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String Graphs. I. The Number of Critical Nonstring Graphs Is Infinite

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String graphs (intersection graphs of curves in the plane) were originally studied in connection with *RC*-circuits. The family of string graphs is closed in the induced minor order, and so it is reasonable to study critical nonstring graphs (nonstring graphs such that all of their proper induced minors are string graphs). The question of whether there are infinitely many nonisomorphic critical nonstring graphs has been an open problem for some time. The main result of this paper settles this question. In a later paper of this series we show that recognizing string graphs is NP-hard.

Intersection graphs of curves in the plane were introduced by Sinden [16] and were also studied in [1]. The notion of "string graphs" was first used by Graham while posing the problem of their characterization [7]. We have proved in [9], besides other results, that the smallest possible nonstring graph has 12 vertices and that complements of planar graphs are string graphs.

The notations and basic definitions used in this paper are presented in Section 1, which also contains a brief review of the concept of the class of forbidden (induced) minors for a given class of graphs. In Section 2, several equivalent definitions of string graphs are given as well as relationships between string graphs and several other intersection-defined classes of graphs. Some subclasses of the class of string graphs that are defined by imposing further topological constraints are discussed in Section 3. The main result of the paper (Theorem 4) is proved in Section 4.

1. PRELIMINARIES

All graphs considered are finite, undirected, and without loops or multiple edges. The vertex set and the edge set of a graph G are denoted by

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$V(G)$ and $E(G)$, respectively. Edges are considered to be two-element subsets of the vertex set. All classes of graphs we deal with are closed under graph isomorphism.

If G is a graph and $A \subset V(G)$, we denote by $G-A$ the graph obtained from G by deleting the vertices of A. We write $G-v$ instead of $G-\{v\}$. The subgraph of G induced by A is denoted by $G|A$ (and so $G|A = G-(V(G)-A)$).

If G is a graph and $e \in E(G)$, we denote by $G \cdot e$ the graph obtained from G by contracting the edge e. If $A = \{e_1, e_2, ..., e_k\} \subset E(G)$, we define $G \cdot A = (G \cdot \{e_1, ..., e_{k-1}\}) \cdot e_k$ recursively.

We say that G is a *subgraph* of H (denoted by $G \subset H$) if $V(G) \subset V(H)$ and $E(G) \subset E(H)$, while G is an *induced subgraph* of H (denoted by $G \leqslant H$) if $G = H - A$ for a suitable $A \subset V(H)$. We write $G \sqsubset H$ when $G \subset H$ and $V(G) = V(H)$.

We say that G is an *induced minor* of H if $G = H' \cdot A$ for suitable $H' \leqslant H$ and $A \subset E(H')$ (i.e., when G can be obtained from H by vertex deletions and edge contractions), while G is a *minor* of H if $G = H' \cdot A$ for suitable $H' \subset H$ and $A \subset E(H')$ (i.e., when G can be obtained from H by vertex deletions, edge deletions, and edge contractions). We write $G \prec H$ (resp. $G \lll H$) when G is isomorphic to a minor (resp. to an induced minor) of H. (We write $G \cong H$ when G is isomorphic to H.)

Suppose M is a minor closed class of graphs.[1] Then M can be characterized by forbidden minors in the following way. Let $F(M)$ be the class of all graphs G not belonging to M such that every proper minor of G is in M. Put $\mathrm{Forb}(M) = F(M)/\cong$; we can imagine that $\mathrm{Forb}(M)$ arises by taking one representant of every isomorphism class of $F(M)$ (thus $\mathrm{Forb}(M) \subset F(M)$, the graphs in $\mathrm{Forb}(M)$ are pairwise nonisomorphic, and every graph in $F(M)$ is isomorphic to some graph in $\mathrm{Forb}(M)$). Then a graph H is in the complement of M if and only if $G \prec H$ for a suitable $G \in \mathrm{Forb}(M)$. It follows from the Robertson–Seymour theorem (formerly known as the Wagner conjecture) [15] that $\mathrm{Forb}(M)$ is finite for every minor closed class M. Another deep result of Robertson and Seymour says that every such class is recognizable in polynomial time [14].

Let us now consider the induced minor order of graphs. If M is an induced minor closed class of graphs, M can be characterized by forbidden induced minors. (Let $Fi(M)$ be the class of all graphs G not belonging to M such that every proper induced minor of G is in M, and let $\mathrm{Forbi}(M) = Fi(M)/\cong$, similarly as above. Then a graph H is in the complement of M if and only if $G \lll H$ for a suitable $G \in \mathrm{Forbi}(M)$.) If M is also minor closed, it follows from the finiteness of $\mathrm{Forb}(M)$ that $\mathrm{Forbi}(M)$ is finite as

[1] A class M is said to be minor closed if $G \prec H$ and $H \in M$ imply $G \in M$. Similarly, M is induced minor closed if $G \lll H$ and $H \in M$ imply $G \in M$.

well (one can easily show that $\mathrm{Forbi}(M) \subset \{G \mid H \sqsubset G$ for a suitable $H \in \mathrm{Forb}(M)\}$). However, it is known that Forbi(M) may be infinite if M is not minor closed, and it is always interesting to determine the cardinality of Forbi(M) for an induced minor closed class M.

Note that the connection between the size of Forbi(M) and the existence of a polynomial recognition algorithm for M is not as clear as in the case of the minor order. If Forbi(M) is infinite, M may be polynomially recognizable (e.g., the classes $\mathcal{DOS}tring$ and $\mathcal{SOS}tring$ defined in Section 3). It is also possible that membership in M may be NP-complete or even undecidable [12]. Moreover, it is not known whether for every graph H, there is a polynomial algorithm determining whether $H \lll G$ holds for an input graph G. If this were true, the finiteness of Forbi(M) would imply the existence of a polynomial recognition algorithm for M. However, the existence of a graph H such that testing $H \lll G$ is NP-complete is not impossible.

The class of all string graphs is induced minor closed but not minor closed. Hence the question of the size of Forbi($\mathcal{S}tring$) was raised and remained open for some time. We prove that Forbi($\mathcal{S}tring$) is infinite. As mentioned above, this does not say anything about the computational complexity of recognizing string graphs. However, we prove this to be NP-hard in the companion paper [11].

2. Definitions of String Graphs

String graphs were first defined by Sinden [16] as intersection graphs of curves in the plane. It is also observed in [16] that intersection graphs of connected regions in the plane describe the same class of graphs. Let us state the definition explicitly:

Definition 1. A graph G is said to be a string graph if there exists a system R of curves (called strings) in the plane such that the intersection graph $I(R) = (R, \{\{r, s\} \mid r \neq s, r, s \in R, r \cap s \neq \emptyset\})$ is isomorphic to G. We call R a *string representation* of G. The class of all string graphs is denoted by $\mathcal{S}tring$.

Remarks. By a curve we mean a homeomorphic image of a closed interval, i.e., a set of points $c = \{h(x) \mid x \in [0, 1]\}$ in the plane E_2 such that $h\colon [0, 1] \to E_2$ is a homeomorphism. Note that we allow multiple intersections of the curves, represented by single edges in the intersection graph.

As already mentioned, if we replace the word "curves" in Definition 1 by "connected regions" or "arc-connected sets," we obtain exactly the same class of intersection graphs. It follows that $\mathcal{S}tring$ is closed under edge contractions and hence under induced minors.

On the other hand it is slightly surprising that every graph is the intersection graph of a system of connected (not necessarily arc-connected) sets in the plane [17].[2]

Since we consider string representations of finite graphs, it is not difficult to see the following

LEMMA 1. *Every string graph has a string representation satisfying the following*:

(i) *each string is a piecewise linear curve consisting of a finite number of straight line segments,*

(ii) *any two strings share a finite number of common intersecting points,*

(iii) *no three or more strings pass through the same point, and*

(iv) *every string representing a vertex of degree* $\leqslant 2$ *shares just one intersecting point with each of its neighbours.*

A purely combinatorial definition of string graphs follows from Lemma 1 and the remarks after Definition 1.

THEOREM 1 [9]. *String graphs are exactly the intersection graphs of paths* (*subtrees, connected subgraphs*) *in planar graphs.*

Remark. Obviously, $\mathscr{String}$ contains several formerly defined and studied classes of graphs, such as, e.g., interval graphs, permutation graphs, circle graphs, and intersection graphs of convex sets in the plane. Theorem 1 suggests how to include string graphs into one fragment of the hierarchy of types of intersection-defined classes of graphs:

Let the class of all (finite) paths, trees, connected planar graphs, and connected graphs be denoted by $\mathscr{P}$, $\mathscr{T}$, $\mathscr{P\ell}$, $\mathscr{G\imath}$, respectively. For $X, Y \in \{\mathscr{P}, \mathscr{T}, \mathscr{P\ell}, \mathscr{G\imath}\}$, $X \subset Y$, put $IXY = \{G \mid \text{there exist } H \in Y \text{ and a system } R \text{ of subgraphs of } H, R \subset X \text{ such that } G \cong I(R)\}$. Clearly, $IXY \subset IX'Y'$ for $X \subset X'$ and $Y \subset Y'$, and hence

$$\begin{array}{ccccccc} I\mathscr{P}\mathscr{P} & \subset & I\mathscr{P}\mathscr{T} & \subset & I\mathscr{P}\mathscr{P\ell} & \subset & I\mathscr{P}\mathscr{G\imath} \\ & & \cap & & \cap & & \cap \\ & & I\mathscr{T}\mathscr{T} & \subset & I\mathscr{T}\mathscr{P\ell} & \subset & I\mathscr{T}\mathscr{G\imath} \\ & & & & \cap & & \cap \\ & & & & I\mathscr{P\ell}\mathscr{P\ell} & \subset & I\mathscr{P\ell}\mathscr{G\imath} \\ & & & & & & \cap \\ & & & & & & I\mathscr{G\imath}\mathscr{G\imath}. \end{array} \tag{1}$$

[2] A set $S \subset E_2$ is said to be arc connected if for every $x, y \in S$, there exists a curve $c \subset S$ starting in x and ending in y, while S is connected in the topological sense if it cannot be split into two disjoint open subsets.

Now $I\mathscr{P}\mathscr{P}$ are interval graphs, $I\mathscr{T}\mathscr{T}$ are chordal graphs [4], and it is an easy exercise to show that $I\mathscr{P}\mathscr{G}r = \mathscr{G}raphs$, the class of all finite (not necessarily connected) graphs. Therefore (1) may be rewritten

$$
\begin{array}{ccccccc}
\text{Interval graphs} & \subsetneqq & I\mathscr{P}\mathscr{T} & \subsetneqq & \mathscr{S}tring & \subsetneqq & \mathscr{G}raphs \\
 & & \not\cap & & \| & & \| \\
 & & \text{Chordal graphs} & \subsetneqq & \mathscr{S}tring & \subsetneqq & \mathscr{G}raphs \\
 & & & & \| & & \| \\
 & & & & \mathscr{S}tring & \subsetneqq & \mathscr{G}raphs \\
 & & & & & & \| \\
 & & & & & & \mathscr{G}raphs.
\end{array}
\qquad (2)
$$

Note that except for $\mathscr{S}tring$, all of the classes depicted in (2) are known to be polynomially recognizable [3, 4, 5].

3. Some Subclasses of $\mathscr{S}tring$

In this section, we define several subclasses of $\mathscr{S}tring$ by imposing certain constraints on the location of the strings in string representations rather than on their form. It turns out that the strings may again be considered to be curves or arc-connected sets yielding the same classes of intersection graphs.

Definition 2. A system of curves is called

(i) an *outerstring representation* if all the curves lie inside a disc and each curve intersects the boundary of the disc in one of its endpoints,

(ii) a *double outerstring representation* if all the curves lie between two parallel lines and each curve intersects each of the border lines in one of its endpoints. (See Fig. 1).

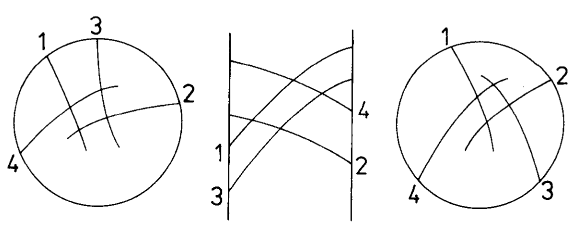

Fig. 1. Left, an outerstring representation of C_4; middle a double outerstring representation of C_4; right, a constrained outerstring representation of C_4.

Remark. Note that a graph has an outerstring representation iff it has a string representation such that all strings penetrate the outer planar region determined by the strings.

DEFINITION 3. We denote by

(i) $\mathcal{O}uterstring$ (in short $\mathcal{OS}tring$) the class of string graphs having an outerstring representation;

(ii) $\mathcal{D}ouble\ \mathcal{O}uterstring$ (in short $\mathcal{DOS}tring$) the class of string graphs having a double outerstring representation;

(iii) $\mathcal{C}onstrained\ \mathcal{O}uterstring$ (in short $\mathcal{COS}tring$) the class of pairs (G, π), where G is a graph and π is a cyclic permutation of its vertices, such that G has an outerstring representation such that the strings meet the outerface of the representation in the (clockwise) order u, $\pi(u)$, $\pi^2(u)$, ... (u is an arbitrary vertex of G); we write $G = (G, \pi)$ when G has vertices $1, 2, ..., n$ and π is such that $\pi(i) = i + 1 \bmod n$ for every i;

(iv) $\mathcal{S}trongly\ \mathcal{O}uterstring$ (in short $\mathcal{SOS}tring$) the class of graphs having a constrained outerstring representation for every cyclic permutation of their vertices.

Remarks. The class $\mathcal{OS}tring$ was defined independently by Abello and Fellows [2] (and we borrow the notation from them), $\mathcal{DOS}tring$ coincides with the class of function graphs [6], and $\mathcal{COS}tring$ coincides with the "constrained case" of Sinden [16]. It is straightforward to prove that the classes $\mathcal{OS}tring$, $\mathcal{DOS}tring$, and $\mathcal{SOS}tring$ are induced minor closed.

The class $\mathcal{DOS}tring$ was independently characterized in [6, 9] (in [6] implicitly in the concept of the function graphs):

THEOREM 2. *A graph G is in $\mathcal{DOS}tring$ if and only if its complement $\bar{G}$ is a comparability graph.*[3]

Note that since comparability graphs are recognizable in polynomial time [5, 13], double outerstring graphs are also easy to recognize. We have proved other properties of $\mathcal{OS}tring$ and $\mathcal{DOS}tring$ in [9]. Namely, a graph G is in $\mathcal{OS}tring$ iff every graph H which admits a partition $V(H) = V_1 \cup V_2$, with $H | V_1 \cong G$ and $H | V_2$ being a complete graph, is a string graph. Similarly, G is in $\mathcal{DOS}tring$ iff every graph H which admits a partition $V(H) = V_1 \cup V_2 \cup V_3$, with $H | V_1 \cong G$ and $H | V_2$ and $H | V_3$ being complete graphs, is a string graph. However, this does not yield a

[3] A comparability graph is a graph expressing the comparability relation between the elements of a partially ordered set.

polynomial-time characterization. Actually, while membership in $\mathscr{String}$ is NP-hard and membership in $\mathscr{DOString}$ is decidable in polynomial time, nothing is known about the computational complexity of the recognition of outerstring graphs.

We conclude this section with a characterization of the class $\mathscr{SOString}$, which also yields a polynomial-time recognition algorithm.

THEOREM 3. *A graph G is in $\mathscr{SOString}$ if and only if its complement $\bar{G}$ is chordal.*[4]

Proof. (1) Suppose $\bar{G}$ is not chordal. Then there exist vertices $v_1, v_2, ..., v_k$, $k \geqslant 4$, such that $\{v_i, v_j\} \in E(G)$ iff $|i-j| > 1$ (subtraction modulo k). Denote $G_k = G \mid \{v_1, v_2, ..., v_k\}$. Let π_k be the cyclic permutation of $v_1, ..., v_k$ with $\pi_k(v_i) = v_{i+1}$ for $i = 1, 2, ..., k$ (addition modulo k) and let π be any cyclic permutation of $V(G)$ satisfying $\pi(v_i) = v_{i+1}$ for $i = 1, 2, ..., k-1$. For $k \geqslant 4$, $(G_k, \pi_k) \notin \mathscr{COString}$ ([16]; cf. also Lemma 4 in the next section). But every constrained outerstring representation of (G, π) would provide a constrained outerstring representation of (G_k, π_k). Hence $(G, \pi) \notin \mathscr{COString}$.

(2) We prove by induction on the number of vertices of G that if $\bar{G}$ is chordal, (G, π) is in $\mathscr{COString}$ for every cyclic permutation π of its vertices. The statement is obvious if G has $\leqslant 3$ vertices.

Let G have at least 4 vertices. It is well known that a chordal graph contains a vertex such that its neighbours form a complete subgraph. So there is $v \in V(G)$ such that the set $A = \{u \mid \{u, v\} \notin E(G)\}$ is independent in G. Denote $G' = G - v$ and put

$$\pi'(u) = \begin{cases} \pi(u) & \text{if } \pi(u) \neq v \\ \pi(v) & \text{if } \pi(u) = v. \end{cases}$$

By induction hypothesis, G' is in $\mathscr{SOString}$ and so it has an outerstring representation with respect to the permutation π'. Without loss of generality we may suppose that the curves representing the vertices of A are straight (note that they are pairwise disjoint). A constrained outerstring representation of (G, π) is now obtained by inserting the curve v so that it follows the boundary of the disc (and is sufficiently close to it) and avoids intersections with the curves representing the vertices of A (see the illustration in Fig. 2). ∎

[4] A graph is chordal iff it does not contain an induced cycle of length greater than 3.

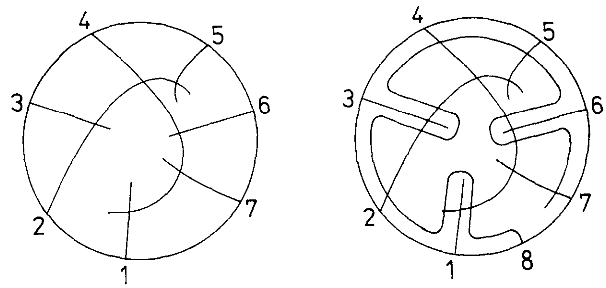

FIG. 2. An illustration with $v = 8$ and $A = \{1, 3, 6\}$.

4. INFINITELY MANY CRITICAL NONSTRING GRAPHS

Since $\mathcal{String}$ is induced minor closed but not minor closed, the question of the size of Forbi($\mathcal{String}$) is raised. The graphs from Forbi($\mathcal{String}$), i.e., the critical nonstring graphs, are called minnegs in accordance with [9]. Recall that to prove that a graph G is a minneg, it suffices to show that $G \notin \mathcal{String}$ and that every vertex deletion and edge contraction in G yields a string graph. The task of this section is to prove that the number of minnegs is infinite. First we state several lemmas.

LEMMA 2. *Let G be a connected graph with the vertex set* $1, 2, ..., n$ *and let the graph C^*G be defined*

$$V(C^*G) = V(G) \cup \bigcup_{i=1}^{n} \{v_i, w_i, z_i\},$$

$$E(C^*G) = E(G) \cup \bigcup_{i=1}^{n} \{\{i, v_i\}, \{v_i, w_i\}, \{w_i, z_i\}, \{w_i, z_{i+1}\}\}$$

(addition in subscripts modulo n). Then $C^*G \in \mathcal{String}$ if and only if $G \in \mathcal{COString}$.

Proof. (1) Suppose $C^*G \in \mathcal{String}$. By Lemma 1, C^*G has a string representation in which the strings w_i, z_i, $i = 1, 2, ..., n$, are forming a cycle (called a bounding cycle) dividing the plane into two regions. Since the strings $i \in V(G)$ do not cross the bounding cycle, we may suppose that all of them lie inside (G is connected). The strings v_i, $i = 1, 2, ..., n$, are tying them to the bounding cycle with respect to the order 1 to n, and so the unions of the strings i and v_i, $i = 1, 2, ..., n$, form a constrained outerstring representation of G (here the bounding cycle plays the role of the boundary of the disc in Definition 2(i)).

(2) The reverse implication is now straightforward. (See Fig. 3.) ∎

LEMMA 3. (i) *Let G be a graph and v one of its vertices such that* $\deg v = 1$. *Then* $G \in \mathcal{S}\mathit{tring}$ *iff* $G - v \in \mathcal{S}\mathit{tring}$.

(ii) *Let G be a graph and u, v two of its vertices such that* $\deg u = \deg v = 2$ *and* $\{u, v\} \in E(G)$. *Then* $G \in \mathcal{S}\mathit{tring}$ *iff* $G \cdot \{u, v\} \in \mathcal{S}\mathit{tring}$. ∎

LEMMA 4. *Let C_n denote the cycle of length n on vertices* $1, 2, ..., n$, $n \geq 5$ *(i.e.,* $C_n = (\{1, 2, ..., n\}, \{\{1, 2\}, \{2, 3\}, ..., \{n, 1\}\})$ *and the complement* $\bar{C}_n$ *of* C_n *is connected). Then*

(i) $\bar{C}_n \notin \mathcal{COS}\mathit{tring}$,

(ii) *if π is a cyclic permutation of* $1, 2, ..., n$ *such that* $\pi(i) = i + 2$, $\pi(i + 1) = i + 3$, $\pi(i + 2) = i + 1$ *for a certain i, and* $\pi(j) = j + 1$ *for* $j \neq i, i + 1, i + 2$, *then* $(\bar{C}_n, \pi) \in \mathcal{COS}\mathit{tring}$. *(That is, permuting any two consecutive vertices in $\bar{C}_n$ yields a graph belonging to* $\mathcal{COS}\mathit{tring}$.)

Proof. (i) This is proved already in [16, Theorem 1]. See also Claim 3.

(ii) A constrained outerstring representation of $(\bar{C}_n, \pi)$ is easily constructed, as is illustrated in Fig. 4 (with $i = n$). ∎

LEMMA 5. (i) *For* $n \geq 5$, $C^*\bar{C}_n \notin \mathcal{S}\mathit{tring}$.

(ii) *Deleting any vertex from* $C^*\bar{C}_n$ *yields a string graph.*

(iii) *Contracting every edge e such that* $e \in E(\bar{C}_n)$ *or* $e = \{w_i, z_i\}$ *(resp.* $e = \{w_i, z_{i+1}\}$) *yields a string graph.*

(iv) *For every i,* $C^*\bar{C}_n \cdot \{\{i, v_i\}, \{v_i, w_i\}\} \in \mathcal{S}\mathit{tring}$.

Proof. (i) follows from Lemma 2 and Lemma 4(i).

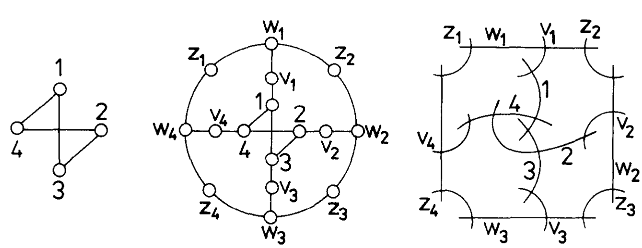

FIG. 3. Left, G; middle, C^*G.

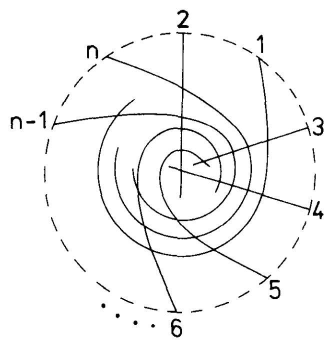

FIGURE 4

We prove (iii) first. If $e=\{i, j\} \in E(\bar{C}_n)$, we have $C^*\bar{C}_n \cdot e = C^*C' \cdot e$, where $C' = (V(\bar{C}_n), E(\bar{C}_n) \cup E')$ and $E' = \{\{i-1, i\}\}$, $E' = \{\{i+1, i\}\}$, or $E' = \{\{i-1, i\}, \{i+1, i\}\}$. In all of these cases the complement of C' is chordal (either a path of length n or a disjoint union of a path of length $n-1$ and one isolated vertex). By Theorem 3, $C' \in \mathcal{SOS}tring$ and hence $C^*C' \in \mathcal{S}tring$ by Lemma 2. Then $C^*\bar{C}_n \cdot e = C^*C' \cdot e \in \mathcal{S}tring$, since edge contraction preserves stringability.

If $e = \{w_i, z_i\}$, a string representation of $C^*\bar{C}_n \cdot e$ is constructed from a constrained outerstring representation of $(\bar{C}_n, \pi)$, where π permutes the vertices i and $i-1$ (cf. Lemma 4(ii)). The edge $\{w_i, w_{i-1}\} \in E(C^*\bar{C}_n \cdot e)$ enables us to switch the order of the ends of the curves w_i, w_{i-1} on the bounding cycle (see the example in Fig. 5).

(ii) We prove that $C^*\bar{C}_n - v \in \mathcal{S}tring$ for every $v \in V(C^*\bar{C}_n)$.

If $v = v_i$ the vertices w_i and z_i have degree 2 in $C^*\bar{C}_n - v$. By Lemma 3(ii), $C^*\bar{C}_n - v \in \mathcal{S}tring$ iff $(C^*\bar{C}_n - v) \cdot \{w_i, z_i\} \in \mathcal{S}tring$. But $(C^*\bar{C}_n - v) \cdot \{w_i, z_i\} = (C^*\bar{C}_n \cdot \{w_i, z_i\}) - v \in \mathcal{S}tring$ by the preceding paragraph. If $v = z_i$, we argue similarly.

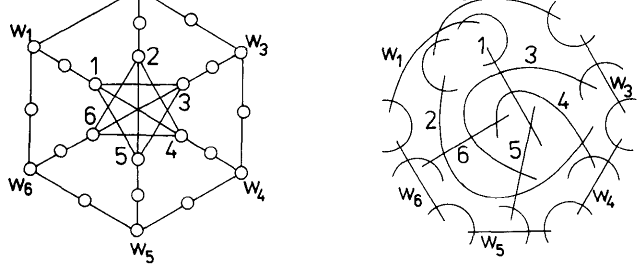

FIG. 5. $C^*\bar{C}_6 \cdot \{w_2, z_2\}$.

If $v = i$ or $v = w_i$, the vertex v has degree 1 in $C^*\bar{C}_n - v$, and the statement follows from Lemma 3(i) and the previous case.

(iv) A string representation of $C^*\bar{C}_n \cdot \{\{i, v_i\}, \{v_i, w_i\}\}$ is constructed from a constrained outerstring representation of $(\bar{C}_n, \pi)$, where π is as in Lemma 4(ii). Since $i \equiv w_i$ in $C^*\bar{C}_n \cdot \{\{i, v_i\}, \{v_i, w_i\}\}$, we can start the curve $i+2$ outside the bounding cycle and cross it through the curve i (see the illustration in Fig. 6 with $i = 1$, $n = 6$). ∎

Now we have proved that $C^*\bar{C}_n$ is almost a minneg. Only the contractions of edges $\{i, v_i\}$ (resp. $\{v_i, w_i\}$) can preserve nonstringability. Hence we may take a maximal set of edges $A_n \subset \{\{i, v_i\} \mid i = 1, 2, ..., n\}$ such that $G_n = C^*\bar{C}_n \cdot A_n$ is a nonstring graph. It follows from Lemma 5 that G_n, $n \geqslant 5$, is an infinite sequence of (pairwise nonisomorphic) minnegs. However, we can describe these minnegs precisely:

PROPOSITION. *Let* $A \subset \{\{i, v_i\} \mid i = 1, 2, ..., n\}$. *Put* $\bar{A} = \{i \mid \{i, v_i\} \in A\}$ *and* $\bar{B} = \{1, 2, ..., n\} - \bar{A}$. *Then the graph* $G_n(A) = C^*\bar{C}_n \cdot A$ *is a minneg if and only if* $\bar{B} = \{j, j+1, k\}$ *for suitable* j *and* $k \neq j-1, ..., j+2$.

Proof.

CLAIM 1. *If* $\bar{B} \subset \{j, j+1, j+2\}$ *for some* j, *then* $G_n(A) \in \mathscr{String}$.

Proof. Let $G_1 = \bar{C}_n - \{j, j+2\}$ and $G_2 = \bar{C}_n - \{j+1\}$. Then the complements of both G_1 and G_2 are chordal, and so both G_1 and G_2 have constrained outerstring representations for all cyclic permutations of their vertices. Thus we can realize G_1 inside the bounding cycle and G_2 outside it; the curves representing the vertices of $\bar{C}_n - \{j, j+1, j+2\}$ cross the bounding cycle and so reach both regions. ∎

CLAIM 2. *If there is no* i *such that* $\{i, i+1\} \subset \bar{B}$, *then* $G_n(A) \in \mathscr{String}$.

Proof. Let $\bar{B} = \{i_1 < i_2 < \cdots < i_m\}$. We may suppose $\bar{B} \neq \emptyset$, since

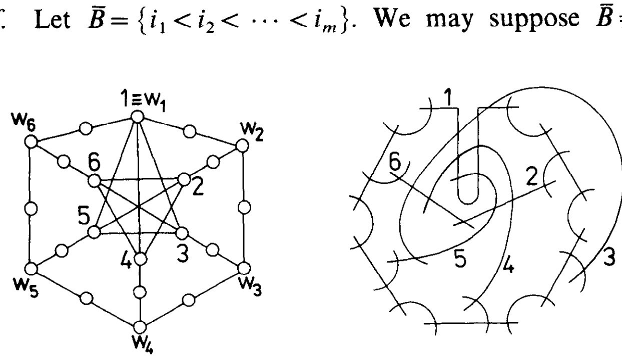

FIG. 6. $C^*\bar{C}_6 \cdot \{\{1, v_1\}, \{v_1, w_1\}\}$.

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otherwise the statement follows from Claim 1. We construct a string representation in three steps. First we place the strings $i_j \in \bar{B}$ inside the bounding cycle so that for every j, all of them meet the region O_j bounded by the strings $i_j, v_{i_j}, w_{i_j}, z_{i_j+1}, w_{i_j+1}, ..., v_{i_{j+1}}, i_{j+1}$ (this is possible, since the strings $i_j \in \bar{B}$ represent a complete subgraph of $G_n(A)$). Then for every $j = 1, 2, ..., m$, we realize the graph $G_n^j = G_n(A) \mid \{i_j, i_j+1, ..., i_{j+1}\} \cup \bar{B}$ inside the region O_j (this is possible since the complement of G_n^j is a disjoint union of a path and isolated vertices and hence is a chordal graph. By Theorem 3, it has a constrained outerstring representation with respect to the order in which the strings meet the region O_j). Finally, we realize the intersections of the strings representing the vertices of $\bar{A}$ outside the bounding cycle (this is possible since all of them cross the bounding cycle and the complement of $G_n \mid \bar{A}$ is chordal). ∎

Claim 3. *Let $n \geqslant 4$ and let H_n be a graph satisfying*

(i) $V(H_n) = \{1, 2, ..., n\}$,

(ii) $\{\{1, i\} \mid i = 3, 4, ..., n-1\} \cup \{\{2, n\}\} \subset E(H_n)$,

(iii) $E(H_n) \cap \{\{i, i+1\} \mid i = 1, 2, ..., n\} = \emptyset$.

Then $H_n \notin \mathscr{COString}$.

Proof. Goes by induction on n.

(1) For $n = 4$, $H_n = \bar{C}_4$ and the statement follows from the Jordan Curve Theorem.

(2) Suppose that no H_k, $k = 4, 5, ..., n$, has a constrained outerstring representation, but some H_{n+1} does. By Lemma 1, there is a string representation of H_{n+1} that has a finite total number of intersecting points of the strings. Consider a constrained outerstring representation R of H_{n+1} that achieves the minimum number of intersecting points among all representations of all graphs H_{n+1} satisfying (i)–(iii). (Note that H_{n+1} is not uniquely defined by the conditions (i)–(iii).) Now shorten the string n so that it misses its last intersecting point. Let us denote the intersection graph of the resulting system R' by H'_{n+1}. We have $E(H'_{n+1}) = E(H_{n+1}) - \{e\}$ for a certain edge $e = \{n, i\} \in E(H_{n+1})$. Since R' has fewer intersecting points than R, H'_{n+1} fails to satisfy one of (i)–(iii). As (i) and (iii) cannot be violated by deleting an edge, it is (ii) which does not hold for H'_{n+1}. Hence $e = \{n, 1\}$. If $\{n, 2\} \in E(H_{n+1})$, it is also $\{n, 2\} \in E(H'_{n+1})$ and $R' \mid \{1, 2, ..., n\}$ is a constrained outerstring representation of some H_n satisfying (i)–(iii), a contradiction. If $\{n, 2\} \notin E(H_{n+1})$, then already $R \mid \{1, 2, n, n+1\}$ provided a constrained outerstring representation of $\bar{C}_4$, again a contradiction. ∎

Remark. It follows from Claim 3 that $\bar{C}_n \notin \mathscr{COString}$ for $n \geqslant 4$. The proof presented above is based on the idea of Sinden's proof of this fact [16, Theorem 1].

CLAIM 4. *If* $\bar{B} = \{j, j+1, k\}$ *for suitable* j, $k \neq j-1, j, j+1, j+2$, *then* $G_n(A) \notin \mathscr{String}$.

Proof. Suppose $G_n(A)$ has a string representation. Without loss of generality we may assume that $j = 1$ and that the string k lies inside the bounding cycle. Since $\{\{1, k\}, \{2, k\}\} \subset E(G_n(A))$, both of strings 1 and 2 must lie inside it as well. We see that each of the strings $3, 4, ..., n-1$ intersects the string 1 inside the bounding cycle. Also the intersecting point of strings 2 and n lies inside, and so inside the bounding cycle; the strings $1, 2, ..., n$ form a constrained outerstring representation of some graph H_n satisfying (i)–(iii), contradicting Claim 3. ∎

The statement of the proposition is now a direct consequence of Claims 1–4 and Lemma 5. ∎

So we have proved

THEOREM 4. *There are infinitely many pairwise nonisomorphic critical nonstring graphs.*

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Note added in proof. Significant progress in related questions was achieved since the submission of the revised version of the paper. First, Middendorf and Pfeiffer gave a new proof of the NP-hardness of string graph recognition [18]. Though not stated there explicitly, their method can be used directly to prove that recognition of outerstring graphs is NP-hard as well. This answers the remark above Theorem 3. Considerable effort was devoted to deciding the computational complexity of the induced minor test. We have finally proved that there indeed is a graph H such that testing $H \lll G$ for an input graph G is NP-complete. The proof will appear in [19].

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