JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **107**, 425–445 (1985)

The Existence of Generalized Solutions for a Class of Quasi-linear Equations of Mixed Type

A. K. AZIZ

Institute for Physical Science and Technology, University of Maryland, and Department of Mathematics, University of Maryland Baltimore County Campus, Baltimore, Maryland 21228

AND

M. SCHNEIDER

Department of Mathematics, University of Karlsruhe, Germany

Submitted by P. D. Lax

1. INTRODUCTION

We consider equations

$$L[u] = k(y)\,u_{xx} + u_{yy} + (a(x, y, u)\,u)_x + (b(x, y, u)\,u)_y + c(x, y, u)\,u = f(x, y, u) \tag{1.1}$$

in a bounded simply connected region G, where $k(y) \gtreqless 0$ for $y \gtreqless 0$ and G is bounded by the curves Γ_0, Γ_1, Γ_2. Hereby Γ_0 is a piecewise smooth curve lying in the half-plane $y > 0$ which intersects the line $y = 0$ in the points $A(-1, 0)$ and $B(0, 0)$. Γ_1 is a piecewise smooth curve through A in $y < 0$ which meets the characteristic of (1.1) issued from B at the point C and Γ_2 consists of the portion CB of the characteristic through B. We assume that Γ_1 either lies in the characteristic triangle formed by the characteristics through A and B (*Frankl problem*) or coincides with the characteristic through A (*Tricomi problem*). We ask for sufficient conditions for the existence of generalized solutions of the problem

$$L[u] = f(x, y, u) \quad \text{in } G, \qquad u|_{\Gamma_0 \cup \Gamma_1} = 0. \tag{1.2}$$

In principle, the case that Γ_1 piecewise has characteristic direction brings no difficulties, but it is not considered.

Remark 1.3. The question arises immediately why in Eq. (1.1) the term

0022-247X/85 $3.00

$c(x, y, u)\, u$ is not combined with the non-linear term $f(x, y, u)$. However, as we shall see, different assumptions on $c(x, y, u)$ and $f(x, y, u)$ are needed for the existence of generalized solutions; e.g., for the function $f(x, y, u)$ a growth condition is essential (Theorem 2.1(iii)) whereas for $c(x, y, u)$ a sign condition is needed (Theorem 2.1(ii)).

To our knowledge there are very few papers in the literature which deal with the question of uniqueness and existence for quasi-linear of mixed type. For uniqueness theorems for the Tricomi problems we refer to the survey paper of Gvazava [3], and the references contained therein, where with use of a maximum principle and under certain growth conditions on $f(x, y, u)$ uniqueness theorems are given.

For existence theorems for generalized solutions for Eq. (2.11) we only know of the paper of Podgaev [8]. In [8] only the special case is treated, in which Γ_1 is not a characteristic and the domain for $y < 0$ is bounded by the curve Γ_1 and the portion CB of the y-axis, where C is the point of intersection of Γ_1 with the y-axis.

2. Notations

If we use Pfaffian forms ([4, 9]) and introduce the operator

$$d_n u = k(y)\, u_x\, dy - u_y\, dx \tag{2.1}$$

Eq. (1.1) becomes

$$\begin{aligned} L[u][dx, dy] = [d, d_n u] + [d, a(x, y, u)\, u\, dy - b(x, y, u)\, u\, dx] \\ + c(x, y, u)\, u[dx, dy] = f(x, y, u)[dx, dy] \end{aligned} \tag{2.2}$$

with

$$u \,|_{\Gamma_0 \cup \Gamma_1} = 0. \tag{2.3}$$

The adjoint boundary conditions for (2.3) [9, p. 248] are

$$\begin{aligned} v|_{\Gamma_0 \cup \Gamma_2} &= 0 \qquad \text{for the Tricomi problem,} \\ v|_{\Gamma_0 \cup \Gamma_1 \cup \Gamma_2} &= 0 \qquad \text{for the Frankl problem.} \end{aligned} \tag{2.4}$$

We introduce the function spaces

$$\begin{aligned} U &= \{u \mid u(x, y) \in C^\infty(\bar{G}),\, u \,|_{\Gamma_0 \cup \Gamma_1} = 0\}, \\ V_T &= \{v \mid v(x, y) \in C^\infty(\bar{G}),\, v \,|_{\Gamma_0 \cup \Gamma_2} = 0\}, \\ V_F &= \{v \mid v(x, y) \in C^\infty(\bar{G}),\, v \,|_{\Gamma_0 \cup \Gamma_1 \cup \Gamma_2} = 0\}. \end{aligned} \tag{2.5}$$

If $u \in U$ and $v \in V_T$ or $v \in V_F$ the formal application of Green's theorem to (2.2) gives

$$\begin{aligned}\iint_G vL[u][dx, dy] &= \int_{\partial G} (vd_n u + v(au\,dy - bu\,dx)) \\ &\quad - \iint_G \{k(y)\, u_x v_x + u_y v_y + a(x, y, u)\, uv_x \\ &\quad + b(x, y, u)\, uv_y - c(x, y, u)\, uv\}[dx, dy] \\ &= \iint_G f(x, y, u)\, v[dx, dy]. \end{aligned} \tag{2.6}$$

From (2.5) we conclude

$$\int_{\partial G} (v\, d_n u + v(au\,dy - bu\,dx)) = 0$$

for $u \in U$ and $v \in V_T$ or $v \in V_F$. Thus we have formally for the problem (2.2) and (2.3) the identity

$$\begin{aligned}B[u, v] = &- \iint_G \{k(y)\, u_x v_x + u_y v_y + a(x, y, u)\, uv_x \\ &+ b(x, y, u)\, uv_y - c(x, y, u)\, uv\}[dx, dy] \\ = &\iint_G f(x, y, u)\, v(dx, dy] \qquad \text{for all } u \in U,\ v \in V_T \text{ or } v \in V_F. \end{aligned} \tag{2.7}$$

As is known, (2.7) gives the basis for the definition of generalized solutions. To this end we introduce the spaces $H_1(bd, k)$ and $H_1(bd^*, k)$, which are obtained by the completion of the function spaces (2.5) with respect to a weighted norm involving the function $k(y)$. More precisely we denote

$$\begin{aligned} H_1(bd, k) &:= \overline{\{u \mid u \in U\}}\ \|u\|_{1,k} = \left(\iint_G \{|k|\, u_x^2 + u_y^2 + u^2\}\, dx\, dy\right)^{1/2} \\ H_1(bd^*, k) &:= \overline{\{v \mid v \in V_T \text{ or } v \in V_F\}}\ \|v\|_{1,k}. \end{aligned} \tag{2.8}$$

We observe the fact that for the Tricomi problem $H_1(bd^*, k)$ is the closure of the space V_T while for the Frankl problem it is the closure of V_F. Nevertheless in the sequel we call the space $H_1(bd^*, k)$ the space adjoint to the space $H_1(bd, k)$.

DEFINITION 2.9. A function $u \in H_1(bd, k) \cap L_\nu(G)$ $(\nu \geqslant 2)$ is called a *generalized solution* of (1.2) if

$$B[u, v] = \iint_G f(x, y, u)\, v\, dx\, dy \qquad \text{for all } v \in H_1(bd^*, k) \cap L_{\nu'}(G), \quad (2.10)$$

where ν and ν' are chosen in such a way that all integrals in (2.10) exist.

We now give an existence theorem for a special case of Eq. (1.1). The proof of this theorem in the following sections indicates the method of approach in dealing with the more general equation (1.1).

THEOREM 2.1. *If*

(i) $k(y) \in C^1_-(\bar{G})$; $k(y) \gtreqqless 0$ *for* $y \gtreqqless 0$; $yk'(y) \geqslant k(y)$ *for* $y \geqslant 0$; $k'/|k|^{1/2} \in L_{sp/(p-s)}(G)$ *with* $p = p + 2$ *and* $s \in (1, 2)$,

(ii) $c_1(x, y) \in C^1(\bar{G})$; $(\alpha + 1)\, c_1(x, y) + xc_{1_x}(x, y) + \alpha y c_{1_y}(x, y) \leqslant 0$ *for* $y \geqslant 0$, $c_1(x, y) + xc_{1_x}(x, y) \leqslant 0$ *for* $y \leqslant 0$, $\alpha \in (\frac{1}{2}, 1)$,

(iii) $f(x, y, u) = |k|^{1/2} f_1(x, y, u)$ *and continuous with respect to* u; $\|f_1\|^2_{L_2(G)} \leqslant K_0 + K_1 \|u\|^{p/2}_{L_p(G)}$, $p = \rho + 2$, $c_1|_{\Gamma_2} \leqslant 0$,

(iv) $k(y)\, n_1^2 + n_2^2 |_{\Gamma_1} \geqslant 0$, $n_1 |_{\Gamma_1} > 0$; $xn_1 + \alpha y n_2 |_{\Gamma_0} \leqslant 0$, *where* (n_1, n_2) *is the inward normal vector*,

then there exists a generalized solution of the boundary value problem

$$\begin{gathered} L[u] = k(y)\, u_{xx} + u_{yy} + c_1(x, y)\, u - |u|^\rho u = f(x, y, u), \\ \rho > 0,\ u|_{\Gamma_0 \cup \Gamma_1} = 0; \end{gathered} \qquad (2.11)$$

i.e., there exists a $u \in H_1(bd, k) \cap L_p(G)$, $p = \rho + 2$, *such that*

$$B[u, v] = -\iint_G \{k(y)\, u_x v_x + u_y v_y - c_1(x, y)\, uv + |u|^\rho uv\}\, dx\, dy$$

$$= \iint_G f(x, y, u)\, v\, dx\, dy \text{ for all } v \in H_1(bd^*, k) \cap L_p(G).$$

It can be seen that under our assumptions the integrals on the left side of (2.12) exist. The existence of the integral $\iint_G f(x, y, u)\, v\, dx\, dy$ follows from (iii) and (4.15).

3. PRELIMINARY LEMMA

LEMMA 3.1. *Suppose*

(i) $k(y) \in C^0(\bar{G}) \cap C^1(G^+)$; $k(y) \gtreqqless 0$ *for* $y \gtreqqless 0$, $yk'(y) \geqslant k(y)$ *for* $y \geqslant 0$,

(ii) $k(y)\,n_1^2 + n_2^2 \mid_{\Gamma_1} \geqslant 0$, $n_1 \mid_{\Gamma_1} > 0$; $xn_1 + \alpha y n_2 \mid_{\Gamma_0} \leqslant 0$, *where* (n_1, n_2) *is the inward normal vector*, $\alpha \in (\frac{1}{2}, 1)$.

(iii) $\{\psi^n\}_{n \in \mathbb{N}} \subset V_{T(F)} \cap H_1(bd^*, k) \cap L_p(G)$, $p = \rho + 2$, *is a complete system of functions,*

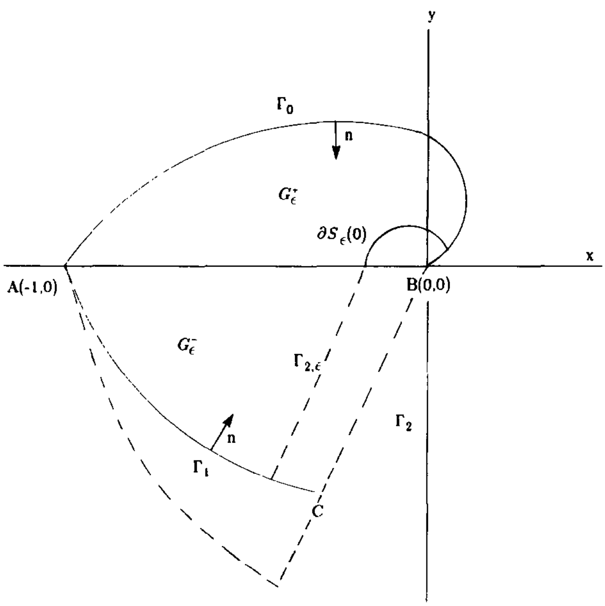
then there exist solutions $\varphi^n \in H_1(bd, k) \cap L_\nu(G)$, $n \in \mathbb{N}$, *for all* $\nu \geqslant 2$ *of the boundary value problems*

$$\begin{aligned} l(\varphi^n) = \alpha^1 \varphi^n_x + \alpha^2 \varphi^n_y = \psi^n, \\ \varphi^n \mid_{\Gamma_0 \cup \Gamma_1} = 0, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \alpha_1 = x,\ \alpha^2 = \alpha y \qquad & \textit{for} \quad y \geqslant 0 \\ = 0 \qquad & \textit{for} \quad y \leqslant 0. \end{aligned}$$

Proof. For fixed $n \in \mathbb{N}$, (3.2) is a partial differential equation whose characteristics by condition (ii) cannot intersect the curves Γ_0 and Γ_1 more than once. The solution φ^n of this equation is a smooth function except, perhaps, at the point $B(0, 0)$. We therefore remove from G^+ a circle with center at B and radius $\varepsilon > 0$ sufficiently small and denote: $\bar{s}_\varepsilon(0) = \{x, y \mid x^2 + y^2 \leqslant \varepsilon^2\} \cap \bar{G}_+$ and $G_\varepsilon^+ = G^+ \setminus \bar{s}_\varepsilon(0)$. Similarly we remove an ε-stripe near Γ_2 from G^- and denote the remaining part of G^- by G_ε^-;



In G_ε^+, omitting the index n, we have $l(\varphi) = x\varphi_x + \alpha y \varphi_y = \psi$, where $\psi \in C^\infty(\bar{G})$. A calculation shows for $v \geqslant 2$

$$\begin{aligned} x\varphi_x \varphi\, |\varphi|^{v-2} &= \frac{1}{v}\,[x\,|\varphi|^v]_x - \frac{1}{v}\,|\varphi|^v, \\ \alpha y \varphi_y \varphi\, |\varphi|^{v-2} &= \frac{\alpha}{v}\,[y\,|\varphi|^v]_y - \frac{\alpha}{v}\,|\varphi|^v, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &-\iint_{G_\varepsilon^+} l(\varphi)\,|\varphi|^{v-2}\,\varphi\,dx\,dy \\ &\quad = \frac{1+\alpha}{v} \iint_{G_\varepsilon^+} |\varphi|^v\,dx\,dy - \frac{1}{v}\int_{\partial G_\varepsilon^+} \{x\,|\varphi|^v\,dy - dy\,|\varphi|^v\,dx\} \\ &\quad \geqslant \frac{1+\alpha}{v} \iint_{G_\varepsilon^+} |\varphi|^v\,dx\,dy = \frac{1+\alpha}{v}\,\|\varphi\|^v_{L_v(G_\varepsilon^+)} \end{aligned} \tag{3.4}$$

because the boundary integral over Γ_0 is zero and along $\partial s_\varepsilon(0) \cap G^+$, non-negative.

In G_ε^-, we have $l(\varphi) = x\varphi_x = \psi$ and conclude in the same way

$$-\iint_{G_\varepsilon^-} l(\varphi)\,|\varphi|^{v-2}\,\varphi\,dx\,dy \geqslant \frac{1}{v}\,\|\varphi\|^v_{L_v(G_\varepsilon^-)}, \qquad v \geqslant 2. \tag{3.5}$$

Letting

$$\left.\begin{aligned} \varphi_\varepsilon &:= \varphi && \text{in} \quad G_\varepsilon^+ \cup G_\varepsilon^- \\ &:= 0 && \text{in} \quad G \setminus \{G_\varepsilon^+ \cup G_\varepsilon^-\} \end{aligned}\right\}, \quad \psi_\varepsilon := l(\varphi_\varepsilon) \tag{3.6}$$

from (3.4) and (3.6) we have

$$\frac{1}{v}\,\|\varphi_\varepsilon\|^v_{L_v(G)} \leqslant -\iint_G l(\varphi_\varepsilon)\,|\varphi_\varepsilon|^{v-2}\,\varphi_\varepsilon\,dx\,dy. \tag{3.7}$$

Using the Hölder inequality and the fact that ψ is a smooth function we get $(p = v,\ q = v/(v-1))$

$$\begin{aligned} \frac{1}{v}\,\|\varphi_\varepsilon\|^v_{L_v(G)} &\leqslant \|\psi_\varepsilon\|_{L_v(G)}\,\|\varphi_\varepsilon\|^{v-1}_{L_v(G)}, \\ \|\varphi_\varepsilon\|_{L_v(G)} &\leqslant v\,\|\psi_\varepsilon\|_{L_v(G)} \leqslant v\,\|\psi\|_{L_v(G)} \leqslant c(v), \end{aligned} \tag{3.8}$$

for all $\varepsilon > 0$. The limit as $\varepsilon \to 0$ gives

$$\varphi \in L_\nu(G), \qquad \text{i.e.,} \quad \varphi^n \in L_\nu(G) \text{ for all } \nu \geqslant 2. \tag{3.9}$$

Now we have to show that the functions $\varphi^n \in L_\nu(G)$, $\nu \geqslant 2$, belong to the space $H_1(bd, k) \cap L_\nu(G)$. For a function $\psi \in V_{T(F)} \cap H_1(bd^*, k) \cap L_p(G)$ (omitting the index n) we introduce the operator

$$T\psi := k(y)\,\psi_{xx} + \psi_{yy}. \tag{3.10}$$

In G_ε^+, we get

$$\begin{aligned} 2\iint_{G_\varepsilon^+} \varphi T\psi[dx, dy] &= 2\iint_{G_\varepsilon^+} \varphi[d, d_n\psi] \\ &= 2\int_{\partial G_\varepsilon^+} \{\varphi\, d_n\psi - \psi\, d_n\varphi\} \\ &\quad + 2\iint_{G_\varepsilon^+} (\alpha^1\varphi_x + \alpha^2\varphi_y)(k(y)\,\varphi_{xx} + \varphi_{yy})[dx, dy] \\ &=: I_{1,\varepsilon}^+ + I_{2,\varepsilon}^+ \end{aligned} \tag{3.11}$$

where we know

$$\varphi\,|_{\Gamma_0 \cup \Gamma_1} = 0, \qquad \psi\,|_{\Gamma_0 \cup \Gamma_2} = 0, \qquad \text{for the Tricomi problem,} \tag{3.12}$$

and

$$\psi\,|_{\Gamma_0 \cup \Gamma_1 \cup \Gamma_2} = 0 \qquad \text{for the Frankl problem,} \tag{3.13}$$

respectively.

Remark 3.14. We have introduced $k(y)$ in (3.10) in order to show the existence of the weighted norm $\|\varphi\|_{1,k} = \|\varphi\|_{H_1(bd,k)} = (\iint_G \{|k|\,\varphi_x^2 + \varphi_y^2 + \varphi^2\}\, dx\, dy)^{1/2}$.

From (3.12) and (3.13) we conclude

$$I_{1,\varepsilon}^+ = 2\int_{x=-1}^{-\varepsilon} (\psi\varphi_y - \varphi\psi_y)\, dx + 2\int_{\partial s_\varepsilon(0) \cap G^+} (\varphi\, d_n\psi - \psi\, d_n\varphi). \tag{3.15}$$

Using Green's theorem [9, p. 256] we have

$$I_{2,\varepsilon}^+ = \int_{\partial G_\varepsilon^+} \Omega - \iint_{G_\varepsilon^+} \{A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2 + D\varphi^2\}[dx, dy] \tag{3.16}$$

where

$$\begin{aligned} \Omega &= (\alpha^1 k(y)\,\varphi_x^2 - \alpha^1\varphi_y^2 + 2\alpha^2 k(y)\,\varphi_x\varphi_y)\,dy \\ &\quad + (\alpha^2 k(y)\,\varphi_x^2 - \alpha^2\varphi_y^2 - 2\alpha^1\varphi_x\varphi_y)\,dx, \\ A &= k(y)(\alpha_x^1 - \alpha_y^2) - \alpha^2 k'(y), \\ B &= k(y)\,\alpha_x^2 + \alpha_y^1, \\ C &= -(\alpha_x^1 - \alpha_y^2); \qquad D = 0. \end{aligned} \tag{3.17}$$

Now using the assumptions (i) and (ii) it follows that

$$\begin{aligned} &-\iint_{G_\varepsilon^+} \{A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2 + D\varphi^2\}[dx, dy] \\ &\qquad \geqslant \iint_{G_\varepsilon^+} \{k(2\alpha - 1)\,\varphi_x^2 + (1-\alpha)\,\varphi_y^2\}[dx, dy] \geqslant 0, \\ &\int_{\Gamma_{0,\varepsilon}} \Omega = -\int_{\Gamma_{0,\varepsilon}} (k\varphi_x^2 + \varphi_y^2)(\alpha^1 n_1 + \alpha^2 n_2)\,ds \geqslant 0, \\ &\qquad\qquad \int_{x=-1}^{-\varepsilon} \Omega = -2\int_{x=-1}^{-\varepsilon} \psi\varphi_y\,dx, \\ &\int_{\partial s_\varepsilon(0)\cap G^+} \Omega = \int_{\partial s_\varepsilon(0)\cap G^+} \{(k\varphi_x^2 + \varphi_y^2)(\alpha^1 n_1 + \alpha^2 n_2)\,ds + 2\psi\,d_n\varphi\} \\ &\qquad \geqslant \int_{\partial s_\varepsilon(0)\cap G^+} 2\psi\,d_n\varphi. \end{aligned} \tag{3.18}$$

The sum of (3.15) and (3.16) gives (omitting the non-negative boundary terms)

$$\begin{aligned} 2\iint_{G_\varepsilon^+} \varphi T\psi[dx, dy] &\geqslant \iint_{G_\varepsilon^+} \{k(2\alpha - 1)\,\varphi_x^2 + (1-\alpha)\,\varphi_y^2[dx, dy] \\ &\quad - 2\int_{x=-1}^{-\varepsilon} \varphi\psi_y\,dx + 2\int_{\partial s_\varepsilon(0)\cap G^+} \varphi\,d_n\psi. \end{aligned} \tag{3.19}$$

Remark 3.20. On Γ_0 we have $\varphi\,|_{\Gamma_0} = \psi\,|_{\Gamma_0} = 0$, i.e., on the smooth parts of Γ_0, $\varphi_x\,dx + \varphi_y\,dx = 0$, $\psi = x\varphi_x + \alpha y\varphi_y = 0$. Thus for $|{}^{dx}_{\ x}\ {}^{dy}_{\alpha y}| \neq 0$ we have $\varphi_x = \varphi_y = 0$.

In G_ε^-, analogously we get

$$
\begin{aligned}
2\iint_{G_\varepsilon^-} & \varphi T\psi[dx, dy] \\
&= 2\int_{\partial G_\varepsilon^-} \{\varphi\, d_n\psi - \psi\, d_n\varphi\} + 2\iint_{G_\varepsilon^-} \alpha^1\varphi_x(k\varphi_{xx} + \varphi_{yy})[dx, dy] \\
&\geqslant \iint_{G_\varepsilon^-} \{(-k)\,\varphi_x^2 + \varphi_y^2\}[dx, dy] \qquad (3.20)\\
&\quad + 2\int_{x=-1}^{-\varepsilon} \varphi\psi_y\, dx + 2\int_{\Gamma_{2,\varepsilon}} \{\varphi\, d_n\psi - \psi\, d_n\varphi\}.
\end{aligned}
$$

Now adding (3.19) and (3.20) we have

$$
\begin{aligned}
2\iint_{G_\varepsilon^+\cup G_\varepsilon^-} \varphi T\psi[dx, dy] \geqslant & \iint_{G_\varepsilon^+} \{k(2\alpha-1)\,\varphi_x^2 + (1-\alpha)\,\varphi_y^2\}[dx, dy] \\
& + \iint_{G_\varepsilon^-} \{(-k)\,\varphi_x^2 + \varphi_y^2\}[dx\, dy] \\
& + 2\int_{\partial s_\varepsilon(0)\cap G^+} \varphi\, d_n\psi \\
& + 2\int_{\Gamma_{2,\varepsilon}} \{\varphi\, d_n\psi - \psi\, d_n\varphi\} = \sum_{i=1}^{4} I_i^\varepsilon. \qquad (21)
\end{aligned}
$$

Introducing once more the function φ_ε (3.6) and using the fact that ψ is a smooth function, we have with (3.9)

$$
\begin{aligned}
\left|2\iint_{G_\varepsilon^+\cup G_\varepsilon^-} \varphi T\psi[dx\, dy]\right| &= \left|2\iint_G \varphi_\varepsilon T\psi_\varepsilon[dx, dy]\right| \\
&\leqslant 2\, \|\varphi_\varepsilon\|_{L_2(G)}\, \|T\psi_\varepsilon\|_{L_2(G)} \leqslant 2c_1\, \|\varphi\|_{L_2(G)}.
\end{aligned}
$$

The line integral $I_3^\varepsilon = 2\int_{\partial s_\varepsilon(0)\cap G^+} \varphi_\varepsilon\, d_n\psi_\varepsilon$ exists almost everywhere for $\varepsilon > 0$ because $\varphi \in L_\nu(G)$ $(\nu \geqslant 2)$ and ψ is a smooth function. From the theorem on the absolute continuity of the integral [5, p. 63] it follows that $I_3^\varepsilon \to 0$ as $\varepsilon \to 0$. Since $\psi\,|_{\Gamma_2} = 0$, we also have $I_4^\varepsilon \to 0$ as $\varepsilon \to 0$.

Consequently we have

$$
2c_1\, \|\varphi\|_{L_2(G)} \geqslant m_0 \iint_G \{|k|\,(\varphi_\varepsilon)_x^2 + (\varphi_\varepsilon)_y^2\}[dx, dy] + I_3^\varepsilon + I_4^\varepsilon
$$

from which, in view of (3.9), we conclude $\|\varphi\|_{H_1(bd,k)} < \infty$ as $\varepsilon \to 0$. But this together with (3.9) implies

$$
\varphi^n \in H_1(bd, k) \cap L_\nu(G), \qquad \nu \geqslant 2. \qquad (3.22)
$$

4. An A Priori Estimate

From Lemma 3.1 we know that for the complete system of functions $\psi^n \in V_{T(F)} \cap H_1(bd^*, k) \cap L_p(G)$, $n \in \mathbb{N}$, there exist functions $\varphi^n \in H_1(bd, k) \cap L_\nu(G)$ $(\nu \geqslant 2)$ such that $l(\varphi^n) = \psi^n$. We now seek an approximate generalized solution $u^r \in H_1(bd, k) \cap L_p(G)$, $p = \rho + 2$, of (2.7) respectively (2.12) in the form

$$u^r = \sum_{i=1}^{r} c_{ir} \varphi^i, \tag{4.1}$$

where the coefficiente c_{ir} are determined from the system of non-linear equations

$$\begin{aligned} B[u^r, \psi^j] = -\iint_G \{&k(y)\, u^r_x \psi^j_x + u^r_y \psi^j_y + a(x, y, u^r)\, u^r \psi^j_x \\ &+ b(x, y, u^r)\, u^r \psi^j_y - c(x, y, u^r)\, u^r \psi^j\}[dx, dy] \\ = \iint_G &f(x, y, u^r)\, \psi^j [dx, dy], \qquad j = 1, 2, \ldots, r. \end{aligned} \tag{4.2}$$

From (4.1) we conclude

$$l(u^r) = \sum_{i=1}^{r} c_{ir} l(\varphi^i) = \sum_{i=1}^{r} c_{ir} \psi^i. \tag{4.3}$$

Multiplying (4.2) by c_{jr} and summing over j we have

$$\begin{aligned} \sum_{j=1}^{r} c_{jr} B[u^r, \psi^j] = -\iint_G \{&k(y)\, u^r_x (l(u^r))_x + u^r_y (l(u^r))_y + a(x, y, u^r)\, u^r (l(u^r))_x \\ &+ b(x, y, u^r)\, u^r (l(u^r))_y - c(x, y, u^r)\, u^r l(u^r)\}[dx, dy] \\ = \iint_G &f(x, y, u^r)\, l(u^r)[dx, dy] = B[u^r, l(u^r)]. \end{aligned} \tag{4.4}$$

Next we seek to obtain from (4.4) an a priori estimate of the form

$$\|u^r\|_{H_1(bd, k)} + \|u^r\|_{L_p(G)} \leqslant c_0 \qquad \text{for all} \quad r \in \mathbb{N}. \tag{4.5}$$

To this end we formulate the following lemma for the special case of problem (2.11) and give hints in the proof of the lemma of how the general case (2.7), i.e., (4.4), may be dealt with.

LEMMA 4.1. *Suppose all the assumptions of Theorem 2.1 hold. Then there exists a constant c_0 independent of r such that*

$$\|u^r\|_{H_1(bd,k)} + \|u^r\|_{L_p(G)} \leqslant c_0 \qquad \textit{for all} \quad r \in \mathbb{N},\ p = \rho + 2.$$

Proof. We treat separately each integral of (4.4). For notational simplicity the index r is omitted. Let

$$2I_1 := -2 \iint_G \{k(y)\, u_x (l(u))_x + u_y (l(u))_y\} [dx, dy]$$

then with $u \in H_1(bd, k) \cap L_p(G)$, we have $l(u) \in H_1(bd^*, k) \cap L_p(G)$. As in (3.18) and (3.16) we have

$$2I_1 = 2 \iint_G l(u)(k(y)\, u_{xx} + u_{yy})[dx, dy]$$

$$= \int_{\partial G} \Omega - \iint_G \{Au_x^2 + 2Bu_x u_y + Cu_y^2 + Du^2\} [dx, dy],$$

where Ω, A, B, C and D are given by (3.17). From the corresponding estimates (3.18) we get, using

$$\tilde{m}_0 = \min \left\{\frac{2\alpha - 1}{2}, \frac{1 - \alpha}{2}, \frac{1}{2}\right\},$$

$$I_1 \geqslant \tilde{m}_0 \iint_G \{|k|\, u_x^2 + u_y^2\} [dx, dy]. \tag{4.6}$$

Since $u\,|_{\Gamma_0 \cup \Gamma_1} = 0$, using Friedrich's inequality, we obtain

$$I_1 = -\iint_G \{k(y)\, u_x (l(u))_x + u_y (l(u))_y\} [dx, dy] \geqslant m_0\, \|u\|^2_{H_1(bd,k)}. \tag{4.7}$$

Let (see (2.11)) $c(x, y, u) = c_1(x, y) + c_2(u)$ and

$$I_2 := \iint_G c(x, y, u)\, ul(u)[dx, dy]$$

$$= \iint_G c_1(x, y)\, ul(u)[dx, dy] + \iint_G c_2(u)\, ul(u)[dx, dy] =: I_{2,1} + I_{2,2}.$$

The application of Green's theorem to $I_{2,1}$ gives (the boundary integrals are zero or non-negative)

$$I_{2,1} \geqslant -\frac{1}{2}\iint_{G^+} [(\alpha+1)\, c_1(x, y) + xc_{1_x}(x, y) + \alpha y c_{1_y}(x, y)]\, u^2[dx, dy]$$

$$-\frac{1}{2}\iint_{G^-} [c_1(x, y) + xc_{1_x}(x, y)]\, u^2[dx, dy] \geqslant \lambda_0 \|u\|^2_{L_2(G)} \tag{4.8}$$

where

$$2\lambda_0 := \min\{-[(\alpha+1)\, c_1 + xc_{1_x} + \alpha y c_{1_y}], -[c_1 + xc_{1_x}]\} \geqslant 0.$$

For the second integral $I_{2,2}$ we notice

$$\frac{\partial}{\partial x}\int_{t=0}^{u} tc_2(t)\, dt = uc_2(u)\, u_x, \qquad \frac{\partial}{\partial y}\int_{t=0}^{u} tc_2(t)\, dt = uc_2(u)\, u_y,$$

and using Green's theorem once again, we obtain

$$I_{2,2} = -(1+\alpha)\iint_{G^+}\int_{t=0}^{u} tc_2(t)\, dt[dx, dy] - \iint_{G^-}\int_{t=0}^{u} tc_2(t)\, dt[dx, dy]. \tag{4.9}$$

If we assume that the function $c_2 = c_2(u)$ satisfies

$$\int_{t=0}^{T} tc_2(t)\, dt \leqslant 0 \qquad \text{for all} \quad T \in (-\infty, +\infty), \tag{4.10}$$

then we have

$$I_{2,2} = \iint_G c_2(u)\, ul(u)[dx, dy] \geqslant -\iint_G \int_{t=0}^{u} tc_2(t)\, dt[dx, dy] \geqslant 0. \tag{4.11}$$

In case of Eq. (2.11) we have $c_2(u) = -|u|^p$, and from (4.11) it follows that

$$\iint_G c_2(u)\, ul(u)[dx, dy] = -\iint_G |u|^p\, ul(u)[dx, dy] = \frac{1}{p}\|u\|^p_{L_p(G)}. \tag{4.12}$$

Thus for the problem (2.11) we have shown ((4.7), (4.8), (4.12))

$$m_0 \|u\|^2_{H_1(bd,k)} + \frac{1}{p}\|u\|^p_{L_p(G)} \leqslant \iint_G f(x, y, u)\, l(u)[dx, dy]. \tag{4.13}$$

By hypothesis we have

$$f(x, y, u) = |k|^{1/2} f_1(x, y, u), \qquad \|f_1\|^2_{L_2(G)} \leqslant K_0 + K_1 \|u\|^{P/2}_{L_P(G)}. \tag{4.14}$$

Thus using Hölder's inequality with $\varepsilon > 0$, $\eta > 0$ gives

$$\begin{aligned}\left|\iint_G f(x, y, u)\, l(u)[dx, dy]\right| &\leqslant \left\|\frac{f}{|k|^{1/2}}\right\|_{L_2(G)} \|u\|_{H_1(bd,k)} \\ &\leqslant \frac{1}{2\varepsilon} \|f_1\|^2_{L_2(G)} + 2\varepsilon \|u\|^2_{H_1(bd,k)} \\ &\leqslant \frac{1}{2\varepsilon} K_0 + \frac{1}{2\varepsilon} K_1 \|u\|^{P/2}_{L_P(G)} + 2\varepsilon \|u\|^2_{H_1(bd,k)} \\ &\leqslant \frac{1}{2\varepsilon} K_0 + 2\varepsilon \|u\|^2_{H_1(bd,k)} + \frac{1}{2\eta}\left(\frac{K_1}{2\varepsilon}\right)^2 + 2\eta \|u\|^P_{L_P(G)}.\end{aligned} \tag{4.15}$$

For ε, η sufficient small, from (4.13) and (4.15) we have

$$\|u\|^2_{H_1(bd,k)} + \|u\|^2_{L_P(G)} \leqslant c_1 \tag{4.16}$$

and thus

$$\|u^r\|_{H_1(bd,k)} + \|u^r\|_{L_P(G)} \leqslant c_0 \qquad \text{for all} \quad r \in \mathbb{N}. \quad \blacksquare \tag{4.17}$$

In conclusion, with suitable constants in (4.17) and in (4.14) we get

$$\|f(x, y, u^r)\|^2_{L_2(G)} \leqslant c_2 \qquad \text{for all} \quad r \in \mathbb{N}. \tag{4.18}$$

Remark 4.19. From the proof of Lemma 4.1 we see that it is essential to estimate the expression (4.11) from below by zero or for a better result by a "L_P-norm" of u. In the latter case we then need for the function $f(x, y, u)$ only a growth condition in the same "L_P-norm" to calculate (4.15). Some functions $c_2(u)$ which fulfill the sign-condition (4.10) are, for example, $-1/(1+u^2)$, $-\sinh u^2$, $-e^{u^2}$.

Therefore in (1.1), $c(x, y, u)$ may be of the form $c(x, y, u) = c_1(x, y) + c_{2,1}(u) + c_{2,2}(u) = c_1(x, y) - |u|^P - 1/(1+u^2)$. In this case for $c_{2,1}(u) = -|u|^P$ we use the estimate (4.12), while for the function $c_{2,2}(u) = -1/(1+u^2)$ we estimate the integral (4.11) from below by zero. It does not appear useful to bring $c_{2,2}(u)$ to the right side of the equation because of the condition (4.14).

Remark 4.20. If we consider the general case of Eq. (1.1) we have to consider in (4.4) the additional integral

$$I_3 = -\iint_G \{a(x, y, u)\, u(l(u))_x + b(x, y, u)\, u(l(u))_y\}[dx, dy].$$

To get a priori estimates of the form (4.17) we now need conditions on a and b such that $I_3 \geq 0$ in G. Moreover the question arises in connection with limit $r \to \infty$ in (4.4). These questions are dealt with in Section 5 (see Lemma 5.2). For example, the following functions a and b are admissible: $b(x, y, u) = 0$ in G, $a(x, y, u) = 0$ in $y \geq 0$, $a(x, y, u) = -|u|^{\rho}$ in $y < 0$.

5. Generalized Solutions

For the problem (2.11) we know from Lemma 4.1

$$\|u^r\|_{H_1(bd,k)} + \|u^r\|_{L_P(G)} \leq c_0 \qquad \text{for all} \quad r \in \mathbb{N}, \tag{5.1}$$

where

$$u^r = \sum_{i=1}^{r} c_{ir} \varphi^i \in H_1(bd, k) \cap L_P(G), \qquad p = \rho + 2, \tag{5.2}$$

and the coefficients c_{ir} are determined by the system of non-linear equations

$$\begin{aligned} B[u^r, \psi^j] &= -\iint_G \{k(y)\, u^r_x \psi^j_x + u^r_y \psi^j_y - c_1(x, y)\, u^r \psi^j + (u^r)^{\rho}\, u^r \psi^j [dx, dy] \\ &= \iint_G f(x, y, u^r)\, \psi^j [dx, dy], \qquad j = 1, 2, ..., r. \end{aligned} \tag{5.3}$$

Since the spaces $L_q(G)$, $1 < q < \infty$, are reflexive and the closed unit sphere of a reflexive space is weakly sequentially compact from (5.1) it follows that there exist a subsequence (which we denote again by u^r) and a function $u \in H_1(bd, k) \cap L_p(G)$ such that

$$\begin{aligned} u^r &\to u \qquad \text{weakly in } H_1(bd, k), \\ u^r &\to u \qquad \text{weakly in } L_p(G) \text{ as } r \to \infty. \end{aligned} \tag{5.4}$$

From (5.4) it follows that we can pass to the limit $r \to \infty$ for the linear terms of (5.3). Thus we know

$$\iint_G \{k(y)\, u^r_x \psi^j_x + u^r_y \psi^j_y - c_1(x, y)\, u^r \psi^j\}[dx, dy]$$

$$\xrightarrow[r \to \infty]{} \iint_G \{k(y)\, u_x \psi^j_x + u_y \psi^j_y - c_1(x, y)\, u \psi^j\}[dx, dy]. \tag{5.5}$$

Now let $v^r := |u^r|^{\rho}\, u^r$, from (5.1) we have $v^r \in L_{p'}(G)$ with $p' = p/(p-1)$ and

$$\|v^r\|_{L_{p'}(G)} \leq c_0 \qquad \text{for all} \quad r \in \mathbb{N}. \tag{5.6}$$

Therefore there exist a subsequence (we denote again with v^r) and a function $\omega^* \in L_{p'}(G)$ such that

$$v^r \to \omega^* \qquad \text{weakly in } L_{p'}(G) \text{ as } r \to \infty. \tag{5.7}$$

LEMMA 5.1. *We have* $\omega^* = |u|^\rho u \in L_{p'}(G)$, $p' = p/(p-1)$ *and*

$$v^r = |u^r|^\rho u^r \to |u|^\rho u \qquad \text{almost everywhere in } G \text{ as } r \to \infty.$$

Proof. We consider the functions $\omega^r := |k|^{1/2} u^r$, $r \in \mathbb{N}$, and calculate for $1 < s < 2$ (Hölder inequality)

$$\begin{aligned}
\|\omega^r\|^s_{w^1_s(G)} &= \iint_G \{(\omega^r_x)^s + (\omega^r_y)^s + (\omega^r)^s\}\, dx\, dy \\
&\leqslant c_0 \left(\iint_G \{|k|\, (u^r_x)^2 + (u^r_y)^2 + (u^r)^2\}\, dx\, dy \right)^{s/2} \\
&\quad + c_1 \iint_G \left| \frac{k'}{|k|^{1/2}} \right|^s |u^r|^s\, dx\, dy.
\end{aligned}$$

Using

$$\iint_G \left| \frac{k'}{|k|^{1/2}} \right|^s |u^r|^s\, dx\, dy \leqslant \left(\iint_G \left| \frac{k'}{|k|^{1/2}} \right|^{ps/(p-s)} dx\, dy \right)^{(p-s)/p} \left(\iint_G |u^r|^p\, dx\, dy \right)^{s/p}$$

we see that the function

$$\omega^r \in w^1_s(G),\ s \in (1, 2) \qquad \text{if} \quad \frac{k'}{|k|^{1/2}} \in L_{ps/(p-s)}(G). \tag{5.8}$$

The Rellich–Kondrachov theorem [1, p. 144; 7, p. 99–101] says that $w^1_s(G)$, $s \in (1, 2)$, is compact in $L_2(G)$. Therefore there exist a subsequence (which we again denote by ω^r) and an element $\omega \in w^1_s(G)$ such that

$$\omega^r = |k|^{1/2} u^r \to \omega \qquad \text{strongly in } L_2(G) \text{ and almost everywhere in } G. \tag{5.9}$$

From (5.4) and (5.9) it follows that $\omega = |k|^{1/2} u$ almost everywhere in G, but this means

$$\begin{aligned}
&u^r \to u, \\
&v^r = |u^r|^\rho u^r \to |u|^\rho u = \omega^* \in L_{p'}(G) \qquad \text{almost everywhere in } G \qquad (5.10) \\
&\text{as } r \to \infty. \quad \blacksquare
\end{aligned}$$

To pass to the limit in the non-linear term on the left side of (5.3) we have to show

$$v^r = |u^r|^\rho u^r \to |u|^\rho u \qquad \text{weakly in } L_{p'}(G)\ (r \to \infty). \tag{5.11}$$

We use

LEMMA 5.2 [6, p. 12]. *If G is a bounded domain and g_n, $g \in L_q(G)$, $1 < q < \infty$ $(n \in \mathbb{N})$, $\|g_n\|_{L_q(G)} \leqslant C$, $g_n \to g$ a.e. in G, then $g_n \to g$ weakly in $L_q(G)$ $(n \to \infty)$.*

In our case we take $g_n := |u^n|^\rho u^n \in L_{p'}(G)$ and we know $\|g_n\|_{L_p(G)} \leqslant C_0$ (5.6), $g_n = |u^n|^\rho u^n \to |u|^\rho u \in L_{p'}(G)$ a.e. in G (5.10). Thus (5.11) follows and

$$\iint_G |u^r|^\rho u^r \psi^j [dx, dy] \to \iint_G |u|^\rho u \psi^j [dx, dy] \qquad \text{as} \quad r \to \infty. \tag{5.12}$$

For the right side of (5.3) we note that the function $f(x, y, u)$ is continuous with respect to u and thus by (5.10) it follows that

$$f(x, y, u^r) \to f(x, y, u) \qquad \text{a.e. in } G\ (r \to \infty).$$

By (4.18) we know $\|f(x, y, u)\|_{L_2(G)} \leqslant c_2$, but then because of $p' = p/(p-1) < 2$ we have $\|f(x, y, u)\|_{L_{p'}(G)} \leqslant c_3$. Using Lemma 5.2 it follows that $f(x, y, u^r) \to f(x, y, u)$ weakly in $L_{p'}(G)$ and

$$\iint_G f(x, y, u^r)\, \psi^j [dx, dy] \to \iint_G f(x, y, u)\, \psi^j [dx, dy] \qquad \text{as} \quad r \to \infty.$$

Taking the limit as $r \to \infty$ in (5.3) and observing that ψ^j is a complete system in $H_1(bd^*, k) \cap L_p(\Omega)$, for $u \in H_1(bd, k) \cap L_p(G)$ we have

$$\begin{aligned} -\iint_G \{k(y)\, u_x v_x + u_y v_y - c_1(x, y)\, uv + |u|^\rho uv\}[dx, dy] \\ = \iint_G f(x, y, u)\, v[dx, dy] \end{aligned} \tag{5.13}$$

for all $v \in H_1(bd^*, k) \cap L_p(G)$, but this implies that u is a generalized solution. ∎

Remark 5.14. Podgaev in [8] uses for his problem in the case of the function space $w_s^1(G)$, $s \in (1, 2)$, in Lemma 5.1 the space $w_2^1(G)$. But then we need the assumption $k'/|k|^{1/2} \in L_{2p/(p-2)}(G)$ (not $L_{p/(p-1)}(G)$ as written in [8]). For the special case $k(y) = \operatorname{sign} y\,|y|^m$, $m \geqslant 1$, then the case $m = 1$ is not allowed. In our theorem we assume

$$\frac{k}{|k|^{1/2}} \in L_{ps/(p-s)}(G), \qquad s \in (1, 2). \tag{5.15}$$

For $k(y) = \operatorname{sign} y\,|y|^m$, $m \geqslant 1$, we write $s = 1 + \varepsilon$, $\varepsilon > 0$, and (5.15) means $m > 2\varepsilon/(1+\varepsilon) + 2/(\rho + 2)$ $(p = \rho + 2)$. The case $m = 1$ is included if $\varepsilon = s - 1 > 0$ is sufficient small. ($m < 1$ is not allowed because in assumption (i) of Theorem 2.1 we have $k(y) \in c^1(\bar{G})$.)

Remark 5.15. If we have a more general equation than (2.11) (see Remark 4.19: $c_2(u) = -|u|^\rho - 1/(1+u^2)$; Remark 4.20: $a = -|u|^\rho$ for $y < 0$, $a = 0$ for $y > 0$, $b \equiv 0$ in G) and further the a priori estimate (4.5) holds, then we still have to show that we can take the limit as $r \to \infty$ in (4.2). In both this follows by Lemma 5.2.

6. The Non-linear System of Equations

To complete the proof of Theorem 2.1, we need to show the solvability of the non-linear system of equations (5.3). To prove the solvability of this non-linear system we assume that the functions φ^n, $n \in \mathbb{N}$, from Lemma 3.1 are normalized in such a way that

$$\iint_G \{|k|\,\varphi^i_x \varphi^j_x + \varphi^i_y \varphi^j_y + \varphi^i \varphi^j\}[dx, dy] = \delta^i_j. \tag{6.1}$$

We first prove for the convenience of the reader the following known result:

Theorem 6.1 (Vicik [10, Lemma 3, p. 15]). *If for the non-linear system of equations*

$$A_j(c_1, c_2, \dots, c_k) = h_j, \qquad j = 1, \dots, k, \tag{6.2}$$

(i) *the* $A_j(C) := A_j(c_1, \dots, c_k)\colon \mathbb{R}^k \to \mathbb{R}^k$ *are continuous functions, and*

(ii) *there exist constants* $a_0 > 0$, $a_1 \geqslant 0$ *such that for* $\varepsilon > 0$ *we have*

$$(A(C), C) = \sum_{j=1}^{k} A_j(C)\,C_j \geqslant a_0\,|C|^{1+\varepsilon} - a_1, \tag{6.3}$$

then there exists at least one solution of (6.2) *for each* $h = (h_1, \dots, h_k)^T$.

Proof. We first prove:

> If $P\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping such that for a real number $\rho > 0$ we have (6.5)
>
> $$Px \cdot x \geqslant 0 \qquad \text{for all} \quad x \in \mathbb{R}^n,\ |x| = \rho,$$

then there exists at least one z with $|z| \leqslant \rho$ and $Pz = 0$. Assuming that $Px \neq 0$ for all $x \in \overline{S_\rho(0)}$, we define by

$$Tx := -Px \frac{\rho}{|Px|}$$

the continuous mapping $T\colon \overline{S_\rho(0)} \to \overline{S_\rho(0)}$. By Brouwer's Fixed-Point Theorem there exists at least one $y \in \overline{S_\rho(0)}$ such that

$$y = -Py \frac{\rho}{|Py|}. \tag{6.6}$$

From (6.6) it follows that $|y| = \rho > 0$ and

$$0 < y \cdot y = |y|^2 = \rho^2 = -\frac{\rho}{|Py|}(Py \cdot y) \leqslant 0,$$

which is a contradiction.

Next we show that:

> If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping and there exists a function $c(t)\colon (0, \infty) \to \mathbb{R}$ with $c(t) \to \infty$ as $t \to \infty$, and such that (6.7)
>
> $$Tx \cdot x \geqslant |x|\, c(|x|) \qquad \text{for all} \quad x \in \mathbb{R}^n \tag{6.8}$$

then $Tx = w$ has a solution for each $w \in \mathbb{R}^n$.

Assume that there exists a $w \in \mathbb{R}^n$ such that $Tx \neq w$ for all $x \in \mathbb{R}^n$. Then we have for the continuous mapping

$$Px := Tx - w\colon \mathbb{R}^n \to \mathbb{R}^n,$$

$$\begin{aligned} Px \cdot x = Tx \cdot x - w \cdot x &\geqslant |x|\, c(|x|) - |w|\, |x| \\ &= |x|\, (c\, |x| - |x|). \end{aligned}$$

Since $c(t) \to \infty$ as $t \to \infty$ there exist $\rho > 0$ such that

$$Px \cdot x \geqslant 0 \qquad \text{for all} \quad |x| \geqslant \rho.$$

The mapping P now fulfills the assumptions in (6.5). Therefore there exists at least one $z \in \mathbb{R}^n$ with $|z| \leqslant \rho$ such that $Pz = 0$, i.e., $Tz = w$, which is a contradiction.

To prove Theorem 6.1 we only need to choose in (6.7) the function $c(t)$ of the form $c(t) = a_0 t^{p-1} - a_1(1/t)$ with $p > 1$, $a_0 > 0$, $a_1 \geqslant 0$. Then (6.8) becomes

$$Tx \cdot x \geqslant |x|\, c(|x|) = a_0 |x|^p - a_1 \qquad \text{for all} \quad x \in \mathbb{R}^n$$
$$\text{with } p = 1 + \varepsilon \geqslant 1.$$

By (6.3) this condition is true for the non-linear system (6.2). Therefore there exists at least one solution of (6.2) for each $h \in \mathbb{R}^n$. ∎

Now we return to our non-linear system (5.3). By (5.2) we have $u^r = \sum_{i=1}^r c_{ir}\varphi^i \in H_1(bd, k) \cap L_p(G)$ and the non-linear system (5.3) takes the form

$$\begin{aligned} A_j(C) := -\sum_{i=1}^r c_{ir} \iint_G \Big\{ & k(y)\, \varphi^i_x \psi^j_x + \varphi^i_y \psi^j_y \\ & - \left(c_1(x, y) + c_2\left(\sum_{i=1}^r c_{ir}\varphi^i \right) \right) \varphi^i \psi^j \Big\} [dx, dy] \\ & - \iint_G f\left(x, y, \sum_{i=1}^r c_{ir}\varphi^i \right) \psi^j [dx, dy] = 0, \qquad j = 1, ..., r. \end{aligned} \tag{6.9}$$

$A_j(C)$ are continuous functions of $c_{1r}, ..., c_{rr}$. To use Theorem 6.1 we have to show

$$(A(C), C) = \sum_{j=1}^r A_j(C)\, C_{jr} \geqslant a_0 |C|^2 - a_1. \tag{6.10}$$

If we use $l(u^r) = \sum_{i=1}^r c_{ir}\psi^i$ we get

$$\begin{aligned} \sum_{j=1}^r A_j(C)\, c_{jr} = -\iint_G & \{ k(y)\, u^r_x (l(u^r))_x + u^r_y (l(u^r))_y \\ & - (c_1(x, y) + c_2(u^r))\, u^r l(u^r) \} [dx, dy] \\ & - \iint_G f(x, y, u^r)\, l(u^r) [dx, dy]. \end{aligned}$$

From (4.7) and (4.8) we get (using (6.1))

$$-\iint_G \{k(y)\, u_x^r(l(u^r))_x + u_y^r(l(u^r))_y - c_1(x, y)\, u^r l(u^r)\}[dx, dy]$$
$$\geqslant a_0 \|u^r\|^2_{H_1(bd,k)}$$
$$= a_0 \sum_{i,j=1}^{r} c_{ir} c_{jr} \iint_G \{|k|\, \varphi_x^i \varphi_x^j + \varphi_y^i \varphi_y^j + \varphi^i \varphi^j\}[dx, dy]$$
$$= a_0 \sum_{i,j=1}^{r} c_{ir} c_{jr} \delta_j^i = a_0\, |C|^2.$$

Equation (4.12) gives

$$\iint_G c_2(u^r)\, u^r l(u^r)[dx, dy] \geqslant 0$$

and from (4.15), (4.17) and (4.18) we have

$$\iint_G f(x, y, u^r)\, l(u^r)[dx, dy] \leqslant \frac{1}{2\varepsilon} \|f_1\|^2_{L_2(G)} + 2\varepsilon\, \|u^r\|^2_{H_1(bd,k)} \leqslant a_1.$$

Thus we know (6.10) and the non-linear system (6.9) admits at least one solution $C = (c_{1r}, ..., c_{rr})$.

Remark 6.11. It can be seen that there always exists a solution of the non-linear system (6.9) if the a priori estimate (4.17) holds. It remains an open question if the generalized solution of (2.11) is unique.

The assumption $k'/|k|^{1/2} \in L_{sp/(p-s)}(G)$, $p = \rho + 2$, $s \in (1, 2)$, is made to prove Lemma 5.1. If we therefore have a linear problem, this condition can be eliminated and we have

THEOREM 6.12. *If*

(i) $k(y) \in C^0(\bar{G}) \cap C^1(G^+)$; $k(y) \gtreqless 0$ *for* $y \gtreqless 0$; $yk'(y) \geqslant k(y)$ *for* $y \geqslant 0$,

(ii) $c_1(x, y) \in C^1(\bar{G})$; $(\alpha + 1)\, c_1(x, y) + xc_{1_x}(x, y) + \alpha y c_{1_y}(x, y) \leqslant 0$ *for* $y \geqslant 0$, $c_1(x, y) + xc_{1_x}(x, y) \leqslant 0$ *for* $y \leqslant 0$, $\alpha \in (\frac{1}{2}, 1)$, $c_1|_{\Gamma_2} \leqslant 0$,

(iii) $f(x, y) = |k|^{1/2} f_1(x, y)$, $f_1(x, y) \in L_2(G)$,

(iv) $k(y)\, n_1^2 + n_2^2\,|_{\Gamma_1} \geqslant 0$, $n_1\,|_{\Gamma_1} > 0$; $xn_1 + \alpha y n_2\,|_{\Gamma_0} \leqslant 0$ *where* (n_1, n_2) *is the inward normal vector, then there exists a generalized solution u of the boundary value problem*

$$L[u] = k(y)\, u_{xx} + u_{yy} + c_1(x, y)\, u = f(x, y) \qquad \text{in } G,\ u\,|_{\Gamma_0 \cup \Gamma_1} = 0;$$

i.e., there exists a $u \in H_1(bd, k)$ *such that*

$$-\iint_G \{k(y)\, u_x v_x + u_y v_y - c_1(x, y)\, uv\}\, dx\, dy = \iint_G f(x, y)\, v\, dx\, dy$$

for all $v \in H_1(bd^*, k)$.

The uniqueness of this solution follows from [2].

References

1. R. A. Adams, "Sobolev Spaces" Academic Press, New York, 1975.
2. A. K. Aziz, Houde Han, and M. Schneider, A finite element method for a boundary value problem of mixed type, *Numer. Math.* **44** (1984), 191–200.
3. D. K. Gvazava, About a class of quasilinear equations of mixed type *Akad. Nauk Gruzinskoi SSR*, (1981), 1–94. [Russian]
4. W. Haack and W. Wendland, "Lectures on Partial and Pfaffian Differential Equations, "Pergamon, Oxford/New York, 1972.
5. A. Kufner, O. John, and S. Fucik, "Function Spaces," Noorhoff, Leyden, 1977.
6. J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier-Villars, Paris, 1969.
7. J. L. Lions and E. Magenes; "Non-Homogenuous Boundary Value Problems and Applications, I," Springer-Verlag, New York, 1972.
8. A. G. Podgaev, On the solvability of the generalized Tricomi problem for a nonlinear equation, *Soviet Math. Dokl.* **18**, No. 5 (1977), 1359–1363.
9. M. Schneider, Funktionalanalytische Methoden zur Behandlung von linearen partiellen Differentialgleichungen vom gemischten Typ, *Oberbliche Mat. BI* 7 (1974), 219–265.
10. M. I. Vicik, Quasi-linear strongly elliptic systems of differential equations in divergence form, *Trans. Moscow Math. Soc.* (1963), 140–208.