclude that $E_1 = E_2$.

To finish the proof of Theorem 1 we shall make the following remark. The above reduction of the PARTITION problem to Instance 1 was carried out under the assumption that successive operations of a job can be processed in the same machine (see routes l^1 and l^2). This assumption is not typical for job-shop problems. Let us modify Instance 1 into the next Instance 2 *to prove the* $3|3|J|C_{\max}$ *problem* NP-*hardness in the case when any successive operations of a job are to be processed on different machines.*

To obtain Instance 2 we introduce $3a$ new operations with sufficiently small durations $\varepsilon/3a > 0$ in Instance 1, where ε is the smallest integer among $\min\{\Delta_{2k+1,\,2k+2} \mid k = 0, \ldots, a-1\}$ and $\min\{e_j \mid j = 1, \ldots, 2a\}$. Recall that both $\Delta_{2k+1,\,2k+2}$ and e_j are strictly positive. For $k = 0, \ldots, a-2$ we join the operations $\langle 1, 6k+6 \rangle$ and $\langle 1, 6k+7 \rangle$, and set

$$y = 6aH + E + \varepsilon,$$

$$l^1 = ([2,3,1,3,1,3]^a), \qquad l^2 = ([1,2,1,2,1,2]^a), \qquad l^3 = ([1,2,3,2]^a),$$

$$t'_{1,\,6k+1} = t_{1,\,6k+2} = H - 2 \sum_{v=-1}^{k-1} \Delta_{2v+1,\,2v+2},$$

$$t'_{1,\,6k+2} = t_{1,\,6k+3} = H + \sum_{v=-1}^{k-1} \Delta_{2v+1,\,2v+2} + e_{2k+2},$$

$$t'_{1,\,6k+3} = t_{1,\,6k+4} = \Delta_{2k+1,\,2k+2},$$

$$t'_{1,\,6k+4} = t_{1,\,6k+5} = 2H - \Delta_{2k+1,\,2k+2},$$

$$t'_{1,\,6k+5} = \varepsilon/3a, \qquad t'_{1,\,6u+6} = t_{1,\,6u+6} + t_{1,\,6u+7} = 2H + \sum_{v=-1}^{u} \Delta_{2v+1,\,2v+2},$$

$$t'_{1,\,6a} = 2H, \qquad t'_{2,\,6k+1} = t_{2,\,4k+1} = H, \qquad t'_{2,\,6k+2} = t_{2,\,4k+2} = 2H,$$

$$t'_{2,\,6k+3} = \varepsilon/3a, \qquad t'_{2,\,6k+4} = t_{2,\,4k+3} = 2H, \qquad t'_{2,\,6k+5} = t_{2,\,4k+4} = H,$$

$$t'_{2,\,6k+6} = \varepsilon/3a, \qquad t'_{3,\,4k+1} = t_{3,\,4k+1} = e_{2k+1},$$

$$t'_{3,\,4k+2} = t_{3,\,4k+2} = e_{2k+2},$$

$$t'_{3,\,4k+3} = t_{3,\,4k+3} = \Delta_{2k+1,\,2k+2}, \qquad t'_{3,\,4k+4} = t_{3,\,4k+4} = H,$$

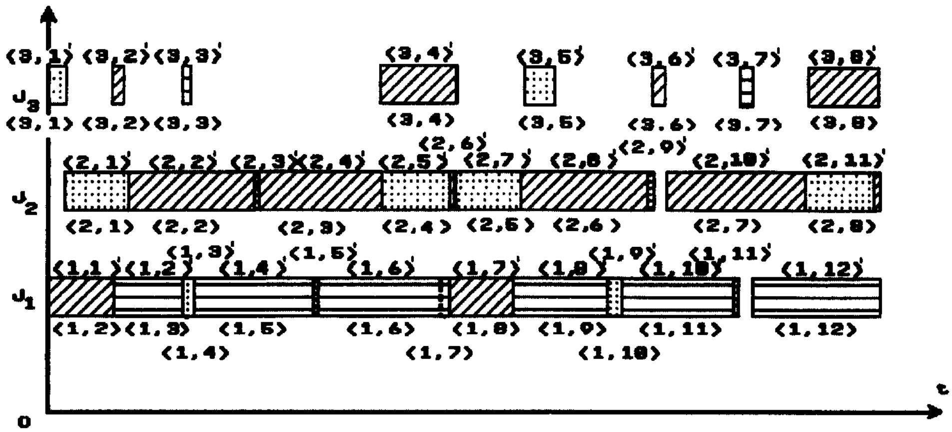

Fig. 11. Initial part of a schedule for Instance 2 similar to the schedule part, represented in Fig. 6.

where $k = 0, \ldots, a-1$, $u = 0, \ldots, a-2$ and the values t_{iq} are defined in Instance 1.

Fig. 11 shows the initial part of schedule for Instance 2 with order like (5) for $k = 0$ and with order like (6) for $k = 1$. This schedule part is similar to the Instance 1 schedule part, represented in Fig. 6. The operations of Instance 2 are denoted by $\langle i, q \rangle'$, $J_i \in \{J_1, J_2, J_3\}$, $1 \leqslant q \leqslant n_i$, and the corresponding operations of Instance 1 are denoted by $\langle i, q \rangle$.

It is easy to see that if there is a job J_1 delay between either operations $\langle 1,6 \rangle$ and $\langle 1,7 \rangle$ or operations $\langle 1,12 \rangle$ and $\langle 1,13 \rangle$, etc., or operations $\langle 1, 6a-6 \rangle$ and $\langle 1, 6a-5 \rangle$ in Instance 1, then such schedule is not active. So having joined these operations we do not restrict all variety of the active schedules. Moreover, new small operations do not restrict or extend the set of active schedules. Introduction of these operations can only increase $C_{\max}$, the value of this increase being not greater than ε.

Considering by analogy fragments of the active schedules for each N_k, $k = 0, \ldots, a-1$, we can be sure that the fragments like (5) and (6) are the best for the Instance 2 and determine a solution of PARTITION problem. The above reduction of the PARTITION problem to $3|3|J|C_{\max}$ problem is a polynomial one, and thus Theorem 1 is proved. □

Corollary 1. *The* $3|3|J|\sum C_i$ *problem is* NP-*hard.*

Proof. Now let us consider the Instance 1 with $\sum C_i$ criterion, i.e., the problem is to find a schedule s^* minimizing a value of mean flow time $(1/n)\sum C_i(s^*)$. We introduce class $\tilde{S}$ of schedules, satisfying the condition

$$\langle 2, 4k+3 \rangle \to \langle 3, 4k+4 \rangle, \quad 0 \leqslant k \leqslant a-1. \tag{9}$$

Remark that for any active schedule $s \in \tilde{S}$ we have $\bar{t}_3(s) \geqslant \bar{t}_2(s)$. But if we are not limited by this class $\tilde{S}$, then it is possible to reduce $\bar{t}_3(s)$. In fact, since job J_3 duration is essentially less in comparison with durations of jobs J_1 and J_2, $(1/n)\sum C_i(s')$ value for $s' \notin \tilde{S}$ can be less than $(1/n)\sum C_i(s)$ for $s \in \tilde{S}$.

Taking into account this fact we construct a new instance (Instance 3), to prove the $3|3|J|\sum C_i$ problem NP-hardness. Note that the machine M_2 is the most "busy" one in Instance 1. For each $k = 0, \ldots, a-1$ we prolong operation $\langle 3, 4k+4 \rangle$ processing time: $t_{3,4k+4} + 10aH = H + 10aH$. To stay in the frameworks of the proof of Theorem 1 we prolong also the processing times of operations $\langle 1, 6k+6 \rangle$ and $\langle 2, 4k+4 \rangle$ by the same value: $t_{1,6k+6} + 10aH = 2H + 10aH$, $t_{2,4k+4} + 10aH = H + 10aH$. The processing times of all other operations N_k, $k = 0, \ldots, a-1$, in Instance 3 remain the same as in Instance 1. Let $y = aH(6 + 10a) + E$.

For any schedule s^*, constructed for Instance 3 according to orders (5) and/or (6), the sum of delays in job J_3 processing is the same as for the corresponding schedule for Instance 1 and it is not greater than $\Theta = a5H$. If for some value k, $0 \leqslant k \leqslant a-1$, conditions (9) are violated, then new delays in jobs J_1 and J_2 processing arise, the duration of the sum of such delays being not less than $H + a10H > \Theta$. Thus, $s \notin \tilde{S}$ implies that s is worse (with respect to $\sum C_i$) than schedule s^* for Instance 3. So we can consider schedules $s \in \tilde{S}$, only.

Since for any such active schedule $s \in \tilde{S}$ the following inequalities are true:

$$\bar{t}_1(s) \geqslant a(6 + 10a)H + \sum_{i \in A_2} e_i, \qquad \bar{t}_2(s) \geqslant a(6 + 10a)H + \sum_{i \in A_1} e_i,$$

$$\bar{t}_3(s) \geqslant \bar{t}_2(s),$$

then

$$\sum C_i(s) \geqslant \bar{t}_1(s) + 2\bar{t}_2(s) = 3a(6 + 10a)H + \sum_{i \in A_2} e_i + 2\sum_{i \in A_1} e_i$$

and value of $\frac{1}{3}\sum C_i(s)$ is not minimal if $\sum_{i \in A_1} e_i = \sum_{i \in A_2} e_i$. Therefore, we modify the Instance 3 fragment for $k = a-1$ by introducing three new operations of job J_3 processing. This final fragment is defined by the operation set $N_{a-1} = \{\langle 1, 6(a-1) + \alpha \rangle \mid \alpha = 1, \ldots, 6\} \cup \{\langle 2, 4(a-1) + \beta \rangle \mid \beta = 1, \ldots, 4\} \cup \{\langle 3, 4(a-1) + \gamma \rangle \mid \gamma = 1, \ldots, 7\}$. Define the route $l^3 = ([1,2,3,2]^{a-1}, 1, 3, 2, 1, 3, 1, 2)$ and the durations of operations $\langle 3, 4(a-1)+1 \rangle, \ldots, \langle 3, 4(a-1)+7 \rangle$:

$$t_{3,4(a-1)+1} = e_{2a-1},$$

$$t_{3,4(a-1)+2} = H - e_{2a-1} - 2\sum_{v=0}^{a-1} \Delta_{2v+1,2v+2}, \qquad t_{3,4(a-1)+3} = e_{2a},$$

$$t_{3,4(a-1)+4} = H - \sum_{v=0}^{a-1} \Delta_{2v+1,2v+2}, \qquad t_{3,4(a-1)+5} = \Delta_{2a-1,2a},$$

$$t_{3,4(a-1)+6} = 3H - \Delta_{2a-1,2a}, \qquad t_{3,4(a-1)+7} = H + 10aH.$$

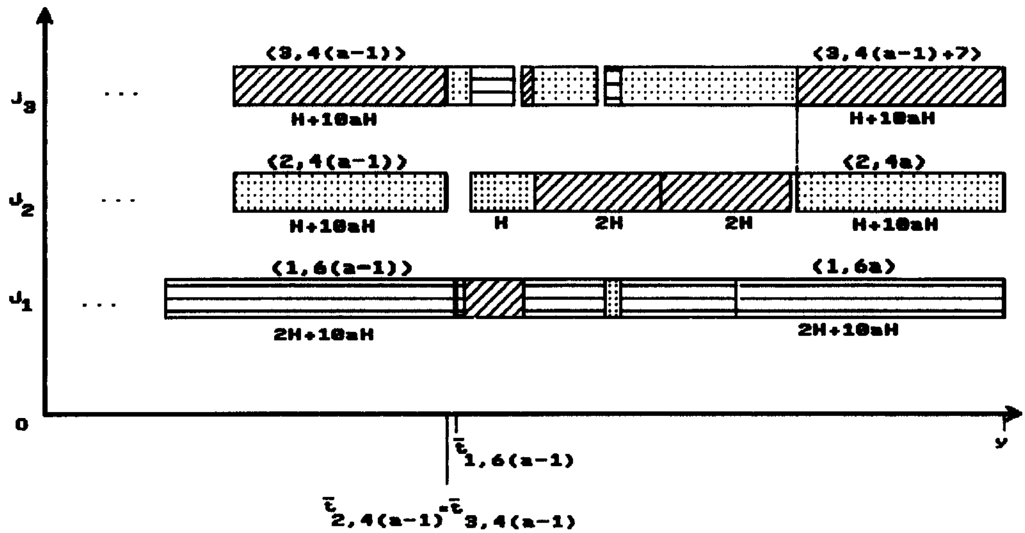

Fig. 12. Final fragment of the schedule for Instance 3.

For $k = a - 1$ we define the following order (10) similar to order (5) for Instance 1:

$$\begin{aligned}
&\langle 3, 4(a-1)+1\rangle \Rightarrow \langle 2, 4(a-1)+1\rangle,\\
&\langle 1, 6(a-1)+2\rangle \Rightarrow \langle 3, 4(a-1)+3\rangle,\\
&\langle 1, 6(a-1)+3\rangle \Rightarrow \langle 3, 4(a-1)+5\rangle,\\
&\langle 2, 4(a-1)+3\rangle \Rightarrow \langle 3, 4(a-1)+7\rangle,\\
&\langle 3, 4(a-1)+6\rangle \Rightarrow \langle 2, 4a\rangle
\end{aligned} \tag{10}$$

(see Fig. 12). It is easy to make sure that these new operations, being scheduled according with order (10), may cause additional delay in job J_2 processing (due to relations $\langle 1, 6(a-1)+4\rangle \Rightarrow \langle 3, 4(a-1)+6\rangle \Rightarrow \langle 2, 4a\rangle$) and so $\bar{t}_2(s) \geqslant \bar{t}_1(s)$. Note that $s \in \tilde{S}$ and $\bar{t}_3(s) \geqslant \bar{t}_2(s)$ also. More precisely,

$$\bar{t}_1(s) = a(6+10a)H + \sum_{i \in A_2} e_i,$$

$$\bar{t}_2(s) = a(6+10a)H + \max\left\{\sum_{i \in A_1} e_i, \sum_{i \in A_2} e_i\right\},$$

$$\bar{t}_3(s) = a(6+10a)H + \max\left\{\sum_{i \in A_1} e_i, \sum_{i \in A_2} e_i\right\}.$$

Thus,

$$\sum C_i(s) = 3a(6+10a)H + \sum_{i \in A_2} e_i + 2\max\left\{\sum_{i \in A_1} e_i, \sum_{i \in A_2} e_i\right\}.$$

Obviously, value of $\frac{1}{3}\sum C_i(s)$ is minimal iff $\sum_{i \in A_1} e_i = \sum_{i \in A_2} e_i = E$ and it is equal to y.

It can be proved that violating of conditions (10) in the schedule s gives $\frac{1}{3}\sum C_i(s) > y$. So order (10) fixes the order like (5) for finish fragment of a schedule for Instance 3.

Note that if the PARTITION problem has a solution, it is possible to construct for Instance 1 with $C_{\max}$ criterion both an optimal schedule s_1, according to orders (5) and/or (6), and another optimal schedule s_2 by interchanging the subsets A_1 and A_2. An optimal schedule for Instance 3 with $\sum C_i$ criterion corresponds to that schedule s_1 or s_2 of Instance 1, which has the final fragment, satisfying the order (5). In this sense, the rules for constructing optimal schedule for Instance 3 based on a solution of the PARTITION problem (and vice versa) is the same as for Instance 1.

Evidently, this modification may be applied to Instance 2 too, and Corollary 1 is proved. □

The validity of the next statement is based on the fact that the operation preemptions do not reduce the values $C_{\max}$ and $\sum C_i$ for Instance 1 (and for Instances 2 and 3 also) because of fixed routes for each job $J_i \in J$.

Corollary 2. *The problems* $3|3|J, \mathrm{Pr}|C_{\max}$ *and* $3|3|J, \mathrm{Pr}|\sum C_i$ *are* NP-*hard.*

3. Flow-shop

Note that NP-hardness of the problems $3|m|F|C_{\max}$ and $3|m|F|\sum C_i$ has been proved in [14–16]. Let us prove analogous results for $3|m|F, \mathrm{Pr}|C_{\max}$ and $3|m|F, \mathrm{Pr}|\sum C_i$ problems.

Theorem 2. *The problems* $3|3|F, \mathrm{Pr}|C_{\max}$ *and* $3|3|F, \mathrm{Pr}|\sum C_i$ *are* NP-*hard.*

Proof. We construct the following Instance 4: $y = 6aH + E$, $m = 9a + 3$, $l^1 = (, 2, , , , , , , 9 \,|\, 10, 11, , , , , , , 18 \,|\, \dots \,|\, 9k + 1, 9k + 2, , , , , , , 9k + 9 \,|\, \dots \,|\, 9a - 8, 9a - 7, , , , , , 9a)$, $l^2 = (, , , 4, , 6, , , , , 11, 12 \,|\, 13, , 15, , , , , 20, 21 \,|\, \dots \,|\, 9k + 4, , 9k + 6, , , , , 9k + 11, 9k + 12 \,|\, \dots \,|\, 9a - 5, , 9a - 3, , , , , 9a + 2, 9a + 3)$ and $l^3 = (, , , 4, 5, 6, , 8, 9, 10, 11 \,|\, , , 13, 14, 15, , 17, 18, 19, 20 \,|\, \dots \,|\, , , 9k + 4, 9k + 5, 9k + 6, , 9k + 8, 9k + 9, 9k + 10, 9k + 11 \,|\, \dots \,|\, , , 9a - 5, 9a - 4, 9a - 3, , 9a - 1, 9a, 9a + 1, 9a + 2)$.

The parts of the routes corresponding to fixed $k \in \{0, \dots, a - 1\}$ (see proof of Theorem 1) are separated by symbol |. Let the operations of Instance 4 be denoted by $\langle i, q \rangle''$, $J_i \in J = \{J_1, J_2, J_3\}$, $1 \leqslant q \leqslant 9a + 3$. As usual, machines listed in l^i are separated by commas. If duration t''_{iq} of operation $\langle i, q \rangle''$ is equal to zero, then machine $M_{l^i_q}$ is omitted (but the comma remains). Let us define nonzero processing times t''_{iq} (recall that the values $t_{iq} > 0$ are defined in the Instance 1):

$$t''_{1,9k+1} = t_{1,6k+1} = \sum_{v=-1}^{k-1} \Delta_{2v+1,2v+2},$$

$$t''_{1,9k+2} = t_{1,6k+2} + t_{1,6k+3} = 2H - \sum_{v=-1}^{k-1} \Delta_{2v+1,2v+2} + e_{2k+2},$$

$$t''_{1,9k+9} = t_{1,6k+4} + t_{1,6k+5} + t_{1,6k+6} = 4H, \qquad t''_{2,9k+4} = t_{2,4k+1} = H,$$

$$t''_{2,9k+6} = t_{2,4k+2} = 2H, \qquad t''_{2,9k+11} = t_{2,4k+3} = 2H,$$

$$t''_{2,9k+12} = t_{2,4k+4} = H, \qquad t''_{3,9k+4} = t_{3,4k+1} = e_{2k+1},$$

$$t''_{3,9k+5} = H - e_{2k+1}, \qquad t''_{3,9k+6} = t_{3,4k+2} = e_{2k+2},$$

$$t''_{3,9k+8} = H - \sum_{v=0}^{k} \Delta_{2v+1,2v+2}, \qquad t''_{3,9k+9} = t_{3,4k+3} = \Delta_{2k+1,2k+2},$$

$$t''_{3,9k+10} = H - \sum_{v=0}^{k} \Delta_{2v+1,2v+2}, \qquad t''_{3,9k+11} = t_{3,4k+4} = H, \quad k = 0,\ldots,a-1.$$

Since preemptions are allowed we add to relations $\rightarrow$ and $\Rightarrow$ two new relations $\overset{\text{Pr}}{\rightarrow}$ and $\overset{\text{Pr}}{\Rightarrow}$. Let us write $\langle i_1,q_1\rangle \overset{\text{Pr}}{\rightarrow} \langle i_2,q_2\rangle$ (provided that $\{J_{i_1}, J_{i_2}\} \subset \{J_1, J_2, J_3\}$, $1 \leqslant q_1 \leqslant n_{i_1}$, $1 \leqslant q_2 \leqslant n_{i_2}$), if $\bar{t}_{i_1,q_1}(s) < \bar{t}_{i_2,q_2}(s)$. If $\bar{t}_{i_1,q_1}(s) < \bar{t}_{i_2,q_2}(s)$ and there is no operation $\langle i_3,q_3\rangle$, $J_{i_3} \in \{J_1, J_2, J_3\}$, $1 \leqslant q_3 \leqslant n_{i_3}$, such that $\bar{t}_{i_1,q_1}(s) < \bar{t}_{i_3,q_3}(s) < \bar{t}_{i_2,q_2}(s)$, then we shall write $\langle i_1,q_1\rangle \overset{\text{Pr}}{\Rightarrow} \langle i_2,q_2\rangle$. Note that if relation $\langle i_1,q_1\rangle \overset{\text{Pr}}{\Rightarrow} \langle i_2,q_2\rangle$ is given, operation $\langle i_1,q_1\rangle$ is started at time $\underline{t}_1$, operation $\langle i_2,q_2\rangle$ is started at time $\underline{t}_2$, $\underline{t}_2 < \underline{t}_1$, and preemptions are allowed, then there is a job J_{i_2} preemption, its duration being equal to $\delta_{i_2} = t_{i_1,q_1}$ (see Fig. 13(a)). Otherwise (if preemptions are forbidden) there is a job J_{i_2} delay, its duration being equal to $\delta_{i_2} = \underline{t}_1 + t_{i_1,q_1} - \underline{t}_2$ (see Fig. 13(b)).

Let us construct the schedule s^0 consisting of a (a means number again) fragments, each kth fragment, $k \in \{0,\ldots,a-1\}$, being associated with the processing operations $N''_k = \{\langle 1,9k+1\rangle'', \langle 1,9k+2\rangle'', \langle 1,9k+9\rangle''\} \cup \{\langle 2,9k+4\rangle'', \langle 2,9k+6\rangle'', \langle 2,9k+11\rangle'', \langle 2,9k+12\rangle''\} \cup \{\langle 3,9k+4\rangle'', \langle 3,9k+5\rangle'', \langle 3,9k+6\rangle'', \langle 3,9k+8\rangle'', \langle 3,9k+9\rangle'', \langle 3,9k+10\rangle'', \langle 3,9k+11\rangle''\}$ on machines M according with order

$$\langle 3,9k+4\rangle'' \Rightarrow \langle 2,9k+4\rangle'', \qquad \langle 3,9k+6\rangle'' \Rightarrow \langle 2,9k+6\rangle'',$$
$$\langle 3,9k+9\rangle'' \Rightarrow \langle 1,9k+9\rangle'', \qquad \langle 2,9k+11\rangle'' \Rightarrow \langle 3,9k+11\rangle'', \tag{11}$$

or according with order

$$\langle 2,9k+4\rangle'' \Rightarrow \langle 3,9k+4\rangle'', \qquad \langle 3,9k+6\rangle'' \overset{\text{Pr}}{\Rightarrow} \langle 2,9k+6\rangle'',$$
$$\langle 3,9k+9\rangle'' \overset{\text{Pr}}{\Rightarrow} \langle 1,9k+9\rangle'', \qquad \langle 2,9k+11\rangle'' \Rightarrow \langle 3,9k+11\rangle''. \tag{12}$$

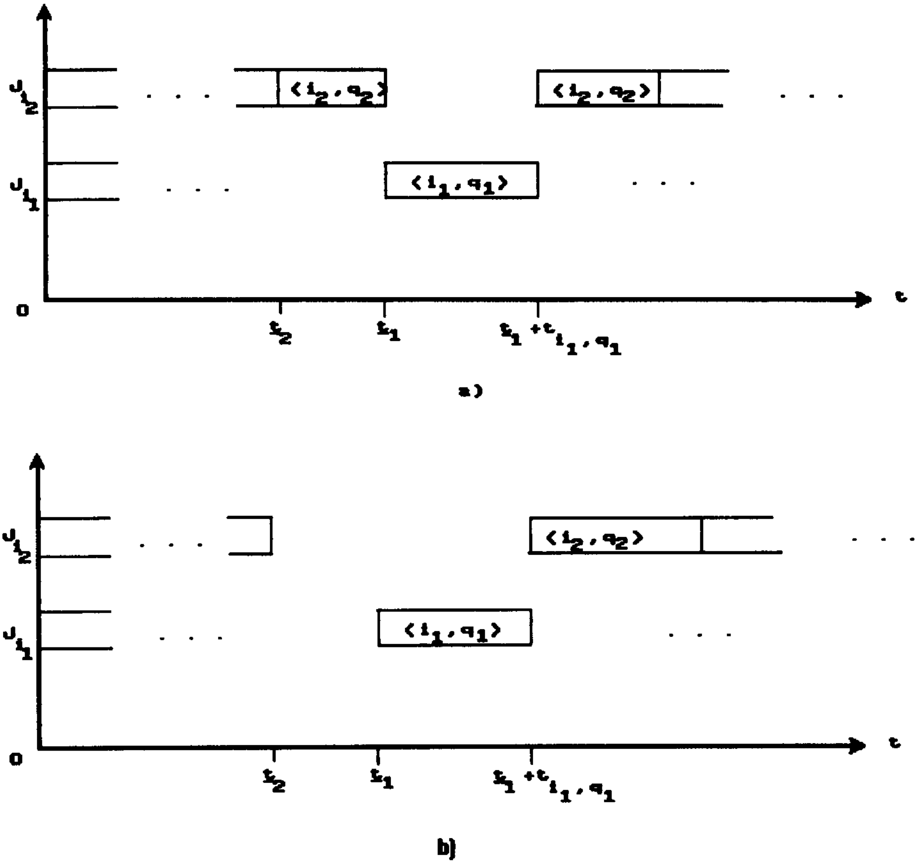

Fig. 13. Part of the schedule for Instance 4.

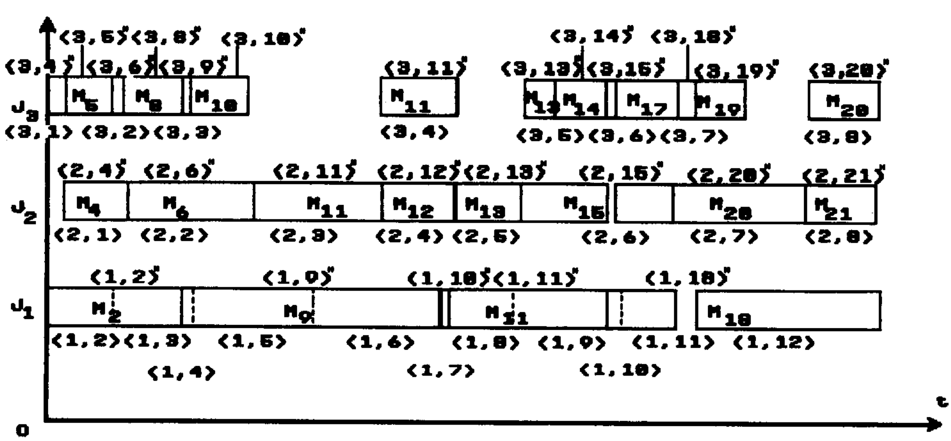

Fig. 14. Initial part of schedule s^0 for Instance 4 with order (11) for $k = 0$ and order (12) for $k = 1$.

Fig. 14 shows the initial part of schedule s^0 (for Instance 4) with order (11) for $k = 0$ and order (12) for $k = 1$. This schedule is similar to the schedule for Instance 1, represented in Fig. 6, and schedule for Instance 2, represented in Fig. 11. The operations of Instance 4 are denoted by $\langle i, q\rangle''$, $J_i \in J = \{J_1, J_2, J_3\}$, $1 \leqslant q \leqslant 9a + 3$, and the corresponding operations of Instance 1 are denoted by $\langle i, q\rangle$ in Fig. 14.

It is easy to see that order (11) causes the same delays in jobs J_1 and J_2 processing as order (5) in Instance 1, and all the operations are processed without preemptions. Order (12) causes the same delays in jobs J_1 and J_2 processing as order (6) in Instance 1, the operations $\langle 2, 9k + 6\rangle''$ and $\langle 1, 9k + 9\rangle''$ are processed with preemptions.

So for $0 \leqslant k \leqslant a - 1$ and conditions (11) the following equalities hold:

$$\bar{t}''_{1,9k+9}(s^0) = \bar{t}''_{1,9k}(s^0) + 6H + e_{2k+2},$$

$$\bar{t}''_{2,9k+12}(s^0) = \bar{t}''_{2,9k+3}(s^0) + 6H + e_{2k+1}, \qquad \bar{t}''_{3,9k+11}(s^0) = \bar{t}''_{2,9k+12}(s^0).$$

Similarly, for conditions (12) we have

$$\bar{t}''_{1,9k+9}(s^0) = \bar{t}''_{1,9k}(s^0) + 6H + e_{2k+1},$$

$$\bar{t}''_{2,9k+12}(s^0) = \bar{t}''_{2,9k+3}(s^0) + 6H + e_{2k+2}, \qquad \bar{t}''_{3,9k+11}(s^0) = \bar{t}''_{2,9k+12}(s^0).$$

So $C_{\max}(s^0) = 6aH + E = y$ if $\sum_{i \in A_1} e_i = \sum_{i \in A_2} e_i = E$. It is not difficult to show that conditions (11) (and conditions (12) also) really specify a part of active schedule s^0 for any $0 \leqslant k \leqslant a - 1$. See proof of Theorem 1.

Using arguments similar to those for operation sets N_k and orders (5) and (6) in Section 2, it is possible to prove that any sequence of operations N''_k, differing from orders (11) and (12), gives schedule s, which is not better than schedule s^0. Thus, we can construct polynomial reduction of a PARTITION problem to the decision problem corresponding to Instance 4 for both criteria $C_{\max}$ and $\sum C_i$. Hence, Theorem 2 is correct. □

Let us show that the problems $3|m|F, \mathrm{Pr}|C_{\max}$ and $3|m|F, \mathrm{Pr}|\sum C_i$ are NP-hard also in the case of strictly positive operation durations. In fact, for each operation $\langle i, q\rangle''$ with zero processing time (see Instance 2) we substitute for its duration $t''_{iq} = 0$ by positive sufficiently small value $\varepsilon/14a > 0$, where ε is defined like in Section 2. We set $y = 6aH + E + \varepsilon$. It is clear that value ε has no effect on the construction of polynomial reduction of the PARTITION problem to the corresponding decision one, and thus the following corollary is true.

Corollary 3. *The problems* $3|m|F, \mathrm{Pr}, t_{iq} > 0|C_{\max}$ *and* $3|m|F, \mathrm{Pr}, t_{iq} > 0|\sum C_i$ *are* NP-*hard.*

4. Open-shop

Recall that Gonzalez and Sahni [7] proved NP-hardness of the $n|3|O|C_{\max}$ problem and developed an O(n) algorithm for $n|2|O|C_{\max}$ problem and an O(n^2m^2) algorithm for $n|m|O, \mathrm{Pr}|C_{\max}$ problem. Due to the possibility of interchanging the set of machines and the set of jobs in $n|m|O|C_{\max}$ and $n|m|O, \mathrm{Pr}|C_{\max}$ problems, we can conclude that the problem $3|m|O|C_{\max}$ is NP-hard and the problem $2|m|O|C_{\max}$ is solvable by O(m) steps.

There is no such symmetry between sets of machines and jobs for an open-shop problem with $\sum C_i$ criterion. Nevertheless, let us show that scheduling instance, used by Strusevich [18] for NP-hardness proof of $n|3|O|\sum C_i$ problem, can be developed to prove NP-hardness of $3|m|O|\sum C_i$ problem too.

Since the routes (machine orders) of jobs may be arbitrary in open-shop, let us agree that notation $\langle i, k\rangle$ means an operation of job $J_i \in J$ on machine $M_k \in M$ (in the case of an open-shop problem).

Theorem 3. *The* $3|m|O|\sum C_i$ *problem is* NP-*hard.*

Proof. Let us reduce polynomially a PARTITION problem from Section 2 to a decision variant of a problem $3|m|O|\sum C_i$. We set $y = 3E$, $m = 2a + 1$ and processing times $t_{ik} = e_k$, $k \in \{1, \ldots, 2a\}$, and $t_{i,\,2a+1} = E$, $i \in \{1, 2, 3\}$, where i denotes the job index and k denotes the machine index.

Sufficiency. If a solution $A = A_1 \cup A_2$, $E_1 = E_2$, of the PARTITION problem exists, we can construct a no-wait schedule s^0 with $\frac{1}{3}\sum C_i = y$ (see Fig. 15). This schedule consists of three parts each of duration E: operation $\langle 1, 2a + 1\rangle$, operations $\langle 2, k'\rangle$ for all machines $M_{k'}$ with $k' \in A_1$, and operations $\langle 3, k''\rangle$ for all machines $M_{k''}$ with $k'' \in A_2$ are completely processed within segment $[0, E]$; operations $\langle 2, 2a + 1\rangle$, $\langle 1, k''\rangle$, and $\langle 3, k'\rangle$ for all $k' \in A_1$ and all $k'' \in A_2$ are completely processed within segment $[E, 2E]$; at last, operations $\langle 3, 2a + 1\rangle$, $\langle 1, k'\rangle$, and $\langle 2, k''\rangle$ for all $k' \in A_1$ and all $k'' \in A_2$ are completely processed within segment $[2E, 3E]$.

Necessity. Since equality $\sum_{k=1}^{2a+1} t_{ik} = 3E$ holds for each job $J_i \in J$, we conclude that any schedule with $\frac{1}{3}\sum C_i \leqslant y = 3E$ has to be a schedule without waits in processing all three jobs within segment $[0, 3E]$.

Let a schedule s^0 with $\frac{1}{3}\sum C_i \leqslant y$ exist. Without loss of generality, we can suppose that machine $M_{2a+1} \in M$ processes job set J in the order (J_1, J_2, J_3). Obviously, to exclude waits in processing job J_2 within segment $[0, 3E]$ is only possible if the total durations of operations $\langle 2, k\rangle$ that have been processed within segment $[0, E]$ is equal to the total durations of operations $\langle 2, l\rangle$ that have been processed within segment $[2E, 3E]$ (see Fig. 15). Thus, the PARTITION problem has a solution $A = A_1 \cup A_2$, $E_1 = E_2$, where subset A_1 is the set of all indexes k of machines $M_k \in M$ processing operations $\langle 2, k\rangle$ of job J_2 within segment $[0, E]$, and subset A_2 is the set of all indexes l of machines $M_l \in M$ processing operations $\langle 2, l\rangle$ of job J_2 within segment $[2E, 3E]$. □

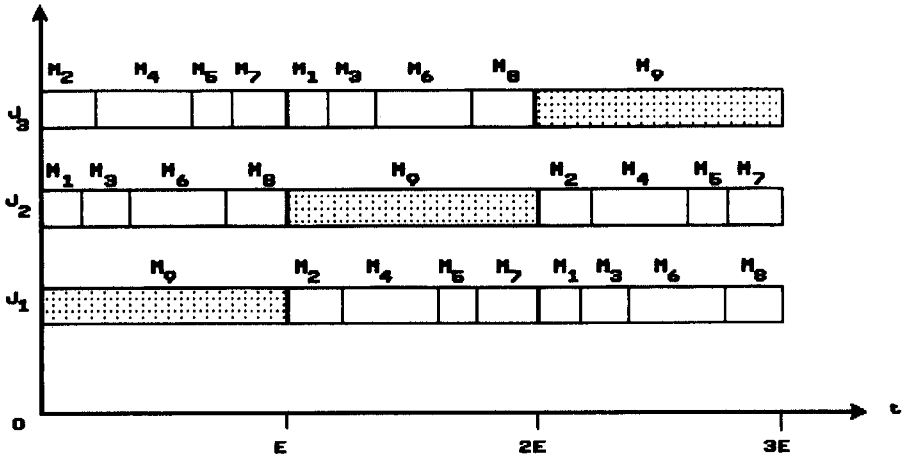

Fig. 15. No-wait schedule s^0 with $\sum C_i(s^0) = y$ for $3|m|O|\sum C_i$ problem.

5. Conclusion

The results obtained in Sections 2–4 as well as known results on the complexity of shop-scheduling problems with $n \leqslant m$ are listed in Table 1. The last column of the table contains reference numbers in square brackets or numbers α of Theorems (Tα) or Corollaries (Cα) from this paper.

As follows from Table 1 the $2|m|J|\Phi$ and $2|m|J,\mathrm{Pr}|\Phi$ problems can be solved by polynomial-time algorithms [13, 16], the complexity depending on $r = \max\{n_1, n_2\}$. Here the parameter Φ denotes an arbitrary regular criterion [10, 11], i.e., a scheduling problem is to minimize a given monotone function $\Phi(\bar{t}_1(s), \ldots, \bar{t}_n(s))$ such that, if $\bar{t}_i(s) \leqslant \bar{t}_i(s')$ for all $J_i \in J$, then $\Phi(\bar{t}_1(s), \ldots, \bar{t}_n(s)) \leqslant \Phi(\bar{t}_1(s'), \ldots, \bar{t}_n(s'))$.

The algorithms [13, 16] are based on the well-known fact that a shop-scheduling problem with $n = 2$ can be formulated as a shortest path problem in the plane with rectangular objects as obstacles. Such graphical algorithms for more simple $2|m|J|C_{\max}$ problem (for the case when operation preemptions are forbidden) have been constructed in [2–4, 20]. The shop-scheduling problems with three jobs are NP-hard even in the case of rather simple criteria $C_{\max}$ and $\sum C_i$. Thus, the NP-hardness of scheduling three jobs take place for all criteria that are usually considered in the scheduling theory [5, 8, 10, 11, 19] (see the reduction of scheduling criteria in [8, pp. 8–10]). The only exclusion is an open-shop scheduling problem with allowed operation preemptions and $C_{\max}$ optimality criterion [7].

Achugbue and Chin [1] have proved NP-hardness of $n|2|O|\sum C_i$ problem. NP-hardness of $n|3|O,C|C_{\max}$ problem is obtained in [14, 16]. Here parameter C indicates that set $J = \{J_1, \ldots, J_n\}$ is ordered: if $i > j$, then $\underline{t}_{iq}(s) \geqslant \bar{t}_{jq'}(s)$ for each schedule s and machine $M_{l^i_q} = M_{l^i_{q'}}$. The linear-time algorithms for $2|m|O|\Phi$ problem and $2|m|O,\mathrm{Pr}|\Phi$ problem (with any given regular criterion Φ) were developed in [12].

Table 1

System type	Job number	Machine number	Additional conditions	Criterion	Complexity	References
J	2	m	$r \leqslant m$	$C_{\max}$	m^2	[4, 20]
J	2	m		$C_{\max}$	$r^2 \log_2 r$	[3]
J	2	m		Φ	$r^2 \log_2 r$	[13, 16]
J	2	m	Pr	Φ	r^3	[13, 16]
J	2	m	$r \leqslant m$	Φ	$m \log_2 m$	[13, 16]
J	2	m	Pr, $r \leqslant m$	Φ	m^2	[13, 16]
J	3	3		$C_{\max}$	NP	T1, [17]
J	3	3	Pr	$C_{\max}$	NP	C2, [17]
J	3	3		$\sum C_i$	NP	C1, [17]
J	3	3	Pr	$\sum C_i$	NP	C2, [17]
F	2	m		Φ	$m \log_2 m$	[13, 16]
F	2	m	Pr	Φ	m^2	[13, 16]
F	3	m		$C_{\max}$	NP	[14–16]
F	3	m	Pr	$C_{\max}$	NP	T2, [17]
F	3	m		$\sum C_i$	NP	[14–16]
F	3	m	Pr	$\sum C_i$	NP	T2, [17]
F	3	m	Pr, $t_{iq} > 0$	$C_{\max}$	NP	C3, [17]
F	3	m	Pr, $t_{iq} > 0$	$\sum C_i$	NP	C3, [17]
O	2	m		$C_{\max}$	$O(m)$	[7]
O	n	m	Pr	$C_{\max}$	$O(n^2m^2)$	[7]
O	2	m		Φ	$O(m)$	[12]
O	2	m	Pr	Φ	$O(m)$	[12]
O	3	m		$C_{\max}$	NP	[7]
O	3	m		$\sum C_i$	NP	T3, [17, 18]
O	n	2		$\sum C_i$	NP	[1]
O	n	3	C	$C_{\max}$	NP	[14–16]

We recall that the first results on the NP-hardness of shop-scheduling problems with $n \leqslant m$ and fixed routes have been obtained in [14–16]. It has been proved there that the problems $3|5|J|C_{\max}$, $3|5|J|\sum C_i$, $3|5|J,\mathrm{Pr}|C_{\max}$, $3|m|F|C_{\max}$ and $3|m|F|\sum C_i$ are NP-hard. The results of Sections 2–4 of this paper were briefly described in [17] in Russian with schematic proofs.

In conclusion, it should be remarked that shop-scheduling problems with fixed number of jobs n are not NP-hard in the strong sense because for each fixed n a shop-scheduling problem may be solved pseudopolynomially.

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