Reverse order laws for generalized inverses of multiple matrix products

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Abstract

In this article we study reverse order laws for generalized inverses and reflexive generalized inverses of the products of multiple matrices $A^{(1)}, \ldots, A^{(n)}$ and the products of generalized inverses and reflexive generalized inverses of $A^{(n)}, \ldots, A^{(1)}$. By applying the multiple product singular value decomposition, we obtain necessary and sufficient conditions for one side inclusion relation of reverse order law for generalized inverses, and necessary and sufficient conditions of reverse order law for reflexive generalized inverses. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Theory and computations of generalized inverses of matrices are important subjects in many branches of applied science, such as matrix analysis, statistics and numerical linear algebra [1,2,8,9]. The concept of generalized inverses of a matrix has a long history. The most commonly used definition of generalized inverses was introduced by Penrose in [7], now is known as the Moore–Penrose conditions, which is a matrix $X$ satisfying some of the following four equations:

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PII: S 0 0 2 4 - 3 7 9 5 ( 9 9 ) 0 0 0 5 3 - 1
in which $B^H$ denotes the conjugate transpose of a matrix $B$. Let $\emptyset \neq \eta \subseteq \{1, 2, 3, 4\}$. Then $A\eta$ denotes the set of all matrices $X$ which satisfy (i) for all $i \in \eta$. Any $X \in A\eta$ is called an $\eta$-inverse of $A$. One usually denotes any $\{1\}$-inverse of $A$ as $A^{(1)}$ or $A^-$ which is also called a $g$-inverse of $A$. Any $\{1, 2\}$-inverse of $A$ is denoted by $A^{(1,2)}$ or $A_r$ which is also called a reflexive $g$-inverse of $A$. The unique $\{1, 2, 3, 4\}$-inverse, or the Moore–Penrose pseudo-inverse of $A$ is denoted by $A^+$.

In this paper we also use the following notation. $C_{m\times n}$ denotes the set of $m$ by $n$ matrices of complex entries, $I = I_n$ denotes the identity matrix of order $n$, $0_{m\times n}$ is the $m$ by $n$ matrix with all zero entries (if no confusion occurs we will omit the subscript). For a matrix $A \in C_{m\times n}$, $\text{rank}(A)$ is the rank of $A$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are respectively the range space and the null space of $A$. Notice that for given matrix $A$, the Moore–Penrose pseudo-inverse $A^+$ is unique, while the number of other kind of generalized inverses of $A$ is in general infinite. Greville [5] first studied the Moore–Penrose pseudo-inverse of the product of two matrices $A$ and $B$, and gave a necessary and sufficient condition for the reverse order law $(AB)^+ = B^+A^+$. Since then, more equivalent conditions for $(AB)^+ = B^+A^+$ have been discovered, and reverse order laws for $g$-inverses and reflexive g-inverses of two-matrix product have also been discussed in the literature. Shinozaki and Sibuya [10,11] studied reverse order laws for $g$-inverses and reflexive g-inverses. Let $A$ and $B$ be given matrices with $AB$ meaningful and $\eta = \{1\}$ or $\eta = \{1, 2\}$. For given generalized inverses $A^\sigma$ and $B^\rho$, they obtained some sufficient conditions under which the relation $B^\rho A^\sigma \in (AB)\eta$ holds. Theory for reverse order laws $B\{1\}A\{1\} = (AB)\{1\}$ and $B\{1, 2\}A\{1, 2\} = (AB)\{1, 2\}$ are more difficult problems. Shinozaki and Sibuya [10] proved that one side inclusion relation $(AB)\{1, 2\} \subseteq B\{1, 2\}A\{1, 2\}$ always holds. Werner [14,15] obtained equivalent conditions for one side inclusion relation $B\{1\}A\{1\} \subseteq (AB)\{1\}$. Wei [13], De Pierro and Wei [4], respectively, derived necessary and sufficient conditions for the reverse order laws $B\{1\}A\{1\} = (AB)\{1\}$ and $B\{1, 2\}A\{1, 2\} = (AB)\{1, 2\}$ by applying the product singular value decomposition (P-SVD) of the matrices $A$ and $B$, and so eventually solved these problems.

On the other hand, the reverse order law for the Moore–Penrose pseudo-inverse of multiple matrix products was considered by Hartwig [6] and Tian [12] for three and $n$ matrices, respectively. To our knowledge, reverse order laws for $g$-inverses and reflexive g-inverses of multiple matrix products have not been studied yet in the literature.

In this paper we will discuss reverse order laws for $g$-inverses and reflexive g-inverses of multiple matrix products. We will apply the multiple singular value decomposition (multiple P-SVD) of products of multiple matrices to study
these problems. The multiple P-SVD was discovered by De Moor and Zha [3], as mentioned in the following lemma.

**Lemma 1.1** (Multiple P-SVD [3]). Suppose that $A^{(j)} \in \mathbb{C}^{m_j \times m_{j+1}}$ for $j = 1, \ldots, n$ with $n \geq 2$ an integer. Then there exist $n + 1$ non-singular matrices $W_j \in \mathbb{C}^{m_j \times m_j}$ with $j = 1, \ldots, n + 1$, such that

$$A^{(j)} = W_j D_j W_{j+1}^{-1},$$

in which

$$D_1 = \begin{pmatrix} r_1^1 & 0 & \cdots & 0 \\ 0 & r_2^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{m_1}^1 \end{pmatrix}, \quad r_1^1 = r_1 = \text{rank}(A^{(1)}),$$

$$D_j = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$r_j = \sum_{k=1}^{j} r_k^j = \text{rank}(A^{(j)})$$

for $j = 2, \ldots, n$, in which each $I$ denotes an identity matrix with noted dimension if the noted number is greater than zero, or is naught when the noted number is zero.

**Remark.** In [3] the matrix $D_n$ has the form

$$D_n = \begin{pmatrix} S_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & S_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S_n & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$s_n^1, s_n^2, \ldots, s_n^n = \text{rank}(A^{(n)}).$$
in which $S_1, \ldots, S_n$ are non-singular diagonal matrices, and $W_1, W_{n+1}$ are unitary matrices. To simplify notation and discussion in this paper, we describe Lemma 1.1 which is different from one in [3]. We replace $W_{n+1}$ in [3] by diag $(S_1, \ldots, S_n, I_{m_{n+1} - r_0})W_{n+1}$ and replace $D_n$ in [3] by one appeared in (3) with $j = n$.

Based on the multiple P-SVD of $n$ matrices $A^{(1)}, \ldots, A^{(n)}$, we can describe structures of any g-inverses $(A^{(j)})^-$ and reflexive g-inverses $(A^{(j)})^-_r$ for $j = 1, \ldots, n$, $B^-$ and $B^-_r$ with $B = A^{(1)}, \ldots, A^{(n)}$ in the following lemma.

**Lemma 1.2.** Suppose that $A^{(j)} \in C^{m_j \times m_{j+1}}$ for $j = 1, \ldots, n$ and the multiple P-SVD of $A^{(j)}$ are given in (2)–(3). Then any $(A^{(1)} \cdots A^{(n)})^- \in (A^{(1)} \cdots A^{(n)})\{1\}$, $(A^{(j)})^- \in A^{(j)}\{1\}$ for $j = 1, \ldots, n$, respectively, have the following forms:

$$(A^{(1)} \cdots A^{(n)})^- = W_{n+1}ZW_1^{-1}, \quad (A^{(j)})^- = W_{j+1}X^{(j)}W_j^{-1}, \quad j = 1, \ldots, n, \quad (4)$$

in which

$$Z = \begin{pmatrix} I & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}^{r_1}_{m_{n+1} - r_0}, \quad X^{(1)} = \begin{pmatrix} I & X^{(1)}_{1,2} \\ X^{(1)}_{2,1} & X^{(1)}_{2,2} \end{pmatrix}^{r_1}_{m_2 - r_1}, \quad (5)$$

and for $j = 2, \ldots, n$,

$$X^{(j)} = \begin{pmatrix} I & X^{(j)}_{1,2} & 0 & X^{(j)}_{1,4} & \cdots & 0 & X^{(j)}_{1,2j} \\ 0 & X^{(j)}_{2,2} & I & X^{(j)}_{2,4} & \cdots & 0 & X^{(j)}_{2,2j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & X^{(j)}_{j,2} & 0 & X^{(j)}_{j,4} & \cdots & I & X^{(j)}_{j,2j} \\ X^{(j)}_{j+1,1} & X^{(j)}_{j+1,2} & X^{(j)}_{j+1,3} & X^{(j)}_{j+1,4} & \cdots & X^{(j)}_{j+1,2j-1} & X^{(j)}_{j+1,2j} \end{pmatrix}^{r_j}_{m_{j+1} - r_j} \begin{pmatrix} r_j \\ r_j \\ \vdots \\ r_j \\ r_j \\ r_j \\ r_j \end{pmatrix}^{r_j}_{m_{j+1} - r_j - r_j} \quad (6)$$

in which $Z_{pq}$ and $X^{(j)}_{pq}$ are arbitrary matrices having noted dimensions. Furthermore, any $(A^{(j)})^-_r \in A^{(j)}\{1, 2\}$ for $j = 1, \ldots, n$ and $(A^{(1)} \cdots A^{(n)})^-_r \in (A^{(1)} \cdots A^{(n)})\{1, 2\}$, respectively, have the forms

$$(A^{(1)} \cdots A^{(n)})^-_r = W_{n+1}ZW_1^{-1}, \quad (A^{(j)})^-_r = W_{j+1}X^{(j)}W_j^{-1}, \quad j = 1, \ldots, n, \quad (7)$$

where $Z$ and $X^{(j)}$ have the same forms as in (5) and (6) in which

$$Z_{2,2} = Z_{2,1}Z_{1,2}, \quad \text{and} \quad X^{(j)}_{j+1,2q} = \sum_{k=1}^{j} X^{(j)}_{j+1,2k-1}X^{(j)}_{k,2q}, \quad q = 1, \ldots, j \quad (8)$$

for $j = 1, \ldots, n$, so that $\text{rank}(Z) = \text{rank}(A^{(1)} \cdots A^{(n)})$ and $\text{rank}(X^{(j)}) = \text{rank}(A^{(j)})$ for $j = 1, \ldots, n$.\]
Proof. For each \( j = 1, \ldots, n \) let \( W_{j+1} Y W_j^{-1} \) be a g-inverse of \( A^{(j)} \). Then from Lemma 1.1, \( Y \) should satisfy \( D_j Y D_j = D_j \). Let \( Y = (X_{ij}^{(j)}) \) be partitioned conforming with the partition of \( D_j \) as in (3) where the row and column partitions of \( Y \) and \( D_j \) should be exchanged. Then from \( D_j Y D_j = D_j \), \( Y = X^{(j)} \) should have the form as in (6). Furthermore, if \( W_{j+1} Y W_j^{-1} \) is also a reflexive g-inverse of \( A^{(j)} \), then from \( Y D_j Y = Y \), \( X_{ij}^{(j)} \) should also satisfy (8). The proof for the structure of \( Z \) is similar. \( \square \)

The paper is organized as follows. In Section 2 we will discuss necessary and sufficient conditions for \( A^{(n)} \{1\} \cdots A^{(1)} \{1\} \subseteq (A^{(1)} \cdots A^{(n)}) \{1\} \); in Section 3 we will discuss necessary and sufficient conditions for \( A^{(n)} \{1, 2\} \cdots A^{(1)} \{1, 2\} \subseteq (A^{(1)} \cdots A^{(n)}) \{1, 2\} \); in Section 4 we will discuss necessary and sufficient condition for \( A^{(n)} \{1, 2\} \cdots A^{(1)} \{1, 2\} = (A^{(1)} \cdots A^{(n)}) \{1, 2\} \); in Section 5 we will conclude the paper with some remarks and further research.

2. Necessary and sufficient conditions for \( A^{(n)} \{1\} \cdots A^{(1)} \{1\} \subseteq A^{(1)} \cdots A^{(n)} \{1\} \)

In this section we will derive necessary and sufficient conditions for one side inclusion relation \( A^{(n)} \{1\} \cdots A^{(1)} \{1\} \subseteq (A^{(1)} \cdots A^{(n)}) \{1\} \). In terms of Lemmas 1.1 and 1.2 we now present the following result.

Theorem 2.1. Let the integer \( n \geq 3 \). Suppose that \( A^{(j)} \in \mathbb{C}^{m_j \times m_{j+1}} \) for \( j = 1, \ldots, n \) and let the multiple P-SVD of \( A^{(1)} \cdots A^{(n)} \) be as in (2) and (3). Then \( A^{(n)} \{1\} \cdots A^{(1)} \{1\} \subseteq (A^{(1)} \cdots A^{(n)}) \{1\} \), if and only if one of the following conditions holds:

(9) \[
\begin{align*}
& (a) \quad r_1^n = 0, \quad \text{or} \\
& (b) \quad r_1^n > 0, \quad m_{i+1} - r_i = r_{i+1}^{i+1} \quad \text{for } i = 1, \ldots, n-1 \quad \text{and} \\
& \quad r_{i+1}^{i+1} = r_{i+2}^{i+1} = \ldots = r_{n+1}^{n+1} \quad \text{for } i = 1, \ldots, n-2.
\end{align*}
\]

Proof. Necessity. In the expressions of \( X^{(j)} \) in (5)–(6) for \( j = 1, \ldots, n \) we use \( X_{ij}^{(j)} \) to denote a special choice of \( X^{(j)} \) in which we set all sub-matrices \( (X_{ij}^{(j)})_{kl} = 0 \) except those identity matrices:

\[
X_{s}^{(1)} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \end{pmatrix}, \quad X_{ij}^{(j)} = \begin{pmatrix} I & 0 & 0 & \ldots & 0 & 0 \\ 0 & I & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \end{pmatrix}.
\]
For given integers $i, j$ with $1 \leq i < j \leq n$, we take $X^{(p)} = X^{(p)}_s$ if $p \neq i$ and $p \neq j$. Let $Y = X^{(n)} \cdots X^{(1)}$. Then we figure out after some calculations

\[
\begin{pmatrix}
I & 0 \\
0 & 1
\end{pmatrix}
Y
\begin{pmatrix}
I & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
r_1^1 \\
m_{n+1}\cdots r_1^1
\end{pmatrix}
\begin{pmatrix}
r_1^1 \\
m_{n+1}\cdots r_1^1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
I & 0 & \cdots & 0
\end{pmatrix}
X^{(n)} \cdots X^{(j)} \cdots X^{(i)} \cdots X^{(1)}
\begin{pmatrix}
I & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
r_1^1 \\
m_{n+1}\cdots r_1^1
\end{pmatrix}
\begin{pmatrix}
r_1^1 \\
m_{n+1}\cdots r_1^1
\end{pmatrix}
\]

\[
= \left[ I \quad \cdots \quad 0 \quad (\hat{X}^{(j)})_{1,2i+2} \quad 0 \right]
\begin{pmatrix}
I & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
r_1^1 \\
m_{n+1}\cdots r_1^1
\end{pmatrix}
\begin{pmatrix}
r_1^1 \\
m_{n+1}\cdots r_1^1
\end{pmatrix}
\]

\[
= I_{r_1^1} + (0 \hat{X}^{(i)})_{1,2i+2} \hat{X}^{(i)}_{1,1}
\]

in which $\hat{X}^{(j)}_{1,2i+2}$ contains the first $r_1^1$ rows of $X^{(j)}_{1,2i+2}$ and $\hat{X}^{(i)}_{1,1}$ contains the first $r_1^1$ columns of $X^{(i)}_{1,1}$,

\[
b_{i,j}^1 = \begin{cases}
r_{j+1}^1 - r_{i+1}^1, & j \geq i + 2; \\
 m_{i+1} - r_i - r_{i+1}^1, & j = i + 1; \\
 0, & j = i + 1.
\end{cases}
\]

If $W_{n+1}YW_{1}^{-1}$ is a g-inverse of $A^{(1)} \cdots A^{(n)}$, then $Y$ should have the same form as $Z$ appeared in (4)–(5). By comparing the left-upper $r_1^1$ by $r_1^1$ principal submatrices of $Y$ and $Z$ we have $I_{r_1^1} = I_{r_1^1} + (0 \hat{X}^{(j)}_{1,2i+2} \hat{X}^{(i)}_{1,1}$ or equivalently, $(0 \hat{X}^{(j)}_{1,2i+2} \hat{X}^{(i)}_{1,1} = 0$. Because both $\hat{X}^{(j)}_{1,2i+2}$ and $\hat{X}^{(i)}_{1,1}$ can be arbitrarily chosen, the above equation holds if and only if $r_1^1 = 0$ or $b_{i,j}^1 = 0$ such that $\hat{X}^{(j)}_{1,2i+2}$ is naught. Notice that these conditions hold for all $1 \leq i < j \leq n$.

(a) If $r_1^1 = 0$, then we obtain the first condition of the theorem.

(b) If $r_1^1 > 0$, let $i = j - 1 = 1, \ldots, n - 1$, then we obtain from $b_{i,j}^1 = 0$ that

\[
m_{i+1} - r_i = r_{i+1}^1 \quad \text{for} \quad i = 1, \ldots, n - 1.
\]

Let $i = 1, \ldots, n - 2$ and $j = i + 2, \ldots, n$, then we obtain $b_{i,j} = 0 = r_{j+1}^1 - r_j^1$, or equivalently,

\[
r_2^2 = r_3^2 = \cdots = r_{n-1}^2,
\]

\[
r_{i+1}^1 = r_{i+2}^1 = \cdots = r_{n}^1,
\]

proving the necessity part.

**Sufficiency.** (a) Suppose $r_1^1 = 0$. Then from (4)–(5) any $m_{n+1}$ by $m_1$ matrix is a generalized inverse of $A^{(1)} \cdots A^{(n)}$, so as $W_{n+1}^*X^{(n)} \cdots X^{(1)}W_{1}^{-1}$. 

(b) Suppose \( r_n^1 > 0 \), \( m_{i+1} - r_i = r_{i+1}^1 \) for \( i = 1, \ldots, n-1 \) and \( r_{i+1}^1 = r_{i+2}^1 = \cdots = r_n^1 \equiv r_n^+ \) for \( i = 1, \ldots, n-2 \). For convenience we define \( m_{n+1} - r_n \equiv r_n^+ \) and \( s_j \equiv r_1^+ + \cdots + r_j^+ \) for \( j = 2, \ldots, n+1 \). Then the matrices \( X^{(j)} \) for \( j = 1, \ldots, n \) have the following forms, see (5)–(6),

\[
X^{(1)} = \begin{pmatrix}
I & X^{(1)}_{12} \\
X^{(1)}_{21} & X^{(1)}_{22}
\end{pmatrix} \begin{pmatrix} r_1^1 \\
r_1^1 \end{pmatrix}, \quad X^{(j)} = \begin{pmatrix}
I & X^{(j)}_{12} & 0 \\
0 & X^{(j)}_{22} & I \\
X^{(j)}_{31} & X^{(j)}_{32} & X^{(j)}_{33}
\end{pmatrix} \begin{pmatrix} r_j^1 \\
r_j^1 \\
r_j^1 - r_j^1 \end{pmatrix}.
\]

Therefore by induction we can easily obtain

\[
X^{(n)} = \begin{pmatrix}
I & \times & 0 \\
0 & \times & I \\
\times & \times & \times
\end{pmatrix} \begin{pmatrix} r_n^1 \\
0 \\
\times
\end{pmatrix}, \quad X^{(n)} \cdots X^{(i)} = \begin{pmatrix}
I & \times & 0 \\
0 & \times & I \\
\times & \times & \times
\end{pmatrix} \begin{pmatrix} r_i^1 \\
r_i^1 \\
r_i^1 - r_i^1 \end{pmatrix} \begin{pmatrix} s_n \\
0 \\
\times
\end{pmatrix}, \quad \text{for } i = n-1, \ldots, 2,
\]

so

\[
Y \equiv X^{(n)} X^{(n-1)} \cdots X^{(1)} = \begin{pmatrix}
I_{r_n^1} & \times \\
\times & \times
\end{pmatrix}
\]

and so \( W_{n+1} Y W_1^{-1} \in (A^{(1)} \cdots A^{(n)}) \{1\} \), proving the sufficiency part. \( \square \)

Now we state the equivalent conditions for \( A^{(n)} \{1\} \cdots A^{(1)} \{1\} \subseteq (A^{(1)} \cdots A^{(n)}) \{1\} \).

**Theorem 2.2.** Let the integer \( n \geq 3 \). Suppose that \( A^{(j)} \in C_{m_j \times m_{j+1}} \) for \( j = 1, \ldots, n \) and let the multiple P-SVD of \( A^{(1)}, \ldots, A^{(n)} \) be as in (2) and (3). Then the following conditions are equivalent:

(i) \( A^{(n)} \{1\} \cdots A^{(1)} \{1\} \subseteq (A^{(1)} \cdots A^{(n)}) \{1\} \),

(ii) \( r_n^1 = 0 \), or

(a) \( r_n^1 > 0 \), \( m_{i+1} - r_i = r_{i+1}^1 \) for \( i = 1, \ldots, n-1 \) and \( r_{i+1}^1 = r_{i+2}^1 = \cdots = r_n^1 \) for \( i = 1, \ldots, n-2 \),

(iii) \( r_n^1 > 0 \) and for \( i = 1, \ldots, n-1 \),

(a) \( \text{rank}(A^{(1)} \cdots A^{(n)}) = 0 \), or

(b) \( \text{rank}(A^{(1)} \cdots A^{(n)}) > 0 \) and for \( i = 1, \ldots, n-1 \), \( \text{rank}(A^{(1)} \cdots A^{(i)}) + \text{rank}(A^{(i+1)}) - \text{rank}(A^{(1)} \cdots A^{(i+1)}) = m_{i+1} \).
(iv) (a) \( R(A^{(i+1)} \cdots A^{(n)}) \subseteq A^{-1}(A^{(i)} \cdots A^{(i)}) \)
for some \( i \) with \( 1 \leq i \leq n - 1 \), or
(b) \( N(A^{(i)} \cdots A^{(i)}) \subseteq R(A^{(i+1)}) \) for \( i = 1, \ldots, n - 1 \).

**Proof.** (i) \( \iff \) (ii). The equivalence between (i) and (ii) is proved in Theorem 2.1.

(ii) \( \iff \) (iii). From the multiple P-SVD of \( A^{(1)}, \ldots, A^{(n)} \) mentioned in Lemma 1.1,

\[
\begin{align*}
\text{rank}(A^{(1)} \cdots A^{(i)}) &= r_n^1, \quad \text{rank}(A^{(1)} \cdots A^{(i)}) = r_i^1, \\
\text{rank}(A^{(i+1)}) &= r_{i+1}, \quad \text{rank}(A^{(i)} \cdots A^{(i+1)}) = r_{i+1}^1
\end{align*}
\]

for \( i = 1, \ldots, n - 1 \). So the equivalence between (ii)-(a) and (iii)-(a) is obvious.

From the above identities, for \( i = 1 \), (ii)-(b) \( \iff \)

\[
\begin{align*}
m_{i+1} - \text{rank}(A^{(1)} \cdots A^{(i)}) - \text{rank}(A^{(i+1)}) + \text{rank}(A^{(1)} \cdots A^{(i)} A^{(i+1)}) \\
= m_{i+1} - r_i^1 - r_{i+1} + r_i^1 = m_2 - r_1 - r_2^2 = 0 \\
\iff \text{(iii)-(b)}.
\end{align*}
\]

For \( i \geq 2 \), (ii)-(b) \( \iff \)

\[
\begin{align*}
m_{i+1} - \text{rank}(A^{(1)} \cdots A^{(i)}) - \text{rank}(A^{(i+1)}) + \text{rank}(A^{(1)} \cdots A^{(i)} A^{(i+1)}) \\
= (m_{i+1} - r_i - r_{i+1}^1) + (r_i^2 - r_{i+1}^2) + \cdots + (r_i^j - r_i^{j+1}) = 0 \\
\iff \text{(iii)-(b)}.
\end{align*}
\]

(ii) \( \iff \) (iv). (iv)-(a) \( \iff \)

\[ A^{(1)} \cdots A^{(i)} A^{(i+1)} \cdots A^{(n)} = 0 \iff \text{(ii)-(a)} \]

For \( i = 1, \ldots, n - 1 \), partition \( W_{i+1} \) conforming with the partition of the rows of \( D_{i+1} \) in (2)-(3):

\[
W_{i+1} = \begin{pmatrix} W_{i+1,1}, W_{i+1,2}, \ldots, W_{i+1,2i+1}, W_{i+1,2i+2} \end{pmatrix},
\]

Then it can be shown that

\[
N(A^{(1)} \cdots A^{(i)}) = R(W_{i+1,3}, W_{i+1,4}, \ldots, W_{i+1,2i+1}, W_{i+1,2i+2}),
\]

\[
R(A^{(i+1)}) = R(W_{i+1,1}, W_{i+1,3}, \ldots, W_{i+1,2i+1}).
\]

So for \( i = 1 \), (iv)-(b) \( \iff \)

\[
N(A^{(1)}) \subseteq R(A^{(2)}) \iff R(W_{2,4}) = \{0\} \iff \text{(ii)-(b)}
\]

and for \( i \geq 2 \), (iv)-(b) \( \iff \)
\[ N(A^{(1)}, \ldots, A^{(i)}) \subseteq R(A^{(i+1)}) \iff R(W_{i+1,4}, \ldots, W_{i+1,2i}, W_{i+1,2i+2}) = \{0\} \iff (ii)-(b). \]

We thus complete the proof of the theorem. \( \square \)

**Remark.** Notice that when \( n = 2 \), the conditions in (9) reduce to the following conditions
\[
 r_1^1 = 0, \quad \text{or} \quad m_2 - r_1 - r_2^2 = 0,
\]
which are exactly the same conditions obtained in Theorem 2.1 of [13]. So the results obtained in this section are generalization of those in [13].

### 3. Necessary and sufficient conditions for \( A^{(n)}\{1,2\} \cdots A^{(1)}\{1,2\} \subseteq A^{(1)} \cdots A^{(n)}\{1,2\} \)

In this section we will derive necessary and sufficient conditions for one side inclusion relation \( A^{(n)}\{1,2\} \cdots A^{(1)}\{1,2\} \subseteq (A^{(1)} \cdots A^{(n)})\{1,2\} \). In light of Section 2, we only need to separately consider the cases listed in (9) because \( (A^{(1)} \cdots A^{(n)})\{1,2\} \subseteq (A^{(1)} \cdots A^{(n)})\{1\} \).

**Theorem 3.1.** *Let the integer \( n \geq 3 \). Suppose that \( A^{(j)} \in C^{m_j \times m_{j+1}} \) for \( j = 1, \ldots, n \) and let the multiple P-SVD of \( A^{(1)}, \ldots, A^{(n)} \) be as in (2) and (3). If \( r_1^1 = 0 \), then* \( A^{(n)}\{1,2\} \cdots A^{(1)}\{1,2\} \subseteq (A^{(1)} \cdots A^{(n)})\{1,2\} \), *if and only if there exists an integer \( p \) with \( 1 \leq p \leq n \), such that \( r_p = 0 \).*

**Proof.** *Sufficiency.* If there exists an integer \( p \) with \( 1 \leq p \leq n \) such that \( r_p = 0 \), then \( A^{(p)} = 0 \) and so \( A^{(p)}\{1,2\} = \{0\} \) because rank\( (A^{(p)}) = \text{rank}(A^{(p)})_{\text{r}} \), and so \( A^{(n)}\{1,2\} \cdots A^{(1)}\{1,2\} = \{0\} \subseteq (A^{(1)} \cdots A^{(n)})\{1,2\} \).

*Necessity.* If \( r_p > 0 \) for all \( p = 1, \ldots, n \), then for each \( p = 1, \ldots, n \), at least one of \( r_p^1, \ldots, r_p^{n} \) is positive. Denote \( i_p = \min\{l : r_p^l > 0, \ l \in \{1, \ldots, p\}\} \). Obviously \( i_1 = 1 \), and for all \( p = 2, \ldots, n \), \( 1 \leq i_p \leq p \) and \( i_{p-1} \leq i_p \). Furthermore, for some \( p \) if \( i_p < p \), then \( i_{p-1} = i_p \).

Notice that from the definitions of \( i_1, \ldots, i_n \) and the expressions of \( X^{(j)} \) appeared in (5)–(6), for each \( j = 2, \ldots, n \) we have
\[
 X_{i_j, j-1}^{(j)} = \begin{cases} 
 r_{i_j}^j, & \text{for } i_j = i_{j-1}, \\
 \text{is naught}, & \text{for } i_j > i_{j-1} \text{ because } r_{i_j}^{j-1} = 0,
\end{cases}
\]
so \((X^{(j)}_{i,j;2j-1}, X^{(j)}_{i,j;2j-1})\) is an \(r_j^i \times r_{j-1}^i\) matrix, and

\[
\begin{pmatrix}
I_{r_j^i}, X^{(j)}_{i,j;2j-1}, X^{(j)}_{i,j;2j-1}
\end{pmatrix},
\text{ for } i = j-1,
\]

\[
X^{(j)}_{i,j;2j-1},
\text{ for } i > j-1.
\]

(15)

For each \(j = 2, \ldots, n\), in (6) we choose a special form for \(X^{(j)}\) such that all submatrices \(X^{(j)}_{pq}\) = 0 except \(X^{(j)}_{i,j;2j-1}\) and those identity matrices. We also choose

\[
X^{(1)} = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then it is not difficult to verify that

\[
Y = X^{(n)} \cdots X^{(1)} = \begin{pmatrix} (X^{(n)}_{i,n;2n,i-1}, X^{(n)}_{i,n;2n,i-1}) \cdots (X^{(2)}_{i,2,1}, X^{(2)}_{i,2,1}) & 0 \\ 0 & 0 \end{pmatrix}.
\]

With the definitions of \(i_1 (= 1), i_2, \ldots, i_n\) and the expressions of (15), we can choose \(X^{(n)}_{i,n;2n,i-1}, X^{(n)}_{i,n;2n,i-1}, \ldots, (X^{(2)}_{i,2,1}, X^{(2)}_{i,2,1})\), such that \((X^{(a)}_{i,n,bn,i-1}, X^{(a)}_{i,n,bn,i-1}, \ldots, (X^{(2)}_{i,2,1}, X^{(2)}_{i,2,1})\) ≠ 0 so \(W_{n+1}W_{n}^{-1} \not\subset (A^{(1)} \cdots A^{(a)})\{1, 2\} = \{0\}\). So when \(r_n^1 = 0\) and \(r_p > 0\) for all \(p = 1, \ldots, n\) we have \(A^{(a)}\{1, 2\} \cdots A^{(1)}\{1, 2\} \not\subset (A^{(1)} \cdots A^{(a)})\{1, 2\}. \square

Now we consider the second case in (9)-(b) and we have the following result.

**Theorem 3.2.** Let the integer \(n \geq 3\). Suppose that \(A^{(j)} \in C^{m_j \times m_{j+1}}\) for \(j = 1, \ldots, n\) and let the multiple P-SVD of \(A^{(1)}, \ldots, A^{(a)}\) be as in (2) and (3). If \(r_n^1 > 0\) and \(m_{i+1} - r_i = r_i^{i+1} + \cdots + r_i^{n-1}, \text{ for } i = 1, \ldots, n - 1\) and

\[
r_i^{i+1} = r_i^{i+2} = \cdots = r_i^{n-1}, \text{ for } i = 1, \ldots, n - 2,
\]

then \(A^{(a)}\{1, 2\} \cdots A^{(1)}\{1, 2\} \subset (A^{(1)} \cdots A^{(a)})\{1, 2\}\), if and only if one of the following conditions holds:

(a) \(r^2 = \cdots = r^{n-1} = r_n^0 = 0\),

(b) \(r^2 > 0\) and \(r_1^i = \cdots = r_{n-1}^i = r_n^i\),

(c) There exists an integer \(q\) with \(2 \leq q < n\) such that \(r^2 = \cdots = r^q = 0\), \(r_q^{q+1} > 0\) and \(r_q^1 = \cdots = r_{n-1}^q = r_n^q\).

(16)

**Proof.** *Necessity.* Under the conditions of the theorem, the matrices \(X^{(j)}\) for \(j = 1, \ldots, n\) have the forms as in (13). We consider the following three cases.

Case (a): \(r^2 = \cdots = r^{n-1} = r_n^0 = 0\). This is the first condition (16)-(a).

Case (b): \(r^2 > 0\). In this case we will show \(r_1^i = r_2^i = \cdots = r_n^i\) by induction. Notice that in this case \(s_j = r^2 + \cdots + r^j > 0\) for \(j = 2, \ldots, n\).
For $j = n$, let $X^{(p)} = X^{(p)}_s$ if $p \neq j = n$. Then

$$Y \equiv X^{(n)} \cdots X^{(1)} = \begin{pmatrix} I & X^{(n)}_{1,2} & 0 \\ 0 & X^{(n)}_{2,2} & 0 \\ X^{(n)}_{3,1} & X^{(n)}_{3,2} & 0 \end{pmatrix} \begin{pmatrix} r^1_n \\ \vdots \\ r^1_{n-1} \\ r^1_n - r^1_{n-1} \end{pmatrix}. $$

If $W_{n+1}W^{-1}_{W_1} \in (A^{(1)} \cdots A^{(n)}) \{1, 2\}$, then we should have $X^{(n)}_{2,2} = 0$ for any choice of $X^{(n)}_{2,2}$ because rank($Y$) = rank($Z$) = $r^1_n$. So we should have $\min\{r^1_{q-1} - r^1_q, s_n\} = 0 = r^1_{n-1} - r^1_n$. So for $j = n$ we have $r^1_{j-1} = r^1_n$. Suppose that $r^1_{j-1} = r^1_j = r^1_n$ for $n \geq j \geq q + 1$. Then by taking $X^{(p)} = X^{(p)}_s$ if $p \neq q$ and by inserting the assumption $r^1_q = r^1_{q+1} = \cdots = r^1_n$ we have

$$Y \equiv X^{(n)} \cdots X^{(1)} = \begin{pmatrix} I & X^{(q)}_{1,2} & 0 \\ 0 & X^{(q)}_{2,2} & 0 \\ \times & \times & 0 \end{pmatrix} \begin{pmatrix} r^1_q \\ \vdots \\ r^1_{q-1} - r^1_q \\ r^1_n - r^1_{q-1} \end{pmatrix}. $$

If $W_{n+1}W^{-1}_{W_1} \in (A^{(1)} \cdots A^{(n)}) \{1, 2\}$, then we should have $X^{(q)}_{2,2} = 0$ for any choice of $X^{(q)}_{2,2}$ because rank($Y$) = rank($Z$) = $r^1_q$. So we should have $\min\{r^1_{q-1} - r^1_q, s_q\} = 0 = r^1_{q-1} - r^1_q$. So for $j = q$ the assumption is also true.

Therefore by induction we have $r^1_n = r^1_{j-1}$ for $j = n = \cdots = 2$ and we obtain the conditions in (16)-(b).

Case (c): There exists an integer $q$ with $2 < q < n$ such that $r^2 = \cdots = r^q = 0$ and $r^{q+1} > 0$. Then the same strategy used in Case (b) can also be applied to show $r^1_n = \cdots = r^1_{q+1} = r^1_q$. Then we obtain the conditions in (16)-(c).

Sufficiency. Notice that under the conditions of the theorem, $X^{(1)}, \ldots, X^{(n)}$ have the forms in (13) with

$$X^{(1)}_{2,2} = X^{(1)}_{2,1}X^{(1)}_{1,2}, \quad \text{and} \quad X^{(j)}_{3,2} = X^{(j)}_{3,1}X^{(j)}_{1,2} + X^{(j)}_{3,3}X^{(j)}_{2,2}. $$

(17)

Case (a). If $r^2 = \cdots = r^{n-1} = r^n = 0$, then from (13) and (17), $X^{(1)}, \ldots, X^{(n)}$ have the forms

$$X^{(j)} = \begin{pmatrix} I_{r^1_j} & X^{(j)}_{1,2} \end{pmatrix}, \quad \text{for } j = 1, \ldots, n - 1, $$

$$X^{(n)} = \begin{pmatrix} I_{r^1_n} \\ X^{(n)}_{3,1} \\ X^{(n)}_{3,2} \end{pmatrix} = \begin{pmatrix} I_{r^1_n} \\ I_{r^1_{n-1}}X^{(n)}_{1,2} \end{pmatrix}, $$

with $r^1_j = r_j \leq r^1_{j-1} = r_{j-1}, j = 2, \ldots, n$. Then by induction we can easily verify that

$$X^{(j)} \cdots X^{(1)} = \begin{pmatrix} I_{r^1_j} \end{pmatrix}. $$
for \( j = 2, \ldots, n - 1 \). So finally

\[
Y \equiv X^{(n)} \cdots X^{(1)} = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right) \left( \begin{array}{c} L_{r_1^n} \\ \times \\ \end{array} \right)
\]

and so \( W_n Y W_1^{-1} \in (A^{(1)} \cdots A^{(n)})\{1, 2\} \).

Case (b). Under the conditions in the theorem and (16)-(b), we have from (13) and (17),

\[
X^{(1)} = \left( \begin{array}{c} I \\ X_{2,1}^{(1)} \\ X_{1,2}^{(1)} \\ r_1^1 \\ r_2^1 \\ r_1^2 \\ r_2^2 \\ m_1 - r_1^1 \\ m_2 - r_1^2 \\ \end{array} \right) = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right) \left( \begin{array}{c} L_{r_1^n} \\ \times \\ \end{array} \right)
\]

for \( j = 2, \ldots, n \) with \( r_1^n = r_1^1 = r_1 < r_2 < \cdots < r_n \). So it is not hard to verify that

\[
Y \equiv X^{(n)} \cdots X^{(1)} = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right) \left( \begin{array}{c} L_{r_1^n} \\ \times \\ \end{array} \right)
\]

and so \( W_n Y W_1^{-1} \in (A^{(1)} \cdots A^{(n)})\{1, 2\} \).

Case (c). Under the conditions in the theorem and (16)-(c), we have from (13) and (17),

\[
X^{(j)} = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right), \quad \text{for } j = 1, \ldots, q - 1
\]

\[
X^{(q)} = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right) \left( \begin{array}{c} L_{r_1^n} \\ \times \\ \end{array} \right), \quad \text{and}
\]

\[
X^{(j)} = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right), \quad \text{for } j = q + 1, \ldots, n,
\]

with \( r_1^n = r_1^q \). So it is not hard to verify that

\[
Y \equiv X^{(n)} \cdots X^{(1)} = \left( \begin{array}{c} I_{r_1^n} \\ \times \\ \end{array} \right) \left( \begin{array}{c} L_{r_1^n} \\ \times \\ \end{array} \right)
\]

and so \( W_n Y W_1^{-1} \in (A^{(1)}(n))\{1, 2\} \).

Thus we complete the proof of the theorem. \( \Box \)

**Remark.** For \( n = 2 \) the necessary and sufficient conditions in Theorem 3.1 reduce to the condition

\[
r_1 = 0, \quad \text{or}, \quad r_2 = 0,
\]

the necessary and sufficient conditions in Theorem 3.2 reduce to the condition

\[
r_1 = m_2, \quad \text{or}, \quad r_2 = m_2.
\]

These are exactly the same conditions derived in [4]. So the results in Theorems 3.1 and 3.2 generalize those in [4].
By combining the results of Theorems 3.1 and 3.2, we now present the main results of this section.

**Theorem 3.3.** Suppose that $A^{(j)} \in \mathbb{C}^{m_j \times m_{j+1}}$ for $j = 1, \ldots, n$ and let the multiple P-SVD of $A^{(1)}, \ldots, A^{(n)}$ be as in (2) and (3). Then the following conditions are equivalent.

(i) $A^{(n)}\{1, 2\} \cdots A^{(1)}\{1, 2\} \subseteq (A^{(1)} \cdots A^{(n)})\{1, 2\},$

(ii) (a) $r_1^n > 0, m_1 \geq \cdots \geq m_n$ and $r_j = m_{j+1}$ for $j = 1, \ldots, n-1$, or
(b) $r_1^n > 0, m_2 \leq \cdots \leq m_{n+1}$, and $r_j = m_j$ for $j = 2, \ldots, n$,
(c) $r_1^n > 0$ and there exists an integer $q$ with $2 \leq q < n$ such that $m_1 \geq \cdots \geq m_q$ and $r_j = m_{j+1}$ for $j = 1, \ldots, q-1$

$\quad \quad m_q \leq \cdots \leq m_{n+1}$ and $r_j = m_j$ for $j = q, \ldots, n$,
(d) There exists an integer $q$ with $1 \leq q \leq n$, such that $r_q = 0$,

(iii) (a) rank $(A^{(1)} \cdots A^{(n)}) > 0$ and $A^{(j)}$ are of full column rank for $j = 1, \ldots, n-1$, or
(b) rank $(A^{(1)} \cdots A^{(n)}) > 0$, and $A^{(j)}$ are of full row rank for $j = 2, \ldots, n$,
(c) rank $(A^{(1)} \cdots A^{(n)}) > 0$ and there exists an integer $q$ with $2 \leq q < n$, such that $A^{(j)}$ are of full column rank for $j = 1, \ldots, q-1$ and $A^{(j)}$ are of full row rank for $j = q, \ldots, n$,
(d) There exists an integer $q$ with $1 \leq q \leq n$, such that rank $(A^{(q)}) = 0$.

**Proof.** The equivalence between (i) and (ii) are implied by Theorems 3.1 and 3.2. In fact, from Theorems 3.1 and 3.2, the conditions in (9)-(b) and (16)-(a) are equivalent to say that the matrices $X^{(j)}$ have the forms expressed in (18), which are just the same conditions as in (ii)-(a) of the theorem; the conditions in (9)-(b) and (16)-(b) are equivalent to say that the matrices $X^{(j)}$ have the forms expressed in (19), which are just the same conditions as in (ii)-(b) of the theorem; the conditions in (9)-(b) and (16)-(c) are equivalent to say that the matrices $X^{(j)}$ have the forms expressed in (20), which are just the same conditions as in (ii)-(c) of the theorem; the condition expressed in (ii)-(d) of the theorem is proved in Theorem 3.1. With the multiple SVD of $A^{(1)}, \ldots, A^{(n)}$ in (2) and (3), the equivalence between (ii) and (iii) is obvious. □

4. Necessary and sufficient conditions for $A^{(n)}\{1, 2\}, \ldots, A^{(1)}\{1, 2\} = (A^{(1)}, \ldots, A^{(n)})\{1, 2\}$

By applying the results of Section 3 and Corollary 9 of [11], we have the following equivalent results.
Theorem 4.1. Suppose that $A^{(j)} \in C_{m_j \times m_{j+1}}$ for $j = 1, \ldots, n$ and let the multiple P-SVD of $A^{(1)}, \ldots, A^{(n)}$ be as in (2) and (3). Then the following conditions are equivalent.

(i) $A^{(n)}\{1, 2\} \cdots A^{(1)}\{1, 2\} = (A^{(1)} \cdots A^{(n)})\{1, 2\}$,
(ii) $A^{(n)}\{1, 2\} \cdots A^{(1)}\{1, 2\} \subseteq (A^{(1)} \cdots A^{(n)})\{1, 2\}$,
(iii) (a) $r^1_n > 0, m_1 > \cdots > m_n$ and $r_j = m_{j+1}$ for $j = 1, \ldots, n - 1$, or
(b) $r^1_n > 0, m_2 \leq \cdots \leq m_{n+1}$ and $r_j = m_j$ for $j = 2, \ldots, n$,
(c) $r^1_n > 0$ and there exists an integer $q$ with $2 \leq q < n$
    such that $m_1 > \cdots > m_q$ and $r_j = m_{j+1}$ for $j = 1, \ldots, q - 1$
    and $m_q \leq \cdots \leq m_{n+1}$ and $r_j = m_j$ for $j = q, \ldots, n$,
(d) There exists an integer $q$ with $1 \leq q \leq n$, such that $r_q = 0$,
(iv) (a) rank$(A^{(1)} \cdots A^{(n)}) > 0$ and $A^{(j)}$ are
    of full column rank for $j = 1, \ldots, n - 1$, or
(b) rank$(A^{(1)} \cdots A^{(n)}) > 0$, and $A^{(j)}$ are
    of full row rank for $j = 2, \ldots, n$,
(c) rank$(A^{(1)} \cdots A^{(n)}) > 0$ and there exists an integer $q$ with $2 \leq q < n$
    such that $A^{(j)}$ are of full column rank for $j = 1, \ldots, q - 1$
    and $A^{(j)}$ are of full row rank for $j = q, \ldots, n$,
(d) There exists an integer $q$ with $1 \leq q \leq n$
    such that rank$(A^{(q)}) = 0$.

Proof. From Theorem 3.3, it suffices to prove the condition (i) of the theorem is equivalent to the condition (ii) of the theorem. Notice that the condition in (i) of the theorem is equivalent to the following two conditions

$$A^{(n)}\{1, 2\} \cdots A^{(1)}\{1, 2\} \subseteq (A^{(1)} \cdots A^{(n)})\{1, 2\}, \quad \text{and}$$

$$(A^{(1)} \cdots A^{(n)})\{1, 2\} \subseteq A^{(n)}\{1, 2\} \cdots A^{(1)}\{1, 2\},$$

so it suffices to prove that in any case the second inclusion relation above always holds. We prove this relation by induction. When $n = 2$ it is well known (Corollary 9 of [11], also see Theorem 3.1 of [4]) that $(A^{(1)}A^{(2)})\{1, 2\} \subseteq A^{(2)}\{1, 2\}A^{(1)}\{1, 2\}$.

Now suppose that for $2 \leq n \leq t$ the assertion is true. For $n = t + 1$, let $A^{(1)} \cdots A^{(t)} \equiv B$. By applying the above relation one more time, we obtain $(BA^{(t+1)})\{1, 2\} \subseteq A^{(t+1)}\{1, 2\}B\{1, 2\}$. Now from the assumption of the induction,

$$(A^{(1)} \cdots A^{(t)}\{1, 2\} \subseteq A^{(t)}\{1, 2\} \cdots A^{(1)}\{1, 2\},$$

so we have

$$(A^{(1)} \cdots A^{(t)}A^{(t+1)})\{1, 2\} \subseteq A^{(t+1)}\{1, 2\}(A^{(1)} \cdots A^{(t)})\{1, 2\}$$

$$\subseteq A^{(t+1)}\{1, 2\}A^{(t)}\{1, 2\} \cdots A^{(1)}\{1, 2\}. $$
So for any integer \( n \geq 2 \), the assertion is true and the equivalence between (i) and (ii) is proved. The equivalences between (ii), (iii) and (iv) are proved in Theorem 3.3. \( \square \)

5. Concluding remarks

For given matrices \( A^{(j)} \in \mathbb{C}^{m_j \times m_{j+1}} \) for \( j = 1, \ldots, n \), in this paper we have derived equivalent conditions for one side inclusion relation of g-inverses
\[
A^{(n)}\{1\} \cdots A^{(1)}\{1\} \subseteq (A^{(1)} \cdots A^{(n)})\{1\}
\]
and equivalent conditions for reverse order law of reflexive g-inverses
\[
A^{(n)}\{1,2\} \cdots A^{(1)}\{1,2\} = (A^{(1)} \cdots A^{(n)})\{1,2\},
\]
by applying the multiple P-SVD of the matrices \( A^{(1)}, \ldots, A^{(n)} \). Therefore in this paper we have established reverse order law for reflexive g-inverses of multiple matrix products and partially established reverse order law for g-inverses of multiple matrix products. Equivalent conditions of one side inclusion relation
\[
(A^{(1)} \cdots A^{(n)})\{1\} \subseteq A^{(n)}\{1\} \cdots A^{(1)}\{1\},
\]
remains an open problem for further investigation.

References