



# Multiple extensions of a finite Euler's pentagonal number theorem and the Lucas formulas

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## Abstract

Motivated by the resemblance of a multivariate series identity and a finite analogue of Euler's pentagonal number theorem, we study multiple extensions of the latter formula. In a different direction we derive a common extension of this multivariate series identity and two formulas of Lucas. Finally we give a combinatorial proof of Lucas' formulas.

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## 1. Introduction

In a recent work [11] we stumbled upon a multivariate identity involving binomial coefficients (see (3.1)), which implies the following identity:

$$\sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \binom{n-r_k}{r_{k+1}} \frac{(-x)^{r_k}}{(1+x)^{2r_k}} = \frac{1-x^{m(n+1)}}{(1-x^m)(1+x)^{mn}}, \quad (1.1)$$

where  $r_{m+1} = r_1$ . It is easy to see that the  $x = \omega := (-1 \pm i\sqrt{3})/2$  case of (1.1) reduces to

$$\sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \binom{n-r_k}{r_{k+1}} (-1)^{r_k} = \begin{cases} (-1)^{mn}(n+1) & \text{if } m \equiv 0 \pmod{3}, \\ \frac{1-\omega^{m(n+1)}}{(1-\omega^m)(1+\omega)^{mn}} & \text{if } m \not\equiv 0 \pmod{3}. \end{cases} \quad (1.2)$$

This paper was motivated by the connection of (1.2) with some classical formulas in the literature.

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First of all, when  $m = 1$ , the formula (1.2) has a known  $q$ -analogue (see [3–5,17]) as follows:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} = \begin{cases} (-1)^{\lfloor n/3 \rfloor} q^{n(n-1)/6} & \text{if } n \not\equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}, \end{cases} \tag{1.3}$$

where the  $q$ -binomial coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}$  is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \prod_{i=1}^k \frac{1 - q^{n-i+1}}{1 - q^i} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing  $n$  by  $3L$  or  $3L + 1$  and  $q$  by  $1/q$  in (1.3) yields

$$\sum_{j=-L}^L (-1)^j q^{j(3j+1)/2} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix} = 1, \tag{1.4}$$

$$\sum_{j=-L}^L (-1)^j q^{j(3j-1)/2} \begin{bmatrix} 2L-j+1 \\ L+j \end{bmatrix} = 1, \tag{1.5}$$

as mentioned in [17]. Both (1.4) and (1.5) reduce to Euler’s pentagonal number theorem [1, p. 11] when  $L \rightarrow \infty$ :

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2} = \prod_{n=1}^{\infty} (1 - q^n). \tag{1.6}$$

It is then natural to look for multiple analogues of (1.3) in light of (1.2). This will be the main object of Section 2.

Secondly, as will be shown, Eq. (1.1) is also related to the two formulas of Lucas (cf. [8]):

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (x+y)^{n-2k} (-xy)^k = \frac{x^{n+1} - y^{n+1}}{x - y}, \tag{1.7}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k} (-xy)^k = x^n + y^n. \tag{1.8}$$

In Section 3 by using the multivariate Lagrange inversion formula we will prove a generalization of the formula (1.1), which is also a common extension of Lucas’ formulas (1.7) and (1.8).

Finally, as Shattuck and Wagner [14] have recently given a combinatorial proof of (1.7) and (1.8) with  $x = 1$  and  $y = \omega$ , we shall give a combinatorial proof of Lucas’ formulas in their full generality in Section 4.

We conclude this section with some remarks. It is known (see [4]) that (1.3) is actually equivalent to an identity due to Rogers (see [1, p. 29, Example 10]). Some modern proofs are given by Ekhad and Zeilberger [5] and Warnaar [17]. The reader is also referred to Cigler’s paper [4] for more information and proofs of (1.3). Some known multiple and finite extensions of Euler’s pentagonal number (1.6) can be found in [2,13,7, (6.2), 12, (1)] and the references therein. Note also that the  $x + y = 1$  and  $xy = z$  cases of (1.7) and (1.8) are sometimes called the Binet formulas (see [10, p. 204]).

## 2. Common extensions of (1.2) and (1.3)

We shall adopt the standard notation of  $q$ -series in [6]. Let

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n = 0, 1, 2, \dots$$

Then the  $q$ -Chu–Vandermonde formula can be written as

$$\sum_{k \geq 0} \frac{(a; q)_k (q^{-N}; q)_k}{(c; q)_k (q; q)_k} \left( \frac{cq^N}{a} \right)^k = \frac{(c/a; q)_N}{(c; q)_N} \tag{2.1}$$

(see [6, p. 354]). We need the following two variations of (2.1).

**Lemma 2.1.** *Let  $n \geq 1$  and  $r, t \leq n$ . Then*

$$\sum_{s=0}^{n-r} \begin{bmatrix} n-r \\ s \end{bmatrix} \begin{bmatrix} n-s \\ t \end{bmatrix} q^{\binom{s}{2}} (-1)^s = q^{(n-r)(n-t)} \begin{bmatrix} r \\ n-t \end{bmatrix}, \tag{2.2}$$

$$\sum_{s=0}^{n-r} \begin{bmatrix} n-r \\ s \end{bmatrix} \begin{bmatrix} n-s \\ t \end{bmatrix} q^{s(s+2r+2t-2n+1)/2} (-1)^s = \begin{bmatrix} r \\ n-t \end{bmatrix}. \tag{2.3}$$

Indeed, Eq. (2.2) follows from (2.1) with  $a = q^{r-n}$ ,  $N = n - t$  and  $c = q^{-n}$ , and (2.3) can be derived from (2.2) by the substitution  $q \rightarrow q^{-1}$ .

**Theorem 2.2.** *Let  $m, n \geq 1$  and  $x_{3k} = -1$  for all  $1 \leq k \leq m$ . Then*

$$\sum_{r_1, \dots, r_{3m} \leq n} \prod_{k=1}^{3m} \begin{bmatrix} n-r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} x_k^{r_k} = \frac{(x_1 x_4 \cdots x_{3m-2})^{n+1} - (x_2 x_5 \cdots x_{3m-1})^{n+1}}{x_1 x_4 \cdots x_{3m-2} - x_2 x_5 \cdots x_{3m-1}} q^{\binom{n}{2}}, \tag{2.4}$$

where  $r_{3m+1} = r_1$ .

**Proof.** By (2.2), the left-hand side of (2.4) equals

$$\begin{aligned} & \sum_{\substack{r_{3i-2}, r_{3i-1} \leq n \\ 1 \leq i \leq m}} \prod_{k=1}^m \begin{bmatrix} n-r_{3k-2} \\ r_{3k-1} \end{bmatrix} q^{\binom{r_{3k-2}}{2} + \binom{r_{3k-1}}{2}} (x_{3k-2})^{r_{3k-2}} (x_{3k-1})^{r_{3k-1}} \\ & \times \sum_{r_3, r_6, \dots, r_{3m} \leq n} \prod_{k=1}^m \begin{bmatrix} n-r_{3k-1} \\ r_{3k} \end{bmatrix} \begin{bmatrix} n-r_{3k} \\ r_{3k+1} \end{bmatrix} q^{\binom{r_{3k}}{2}} (-1)^{r_{3k}} \\ & = \sum_{\substack{r_{3i-2}, r_{3i-1} \leq n \\ 1 \leq i \leq m}} \prod_{k=1}^m \begin{bmatrix} n-r_{3k-2} \\ r_{3k-1} \end{bmatrix} \begin{bmatrix} r_{3k-1} \\ n-r_{3k+1} \end{bmatrix} q^{\binom{r_{3k-2}}{2} + \binom{r_{3k-1}}{2} + (n-r_{3k-1})(n-r_{3k+1})} (x_{3k-2})^{r_{3k-2}} (x_{3k-1})^{r_{3k-1}}. \end{aligned} \tag{2.5}$$

Note that

$$\prod_{k=1}^m \begin{bmatrix} n-r_{3k-2} \\ r_{3k-1} \end{bmatrix} \begin{bmatrix} r_{3k-1} \\ n-r_{3k+1} \end{bmatrix} = \begin{cases} 1 & \text{if } r_{3k-2} + r_{3k-1} = n \text{ and } r_{3k-1} + r_{3k+1} = n \text{ for all } 1 \leq k \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the nonzero terms in the right-hand side of (2.5) are those indexed by  $r_1 = r_4 = \cdots = r_{3m-2}$  and  $r_2 = r_5 = \cdots = r_{3m-1} = n - r_1$ . Finally, since

$$\binom{r_{3k-2}}{2} + \binom{r_{3k-1}}{2} + (n - r_{3k-1})(n - r_{3k+1}) = \binom{n}{2}$$

for  $r_{3k-2} + r_{3k-1} = n$  and  $r_{3k-1} + r_{3k+1} = n$ , we see that the right-hand side of (2.5) equals

$$\sum_{i=0}^n q^{\binom{n}{2}} (x_1 x_4 \cdots x_{3m-2})^i (x_2 x_5 \cdots x_{3m-1})^{n-i},$$

as desired.  $\square$

Letting  $x_k = -1$  for all  $1 \leq k \leq 3m$  in the above theorem yields a  $q$ -analogue of (1.2) for  $m \equiv 0 \pmod{3}$ .

**Corollary 2.3.** *Let  $m, n \geq 1$ . Then*

$$\sum_{r_1, \dots, r_{3m} \leq n} \prod_{k=1}^{3m} \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} = (-1)^{mn} (n+1) q^{m \binom{n}{2}}, \tag{2.6}$$

where  $r_{3m+1} = r_1$ .

The following theorem gives a  $q$ -analogue of (1.2) for  $m \not\equiv 0 \pmod{3}$ .

**Theorem 2.4.** *Let  $m, n \geq 1$  and  $m \not\equiv 0 \pmod{3}$ . Then*

$$\sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} = \begin{cases} (-1)^{\lfloor (m+n-1)m/3 \rfloor} q^{mn(n-1)/6} & \text{if } n \not\equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}, \end{cases} \tag{2.7}$$

where  $r_{m+1} = r_1$ .

**Proof.** Replacing  $q$  by  $q^{-1}$  in (1.3), we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} q^{k^2 + \binom{n-k}{2}} \begin{bmatrix} n - k \\ k \end{bmatrix} = \begin{cases} (-1)^{\lfloor (2n+2)/3 \rfloor} q^{n(n-1)/3} & \text{if } n \not\equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \tag{2.8}$$

By (2.2), we have

$$\sum_{r_1=0}^n \sum_{r_2=0}^n \begin{bmatrix} n - r_1 \\ r_2 \end{bmatrix} \begin{bmatrix} n - r_2 \\ r_1 \end{bmatrix} q^{\binom{r_1}{2} + \binom{r_2}{2}} (-1)^{r_1+r_2} = \sum_{r_1=0}^n \begin{bmatrix} r_1 \\ n - r_1 \end{bmatrix} q^{\binom{r_1}{2} + (n-r_1)^2} (-1)^{r_1}, \tag{2.9}$$

which is the left-hand side of (2.8). This proves the  $m = 2$  case of (2.7).

Again, by (2.2), we see that

$$\begin{aligned} \sum_{r_1, \dots, r_4 \leq n} \prod_{k=1}^4 \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} &= \sum_{r_1=0}^n \sum_{r_3=0}^n \begin{bmatrix} r_1 \\ n - r_3 \end{bmatrix} \begin{bmatrix} r_3 \\ n - r_1 \end{bmatrix} q^{\binom{r_1}{2} + \binom{r_3}{2} + 2(n-r_1)(n-r_3)} (-1)^{r_1+r_3} \\ &= \sum_{r_1=0}^n \sum_{r_3=0}^n \begin{bmatrix} n - r_1 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ r_1 \end{bmatrix} q^{\binom{n-r_1}{2} + \binom{n-r_3}{2} + 2r_1 r_3} (-1)^{r_1+r_3}, \end{aligned}$$

where  $r_5 = r_1$  is the product of the  $q \rightarrow q^{-1}$  case of the left-hand side of (2.9) and  $q^{n(n-1)}$ . This proves the  $m = 4$  case of (2.7).

For  $m > 4$ , by (2.2) and (2.3), there holds

$$\begin{aligned} \sum_{r_1, \dots, r_4 \leq n} \prod_{k=1}^4 \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} &= \sum_{r_1=0}^n \sum_{r_3=0}^n \begin{bmatrix} r_1 \\ n - r_3 \end{bmatrix} \begin{bmatrix} r_3 \\ n - r_5 \end{bmatrix} q^{\binom{r_1}{2} + \binom{r_3}{2} + (2n-r_1-r_5)(n-r_3)} (-1)^{r_1+r_3} \\ &= \sum_{r_1=0}^n \sum_{r_3=0}^n \begin{bmatrix} n - r_1 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ n - r_5 \end{bmatrix} q^{\binom{n-r_1}{2} + \binom{n-r_3}{2} + (n+r_1-r_5)r_3} (-1)^{r_1+r_3} \\ &= q^{\binom{n}{2}} \sum_{r_1=0}^n \begin{bmatrix} r_1 \\ r_5 \end{bmatrix} q^{\binom{n-r_1}{2}} (-1)^{r_1} \\ &= (-1)^n q^{\binom{n}{2}} \sum_{r_1=0}^n \begin{bmatrix} n - r_1 \\ r_5 \end{bmatrix} q^{\binom{r_1}{2}} (-1)^{r_1}. \end{aligned} \tag{2.10}$$

It follows that

$$\sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} = (-1)^n q^{\binom{n}{2}} \sum_{r_1, \dots, r_{m-3} \leq n} \prod_{k=1}^{m-3} \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k}.$$

By induction we can complete the proof based on the  $m = 2, 4$  cases.  $\square$

The following result gives multiple extensions of (1.4) and (1.5).

**Corollary 2.5.** *Let  $L, m \geq 1$ . Then*

$$\sum_{j_1, \dots, j_m = -L}^{2L} \prod_{k=1}^m \begin{bmatrix} 2L - j_k \\ L + j_{k+1} \end{bmatrix} q^{j_k j_{k+1} + \binom{j_k}{2}} (-1)^{j_k} = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{3}, \\ 3L + 1 & \text{if } m \equiv 0 \pmod{3}, \end{cases} \tag{2.11}$$

$$\sum_{j_1, \dots, j_m = -L}^{2L+1} \prod_{k=1}^m \begin{bmatrix} 2L - j_k + 1 \\ L + j_{k+1} \end{bmatrix} q^{j_k j_{k+1} + \binom{j_k}{2}} (-1)^{j_k} = \begin{cases} (-1)^{\lfloor m^2/3 \rfloor} & \text{if } m \not\equiv 0 \pmod{3}, \\ (-1)^{m/3} (3L + 2) & \text{if } m \equiv 0 \pmod{3}, \end{cases} \tag{2.12}$$

where  $j_{m+1} = j_1$ .

**Proof.** Take  $n = 3L$  in (2.6) and (2.7), and replace  $r_k$  by  $j_k + L$  and  $q$  by  $1/q$ . After making some simplifications, we obtain (2.11). In much the same way, when  $n = 3L + 1$  we are led to (2.12).  $\square$

For  $m \geq 4$ , we can further generalize Theorem 2.4 as in the following two theorems.

**Theorem 2.6.** *Let  $m \geq 4, n \geq 1$  and  $m \equiv 1 \pmod{3}$ . Let  $s \leq m$  be a positive integer such that  $s \not\equiv 0 \pmod{3}$ . Then*

$$\sum_{r_1, \dots, r_m \leq n} z^{r_1 - r_s} \prod_{k=1}^m \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} = \begin{cases} (-1)^{\lfloor (m+n-1)m/3 \rfloor} q^{mn(n-1)/6} & \text{if } n \not\equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}, \end{cases} \tag{2.13}$$

where  $r_{m+1} = r_1$ .

**Proof.** We first prove the  $m = 4$  case. By symmetry, we may assume that  $s = 2$ . In this case, the left-hand side of (2.13) equals

$$\begin{aligned} & \sum_{r_1, \dots, r_4 \leq n} z^{r_1 - r_2} \begin{bmatrix} n - r_1 \\ r_2 \end{bmatrix} \begin{bmatrix} n - r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_1 \end{bmatrix} q^{\binom{r_1}{2} + \dots + \binom{r_4}{2}} (-1)^{r_1 + \dots + r_4} \\ &= \sum_{k=-n}^n \sum_{r_2, r_3, r_4 \leq n} z^k \begin{bmatrix} n - r_2 - k \\ r_2 \end{bmatrix} \begin{bmatrix} n - r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_2 + k \end{bmatrix} q^{\binom{r_2+k}{2} + \binom{r_2}{2} + \binom{r_3}{2} + \binom{r_4}{2}} (-1)^{k+r_3+r_4}. \end{aligned} \tag{2.14}$$

By (2.2), for  $k > 0$ , we have

$$\sum_{r_3 \leq n} \begin{bmatrix} n - r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_2 + k \end{bmatrix} q^{\binom{r_3}{2}} (-1)^{r_3} = \begin{bmatrix} r_2 \\ n - r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_2 + k \end{bmatrix} q^{(n-r_2)(n-r_4)} = 0,$$

while for  $k < 0$ , we have

$$\sum_{r_4 \leq n} \begin{bmatrix} n - r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_2 + k \end{bmatrix} q^{\binom{r_4}{2}} (-1)^{r_4} = \begin{bmatrix} n - r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} r_3 \\ n - r_2 - k \end{bmatrix} q^{(n-r_3)(n-r_2-k)} = 0.$$

Therefore, the right-hand side of (2.14) is independent of  $z$ . This completes the proof of (2.13) for  $m = 4$ .

For  $m \geq 7$ , again by symmetry, we may assume that  $s \geq (m + 3)/2 \geq 5$ . We then complete the proof by induction on  $m$  and using (2.10).  $\square$

**Theorem 2.7.** Let  $m \geq 5, n \geq 1$  and  $m \equiv 2 \pmod{3}$ . Let  $s \leq m$  be a positive integer such that  $s \not\equiv 2 \pmod{3}$ . Then

$$\sum_{r_1, \dots, r_m \leq n} z^{r_1 - r_s} \prod_{k=1}^m \begin{bmatrix} n - r_k \\ r_{k+1} \end{bmatrix} q^{\binom{r_k}{2}} (-1)^{r_k} = \begin{cases} (-1)^{\lfloor (m+n-1)m/3 \rfloor} q^{mn(n-1)/6} & \text{if } n \not\equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}, \end{cases} \tag{2.15}$$

where  $r_{m+1} = r_1$ .

**Proof.** For  $m = 5$ , by symmetry, we may assume that  $s = 3$ . In this case, the left-hand side of (2.15) may be written as

$$\sum_{k=-n}^n \sum_{r_2, \dots, r_5 \leq n} z^k \begin{bmatrix} n - r_3 - k \\ r_2 \end{bmatrix} \begin{bmatrix} n - r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_5 \end{bmatrix} \begin{bmatrix} n - r_5 \\ r_3 + k \end{bmatrix} \times q^{\binom{r_3+k}{2} + \binom{r_2}{2} + \binom{r_3}{2} + \binom{r_4}{2} + \binom{r_5}{2}} (-1)^{k+r_2+r_4+r_5}. \tag{2.16}$$

By (2.2), for  $k > 0$ , we have

$$\sum_{r_4 \leq n} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_5 \end{bmatrix} \begin{bmatrix} n - r_5 \\ r_3 + k \end{bmatrix} q^{\binom{r_4}{2}} (-1)^{r_4} = 0,$$

while for  $k < 0$ , we have

$$\sum_{r_5 \leq n} \begin{bmatrix} n - r_3 \\ r_4 \end{bmatrix} \begin{bmatrix} n - r_4 \\ r_5 \end{bmatrix} \begin{bmatrix} n - r_5 \\ r_3 + k \end{bmatrix} q^{\binom{r_5}{2}} (-1)^{r_5} = 0.$$

Therefore, the right-hand side of (2.16) is independent of  $z$ . This completes the proof of the  $m = 5$  case of (2.15).

For  $m \geq 8$ , again by symmetry, we may assume that  $s \geq (m + 3)/2$ . We then complete the proof by induction on  $m$  and using (2.10).  $\square$

### 3. Generalization of (1.1) and Lucas' formulas

The following identity (3.1) was already announced in [11].

**Theorem 3.1.** We have

$$\sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \binom{n - r_k}{r_{k+1}} \frac{(-x_k)^{r_k}}{(1 + x_k)^{r_k + r_{k+1}}} = \frac{1 - x_1^{n+1} \dots x_m^{n+1}}{1 - x_1 \dots x_m} \prod_{k=1}^m \frac{1}{(1 + x_k)^n}, \tag{3.1}$$

where  $r_{m+1} = r_1$ .

To prove this theorem, we need the following form of the multivariate Lagrange inversion formula (see [9, p. 21]).

**Lemma 3.2.** Let  $m \geq 1$  be a positive integer and  $\mathbf{x} = (x_1, \dots, x_m)$ . Suppose that  $x_i = u_i \phi_i(\mathbf{x})$  for  $i = 1, \dots, m$  and  $\phi_i$  is a formal power series in  $\mathbf{x}$  with complex coefficients such that  $\phi_i(0, \dots, 0) \neq 0$ . Then any formal power series  $f(\mathbf{x})$  with complex coefficients can be expanded into a power series in  $\mathbf{u} = (u_1, \dots, u_m)$  as follows:

$$f(\mathbf{x}(\mathbf{u})) = \sum_{\mathbf{r} \in \mathbb{N}^m} \mathbf{u}^{\mathbf{r}} [\mathbf{x}^{\mathbf{r}}] \{ f(\mathbf{x}) \phi_1^{r_1}(\mathbf{x}) \dots \phi_m^{r_m}(\mathbf{x}) \Delta_m \},$$

where  $[\mathbf{x}^{\mathbf{r}}] f(\mathbf{x})$  denotes the coefficient of  $\mathbf{x}^{\mathbf{r}} = x_1^{r_1} \dots x_m^{r_m}$  in the series  $f(\mathbf{x})$  and

$$\Delta_m = \det \left( \delta_{ij} - \frac{x_j}{\phi_i(\mathbf{x})} \frac{\partial \phi_i(\mathbf{x})}{\partial x_j} \right)_{1 \leq i, j \leq m}.$$

**Proof of (3.1).** Let  $\phi_i(\mathbf{x}) = (1 + x_{i-1})(1 + x_i)$  ( $1 \leq i \leq m$ ), where  $x_0 = x_m$ . Then  $\Delta_1 = (1 - x_1)/(1 + x_1)$  and for  $m \geq 2$

$$\Delta_m = \begin{vmatrix} \frac{1}{1+x_1} & 0 & \cdots & 0 & \frac{-x_m}{1+x_m} \\ -x_1 & \frac{1}{1+x_2} & 0 & \cdots & 0 \\ \frac{1}{1+x_1} & \frac{-x_2}{1+x_2} & \frac{1}{1+x_3} & \cdots & 0 \\ 0 & \frac{1}{1+x_2} & \frac{1}{1+x_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{-x_{m-1}}{1+x_{m-1}} & \frac{1}{1+x_m} \end{vmatrix} = \frac{1 - x_1 \cdots x_m}{\prod_{k=1}^m (1 + x_k)}.$$

Now take

$$f(\mathbf{x}) = \frac{1 - x_1^{n+1} \cdots x_m^{n+1}}{1 - x_1 \cdots x_m} \prod_{k=1}^m \frac{1}{(1 + x_k)^n}.$$

Then

$$f(\mathbf{x}) \phi_1^{r_1}(\mathbf{x}) \cdots \phi_m^{r_m}(\mathbf{x}) \Delta_m = \frac{1 - x_1^{n+1} \cdots x_m^{n+1}}{\prod_{k=1}^m (1 + x_k)^{n+1-r_k-r_{k+1}}}.$$

Note that

$$[\mathbf{x}^r] \prod_{k=1}^m \frac{1}{(1 + x_k)^{n+1-r_k-r_{k+1}}} = \prod_{k=1}^m (-1)^{r_k} \binom{n - r_{k+1}}{r_k} = \prod_{k=1}^m (-1)^{r_k} \binom{n - r_k}{r_{k+1}}.$$

Also,

$$[\mathbf{x}^r] \prod_{k=1}^m \frac{x_k^{n+1}}{(1 + x_k)^{n+1-r_k-r_{k+1}}} = \begin{cases} \prod_{k=1}^m (-1)^{r_k} \binom{n-r_k}{r_{k+1}} & \text{if } r_1, \dots, r_m \geq n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

By subtraction we derive from Lemma 4.1 that

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\min\{r_1, \dots, r_m\} \leq n} u_1^{r_1} \cdots u_m^{r_m} \prod_{k=1}^m (-1)^{r_k} \binom{n - r_k}{r_{k+1}} \\ &= \sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \binom{n - r_k}{r_{k+1}} \frac{(-x_k)^{r_k}}{(1 + x_k)^{r_k+r_{k+1}}}, \end{aligned}$$

as desired.  $\square$

**Remark.** Strehl [16] has obtained more binomial coefficient formulas by applying the multivariate Lagrange inversion formula.

Letting  $x_i = x$  for all  $i$  in (3.1) we obtain (1.1), while letting  $x = (\sqrt{5} - 3)/2$  in (1.1) we obtain the following remarkable identity:

**Proposition 3.3.** For  $m, n \geq 1$ , we have

$$\sum_{r_1, \dots, r_m \leq n} \prod_{k=1}^m \binom{n - r_k}{r_{k+1}} = \frac{2^{m(n+1)} - (\sqrt{5} - 3)^{m(n+1)}}{(2^m - (\sqrt{5} - 3)^m)(\sqrt{5} - 1)^{mn}},$$

where  $r_{m+1} = r_1$ .

To see that (1.1) is a common multiple extension of two formulas of Lucas, we first recall the following elementary counting results (see, for example, [15, Lemma 2.3.4]).

**Lemma 3.4.** *The number of ways of choosing  $k$  points, no two consecutive, from a collection of  $n - 1$  points arranged on a line is  $\binom{n-k}{k}$ . The number of ways of choosing  $k$  points, no two consecutive, from a collection of  $n$  points arranged on a cycle is  $(n/(n - k)) \binom{n-k}{k}$ .*

Now, the  $m = 1$  case of (1.1) corresponds to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{(-x)^k}{(1+x)^{2k}} = \frac{1-x^{n+1}}{(1-x)(1+x)^n}. \tag{3.2}$$

On the other hand, for  $r_1, \dots, r_m \in \{0, 1\}$  and  $r_{m+1} = r_1$ , the product  $\prod_{k=1}^m \binom{1-r_k}{r_{k+1}}$  equals 1 if there are no two consecutive 1's in the sequence  $r_1, \dots, r_m, r_{m+1}$ , and 0 otherwise. Thus, by Lemma 3.4, the  $n = 1$  case of (1.1) corresponds to the following identity:

$$\sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m}{m-k} \binom{m-k}{k} \frac{(-x)^k}{(1+x)^{2k}} = \frac{1+x^m}{(1+x)^m}. \tag{3.3}$$

Clearly Lucas' formulas (1.7) and (1.8) are equivalent to (3.2) and (3.3). When  $x = \omega$  the latter formulas (replacing  $m$  by  $n$  in (3.3)) can be written as

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k = \frac{1-\omega^{n+1}}{(1-\omega)(1+\omega)^n} = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{6}, \\ 0 & \text{if } n \equiv 2, 5 \pmod{6}, \\ -1 & \text{if } n \equiv 3, 4 \pmod{6} \end{cases} \tag{3.4}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} (-1)^k = \frac{1+\omega^n}{(1+\omega)^n} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{6}, \\ 1 & \text{if } n \equiv 1, 5 \pmod{6}, \\ -1 & \text{if } n \equiv 2, 4 \pmod{6}, \\ -2 & \text{if } n \equiv 3 \pmod{6}. \end{cases} \tag{3.5}$$

Motivated by the recent combinatorial proof of (3.4) and (3.5) by Shattuck and Wagner [14], we shall give a combinatorial proof of a polynomial version of (3.2) and (3.3) in the next section.

#### 4. Combinatorial proof of Lucas' formulas

Letting  $m = -x/(1+x)$  in (3.2) and (3.3), we obtain

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} m^k (m+1)^k = \frac{1}{2m+1} ((m+1)^{n+1} - (-m)^{n+1}), \tag{4.1}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} m^k (m+1)^k = (m+1)^n + (-m)^n. \tag{4.2}$$

We now give a bijective proof of (4.1) and (4.2) assuming that  $m$  is a positive integer. Obviously this is sufficient to prove their validity.

- For any positive integer  $n$ , let  $[n] := \{1, \dots, n\}$ . Given  $n > 1$ , let  $\mathcal{S}$  be the set of all triples  $(A; f, g)$  such that  $A$  is a subset of  $[n - 1]$  without consecutive integers,  $f: A \rightarrow [m]$  and  $g: A \rightarrow [m + 1]$  are two mappings (or colorings). By Lemma 3.4 the left-hand side of (4.1) is the cardinality of  $\mathcal{S}$ .



A chain is a set of consecutive integers, the cardinality being called its length. Let  $X$  be a set of integers. A chain  $Y \subseteq X$  is called *maximal* if there is no other chain  $Y'$  in  $X$  such that  $Y \subset Y'$ . It is clear that  $X$  can be decomposed uniquely as a union of its disjoint maximal chains. Let  $\mathcal{T}$  be the set of all pairs  $(X; h)$  where  $X \subseteq [n]$  such that the maximal chain containing  $n$  in  $X$  (if exists) is of *even* length and  $h: X \rightarrow [m]$  is a mapping. Since the number of all pairs  $(X; h)$  with  $X \subseteq [n]$  and  $h: X \rightarrow [m]$  is equal to  $(m + 1)^n$ , the number of all such pairs  $(X; h)$  with the maximal chain containing  $n$  being of even length, say  $2k$ , is given by

$$m^{2k}(m + 1)^{n-2k-1} = m^{2k}(m + 1)^{n-2k} - m^{2k+1}(m + 1)^{n-2k-1}$$

if  $2k < n$ , and  $m^n$  if  $2k = n$ . Summing up, the cardinality of  $\mathcal{T}$  equals

$$\sum_{k=0}^{\lfloor n/2 \rfloor} m^{2k}(m + 1)^{n-2k} - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} m^{2k+1}(m + 1)^{n-2k-1} = \sum_{k=0}^n (-m)^k (m + 1)^{n-k},$$

i.e., the right-hand side of (4.1).

It remains to establish a bijection  $\theta: \mathcal{S} \rightarrow \mathcal{T}$ . For each  $(A; f, g) \in \mathcal{S}$ , let  $B = \{i + 1 : i \in A \text{ and } g(i) \in [m]\}$  and define  $\theta(A; f, g) = (X; h)$  by  $X = A \cup B$  and  $h|_A = f$  and  $h(i) = g(i - 1)$  for  $i \in B$ . It is easy to see that  $(X; h) \in \mathcal{T}$ . Conversely, let  $(X; h) \in \mathcal{T}$ , suppose  $X = X_1 \cup \dots \cup X_s$ , where  $X_i$  is a maximal chain of  $X$  for each  $i = 1, \dots, s$ . Write  $X_i = \{x_{i,1}, x_{i,2}, x_{i,3}, \dots\}$  in increasing order. Define the triple  $(A; f, g) \in \mathcal{S}$  by  $A = \bigcup_{i=1}^s \{x_{i,1}, x_{i,3}, x_{i,5}, \dots\}$ ,  $f = h|_A$  and  $g(i) = h(i + 1)$  if  $i + 1 \in X \setminus A$  and  $g(i) = m + 1$  if  $i + 1 \notin X \setminus A$ . Then  $(A; f, g)$  is the unique preimage of  $(X; h)$  under the mapping  $\theta$ . This completes the proof of (4.1).

- Next consider the cyclic group  $\mathbb{Z}_n = \{0, 1, \dots, n - 1\}$ . Let  $\mathcal{U}$  be the set of triples  $(A; f, g)$ , where  $A$  is a subset of  $\mathbb{Z}_n$  without consecutive elements of  $\mathbb{Z}_n$ ,  $f: A \rightarrow [m]$  and  $g: A \rightarrow [m + 1]$  are two mappings. By Lemma 3.4 the left-hand side of (4.2) is equal to the cardinality of  $\mathcal{U}$ .

Let  $\mathcal{V}$  be the set of all pairs  $(X; h)$  where  $X \subseteq \mathbb{Z}_n$  and  $h: X \rightarrow [m]$  is a mapping. We define a mapping  $\varphi: \mathcal{U} \rightarrow \mathcal{V}$  as follows.

For each  $(A; f, g) \in \mathcal{U}$ , let  $B = \{i + 1 : i \in A \text{ and } g(i) \in [m]\}$ ,  $X = A \cup B$ ,  $h|_A = f$  and  $h(i) = g(i - 1)$  for  $i \in B$ . Then  $\varphi(A; f, g) = (X; h) \in \mathcal{V}$ . Conversely, each  $(X; h) \in \mathcal{V}$  with  $X \subsetneq \mathbb{Z}_n$  has a unique preimage under the mapping  $\varphi$ . However each  $(\mathbb{Z}_n; h) \in \mathcal{V}$  has no preimage if  $n$  is odd, and has two preimages if  $n$  is even:  $(A_1; f_1, g_1)$  and  $(A_2; f_2, g_2)$ , where  $A_1 = \{0, 2, 4, \dots, n - 2\}$ ,  $A_2 = \{1, 3, 5, \dots, n - 1\}$ ,  $f_1(i) = h(i)$  and  $g_1(i) = h(i + 1)$  for  $i \in A_1$ ;  $f_2(i) = h(i)$  and  $g_2(i) = h(i + 1)$  for  $i \in A_2$ . Thus, the cardinality of  $\mathcal{U}$  is equal to  $(m + 1)^n + (-m)^n$ . This completes the proof of (4.2).

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