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Existence, multiplicity, and nonexistence of positive solutions to a differential equation on a measure chain

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Abstract

We are concerned with proving existence of one or more than one positive solution of a general two point boundary value problem for the nonlinear equation $Lx(t) := -[p(t)x^{\Delta}(t)]^{\Delta} + q(t)x^{\sigma}(t) = \lambda a(t) f(t, x^{\sigma}(t))$. We shall also obtain criteria which leads to nonexistence of positive solutions. Here the independent variable t is in a "measure chain". We will use fixed point theorems for operators on a Banach space. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

We are concerned with proving the existence, nonexistence, and multiplicity results for positive solutions to the boundary value problem

$$Lx(t) := -[p(t)x^{\Delta}(t)]^{\Delta} + q(t)x^{\sigma}(t) = \lambda a(t) f(t, x^{\sigma}(t)), \tag{1}$$

$$\alpha x(a) - \beta x^{\Delta}(a) = 0, \tag{2}$$

$$\gamma x(\sigma^2(b)) + \delta x^{\Delta}(\sigma(b)) = 0. \tag{3}$$

To understand this so-called differential equation (1) on a measure chain (time scale) $\mathbb{T}$ we need some preliminary definitions.

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Definition. Let $\mathbb{T}$ be a nonempty closed subset of the real numbers $\mathbb{R}$ and define the forward jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ by

$$\sigma(t) := \inf\{\tau > t: \tau \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ at t for $t > \inf \mathbb{T}$ by

$$\rho(t) := \sup\{\tau < t: \tau \in \mathbb{T}\}$$

for all $t \in \mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say t is left scattered. If $\sigma(t) = t$ we say t is right dense, while if $\rho(t) = t$ we say t is left dense.

Throughout this paper we make the blanket assumption that $a \leqslant b$ are points in $\mathbb{T}$.

Definition. Define the interval in $\mathbb{T}$

$$[a, b] := \{t \in \mathbb{T} \text{ such that } a \leqslant t \leqslant b\}.$$

Other types of intervals are defined similarly.

We are concerned with calculus on measure chains which is a unified approach to continuous and discrete calculus. An excellent introduction is given by Hilger [17]. Agarwal and Bohner [1] refer to it as calculus on time scales. Other papers in this area include Agarwal and Bohner [3], Agarwal, Bohner, and Wong [4], Hilger and Erbe [9], and Erbe and Peterson [11,12] and the book [18].

Definition. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$ such that $t < \sup \mathbb{T}$, then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We call $x^{\Delta}(t)$ the delta derivative of $x(t)$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$, $t < \sup \mathbb{T}$, and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, then

$$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t).$$

In particular if $\mathbb{T} = \mathbb{Z}$, then the equation $Lx(t) = 0$ is the self-adjoint difference equation

$$Lx(t) = -\Delta[p(t)\Delta x(t)] + q(t)x(t+1) = 0.$$

See the books [2,19] and the references there for results concerning this self-adjoint difference equation. Of course if $T = \mathbb{R}$, then the equation $Lx(t) = 0$ reduces to the self-adjoint differential equation

$$Lx(t) = -[p(t)x'(t)]' + q(t)x(t) = 0,$$

which has been studied extensively over the years. See Coppel [6] for a good presentation of results for this differential equation.

Definition. If $F^{\Delta}(t)=f(t)$, then we define an integral by

$$\int_a^t f(\tau)\Delta\tau = F(t)-F(a).$$

In this paper we will use elementary properties of this integral that either are in Refs. [2–6] or are easy to verify.

2. Main results

Throughout the rest of this paper we make the following assumptions: $p(t)>0$ is delta differentiable on $[a,\sigma^2(b)]$, $q(t)\geqslant 0$ is right-dense continuous on $[a,b]$, $a(t)\geqslant 0$ is right-dense continuous on $[a,b]$, $f:\mathbb{T}\times\mathbb{R}\to[0,\infty)$, is continuous, and

$$\alpha,\beta,\gamma,\quad \delta\geqslant 0,\quad \alpha^2+\beta^2>0,\quad \gamma^2+\delta^2>0.$$

Let $\mathbb{D}:=\{x:[a,\sigma^2(b)]\to\mathbb{R}:x^{\Delta}(t)$ *is continuous on* $[a,\sigma^2(b)]$ *and* $[p(t)x^{\Delta}(t)]^{\Delta}$ *is right-dense continuous on* $[a,b]\}$. By a solution $x(t)$ of (1) on $[a,\sigma^2(b)]$ we mean $x\in\mathbb{D}$ and Eq. (1) holds for $t\in[a,b]$. Before we define the cone we will be working with we need some preliminary results.

Fix $\tau\in[a,\sigma^2(b)]$. For $a<\tau\leqslant\sigma^2(b)$, define $g_\tau(t)$ to be the solution of the boundary value problem

$$Lg_\tau(t)=0,$$

$$g_\tau(a)=0,\quad g_\tau(\tau)=1.$$

(It can be shown that $Lx(t)=0$ is disconjugate on $[a,\sigma^2(b)]$ and hence the above boundary value problem has a unique solution.) Similarly if $a\leqslant\tau<\sigma^2(b)$, then we let $h_\tau(t)$ be the unique solution of the boundary value problem

$$Lh_\tau(t)=0,$$

$$h_\tau(a)=0,\quad h_\tau(\tau)=1.$$

Then we define $\omega(t,\tau)=g_\tau(t)$, if $\tau=\sigma^2(b)$, $\omega(t,\tau)=h_\tau(t)$, if $\tau=a$, and if $a<\tau<\sigma^2(b)$,

$$\omega(t,\tau)=\begin{cases} g_\tau(t), & a\leqslant t\leqslant\tau,\\ h_\tau(t), & \tau\leqslant t\leqslant\sigma^2(b).\end{cases}$$

In the proof of the next lemma we will use the following maximum principle which appears in [13].

Theorem 1 (Maximum principle). *Assume* $p(t)>0$ *is delta differentiable on* $[a,\sigma(b)]$ *and* $q(t)\geqslant 0$ *on* $[a,b]$. *If* $z\in\mathbb{D}$ *is a solution of the differential inequality*

$$Lz(t)\leqslant 0$$

for $t\in[a,b]$ *such that* $z(t)$ *has a nonnegative maximum on* $[a,\sigma^2(b)]$, *then the maximum of* $z(t)$ *occurs at* a *or* $\sigma^2(b)$.

Lemma 2. *If $u \in E$, $u(t) \geqslant 0$ for $t \in [a, \sigma^2(b)]$, $Lu(t) \geqslant 0$ for $t \in [a, b]$ and we choose $\tau_0 \in [a, \sigma^2(b)]$ so that $u(\tau_0) = ||u||$, then*

$$u(t) \geqslant \omega(t, \tau_0)||u|| \geqslant \min\{g_{\tau_0}(t), h_{\tau_0}(t)\}||u|| \geqslant k(t)||u||,$$

where

$$k(t) := g_{\sigma^2(b)}(t) h_a(t) \tag{4}$$

for $t \in [a, \sigma^2(b)]$.

Proof. First assume $a < \tau_0 \leqslant \sigma^2(b)$. Then for $t \in [a, \tau_0]$, let

$$z_1(t) := \omega(t, \tau_0)||u|| - u(t) = g_{\tau_0}(t)||u|| - u(t).$$

Note that

$$Lz_1(t) = ||u|| L g_{\tau_0}(t) - Lu(t) = -Lu(t) \leqslant 0, \quad t \in [a, \rho^2(\tau_0)].$$

Also,

$$z_1(a) = g_{\tau_0}(a)||u|| - u(a) = -u(a) \leqslant 0$$

and

$$z_1(\tau_0) = g_{\tau_0}(\tau_0)||u|| - u(\tau_0) = ||u|| - u(\tau_0) = 0.$$

Hence $z_1(t)$ has a nonnegative maximum in $[a, \tau_0]$. By the maximum principle $z_1(t)$ has a maximum at the end point τ_0. Therefore $z_1(t) \leqslant 0$ on $[a, \tau_0]$ which gives us the desired result

$$u(t) \geqslant \omega(t, \tau_0), \quad t \in [a, \tau_0].$$

Next assume that $a \leqslant \tau_0 < \sigma^2(b)$. Then for $t \in [\tau_0, \sigma^2(b)]$, let

$$z_2(t) := \omega(t, \tau_0)(t)||u|| - u(t) = h_{\tau_0}(t)||u|| - u(t).$$

Note that

$$Lz_2(t) = ||u|| L h_{\tau_0}(t) - Lu(t) = -Lu(t) \leqslant 0, \quad t \in [\tau_0, b],$$

Also,

$$z_2(\tau_0) = h_{\tau_0}(\tau_0)||u|| - u(\tau_0) = ||u|| - u(\tau_0) = 0$$

and

$$z_2(\sigma^2(b)) = h_{\tau_0}(\sigma^2(b))||u|| - u(\sigma^2(b)) = -u(\sigma^2(b)) \leqslant 0.$$

Hence $z_2(t)$ has a nonnegative maximum in $[\tau_0, \sigma^2(b)]$. By the maximum principle $z(t)$ has a maximum at the end point τ_0, Therefore $z_2(t) \leqslant 0$ *on* $[a, \tau_0]$ which gives us the desired result

$$u(t) \geqslant \omega(t, \tau_0)||u||, \quad t \in [\tau_0, \sigma^2(b)].$$

Putting these two cases together we have

$$u(t) \geqslant \omega(t, \tau_0)||u||, \quad t \in [a, \sigma^2(b)],$$

which is the first inequality in the statement of the theorem that we wanted to prove. Using elementary arguments one can prove that for each fixed $\tau \in (a, \sigma^2(b)]$, $g_\tau(t)$ is a strictly increasing function of t for $t \in [a,\sigma^2(b)]$ and for each fixed $\tau \in [a,\sigma^2(b))$, $h_{\tau_0}(t)$ is a strictly decreasing function of t for $t \in [a,\sigma^2(b)]$. It follows that we get the second inequality in the statement of the theorem

$$u(t) \geqslant \min\{g_{\tau_0}(t), h_{\tau_0}(t)\}||u||.$$

Since $Lx(t)=0$ is disconjugate on $[a,\sigma^2(b)]$, we get that if $a \leqslant \tau_1 < \tau_2 \leqslant \sigma^2(b)$, then

$$g_{\tau_1}(t) < g_{\tau_2}(t)$$

for $t \in (a,\sigma^2(b)]$ and

$$h_{\tau_1}(t) > h_{\tau_2}(t)$$

for $t \in [a,\sigma^2(b))$. Using this we get that

$$u(t) \geqslant \min\{g_{\tau_0}(t), h_{\tau_0}(t)\}||u|| \geqslant g_{\sigma^2(b)}(t)h_a(t)||u|| = k(t)||u||$$

for $t \in [a,\sigma^2(b)]$, which is the last inequality in the statement of this theorem. □

Define E to be the Banach space

$$E = C[a,\sigma^2(b)],$$

where the norm on E is the sup norm. We then define a cone P in E in terms of the function $k(t)$ given by (4) that appears in the above lemma by

$$P = \{x \in E\colon x(t) \geqslant k(t)||x||,\ t \in [a,\sigma^2(b)]\}.$$

Let $G(t,s)$ be the Green's function for the boundary value problem (BVP) $Lx(t)=0$, (2), (3). Under the above assumptions it can be shown (see [11]) that this Green's function is of the form

$$G(t,s) = \begin{cases} \frac{1}{c}\phi(t)\psi(\sigma(s)), & t \leqslant s, \\ \frac{1}{c}\phi(\sigma(s))\psi(t), & \sigma(s) \leqslant t, \end{cases}$$

where $\phi(t)$ and $\psi(t)$ are the solutions of the "initial value problems"

$$L\phi(t) = 0, \qquad \phi(a) = \beta, \qquad \phi^{\Delta}(a) = \alpha, \tag{5}$$

$$L\psi(t) = 0, \qquad \psi(\sigma^2(b)) = \delta, \qquad \psi^{\Delta}(\sigma(b)) = -\gamma, \tag{6}$$

respectively, and

$$c := p(t)[\psi(t)\phi^{\Delta}(t) - \psi^{\Delta}(t)\phi(t)]. \tag{7}$$

It can be shown that $\phi(t) \geqslant 0$ on $[a,\sigma^2(b)]$ and is nondecreasing on $[a,\sigma^2(b)]$ and that $\psi(t) \geqslant 0$ on $[a,\sigma^2(b)]$ and is nonincreasing on $[a,\sigma^2(b))$. This and the assumption that 0 is not an eigenvalue of $Lx(t) = \lambda x^{\sigma}(t)$, (2), (3) implies that c is a positive constant and

$$0 \leqslant G(t,s) \leqslant G(\sigma(s),s) \tag{8}$$

for $a \leqslant t \leqslant \sigma^2(b),\ a \leqslant s \leqslant b$.

It is not difficult to show that the eigenvalue problem (1)–(3) having a solution is equivalent to the fixed point equation

$$u = Au, \qquad u \in E := C[a, \sigma^2(b)], \tag{9}$$

having a solution, where the operator A is defined by

$$Au(t) = \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s, x^\sigma(s))\,\Delta s, \tag{10}$$

$t \in [a, \sigma^2(b)]$.

Also define, for r a positive number, P_r by

$$P_r = \{x \in P\colon ||x|| \leqslant r\}.$$

We refer to [7,20] for a discussion of the fixed point index that we use below. In particular, we will make frequent use of the following lemma.

Lemma 3. *Let E be a Banach space, $P \subset E$ a cone in E. Assume $r > 0$ and that A: $P_r \to P$ is a compact operator such that $Ax \neq x$ for $x \in \partial P_r := \{x \in P : ||x|| = r\}$.*

(a) *If $||x|| \leqslant ||Ax||$ for $x \in \partial K_r$, then*

$$i(A, P_r, P) = 0.$$

(b) *If $||x|| \geqslant ||Ax||$ for $x \in \partial K_r$, then*

$$i(A, P_r, P) = 1.$$

As a consequence of this lemma it follows that if there exist distinct $r_1, r_2 > 0$ such that condition (a) holds for $x \in \partial K_{r_1}$ and (b) holds for $x \in \partial K_{r_2}$, then A has a fixed point (nonzero) whose norm is between r_1 and r_2.

Define the nonnegative extended real numbers f_0, f^0, f_∞, and f^∞ by

$$f_0 := \liminf_{u \to 0+} \min_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u},$$

$$f^0 := \limsup_{u \to 0+} \max_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u},$$

$$f_\infty := \liminf_{u \to \infty} \min_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u},$$

$$f^\infty := \limsup_{u \to \infty} \max_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u},$$

respectively.

These numbers can be regarded as generalized super or sublinear conditions on the function $f(t,u)$ at $u = 0$ and $u = \infty$. Thus, if $f_0 = f^0 = 0$ $(+\infty)$, then $f(t,u)$ is superlinear (sublinear) at $u = 0$ and if $f_\infty = f^\infty = 0$ $(+\infty)$, then $f(t,u)$ is sublinear (superlinear) at $u = +\infty$. Theorem 4 below applies to the case when these numbers may be finite. In addition, Theorem 5 applies when one (or more than one) of the numbers f_0, f^0, f_∞, f^∞ is 0 or ∞. (See also the remark following the proof of Theorem 5.)

Let

$$A_1 := \max_{t\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)a(s)\,\Delta s$$

We assume that $\mathbb{T}$ is a measure chain so that we can pick $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$a+\varepsilon_1 < \sigma^2(b)-\varepsilon_2$$

are in $\mathbb{T}$. Let

$$I := [a+\varepsilon_1, d],$$

where d is a point in $\mathbb{T}$ such that $\sigma(d) = \sigma^2(b)-\varepsilon_2$. Then let

$$K(t) := K(t,\varepsilon_1,\varepsilon_2) = \int_I G(t,s)a(s)k^\sigma(s)\,\Delta s,$$

where $k(t)$ is given by (4). We now define

$$A_2 := \min\{K(t)\colon t\in I\}.$$

Theorem 4. *If* $a(t) > 0$ *on* I *and either*

(a) $\frac{1}{A_2 f_\infty} < \lambda < \frac{1}{A_1 f^0}$

or

(b) $\frac{1}{A_2 f_0} < \lambda < \frac{1}{A_1 f^\infty}$,

then the eigenvalue problem (1)–(3) *has a positive solution.*

Proof. Assume (a) holds. Since

$$\lambda < \frac{1}{A_1 f^0},$$

there is an $\varepsilon > 0$ so that

$$A_1(f^0+\varepsilon)\lambda \leqslant 1.$$

Using the definition of f^0, there is a $\delta_0 > 0$, sufficiently small, so that

$$\max\left\{\frac{f(t,u)}{u}\colon t\in[a,\sigma(b)]\right\} < f^0+\varepsilon$$

for $0 < u \leqslant \delta_0$. It follows that

$$f(t,u) \leqslant (f^0+\varepsilon)u \tag{11}$$

for $0\leqslant u\leqslant \delta_0$, $t\in[a,\sigma(b)]$.

Assume $u \in P_{\delta_0}$, then

$$Au(t) = \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,u^\sigma(s))\Delta s$$
$$\leqslant \lambda(f^0 + \varepsilon)||u|| \int_a^{\sigma(b)} G(t,s)a(s)\Delta s \leqslant \lambda(f^0 + \varepsilon)||u||A_1$$
$$\leqslant ||u||.$$

Hence we have shown that if $u \in P_{\delta_0}$, then

$$||Au|| \leqslant ||u||.$$

Next, we use the assumption

$$\frac{1}{A_2 f_\infty} < \lambda.$$

First, we consider the case when $f_\infty < \infty$. In this case pick an $\varepsilon_0 > 0$ so that

$$\lambda A_2(f_\infty - \varepsilon_0) \geqslant 1.$$

Using the definition of f_∞ there is a $\delta_1 > \delta_0$, sufficiently large, so that

$$\min_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u} \geqslant f_\infty - \varepsilon_0$$

for $u \geqslant \delta_1$. It follows that

$$f(t,u) \geqslant (f_\infty - \varepsilon_0)u \tag{12}$$

for $u \geqslant \delta_1$, $t \in [a,\sigma(b)]$. We now show there is a $\delta_2 \geqslant \delta_1$ such that if $u \in P_{\delta_2}$ then

$$||Au|| \geqslant ||u||.$$

Assume $u \in P$. By Lemma 2

$$u(t) \geqslant k(t)||u||$$

for $t \in [a, \sigma^2(b)]$. Hence,

$$u^\sigma(t) \geqslant k^\sigma(t)||u||$$

for $t \in [a,\sigma(b)]$. It follows that

$$u^\sigma(t) \geqslant D||u||$$

for $t \in I$, where

$$D := h_a^\sigma(a + \varepsilon_1)g_{\sigma^2(b)}(\sigma^2(b) - \varepsilon_2) \leqslant 1.$$

Pick

$$\delta_2 \geqslant \frac{\delta_1}{D} \geqslant \delta_1.$$

Now assume $u \in \partial P_{\delta_2}$ and consider

$$Au(t) = \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,u^\sigma(s))\Delta s \geqslant \lambda \int_I G(t,s)a(s)f(s,u^\sigma(s))\Delta s.$$

Since for $t \in I$,

$$u^\sigma(t) \geqslant D||u|| = D\delta_2 \geqslant \delta_1,$$

we get that

$$\begin{aligned} Au(t) &\geqslant \lambda(f_\infty - \varepsilon_0)\int_I G(t,s)a(s)u^\sigma(s)\Delta s \\ &\geqslant \lambda(f_\infty - \varepsilon_0)||u|| \int_I G(t,s)a(s)k^\sigma(s)\Delta s \geqslant \lambda(f_\infty - \varepsilon_0)A_2||u|| \\ &\geqslant ||u||. \end{aligned}$$

Hence if $f_\infty < \infty$, and if $u \in \partial P_{\delta_2}$, then

$$||Au|| \geqslant ||u||.$$

Finally, we consider the case $f_\infty = \infty$. In this case the hypothesis

$$\frac{1}{A_2 f_\infty} < \lambda$$

means $\lambda > 0$. So fix $\lambda > 0$ and pick $M > 0$ sufficiently large so that

$$\lambda M \int_I G(t,s)a(s)k^\sigma(s)\,\Delta s \geqslant 1,$$

for any $t \in I$. Now define $g : [\delta_0, \infty) \to \mathbb{R}^+$ by

$$g(u) = \inf\left\{\frac{f(t,v)}{v} \colon t \in [a,\sigma(b)], u \geqslant v\right\}.$$

Then $g(u)$ is nondecreasing on $[\delta_0, \infty)$ and

$$\lim_{u\to\infty} g(u) = \infty.$$

Hence in this case we can choose $\delta_1 > \delta_0$ so that

$$g(u) \geqslant M$$

for $u \geqslant \delta_1$. But then by the definition of $g(u)$ we get that

$$f(t,u) \geqslant Mu$$

for all $t \in [a,\sigma(b)], u \geqslant \delta_1$. Now choose δ_2 as in the first case. Then assume $u \in \partial P_{\delta_2}$ and consider for any $t \in I$

$$\begin{aligned} Au(t) &= \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,u^\sigma(s))\,\Delta s \\ &\geqslant \lambda \int_I G(t,s)a(s)f(s,u^\sigma(s))\,\Delta s \geqslant \lambda M \int_I G(t,s)a(s)u^\sigma(s)\,\Delta s \\ &\geqslant \lambda M ||u|| \int_I G(t,s)a(s)k^\sigma(s)\,\Delta s \geqslant ||u||. \end{aligned}$$

Hence again we have for any $u \in \partial P_{\delta_2}$,

$$||Au|| \geqslant ||u||.$$

Therefore by Lemma 3, A has a fixed point u with

$$\delta_0 < ||u|| < \delta_2.$$

This shows that condition (a) yields the existence of a positive solution of the eigenvalue problem (1)–(3). Similar arguments to these show that condition (b) also gives the existence of a positive solution to the eigenvalue problem (1)–(3). This completes the proof of the theorem. □

Our next result gives criteria for the existence of one (or none, or more than one) positive solution of the eigenvalue problem (1)–(3) in terms of the superlinear or sublinear behavior of $f(t,u)$.

Theorem 5. *In addition to our earlier assumptions assume $a(t) > 0$ on $[a,b]$ and $f(t,u) > 0$ on $[a,\sigma^2(b)] \times \mathbb{R}^+$.*

(a) *If $f^0 = 0$ or $f^\infty = 0$, then there is a $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ the eigenvalue problem* (1)–(3) *has a positive solution.*

(b) *If $f^0 = f^\infty = 0$, then there is a $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ the eigenvalue problem* (1)–(3) *has two positive solutions.*

(c) *If $f_0 = \infty$ or $f_\infty = \infty$, then there is a $\lambda_0 > 0$ such that for all $0 < \lambda \leqslant \lambda_0$ the eigenvalue problem* (1)–(3) *has a positive solution.*

(d) *If $f_0 = f_\infty = \infty$, then there is a $\lambda_0 > 0$ such that for all $0 < \lambda \leqslant \lambda_0$ the eigenvalue problem* (1)–(3) *has two positive solutions.*

(e) *If there is a constant $c > 0$ such that $f(t,u) \geqslant cu$ for $u \geqslant 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem* (1)–(3) *has no positive solutions for $\lambda \geqslant \lambda_0$.*

(f) *If there is a constant $c > 0$ such that $f(t,u) \leqslant cu$ for $u \geqslant 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem* (1)–(3) *has no positive solutions for $0 < \lambda \leqslant \lambda_0$.*

Proof. Part (a). Let $t_0 \in I$ and for all $p > 0$ define

$$m(p) = \min\left\{ \int_I G(t_0,s)a(s)f(s,x^\sigma(s))\,\Delta s \colon x \in \partial P_p \right\}.$$

It can be shown that $m(p) > 0$ for all $p > 0$. We now show that for any $p_0 > 0$ that for all $\lambda \geqslant \lambda_0$, where

$$\lambda_0 := \frac{p_0}{m(p_0)}, \tag{13}$$

we have that if $x \in \partial P_{p_0}$, then $||Ax|| \geqslant ||x||$. To prove this let $x \in \partial P_{p_0}$. Then for $\lambda \geqslant \lambda_0$

$$\begin{aligned} Ax(t_0) &= \lambda \int_a^{\sigma(b)} G(t_0,s)a(s)f(s,x^\sigma(s))\,\Delta s \\ &\geqslant \lambda \int_I G(t_0,s)a(s)f(s,x^\sigma(s))\,\Delta s \geqslant \lambda m(p_0) \\ &\geqslant \lambda_0 m(p_0) = p_0 = ||x||. \end{aligned}$$

Hence it follows that $||Ax|| \geqslant ||x||$ for all $x \in \partial P_{p_0}$, for all $\lambda \geqslant \lambda_0$.

We now show that the condition $f_0 = 0$ implies that given any $p_0 > 0$ there is an h_0 such that $0 < h_0 < p_0$ and for any $x \in \partial P_{h_0}$ it follows that $||Ax|| \leqslant ||x||$, for all $\lambda \geqslant \lambda_0$. To prove this fix $\lambda \geqslant \lambda_0$ and pick $\eta_0 > 0$ so that

$$\eta_0 \lambda \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \leqslant 1. \tag{14}$$

Since

$$f^0 := \limsup_{u \to 0+} \max_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u} = 0,$$

there is an $h_0 < p_0$ such that

$$\max_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u} \leqslant \eta_0$$

for $0 < u \leqslant h_0$. Hence we have that

$$f(t,u) \leqslant \eta_0 u \tag{15}$$

for $t \in [a,\sigma(b)]$, $0 \leqslant u \leqslant h_0$.

Let $x \in \partial P_{h_0}$ and consider

$$\begin{aligned} Ax(t) &= \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,x^\sigma(s))\,\Delta s \\ &\leqslant \lambda \int_a^{\sigma(b)} G(\sigma(s),s)a(s)f(s,x^\sigma(s))\,\Delta s \leqslant \lambda \eta_0 \int_a^{\sigma(b)} G(\sigma(s),s)a(s)x^\sigma(s)\,\Delta s \\ &\leqslant \lambda \eta_0 ||x|| \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \leqslant ||x||. \end{aligned}$$

It follows that if $x \in \partial P_{h_0}$, then $||Ax|| \leqslant ||x||$, and hence, the eigenvalue problem (1)–(3) has a positive solution and the first part of (a) has been proven.

We now prove the second part of part (a) of this theorem. Fix $\lambda \geqslant \lambda_0$ where λ_0 is given by (13). Pick η_0 so that (14) holds. Since $f^\infty = 0$ there is a $H_0 > p_0$ so that

$$\max_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u} \leqslant \eta_0$$

for $u \geqslant H_0$. Hence we have that

$$f(t,u) \leqslant \eta_0 u \tag{16}$$

for $t \in [a,\sigma(b)]$, $u \geqslant H_0$.

We consider two cases. The first case is that $f(t,x)$ is bounded on $[a,\sigma(b)] \times \mathbb{R}^+$. In this case there is a positive number N such that

$$|f(t,u)| \leqslant N$$

for $t \in [a,\sigma(b)]$, $u \in \mathbb{R}^+$. Choose $H_1 \geqslant H_0$ so that

$$N\lambda \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \leqslant H_1.$$

Then for $x \in \partial P_{H_1}$ we have

$$\begin{aligned} Ax(t) &= \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,x^\sigma(s))\,\Delta s \\ &\leqslant \lambda N \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \leqslant H_1. \end{aligned}$$

It follows that if $x \in \partial P_{H_1}$, then $||Ax|| \leqslant ||x||$. Since at the beginning of the proof of this theorem we proved that if $x \in \partial P_{p_0}$, then $||Ax|| \geqslant ||x||$, and since $p_0 < H_1$ it follows from Lemma 3 that A has a fixed point and hence the eigenvalue problem (1)–(3) has a positive solution.

Next, we consider the case where $f(t,u)$ is unbounded on $[a,\sigma(b)] \times \mathbb{R}^+$. Let

$$g(h) := \max\{f(t,u):\ t \in [a,\sigma(b)],\ 0 \leqslant u \leqslant h\}.$$

Then $g(h)$ is nondecreasing and

$$\lim_{h\to\infty} g(h) = \infty.$$

Choose $H_2 \geqslant H_0$ so that

$$g(H_2) \geqslant g(h),$$

for $0 \leqslant h \leqslant H_2$.

Then for $x \in \partial P_{H_2}$,

$$\begin{aligned} Ax(t) &= \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,x^\sigma(s))\,\Delta s \\ &\leqslant \lambda g(H_2) \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \leqslant \lambda\eta_0 H_2 \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \\ &\leqslant H_2 = ||x||. \end{aligned}$$

It follows that the eigenvalue problem (1)–(3) has a positive solution $x_0(t)$ satisfying $p_0 \leqslant ||x_0|| \leqslant H_2$ and the proof of part (a) of this theorem is complete.

Part (b). Clearly if $f^0 = f^\infty = 0$, then by the proof of part (a) we get for any $p_0 > 0$ that for each fixed $\lambda \geqslant \lambda_0 := p_0/m(p_0)$ there are numbers $h_0 < p_0 < H_2$ such that there are two positive solutions of the eigenvalue problem (1)–(3) satisfying

$$h_0 < ||x_1|| < p_0 < ||x_2|| < H_2.$$

The proof of part (c) will be easy to see when we prove part (d) so we will only prove part (d) here.

Part (d). Assume $f_0 = f_\infty = \infty$ and $0 < r_1 < r_2$ are given numbers. Let

$$M_i := \max\{f(t,u):\ (t,u) \in [a,\sigma(b)] \times [0,r_i]\}$$

for $i = 1,2$. Then if $u \in \partial P_{r_i}$, it follows that

$$Au(t) \leqslant M_i \lambda \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s.$$

It follows that we can pick $\lambda_0 > 0$ sufficiently small so that for all $0 < \lambda \leqslant \lambda_0$

$$||Au|| \leqslant ||u|| \quad \text{for all } u \in \partial P_{r_i},$$

$i = 1,2$.

Fix $\lambda \leqslant \lambda_0$. Choose $M > 0$ sufficiently large so that

$$\lambda k_0 M \int_I G(t_0,s)a(s)\,\Delta s \geqslant 1, \tag{17}$$

where $t_0 \in I$ and

$$k_0 := g_{\sigma^2(b)}(\sigma^2(b) - \varepsilon_2)h_a(a + \varepsilon_1).$$

Since $f_0 = \infty$, there is $u_1 < r_1$ such that

$$\min_{t \in [a,\sigma(b)]} \frac{f(t,u)}{u} \geqslant M$$

for $0 < u \leqslant u_1$. Hence we have that

$$f(t,u) \geqslant Mu$$

for $t \in [a,\sigma(b)]$, $0 < u \leqslant u_1$. We next show that if $u \in \partial P_{u_1}$, then $||Au|| \geqslant ||u||$. To show this assume $u \in \partial P_{u_1}$. Then

$$\begin{aligned} Au(t_0) &= \lambda \int_a^{\sigma(b)} G(t_0,s)a(s)f(s,u^\sigma(s))\,\Delta s \\ &\geqslant M\lambda \int_a^{\sigma(b)} G(t_0,s)a(s)u^\sigma(s)\,\Delta s \geqslant M\lambda \int_I G(t_0,s)a(s)u^\sigma(s)\,\Delta s. \end{aligned}$$

Since $u \in \partial P_{u_1}$, we have by Lemma 2 that

$$u(t) \geqslant k(t)||u|| = g_{\sigma^2(b)}(t)h_a(t)||u|| \geqslant k_0||u||$$

for $t \in [a + \varepsilon_1, \sigma^2(b) - \varepsilon_2]$. Hence

$$u^\sigma(t) \geqslant k_0||u||$$

for $t \in I$. Hence from above we get that

$$\begin{aligned} Au(t_0) &\geqslant M||u||\lambda \int_I G(t_0,s)a(s)\,\Delta s \\ &\geqslant ||u|| \end{aligned}$$

by (17). Hence we have shown that if $u \in \partial P_{u_1}$, then $||Au|| \geqslant ||u||$.

Next, we use the assumption that $f_\infty = \infty$. Since $f_\infty = \infty$ there is a $u_2 > r_2$ such that

$$\min_{t\in[a,\sigma(b)]} \frac{f(t,u)}{u} \geqslant M$$

for $u \geqslant u_2$ and M is chosen so that (17) holds. It follows that

$$f(t,u) \geqslant Mu$$

for $t \in [a,\sigma(b)]$, $u \geqslant u_2$. Let

$$u_3 := \frac{u_2}{k_0}.$$

We next show that if $u \in \partial P_{u_3}$, then $||Au|| \geqslant ||u||$. To show this assume $u \in \partial P_{u_3}$. Then by Lemma 2

$$u(t) \geqslant k(t)||u|| = g_{\sigma^2(b)}(t)h_a(t)||u|| \geqslant k_0||u||$$

for $t \in [a+\varepsilon_1, \sigma^2(b)-\varepsilon_2]$. Hence

$$u^\sigma(t) \geqslant k_0||u|| = k_0u_3 = k_0\frac{u_2}{k_0} = u_2$$

for $t \in I$. Using this we get that

$$\begin{aligned} Au(t_0) &= \lambda \int_a^{\sigma(b)} G(t_0,s)a(s)f(s,u^{\sigma(s)})\,\Delta s \\ &\geqslant \lambda M \int_I G(t_0,s)a(s)u^{\sigma(s)}\,\Delta s \geqslant \lambda M||u|| \int_I G(t_0,s)a(s)\,\Delta s \\ &\geqslant ||u||. \end{aligned}$$

Hence we have shown that if $u \in \partial P_{u_3}$, then $||Au|| \geqslant ||u||$. It follows from Lemma 3 that the operator A has two fixed points $x_1(t)$ and $x_2(t)$ satisfying

$$u_1 < ||x_1|| < r_1 < r_2 < ||x_2|| < u_3.$$

This implies that the eigenvalue problem (1)–(3) has two positive solutions for $0 < \lambda \leqslant \lambda_0$. Part (e). Assume there is a constant $c > 0$ such that $f(t,u) \geqslant cu$ for $u \geqslant 0$. Assume $v(t)$ is a positive solution of the eigenvalue problem (1)–(3). We will show that for λ sufficiently large that this leads to a contradiction. Since $Av(t) = v(t)$ for $t \in [a,\sigma^2(b)]$, we have for $t_0 \in I$

$$\begin{aligned} v(t_0) &= \lambda \int_a^{\sigma(b)} G(t_0,s)a(s)f(s,v^\sigma(s))\,\Delta s \\ &\geqslant \lambda \int_I G(t_0,s)a(s)f(s,v^\sigma(s))\,\Delta s \geqslant c\lambda \int_I G(t_0,s)a(s)v^\sigma(s)\,\Delta s. \end{aligned}$$

Similar to arguments we did above we have that

$$v^\sigma(t) \geqslant k_0||v||$$

for $t \in I$. Using this we get that

$$v(t_0) \geqslant c\lambda \int_I G(t_0,s)a(s)v^\sigma(s)\,\Delta s \geqslant c\lambda k_0 ||v|| \int_I G(t_0,s)a(s)\,\Delta s.$$

If we pick λ_0 sufficiently large so that

$$\lambda k_0 c \int_I G(t_0,s)a(s)\,\Delta s > 1$$

for all $\lambda \geqslant \lambda_0$, then we have that

$$v(t_0) > ||v||$$

which is a contradiction.

Part (f). Assume there is a constant $c > 0$ such that $f(t,u) \leqslant cu$ for $u \geqslant 0$. Assume $v(t)$ is a positive solution of the eigenvalue problem (1)–(3). We will show that for λ sufficiently small and positive that this leads to a contradiction. Since $Av(t) = v(t)$ for $t \in [a, \sigma^2(b)]$,

$$\begin{aligned} v(t) &= \lambda \int_a^{\sigma(b)} G(t,s)a(s)f(s,v^\sigma(s))\,\Delta s \\ &\leqslant c\lambda \int_a^{\sigma(b)} G(\sigma(s),s)a(s)v^\sigma(s)\,\Delta s \\ &\leqslant c\lambda ||v|| \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s \end{aligned}$$

for $t \in [a, \sigma^2(b)]$. Pick λ_0 sufficiently small so that for $0 < \lambda \leqslant \lambda_0$,

$$c\lambda \int_a^{\sigma(b)} G(\sigma(s),s)a(s)\,\Delta s < 1,$$

then we have $v(t) < ||v||$ for $t \in [a, \sigma(b)]$ which is a contradiction. □

We wish to note that Theorem 5 applies in many cases to which Theorem 4 does not apply. For example, if $f_0 = f^0 = f_\infty = f^\infty = 0$, then Theorem 4 does not apply but Theorem 5, part (b) yields the existence of two positive solutions for $\lambda \geqslant \lambda_0 \geqslant 0$. Likewise Theorem 5 part (d) gives the existence of two positive solutions if $f_0 = f^0 = f_\infty = f^\infty = \infty$ and $0 < \lambda \leqslant \lambda_0$, for some $\lambda_0 > 0$. The other parts of Theorem 5 give existence or nonexistence criteria different from those of Theorem 4 also. Related results may also be found in [8,10,12,14–16,21].

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