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Demazure resolutions as varieties of lattices with infinitesimal structure

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ABSTRACT

Let k be a field of positive characteristic. We construct, for each dominant cocharacter λ of the standard maximal torus in Sl_n, a closed subvariety $\mathcal{D}(\lambda)$ of the multigraded Hilbert scheme of an affine space over k, such that the k-valued points of $\mathcal{D}(\lambda)$ can be interpreted as lattices in $k((z))^n$ endowed with infinitesimal structure. The variety $\mathcal{D}(\lambda)$ carries a natural $\mathrm{Sl}_n(k[\![z]\!])$-action. Moreover, for any λ we construct an $\mathrm{Sl}_n(k[\![z]\!])$-equivariant universal homeomorphism from $\mathcal{D}(\lambda)$ to a Demazure resolution of the Schubert variety $\mathcal{S}(\lambda)$ associated with λ in the affine Grassmannian. Lattices in $\mathcal{D}(\lambda)$ have non-trivial infinitesimal structure if and only if they lie over the boundary of the big cell of $\mathcal{S}(\lambda)$.

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1. Introduction

Let k be a field and consider the group $G = \mathrm{Sl}_n$ over k. Associated with G we have the affine Grassmannian $\mathcal{G}$ which we regard as a functor from k-algebras to sets, and whose set of k-valued points is the set of lattices $\mathcal{L} \subset k((z))^n$ such that $\bigwedge^n \mathcal{L} = k[\![z]\!]$. It is well known, and we will explain this in Section 2.1, that the affine Grassmannian is an ind-scheme over k, defined as the inductive limit of the k-schemes $\mathcal{G}^{(N)}$, where $\mathcal{G}^{(N)}(k) = \{\mathcal{L} \in \mathcal{G}$ such that $z^N k[\![z]\!]^n \subset \mathcal{L} \subset z^{-N} k[\![z]\!]^n\}$. Each 'finite piece' $\mathcal{G}^{(N)}$ is seen to be a closed subscheme of the ordinary Grassmannian $\mathrm{Grass}(nN, 2nN)$ by identifying a lattice $\mathcal{L}$ with its image in $z^{-N}k[\![z]\!]^n / z^N k[\![z]\!]^n \simeq k^{2nN}$. From this point of view, $\mathcal{G}^{(N)}$ is just the closed subvariety of $\mathrm{Grass}(nN, 2nN)$ which parametrizes those nN-dimensional subspaces of k^{2nN} which are stable under the endomorphism of k^{2nN} corresponding to multiplication by z in $k((z))^n$. Equivalently, one could say that $\mathcal{G}^{(N)}$ parametrizes nN-dimensional linear subvarieties of $\mathbb{A}_k^{2nN}$ which are stable under the endomorphism of $\mathbb{A}_k^{2nN}$ 'given by multiplication by z'. Moreover, $\mathcal{G}$ carries a natural $\mathrm{Sl}_n(k[\![z]\!])$-

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operation whose orbit closures are by definition the Schubert varieties in $\mathcal{G}$. Any Schubert variety $\mathcal{S} \subset \mathcal{G}$ can be realized as a closed subvariety of a finite piece $\mathcal{G}^{(N)}$ of the affine Grassmannian.

In this paper we are going to investigate a variant of the finite pieces of the affine Grassmannian by considering parameter spaces for certain *non-linear* subvarieties of $\mathbb{A}_k^{nN}$. This will lead to objects which are closely related to Demazure varieties.

From now on, let k have positive characteristic p.

For a k-algebra R we construct in Section 3 the *ring of Frobenius-twisted power series* in the following way: we identify $\mathrm{W}(R)$, the ring of Witt vectors over R, and $R^{\mathbb{N}}$ as sets and let $I_n \subset \mathrm{W}(R)$ be the ideal with underlying set $\{0\}^n \times R^{\mathbb{N}} \subset R^{\mathbb{N}}$. The sequence of ideals $(I_n)_{n\in\mathbb{N}}$ defines a filtration on $\mathrm{W}(R)$. The ring $R[\![z]\!]^F$ of Frobenius-twisted power series over R is by definition the completion of the graded ring associated with this filtration. As a set, $R[\![z]\!]^F \simeq R^{\mathbb{N}}$, and, if we write $z := \tilde{p} \in I_1/I_2 \simeq R$, then multiplication in $R[\![z]\!]^F$ is given by the formula

$$\begin{aligned}&\big(a_0 + a_1 z + a_2 z^2 + \cdots\big)\big(b_0 + b_1 z + b_2 z^2 + \cdots\big)\\ &\quad = a_0 b_0 + \big(a_0^p b_1 + a_1 b_0^p\big) z + \big(a_0^{p^2} + a_1^p b_1^p + a_2 b_0^{p^2}\big) z^2 + \cdots.\end{aligned}$$

This explains the term 'Frobenius-twisted'.

Let us regard $\mathbb{A}_k^{nN} = (\mathbb{A}_k^N)^n$ as a scheme endowed with the obvious structure of $k[\![z]\!]^F$-module scheme. On its coordinate ring $k[x_{i,j};\ i = 1, \ldots, n;\ j = 0, \ldots, N-1]$ we fix the grading given by $\deg x_{i,j} = p^j$. Haiman and Sturmfels have shown in [1] that for any Noetherian $\mathbb{N}$-graded k-algebra A and any function $h : \mathbb{N} \to \mathbb{N}$ there exists a k-scheme, called the multigraded Hilbert scheme, whose R-valued points are the 'admissible' ideals $I \subset A \otimes_k R$ with Hilbert function h (see Section 4.1). In particular, there exists a multigraded Hilbert scheme for $\mathbb{A}_k^{nN}$ with the above grading, and we will see in Section 4.2 that it carries a natural action of $\mathrm{Sl}_n(k[\![z]\!]^F)$. We will call a closed subscheme $V \subset \mathbb{A}_k^{nN}$ defined by an admissible ideal a *lattice scheme*, if V is stable under the module operations on $\mathbb{A}_k^{nN}$, and we will show that the $\mathrm{Sl}_n(k[\![z]\!]^F)$-action on the multigraded Hilbert scheme restricts to an action on the *closed subvariety* of lattice schemes.

There are certain orbit closures which will be of particular interest for us: let $\check{\mathrm{X}}(T) \subset \mathbb{Z}^n$ be the set of cocharacters of the standard maximal torus $T \subset \mathrm{Sl}_n$, and let $\check{\mathrm{X}}_+(T) \subset \check{\mathrm{X}}(T)$ be the set of dominant cocharacters with respect to the standard Borel subgroup of upper triangular matrices. Then, for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \check{\mathrm{X}}_+(T)$, we set $\tilde{\lambda} = (\lambda_1 - \lambda_n, \ldots, \lambda_n - \lambda_n)$ and consider the lattice scheme $V(\lambda) \subset \mathbb{A}_k^{nN}$ defined by the ideal

$$I(\lambda) = \langle x_{1,0}, \ldots, x_{1,\tilde{\lambda}_1 - 1}, \ldots, x_{n-1,0}, \ldots, x_{n-1,\tilde{\lambda}_{n-1}-1} \rangle.$$

By looking at k-valued points we interpret $V(\lambda)$ as the 'standard lattice' with elementary divisors $z^{\tilde{\lambda}_1}, \ldots, z^{\tilde{\lambda}_n}$. Let us denote by $\mathcal{D}(\lambda)$ its orbit closure in the multigraded Hilbert scheme.

On the other hand, for $\lambda \in \check{\mathrm{X}}_+(T)$ we define a sequence $\mu_1, \ldots, \mu_N \in \mathbb{Z}^n$ by setting $\mu_j := (1, \ldots, 1, 0, \ldots, 0)$ where the number of 1's is equal to the number of indices $1 \leqslant i \leqslant d$ with $\lambda_i - \lambda_n \geqslant j$. In particular, we have $\sum_{j=1}^N \mu_i = \lambda - \lambda_n(1, \ldots, 1)$. As we will explain in Section 2, this sequence of μ_j's defines a sequence of minuscule dominant cocharacters of the standard maximal torus in $\mathbb{P}\mathrm{Gl}_n$ (with respect to the standard Borel), and hence determines a Demazure resolution $\pi(\lambda) : \Sigma(\mu_1, \ldots, \mu_N) \to \mathcal{S}(\lambda)$ of the Schubert variety $\mathcal{S}(\lambda) \subset \mathcal{G}$ associated with λ.

The varieties $\mathcal{D}(\lambda)$ and $\Sigma(\mu_1, \ldots, \mu_N)$ are related according to the following

Theorem 1.1. *The k-variety $\mathcal{D}(\lambda)$ is an iterated bundle of ordinary Grassmannians, and there is a universal homeomorphism*

$$\sigma : \mathcal{D}(\lambda) \to \Sigma(\mu_1, \ldots, \mu_N).$$

Furthermore, let $\pi'(\lambda) : \mathcal{D}(\lambda) \to \Sigma(\mu_1, \ldots, \mu_N) \to \mathcal{S}(\lambda)$ denote the composition of σ with the Demazure resolution $\pi(\lambda)$ of $\mathcal{S}(\lambda)$. Then a lattice scheme given by a k-valued point in $\mathcal{D}(\lambda)$ is reduced if and only if it is mapped under $\pi'(\lambda)$ to the big cell of $\mathcal{S}(\lambda)$.

Hence the fiber of the Demazure resolution $\pi(\lambda)$ over a lattice $\mathcal{L}$ in the boundary of $\mathcal{S}(\lambda)$ is universally homeomorphic to a variety of non-trivial infinitesimal structures on $\mathcal{L}$.

Undoubtedly, both Demazure resolutions and (multigraded) Hilbert schemes are important objects which are interesting in their own right. This already might justify the effort to study a connection between them, at least when working over positive characteristic. However, the main motivation and application I have in mind is the attempt to understand the analogous situation where one replaces the ring of (Frobenius-twisted) power series over k by the ring of Witt vectors $\mathrm{W}(k)$. Indeed, it is not so clear how (or even impossible?) to put a structure of ind-scheme on the set

$$\left\{\mathcal{L} \subset \mathrm{W}(k)[1/p]^n \text{ a } \mathrm{W}(k)\text{-submodule} \;\middle|\; \bigwedge^n \mathcal{L} = \mathrm{W}(k)\right\} \quad (k \text{ perfect}), \tag{1.1}$$

i.e. to construct a p-adic analog of the affine Grassmannian for Sl_n as an ind-scheme. In particular, there is also no (obvious) way to define Schubert *varieties* in this setting. A first naive attempt to construct the analog of Schubert varieties in the p-adic setting would be the analog of the construction explained above: consider the ring $\mathrm{W}_N(k)$ of Witt vectors of length N. Identify $\mathrm{W}_N(k) \simeq \mathbb{A}_k^N(k) = \operatorname{Spec} k[x_1, \ldots, x_N]$ and endow the polynomial ring with a grading that makes the polynomials, which define addition and multiplication in $\mathrm{W}_N(k)$, homogeneous. Then consider the closed subscheme $\mathcal{M}$ of the multigraded Hilbert scheme of $\mathbb{A}_k^{nN}$, which parametrizes those homogeneous subschemes of the $\mathbb{A}_k^{nN}$ which are stable under addition and multiplication by scalars in $\mathrm{W}_N(k)$ (the 'lattice schemes'). The scheme $\mathcal{M}$ carries an action of $\mathrm{Sl}_n(\mathrm{W}(k))$, and we may study the orbit closure $\mathcal{D}_p(\lambda)$ of the lattice scheme which corresponds to the lattice with elementary divisors λ.

In fact, there is a paper by Haboush [2] in which he constructs spaces of lattices over the ring of Witt vectors as closed subschemes of certain Hilbert schemes – however, in his paper he does not account for the following phenomenon, which appears to me to be an unavoidable consequence of the construction by means of a Hilbert scheme.

Similar to the case of Frobenius-twisted power series, the lattices in the boundary of such an orbit closure $\mathcal{D}_p(\lambda)$ will in general have infinitesimal structure. In fact, one can show by simple calculations: *in the special case $n = 2$, $\lambda = (1, -1)$ and $p \geqslant 3$ the varieties $\mathcal{D}(\lambda)$ and $\mathcal{D}_p(\lambda)$ are equal.* We are going to calculate $\mathcal{D}(\lambda)$ in Section 6 – the calculation in the Witt vector case is completely parallel.

On the other hand, in a subsequent paper we are goint to show: one can define an fpqc-sheaf $\mathcal{G}_p$ whose R-valued points, for R a perfect k-algebra, can be interpreted as lattices over $\mathrm{W}(R)$ in a similar way as in the function field case. In particular the set $\mathcal{G}_p(k)$ is just the set of lattices in (1.1). Moreover, for every vector λ of elementary divisors, there are $\mathrm{Sl}_n(\mathrm{W}(k))$-equivariant maps

$$\mathcal{D}_p(\lambda) \to \mathcal{G}_p,$$

which induce isomorphisms on the 'big cells'. The present paper suggests the following interpretation: we are not able to construct the p-adic affine Grassmannian and its Schubert varieties as (ind-) schemes, but we may construct Demazure resolutions of its Schubert varieties as honest k-schemes. Moreover, the situation of Frobenius-twisted power series is certainly related to the Witt vector situation, whence the study of the former might lead to insight into the latter, in particular into the geometry of the varieties $\mathcal{D}_p(\lambda)$.

We will start this paper by recalling in Section 2 the notions of affine Grassmannian, Schubert varieties and their Demazure resolutions. The main purpose of this section is to clarify the connection between the most common description of a Demazure variety in terms of quotients of products of parahorics, and an equivalent description as a variety of lattice chains. In this paper we will work with the latter description. In Section 3 we introduce the ring of Frobenius-twisted power series. This ring structure gives rise to a structure of ring scheme on the affine space $\mathbb{A}_k^N$, and to a structure

of (n-dimensional) $\mathbb{A}_k^N$-module scheme on $\mathbb{A}_k^{nN}$. Flat subschemes of $\mathbb{A}_k^{nN}$, which are stable with respect to the addition- and scalar multiplication-morphisms, will be called lattice schemes, and are parametrized by a closed subscheme $\mathcal{M}$ of the multigraded Hilbert scheme. This is shown in Section 4. The technical heart of the paper is Section 5. Here we introduce the notion of 'twisted-linear' ideals in a polynomial ring of positive characteristic, and prove functoriality and flatness results for twisted linear ideals. Moreover we prove that twisted linearity is a 'closed condition' on the multigraded Hilbert scheme. Finally, the last section is devoted to the construction of an isomorphism from $\mathcal{D}(\lambda)$ to an iterated bundle of ordinary Grassmannians (Theorem 6.2 and Corollary 6.5) and to the construction of the universal homeomorphism $\sigma : \mathcal{D}(\lambda) \to \Sigma(\mu_1, \dots, \mu_N)$ (Theorem 6.4). As an easy corollary (Corollary 6.8) we obtain that a lattice scheme $V \in \mathcal{D}(\lambda)(k)$ maps to the big cell of $\mathcal{S}(\lambda)$ if and only if V is reduced, which proves the last assertion of Theorem 1.1. Also we illustrate these results by explicitly calculating the respective objects and morphisms for the simplest non-trivial example, given by $n = 2$, $N = 2$, $\lambda = (1, -1)$.

2. The affine Grassmannian and Demazure varieties

In this section we describe the construction of the affine Grassmannian associated to the special linear group Sl_n, as well as the Demazure–Hansen–Bott–Samelson desingularization of its Schubert varieties. For both objects we recall the 'standard'-definition in terms of quotients of loop groups, and give (also well-known) equivalent descriptions as a variety of lattices, and a variety of lattice chains, respectively. These latter descriptions are those which we are going to use in the rest of the paper.

Throughout this section, we denote by k an arbitrary algebraically closed field.

2.1. *The affine Grassmannian for* Sl_n

Our reference for this section is the article [3] by Beauville and Laszlo. Let $G = \mathrm{Sl}_n$, let $T \subset B \subset G$ be the standard maximal torus (of diagonal matrices) and the standard Borel subgroup (of upper triangular matrices) of G. Moreover, denote by $\check{\mathrm{X}}(T)$ the group of cocharacters of T, and by $\check{\mathrm{X}}_+(T)$ the subset of dominant cocharacters (with respect to the fixed Borel B). Consider the following (set)-valued functors on the category of k-algebras:

$$\mathrm{L}^{\geqslant 0}G : R \mapsto G\big(R[\![z]\!]\big) \qquad \text{(the 'positive loop group' associated to } G\text{)},$$
$$\mathrm{L}G : R \mapsto G\big(R((z))\big) \qquad \text{(the 'loop group' associated to } G\text{)}.$$

It is easy to see that the positive loop group is representable by an (infinite dimensional) affine k-scheme, while the loop group is representable by an ind-scheme over k (i.e. an inductive limit of k-schemes in the category of k-spaces). In both cases, we will not further distinguish between the (ind-) schemes and the functors which they represent. We identify $\check{\mathrm{X}}(T)$ with $\mathbb{Z}^n$ and hence obtain the injection

$$\check{\mathrm{X}}(T) \to \mathrm{L}G; \qquad \lambda \mapsto z^\lambda := \mathrm{diag}\big(z^{\lambda_1}, \dots, z^{\lambda_n}\big). \tag{2.1}$$

Thus we may regard the group of cocharacters of T as a subset of $\mathrm{L}G(R)$ for any R.

Definition 2.1. The quotient $\mathrm{L}G/\mathrm{L}^{\geqslant 0}G$ in the category of k-spaces (i.e. the fpqc-sheaf associated with the functor $R \mapsto G(R((z)))/G(R[\![z]\!])$) is called the *affine Grassmannian* for G.

In the sequel we abbreviate $\mathcal{G} := \mathrm{L}G/\mathrm{L}^{\geqslant 0}G$. On the k-space $\mathcal{G}$ we have a natural action of $\mathrm{L}^{\geqslant 0}G$ by multiplication on the left. On the level of k-valued points, this action induces a double coset decomposition for $\mathrm{L}G(k)$, namely

$$\mathrm{L}G(k) = \bigcup_{\lambda \in \check{\mathrm{X}}_+(T)} \mathrm{L}^{\geqslant 0}G(k) z^\lambda \mathrm{L}^{\geqslant 0}G(k). \tag{2.2}$$

This shows that the $\mathrm{L}^{\geqslant 0}G(k)$-orbits in $\mathcal{G}(k)$ are parametrized by the dominant cocharacters of T. These orbits are called the *Schubert cells* of $\mathcal{G}$, and denoted in the sequel by $\mathcal{C}(\lambda)$ for $\lambda \in \check{\mathrm{X}}_+(T)$.

It is known that $\mathcal{G}$ is in fact representable by an ind-scheme over k. Before we state the precise result, let us recall the notion of lattice, which is essential for the proof (and the rest of this paper).

Definition 2.2. An $R[\![z]\!]$-submodule $\mathcal{L} \subset R((z))^n$ is called a *lattice*, if

1. there exists $N \in \mathbb{N}$, such that $z^N R[\![z]\!]^n \subset \mathcal{L} \subset z^{-N} R[\![z]\!]$, and
2. the quotient $z^{-N} R[\![z]\!]/\mathcal{L}$ is a projective R-module.

The lattice $\mathcal{L}$ is called *special*, if $\bigwedge^n \mathcal{L} = R[\![z]\!]$, i.e. $\mathcal{L}$ has fpqc-locally a basis with determinant 1.

Theorem 2.3. *(See Beauville and Laszlo [3].) The affine Grassmannian $\mathcal{G}$ is isomorphic to the functor, which associates to every k-algebra R the set of special lattices in $R((z))^n$. Moreover, it is representable by an inductive limit of closed subschemes of usual Grassmannians, i.e. it is an ind-scheme and even ind-projective.*

The idea of the proof is to write $\mathcal{G} = \bigcup_{N \in \mathbb{N}} \mathcal{G}^{(N)}$, where

$$\mathcal{G}^{(N)}(R) = \left\{\text{special lattices } \mathcal{L} \text{ such that } z^N R[\![z]\!]^n \subset \mathcal{L} \subset z^{-N} R[\![z]\!]^n\right\}.$$

One can then show that for any N the inclusion

$$\mathcal{G}^{(N)}(R) \hookrightarrow \mathrm{Grass}\big(nN, z^{-N}k[\![z]\!]^n/z^N k[\![z]\!]^n\big)(R); \qquad \mathcal{L} \mapsto \mathcal{L}/z^N R[\![z]\!]^n$$

defines an isomorphism of functors from $\mathcal{G}^{(N)}$ to a closed subscheme of the Grassmannian $\mathrm{Grass}(nN, z^{-N}k[\![z]\!]^n/z^N k[\![z]\!]^n)$ of nN-dimensional k-linear subspaces of $z^{-N}k[\![z]\!]^n/z^N k[\![z]\!]^n$.

It is easy to see that $\mathrm{L}^{\geqslant 0}G$ acts algebraically on every $\mathcal{G}^{(N)}$, whence the Schubert cells are quasi-projective k-schemes, each lying in a suitable $\mathcal{G}^{(N)}$. The closure of the Schubert cell $\mathcal{C}(\lambda)$ in $\mathcal{G}$, denoted by $\mathcal{S}(\lambda)$ in the sequel, is called a *Schubert variety*.

Remark 2.4. For $G = \mathrm{Sl}_n$ we have defined the affine Grassmannian as the fpqc-quotient $\mathrm{L}G/\mathrm{L}^{\geqslant 0}G$. However, the same definition applies to other reductive groups G; let us consider the case $G = \mathrm{Gl}_n$. We denote by $\mathcal{G}_{\mathrm{Gl}_n}$ the *affine Grassmannian for* Gl_n, i.e. the quotient $\mathrm{L}\mathrm{Gl}_n/\mathrm{L}^{\geqslant 0}\mathrm{Gl}_n$ in the category of k-spaces. Then, in the same way as for Sl_n, we obtain (see [3]): the affine Grassmannian $\mathcal{G}_{\mathrm{Gl}_n}$ is isomorphic to the functor which associates to every k-algebra R the set of lattices in $R((z))^n$. Furthermore, $\mathcal{G}_{\mathrm{Gl}_n}$ is an ind-scheme over k which is representable by an inductive limit of closed subschemes of usual Grassmannians.

2.2. Iwahori and parahoric subgroups

We return to our notation $G = \mathrm{Sl}_n$. Let $\epsilon : G(k[\![z]\!]) \to G(k)$ be the map induced by $z \mapsto 0$. The *standard Iwahori subgroup* $\mathcal{I} \subset G(k[\![z]\!])$ is by definition the preimage of the standard Borel subgroup $B \subset G$. Any bounded (for the z-adic valuation) subgroup of $G(k((z)))$ containing $\mathcal{I}$ is called a *standard parahoric subgroup* of $G(k((z)))$. In general, an *Iwahori-* (*resp. parahoric*) *subgroup* of $G(k((z)))$ is a $G(k((z)))$-conjugate of $\mathcal{I}$ (resp. a standard parahoric subgroup). Note that Iwahori- (resp. parahoric) subgroups are exactly the stabilizers of complete (resp. partial) lattice chains

$$z\mathcal{L} \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_l \subset \mathcal{L}, \quad l \leqslant n-1.$$

In particular, the standard Iwahori subgroup is the stabilizer of the *standard lattice chain*

$$zk[\![z]\!]^n \subset k[\![z]\!] \oplus zk[\![z]\!]^{n-1} \subset \cdots \subset k[\![z]\!]^{n-1} \oplus zk[\![z]\!] \subset k[\![z]\!]^n,$$

while the standard parahoric subgroups are the stabilizers of its respective subflags. A maximal standard parahoric subgroup is the stabilizer of a single lattice in the standard lattice chain.

We are going to relate the set of maximal parahoric subgroups of Sl_n to the set of minuscule dominant cocharacters of the standard maximal torus of its adjoint $\mathbb{P}\mathrm{Gl}_n$: let $A \subset \mathbb{P}\mathrm{Gl}_n$ be the standard maximal torus and let $\check{\mathrm{X}}(A) \simeq \mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$ be the set of cocharacters of A. There is an obvious *inclusion*

$$\iota : \check{\mathrm{X}}(T) \hookrightarrow \check{\mathrm{X}}(A),$$

induced by the canonical map $T \to A$. With the identifications $\check{\mathrm{X}}(T) \subset \mathbb{Z}^n$ and $\check{\mathrm{X}}(A) \simeq \mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$ the map ι is given by the quotient map $\mathbb{Z}^n \to \mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$. Moreover, the image of ι is precisely the kernel of the map

$$\mathbb{Z}^n/(1,\dots,1)\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}; \qquad \lambda = (z_1,\dots,z_n) \mapsto |\lambda| := z_1 + \cdots + z_n. \tag{2.3}$$

Note that ι sends $\check{\mathrm{X}}_+(T)$ to $\check{\mathrm{X}}_+(A)$. Furthermore we denote by $\check{\mathrm{X}}_{+,\min}(A)$ the set of minuscule dominant cocharacters of A – these are the classes in $\mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$ of the vectors of the form $(1,\dots,1,0,\dots,0)$. Then we have the bijection

$$\begin{aligned} \check{\mathrm{X}}_{+,\min}(A) &\to \{\text{maximal standard parahoric subgroups of } G = \mathrm{Sl}_n\}, \\ \mu &\mapsto P_\mu := z^{-\tilde{\mu}} G\big(k[\![z]\!]\big) z^{\tilde{\mu}}, \end{aligned} \tag{2.4}$$

where $\tilde{\mu} \in \mathbb{Z}^n$ is a representative of μ. To abbreviate notation in the following section, we set $P_{\mu_1,\mu_2} := P_{\mu_1} \cap P_{\mu_2}$, the stabilizer of the two respective lattices in the standard flag, for μ_i minuscule dominant cocharacters of A.

2.3. Demazure–Hansen–Bott–Samelson desingularization of Schubert varieties

The presentation of the material in this section follows roughly the exposition by Gaussent and Littelmann in [4], and complements it with an alternative description of Demazure–Hansen–Bott–Samelson varieties in terms of lattice chains. The results are essentially due to Contou-Carrère [5]. Note that instead of 'Demazure–Hansen–Bott–Samelson variety' we will often simply speak of 'Demazure variety'. This is common in the literature.

Definition 2.5 *(Demazure variety).* Let $\nu_1,\dots,\nu_m$ be a sequence of minuscule dominant cocharacters of $\mathbb{P}\mathrm{Gl}_n$. The *Demazure–Hansen–Bott–Samelson variety* $\Sigma(\nu_1,\dots,\nu_m)$ is defined as

$$\Sigma(\nu_1,\dots,\nu_m) = P_0 \times^{P_{0,\nu_1}} P_{\nu_1} \times^{P_{\nu_1,\nu_2}} \cdots \times^{P_{\nu_{m-1},\nu_m}} P_{\nu_m}/P_{\nu_m,0},$$

i.e. the variety $P_0 \times P_{\nu_1} \times \cdots \times P_{\nu_m}$ modulo the right-action of the subgroup $P_{0,\nu_1} \times \cdots \times P_{\nu_{m-1},\nu_m} \times P_{\nu_m,0}$ given by

$$(g_0,\dots,g_m)\cdot(q_0,\dots,q_m) = \big(g_0q_0, q_0^{-1}g_1q_1,\dots,q_{m-1}^{-1}g_mq_m\big).$$

There is a natural morphism

$$\begin{aligned} \pi : \Sigma(\nu_1,\dots,\nu_m) &\to \mathcal{G} \\ [g_0,\dots,g_m] &\mapsto g_0\cdots g_m G\big(k[\![z]\!]\big)/G\big(k[\![z]\!]\big), \end{aligned} \tag{2.5}$$

whose image is a closed $G(k[\![z]\!])$-invariant subvariety of $\mathcal{G}$.

Let us remark here that in [4] Gaussent and Littelmann define a more general notion of Demazure variety in terms of sequences of 'types' (admitting not only maximal, but any standard parahoric subgroups in the above definition). In our situation however, to give a sequence of types in the sense of [4] is the same as to give a sequence of minuscule dominant cocharacters.

We are now going to describe Demazure–Hansen–Bott–Samelson varieties in terms of descending lattice chains.

Choose a sequence $\mu_1, \dots, \mu_{m+1} \in \{0,1\}^n$ such that every μ_i represents a minuscule dominant cocharacter of A and $\mu := \mu_1 + \cdots + \mu_{m+1}$ represents a dominant cocharacter in $\iota(\check{\mathrm{X}}_+(T))$. We denote by $\lambda \in \check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ its preimage under ι. Then $\frac{1}{n}|\lambda|$ is an integer by the description of $\iota(\check{\mathrm{X}}_+(T))$ in (2.3).

To the sequence $\mu_1, \dots, \mu_{m+1}$ we associate the subvariety $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}) \subset \prod_{j=1}^{m+1} \mathcal{G}_{\mathrm{Gl}_n}$, obtained by endowing the subset

$$\left\{\left(\mathcal{L}_0 = k[\![z]\!]^n, \mathcal{L}_1, \dots, \mathcal{L}_{m+1}\right) \,\middle|\, \mathrm{inv}(\mathcal{L}_i, \mathcal{L}_{i+1}) = \mu_{i+1}\right\} \subset \prod_{j=1}^{m+1} \mathcal{G}_{\mathrm{Gl}_n}(k) \tag{2.6}$$

with the reduced induced scheme-structure. Here, by $\mathrm{inv}(\mathcal{L}_i, \mathcal{L}_{i+1})$ we denote the vector of elementary divisors of $\mathcal{L}_{i+1}$ relative to $\mathcal{L}_i$, ordered by decreasing size.

Remark 2.6. This definition implies that for any i we have $z\mathcal{L}_i \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i$. The points of $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$ thus correspond to sequences of vertices in the affine building of Sl_n, where two subsequent vertices are joint by a 1-dimensional face in the building.

Similar to the case of $\Sigma(\nu_1, \dots, \nu_m)$, there is a morphism

$$\begin{aligned} \tilde{\pi} : \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}) &\to \mathcal{G} \\ (\mathcal{L}_0, \dots, \mathcal{L}_{m+1}) &\mapsto \mathcal{L}_{m+1} z^{-\frac{1}{n}|\lambda|} \end{aligned} \tag{2.7}$$

(recall that $-\frac{1}{n}|\lambda|$ is an integer, by our assumptions). It is easy to check that the lattice $\mathcal{L}_{m+1} z^{-\frac{1}{n}|\lambda|}$ is special by looking at the elementary divisors of the corresponding lattices.

In order to relate the previous two constructions, let $\nu_i, i = 1, \dots, m$, be as in the definition of Demazure variety, and for $i = 1, \dots, m$ choose μ_i to be a representative of the unique minuscule dominant cocharacter lying in the W-orbit of $\nu_{i-1} - \nu_i$ ($W = \mathrm{S}_n$, the Weyl group of $\mathbb{P}\mathrm{Gl}_n$). Moreover, set $\nu_{m+1} = 0$, and for every $i = 1, \dots, m+1$ choose a representative $\tilde{\nu}_i$ of ν_i such that $\tilde{\nu}_{i-1} - \tilde{\nu}_i \in \{0,1\}^n$. Of course, this construction can be reversed: to any sequence of μ_i's we can find the corresponding ν_i's. With this notation we have the following

Proposition 2.7. *Let $[g_0, \dots, g_m] \in \Sigma(\nu_1, \dots, \nu_m)$. The assignment*

$$\mathcal{L}_i := g_0 \cdots g_{i-1} z^{-\tilde{\nu}_i} \mathcal{L}_0, \quad i = 1, \dots, m+1, \tag{2.8}$$

defines an isomorphism of varieties

$$\varphi : \Sigma(\nu_1, \dots, \nu_m) \xrightarrow{\simeq} \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}).$$

Furthermore, we have $\varphi \circ \tilde{\pi} = \pi$, and the morphisms (2.5) *and* (2.7) *are desingularizations of the Schubert variety $\mathcal{S}(\lambda)$ in the affine Grassmannian of $G = \mathrm{Sl}_n$.*

Proof. Let $(g_0, \dots, g_m)$ be a representative of an R-valued point in $\Sigma(\nu_1, \dots, \nu_m)$ (R any k-algebra) and let $\mathcal{L}_i$ be as in the statement of the proposition, i.e. the lattice generated by the columns of the matrix $g_0 \cdots g_{i-1} z^{-\tilde{\nu}_i}$ for $i = 1, \dots, m+1$. The calculation

$$\begin{aligned}\operatorname{inv}(\mathcal{L}_i, \mathcal{L}_{i+1}) &= \operatorname{inv}\bigl(\mathcal{L}_0, z^{\tilde{\nu}_i} g_i z^{-\tilde{\nu}_{i+1}} \mathcal{L}_0\bigr) \\ &= \operatorname{inv}\bigl(\mathcal{L}_0, h z^{\tilde{\nu}_i - \tilde{\nu}_{i+1}} \mathcal{L}_0\bigr) \quad \bigl(\text{for some } h \in G\bigl(k[\![z]\!]\bigr)\bigr) = \mu_{i+1}\end{aligned}$$

shows that (2.8) indeed gives a point in $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$. It is independent of the representative $(g_0, \dots, g_m)$, and indeed $\varphi \circ \tilde{\pi} = \pi$. We are left with the construction of an inverse map, which we will first explain on the level of lattices in $k((z))^n$. Let $(\mathcal{L}_0, \dots, \mathcal{L}_{m+1}) \in \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})(k)$, and assume we have constructed $g_0 \in P_{\nu_0}, \dots, g_{i-1} \in P_{\nu_{i-1}}$ up to the right action of $P_{0,\nu_1} \times \cdots \times P_{\nu_{i-1},\nu_i}$, such that $\mathcal{L}_j = g_0 \cdots g_{j-1} z^{-\tilde{\nu}_j} \mathcal{L}_0$ for $j \leqslant i$. Choose $h_i \in \mathrm{Sl}_n(k((z)))$ such that $\mathcal{L}_{i+1} = h_i \mathcal{L}_0$, and set $g_i = g_{i-1}^{-1} \cdots g_0^{-1} h_i z^{\tilde{\nu}_{i+1}}$. Then the equation

$$\operatorname{inv}\bigl(\mathcal{L}_0, z^{\tilde{\nu}_i} g_i z^{-\tilde{\nu}_{i+1}} \mathcal{L}_0\bigr) = \operatorname{inv}(\mathcal{L}_i, \mathcal{L}_{i+1}) = \mu_{i+1}$$

shows that $g_i \in P_{\nu_i} P_{\nu_{i+1}}$. Moreover, for an appropriate choice of representative h_i we even get $g_i \in P_{\nu_i}$, which determines g_i up to right action by $P_{\nu_i,\nu_{i+1}}$. Hence, by induction, we get a unique preimage for any sequence of lattices in $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$. In order to obtain a true morphism, we have to describe the map on the level of R-valued points for any local k-algebra R. However, the quotient $\mathrm{L}\mathrm{Gl}_n \to \mathrm{L}\mathrm{Gl}_n / \mathrm{L}^{\geqslant 0}\mathrm{Gl}_n$ is locally trivial for the Zariski-topology (see e.g. Faltings [6]), which shows that the above construction can indeed be carried out for R-valued points, R any local k-algebra. This proves the first claim. Finally note that by Remark 2.6 $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$ is a twisted product of ordinary Grassmannians and thus smooth and projective. Since the dimensions of source and target of the morphisms in question are equal and the involved schemes are projective over k, the second claim follows. □

For later use we state explicitly a well-known formula for the dimension of $\Sigma(\nu_1, \dots, \nu_m)$ (see e.g. Gaussent and Littelmann, [4]): let $\mu_1, \dots, \mu_{m+1}$ and λ be determined by the ν_i as before, and let ρ be half the sum of the positive roots $\{\alpha_{i,j}\}$ of $T \subset \mathrm{Sl}_n$. Then

$$\dim \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}) = \dim \Sigma(\nu_1, \dots, \nu_m) = 2\langle \lambda, \rho \rangle. \tag{2.9}$$

As a sketch of proof, we just note the following line of equations:

$$\begin{aligned}2\langle \lambda, \rho \rangle &= \sum_{j>i} \langle \lambda, \alpha_{i,j} \rangle = \sum_{i=1}^{m+1} |\mu_i| \bigl(n - |\mu_i|\bigr) \\ &= \sum_{i=1}^{m+1} \dim \operatorname{Grass}\bigl(|\mu_i|, n\bigr) = \dim \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}),\end{aligned}$$

where $\operatorname{Grass}(|\mu_i|, n)$ denotes the Grassmannian of $|\mu_i|$-dimensional subspaces in k^n.

3. Frobenius-twisted power series rings

For this and the following sections let k denote a perfect field of positive characteristic p, and let R be a k-algebra.

Let $\mathrm{W}(R)$ be the ring of Witt vectors over R and identify it as a set with $R^{\mathbb{N}}$. On $\mathrm{W}(R)$ we consider the filtration $\mathcal{I} : \mathrm{W}(R) = I_0 \supset I_1 \supset I_2 \supset \cdots$, where, for every $n \in \mathbb{N}$, I_n is the ideal with underlying set $\{0\}^n \times R^{\mathbb{N}} \subset R^{\mathbb{N}}$.

Definition 3.1.

(1) The *ring of Frobenius-twisted power series* over R is the completion of the associated graded ring $\mathrm{gr}_{\mathcal{I}}\,\mathrm{W}(R)$ with respect to the filtration given by the ideals $\bigoplus_{i\geqslant N} I_i/I_{i+1}$. We denote this ring by $R[\![z]\!]^F$.
(2) The *ring of (truncated) Frobenius-twisted power series of length N* is the quotient of $\mathrm{gr}_{\mathcal{I}}\,\mathrm{W}(R)$ by the ideal $\bigoplus_{i\geqslant N} I_i/I_{i+1}$. We denote it by $R[\![z]\!]^F_N$.

Note that the ring $R[\![z]\!]^F$ contains $R = \mathrm{W}(R)/I_1$ as a subring, and its underlying additive group is isomorphic to $R^{\mathbb{N}}$ with addition given componentwise. Furthermore, if we write an element $(a_0, a_1, a_2, \ldots) \in R[\![z]\!]^F$ as $a_0 + a_1 z + a_2 z^2 + \cdots$, then multiplication in $R[\![z]\!]^F$ is as follows:

$$\begin{aligned}&\big(a_0 + a_1 z + a_2 z^2 + \cdots\big)\cdot\big(b_0 + b_1 z + b_2 z^2 + \cdots\big)\\ &\quad = a_0 b_0 + \big(a_0^p b_1 + a_1 b_0^p\big) z + \big(a_0^{p^2} b_2 + a_1^p b_1^p + a_2 b_0^{p^2}\big) z^2 + \cdots. \end{aligned} \tag{3.1}$$

By the way this gives us a morphism of rings

$$\begin{aligned}\mathcal{F} = 1 \times F \times F^2 \times \cdots : R[\![z]\!] &\to R[\![z]\!]^F\\ a_0 + a_1 z + a_2 z^2 + \cdots &\mapsto a_0 + a_1^p z + a_2^{p^2} z^2 + \cdots,\end{aligned} \tag{3.2}$$

which is an isomorphism if and only if R is perfect. Note furthermore, that $R[\![z]\!]^F_N = R[\![z]\!]^F / z^N R[\![z]\!]^F$ is only true if R is perfect or if $N = 0$.

On $\mathbb{A}^N_R = \operatorname{Spec} R[x_0, x_1, \ldots, x_{N-1}]$ the ring structure of $R[\![z]\!]^F_N$ induces a structure of ring scheme, such that for any R-algebra S we have $\mathbb{A}^N_R(S) = S[\![z]\!]^F_N$. Similarly, we obtain a structure of $R[\![z]\!]^F_N$-module scheme on $\mathbb{A}^{nN}_R = (\mathbb{A}^N_R)^n$.

A grading on the coordinate ring of $\mathbb{A}^{nN}_k$. The crucial observation is the following: if we fix the grading

$$\deg x_{i,j} = p^j, \quad i = 1, \ldots, n;\ j = 0, \ldots, N-1,$$

on the coordinate ring of $\mathbb{A}^{nN}_k$, then addition,

$$a : \mathbb{A}^{nN}_k \times_{\operatorname{Spec} k} \mathbb{A}^{nN}_k \to \mathbb{A}^{nN}_k,$$

as well as multiplication by a scalar $\in k[\![z]\!]^F_N$, are given by graded homomorphisms of the respective coordinate rings (see Eq. (3.1)). *From now on, we will consider the coordinate ring of $\mathbb{A}^{nN}_k$ endowed with this grading*. In the following section we are going to construct a moduli space for closed subschemes $V \subset \mathbb{A}^{nN}_k$ with the property that addition and scalar multiplication restrict to V; that is, V inherits the module structure from $\mathbb{A}^{nN}_k$.

4. A moduli space for lattice schemes

We have seen in Section 2 that the R-valued points of the affine Grassmannian (for Sl_n) are just the special lattices in $R((z))^n$. One can view lattices lying between $z^N R[\![z]\!]$ and $z^{-N} R[\![z]\!]$ as linear subspaces of $\mathbb{A}^{2nN}_R$, which are stable under the endomorphism $\mathbb{A}^{2nN}_R \to \mathbb{A}^{2nN}_R$ induced by multiplication by z on $R((z))^n$. These subspaces are parametrized by a closed subvariety of a finite dimensional Grassmannian.

In the sequel we are going prove a similar statement in the case of Frobenius-twisted power series. This is, roughly speaking: subschemes of $\mathbb{A}^{2nN}_k$ (endowed with the modules structure coming

from $k[\![z]\!]^F$) which satisfy certain flatness conditions and are stable under the module operations, are parametrized by a closed subscheme of the multigraded Hilbert scheme of $\mathbb{A}_k^{2nN}$.

4.1. Multigraded Hilbert schemes

Let us briefly summarize what we will need about multigraded Hilbert schemes. Our reference for this section is Haiman and Sturmfels [1].

Let S be a scheme and let $\mathbb{A}_S^n$ be the n-dimensional affine space over S, given by the sheaf $\mathcal{A} = \mathcal{O}_S[x_1, \dots, x_n]$ of polynomial rings. A grading of $\mathcal{A}$ by a semigroup A is a semigroup homomorphism $\deg : \mathbb{N}^n \to A$. Identifying (over any open set $U \subset S$) the vector $u \in \mathbb{N}^n$ with the monomial $x_1^{u_1} \cdots x_n^{u_n} \in \mathcal{A}$ induces a (multi-) grading

$$\mathcal{A} = \bigoplus_{a \in A} \mathcal{A}_a,$$

where, over any open set $U \subset S$, $\mathcal{A}_a(U)$ is the $\mathcal{O}_S(U)$-span of the monomials of degree a.

A quasi-coherent sheaf of homogeneous ideals $\mathcal{I} \subset \mathcal{A}$ is called *admissible* over S (for short: an admissible ideal), if $(\mathcal{A}/\mathcal{I})_a$ is a locally free sheaf of constant rank on S for all $a \in A$. With any admissible ideal $\mathcal{I} \subset \mathcal{A}$ we associate its *Hilbert function*, given by

$$h_{\mathcal{I}} : A \to \mathbb{N}, \qquad a \mapsto \operatorname{rk}(\mathcal{A}/\mathcal{I})_a.$$

Subschemes defined by admissible ideals will be called admissible as well, and by the Hilbert function of an admissible subscheme we mean the Hilbert function of its structure sheaf.

Let $h : A \to \mathbb{N}$ be any function supported on $\deg(\mathbb{N}^n)$, and define the *Hilbert functor* $\mathcal{H}_S^h$ from the category of S-schemes to the category of sets by

$$\begin{aligned} \mathcal{H}_S^h(X) = \big\{ &\text{admissible ideals } \mathcal{I} \subset \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_X \\ &\text{such that } \operatorname{rk}(\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_X/\mathcal{I})_a = h(a) \text{ for all } a \in A \big\}. \end{aligned} \tag{4.1}$$

This functor is representable by a scheme:

Theorem 4.1 *(Haiman, Sturmfels). There exists a quasiprojective scheme H_S^h over S which represents the functor $\mathcal{H}_S^h$. If the grading of $\mathcal{A}$ is positive, i.e.* 1 *is the only monomial of degree* 0*, then this scheme is even projective over S.*

The scheme H_S^h is called the *multigraded Hilbert scheme* (associated with the Hilbert function h).

4.2. Lattice schemes

Recall that on $\mathbb{A}_R^{nN}$ we have a structure of $R[\![z]\!]_N^F$-module scheme (Section 3). By $a : \mathbb{A}_R^{nN} \times_{\operatorname{Spec} R} \mathbb{A}_R^{nN} \to \mathbb{A}_R^{nN}$, resp. $m_x : \mathbb{A}_R^{nN} \to \mathbb{A}_R^{nN}$, denote the morphisms defining addition, resp. multiplication by the scalar $x \in R[\![z]\!]_N^F$.

Definition 4.2. An admissible R-subscheme $V \subset \mathbb{A}_R^{nN}$ is a *lattice scheme* over R, if the scheme-theoretic image under a of $V \times_R V$, as well as the scheme-theoretic image under m_x of V (any $x \in R[\![z]\!]_N^F$), factor through $V \subset \mathbb{A}_R^{nN}$. In other words, V is required to be stable under the module operations on $\mathbb{A}_R^{nN}$.

Remark 4.3. Let I be the ideal defining an admissible R-subscheme $V \subset \mathbb{A}_R^{nN}$. Then the conditions of the definition can be restated by requiring that the composite morphisms

$$R[x_{i,j}] \xrightarrow{a^\#} R[x_{i,j}] \otimes_R R[x_{i,j}] \to R[x_{i,j}]/I \otimes_R R[x_{i,j}]/I,$$

$$R[x_{i,j}] \xrightarrow{m_x^\#} R[x_{i,j}] \to R[x_{i,j}]/I$$

factor through $R[x_{i,j}]/I$. Note furthermore that any admissible V contains the 0-section and is trivially stable under taking additive inverses.

Since we are considering admissible ideals, the target rings of the above morphisms are graded, with each graded component R-locally free of finite rank. Hence, for any of these morphisms, the image of the ideal I defines a set of equations over R which describe precisely the locus in $\operatorname{Spec} R$ where the respective morphism factors in the above manner. As a consequence of Theorem 4.1 we thus obtain

Corollary 4.4. *For any numerical function $h : \mathbb{N} \to \mathbb{N}$ there exists a projective k-scheme $\mathcal{M}_n^{(N),h}$ which parametrizes those admissible ideals of $\mathcal{O}_{\mathbb{A}_k^{nN}}$ which define closed $\mathbb{A}_k^N$-submodule schemes of $\mathbb{A}_k^{nN}$ with Hilbert function h. More precisely: for any k-algebra R, the R-valued points of $\mathcal{M}_n^{(N),h}$ are exactly the locally free closed R-subschemes of $\mathbb{A}_R^{nN}$ with Hilbert function h, which are $\mathbb{A}_R^N$-submodule schemes of $\mathbb{A}_R^{nN}$.*

Proof. By Theorem 4.1 we obtain a projective k-scheme H_k^h parametrizing admissible ideals with Hilbert function h. The discussion above shows that the locus in H_k^h, where the universal family is stable under the module operations on $\mathbb{A}_{H_k^h}^{nN}$, is a closed subscheme of H_k^h. This is the scheme $\mathcal{M}_n^{(N),h}$. $\square$

It is easy to see that the (set)-valued functor on the category of k-algebras

$$\mathrm{L}^{F,\geqslant 0}G : R \mapsto G\big(R[\![z]\!]^F\big)$$

is representable by an infinite dimensional affine group scheme over k. Similarly, the functor

$$\mathrm{L}_N^{F,\geqslant 0}G : R \mapsto G\big(R[\![z]\!]_N^F\big)$$

is representable by a (finite dimensional) algebraic group over k. We have $\mathrm{L}_N^{F,\geqslant 0}G = \operatorname{Res}_{k[\![z]\!]_N^F/k}(G \times_{\operatorname{Spec} k} \operatorname{Spec} k[\![z]\!]_N^F)$, the Weil restriction of $G \times_{\operatorname{Spec} k} \operatorname{Spec} k[\![z]\!]_N^F$. There are canonical morphisms of k-groups

$$\mathrm{L}^{\geqslant 0}G \to \mathrm{L}^{F,\geqslant 0}G, \qquad \mathrm{L}^{F,\geqslant 0}G \to \mathrm{L}_N^{F,\geqslant 0}G, \tag{4.2}$$

where the first of these morphisms is induced by the map (3.2). Moreover, there is an obvious algebraic operation of the k-group $\mathrm{L}^{F,\geqslant 0}G$ on $\mathbb{A}_k^{nN}$ given by multiplication of an $n \times n$-matrix and an $n \times 1$-vector over the ring of twisted truncated power series of length N. This operation factors through $\mathrm{L}_N^{F,\geqslant 0}G$ and respects the grading of the structure sheaf of $\mathbb{A}_k^{nN}$. Hence, by the universal property of $\mathcal{M}_n^{(N),h}$, we obtain an operation

$$\mathrm{L}_N^{F,\geqslant 0}G \times_{\operatorname{Spec} k} \mathcal{M}_n^{(N),h} \to \mathcal{M}_n^{(N),h}.$$

Certain orbits of this action and their closures will be the object of our interest.

4.2.1. The standard lattice scheme associated to a dominant cocharacter

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \check{\mathrm{X}}_+(T)$, set $\tilde{\lambda} = \lambda - (\lambda_n, \dots, \lambda_n) \in \mathbb{N}^n$ and set $N := \tilde{\lambda}_1$. Then $\tilde{\lambda}$ defines a lattice scheme in $\mathbb{A}_k^{nN}$, given by the ideal $I(\lambda) \subset k[x_{i,j};\ i = 1, \dots, n;\ j = 0, \dots, N-1]$, where

$$I(\lambda) = (x_{1,0}, \dots, x_{1,\tilde{\lambda}_1 - 1}, \dots, x_{n-1,0}, \dots, x_{n-1,\tilde{\lambda}_{n-1}-1}). \tag{4.3}$$

Let us call this lattice scheme the *standard lattice scheme* associated with λ. We denote its $\mathrm{L}^{F,\geqslant 0}G$-orbit in $\mathcal{M}_n^{(N),h}$ by $\mathcal{O}(\lambda)$, and its orbit-closure by $\mathcal{D}(\lambda)$. The latter will turn out to be closely related to a Demazure resolution of the Schubert variety $\mathcal{S}(\lambda)$ with respect to the 'standard decomposition of λ' into minuscule dominant cocharacters, which we describe below.

4.2.2. The standard decomposition of a dominant cocharacter

For $\lambda \in \check{\mathrm{X}}_+(T)$ and for $i = 1, \dots, N$ choose $\mu_i = (1, \dots, 1, 0, \dots, 0)$ such that the number of 1's in this expression equals the number of entries in $\tilde{\lambda}$ which are $\geqslant i$. This defines a decomposition of λ into minuscule dominant cocharacters $\bar{\mu}_i$ as described in Section 2. Obviously we have $\tilde{\lambda} = \mu_1 + \dots + \mu_N$, and the μ_i's are ordered 'by size':

$$|\mu_1| \geqslant \dots \geqslant |\mu_N|.$$

We call this decomposition of λ into minuscule dominant cocharacters the *standard decomposition* of λ. For the rest of the paper we will assume λ, $\tilde{\lambda}$ and the μ_i's chosen in this way.

5. Twisted linearity and flatness results

To give an idea of the relation of $\mathcal{D}(\lambda)$ to Demazure resolutions and thereby motivate the subsequent technical section, consider

Example 5.1. Choose $n = 2$ (hence $G = \mathrm{Sl}_2$) and $\lambda = (1, -1) \in \check{\mathrm{X}}_+(T)$. Then $N = 2$, $\tilde{\lambda} = (2, 0)$, and the standard lattice scheme associated with λ is given by $I(\lambda) = \langle x_{1,0}, x_{1,1} \rangle$. For convenience we rename the variables $x_{1,j} \mapsto x_j$, $x_{2,j} \mapsto y_j$, whence $I(\lambda) = \langle x_0, x_1 \rangle$.

Claim: The $\mathrm{L}^{F,\geqslant 0}G$-orbit of $I(\lambda)$ consists of all ideals of the form

$$I := A \cdot I(\lambda) = \left\langle a_0 x_0 + b_0 y_0, a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p \right\rangle, \quad \text{with } a_0 \neq 0 \text{ or } b_0 \neq 0.$$

To see this, one calculates the effect of a matrix $A = \begin{pmatrix} a_0 + a_1 z & b_0 + b_1 z \\ c_0 + c_1 z & d_0 + d_1 z \end{pmatrix} \in G(k[\![z]\!]^F)$ on the vector $x = \begin{pmatrix} x_0 + x_1 z \\ y_0 + y_1 z \end{pmatrix} \in (k[x_i, y_i][\![z]\!]^F)^2$: we obtain

$$A \cdot x \equiv \begin{pmatrix} (a_0 x_0 + b_0 y_0) + (a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p) z \\ (c_0 x_0 + d_0 y_0) + (c_0^p x_1 + c_1 x_0^p + d_0^p y_1 + d_1 y_0^p) z \end{pmatrix} \mod z^2.$$

Since the operation of $\mathrm{L}^{F,\geqslant 0}G$ on $k[x_i, y_i]$ is given by the transpose of this action, we see that the images of x_0 resp. x_1 are of the form $a_0 x_0 + b_0 y_0$ resp. $a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p$, with $a_0 \neq 0$ or $b_0 \neq 0$. Thus the claim.

Claim: The ideal I is graded, and denoting the i-th graded component by I_i, we have

$$I_1 \cap \langle x_0, y_0 \rangle = \langle a_0 x_0 + b_0 y_0 \rangle$$

and

$$I_p \cap \left\langle x_0^p, y_0^p, x_1, y_1 \right\rangle = \left\langle a_0^p x_0^p + b_0^p y_0^p, a_1 x_0^p + b_1 y_0^p + a_0^p x_1 + b_0^p y_1 \right\rangle.$$

Only the latter equation requires an argument: we have to verify that, if $(a_0x_0 + b_0y_0) \cdot P(x_0, y_0) \in \langle x_0^p, y_0^p\rangle$, then $P(x_0, y_0) = \gamma(a_0x_0 + b_0y_0)^{p-1}$, for $\gamma \in k$. Assume $a_0 \neq 0$ and consider the linear transformation of variables $x_0 \mapsto (1/a_0)x_0 - (b_0/a_0)y_0$, which stabilizes $\langle x_0^p, y_0^p\rangle$. Hence $(a_0x_0 + b_0y_0) \cdot P(x_0, y_0) \in \langle x_0^p, y_0^p\rangle$ if and only if $x_0 \cdot P((1/a_0)x_0 - (b_0/a_0)y_0, y_0) \in \langle x_0^p, y_0^p\rangle$. Thus we must have $P((1/a_0)x_0 - (b_0/a_0)y_0, y_0) = \gamma' x_0^{p-1}$ for some $\gamma' \in k$, whence $P(x_0, y_0) = \gamma(a_0x_0 + b_0y_0)^{p-1}$. The case $b_0 \neq 0$ is similar, which proves the claim.

Applying the absolute Frobenius morphism to I_1 yields a submodule of I_p, and identifying x_0^p, y_0^p, x_1, y_1 with the standard basis of $k^{2\times 2}$, these define a descending sequence of lattices in $k^{2\times 2}$ in the sense of Section 2:

$$\begin{aligned}\mathcal{L}_0 = \operatorname{Hom}_k\bigl(k^{2\times 2}, k\bigr) \supset \mathcal{L}_1 &= \operatorname{Hom}_k\bigl(k^{2\times 2}/\bigl\langle a_0^p x_0^p + b_0^p y_0^p\bigr\rangle, k\bigr)\\ \supset \mathcal{L}_2 &= \operatorname{Hom}_k\bigl(k^{2\times 2}\bigl/\bigl\langle a_0^p x_0^p + b_0^p y_0^p, a_1 x_0^p + b_1 y_0^p + a_0^p x_1 + b_0^p y_1\bigr\rangle, k\bigr).\end{aligned}$$

This corresponds to a point in the Demazure resolution of $\mathcal{S}(\lambda)$ (in the situation of this example there is only one Demazure resolution, since there is only one decomposition of λ into minuscule dominant cocharacters).

Remark 5.2. From the example it is clear that we will have to deal with certain Frobenius twists when relating points of $\mathcal{D}(\lambda)$ to points of a Demazure variety.

In the following we are going to prove auxiliary results, which will later be used to study the general case $G = \mathrm{Sl}_n$.

As always, let R be an arbitrary k-algebra, denote by $\mathfrak{X} = \{x_1, \ldots, x_m\}$ a set of indeterminates with $\deg x_i = p^{d_i}$ for any i, and set

$$F_{p^l}(R) = \bigoplus_{d_i \leqslant l} R x_i^{p^{l-d_i}}.$$

To carry out the construction of the example for any (R-valued) point in $\mathcal{D}(\lambda)$, possibly meeting the boundary of $\mathcal{O}(\lambda)$, we need to know that the corresponding ideals are well behaved, in a sense to be made precise, with respect to intersections with the R-submodules $F_{p^l}(R)$ of $R[x_{i,j}]$. To show this is the goal of this section.

The crucial property that the ideals which we consider will turn out to have, is subject of the following

Definition 5.3. Let R be any k-algebra. We call an element $f \in R[\mathfrak{X}]$ *twisted-linear*, if $f \in \bigcup_l F_{p^l}(R)$. We call an ideal $I \subset R[\mathfrak{X}]$ *twisted-linear*, if I is generated by a finite subset consisting of twisted-linear elements. Equivalently, a finitely generated ideal I is twisted-linear, if and only if it is generated by $\bigcup_l (F_{p^l}(R) \cap I)$. Such an ideal is obviously graded.

In the following, if M is a graded R-module, we denote by M_d its degree d-part. Note that $F_{p^N}(R)$ is a direct summand of $R[\mathfrak{X}]_{p^N}$. Note furthermore that, if I is finitely generated, then $R[\mathfrak{X}]/I$ is finitely presented as an R-algebra, whence every homogeneous component of $R[\mathfrak{X}]/I$ is finitely presented as an R-module. Thus every such homogeneous component is R-flat if and only if it is R-projective if and only if it is locally free over R. In this case, the same properties hold for the homogeneous components of I.

Lemma 5.4. *Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal such that $R[\mathfrak{X}]/I$ is R-flat. Assume that I is generated by twisted-linear elements f_j with $\deg f_j = p^{e_j}$ for each j. Then we have for any N*

$$\sum_{e_j \leqslant N} R f_j^{p^{N-e_j}} = F_{p^N}(R) \cap I.$$

Before proving this lemma, we state two corollaries:

Corollary 5.5. *Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal such that $R[\mathfrak{X}]/I$ is flat over R. Then for any R-algebra S:*

$$\left(F_{p^N}(R) \cap I\right) \cdot S = F_{p^N}(S) \cap (I \otimes_R S)$$

(*where the left expression denotes the image of $(F_{p^N}(R) \cap I) \otimes_R S$ in $S[\mathfrak{X}]$*).

Proof. The ideal $I \otimes_R S \subset S[\mathfrak{X}]$ satisfies the assumptions of the lemma, with R replaced by S. In particular, it is generated by the images of the f_j in $S[\mathfrak{X}]$. Hence we have

$$F_{p^N}(S) \cap (I \otimes_R S) = \sum_{e_j \leqslant N} S f_j^{p^{N-e_j}} = \left(F_{p^N}(R) \cap I\right) \cdot S,$$

where the first equality results from the lemma applied to the ideal $I \otimes_R S$, while the second is a consequence of the lemma applied to I itself. □

Corollary 5.6. *Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal such that $R[\mathfrak{X}]/I$ is flat over R. Then the R-module $F_{p^N}(R)/(F_{p^N}(R) \cap I)$ is R-flat, as well as the quotient of degree p^N-components $R[\mathfrak{X}]_{p^N}/(F_{p^N}(R) + I_{p^N})$.*

Proof. The second module in question is the cokernel of the injection of R-modules

$$F_{p^N}(R)/\left(F_{p^N}(R) \cap I\right) \hookrightarrow \left(R[\mathfrak{X}]/I\right)_{p^N}.$$

Corollary 5.5 states that this map is still injective after tensoring with any R-algebra S, so the long exact sequence of Tor's tells us that the cokernel is flat. And thus also $F_{p^N}(R)/(F_{p^N}(R) \cap I)$. □

Hence we see that we can in fact restate the equation in Corollary 5.5 in the form

$$\left(F_{p^N}(R) \cap I\right) \otimes_R S = F_{p^N}(S) \cap (I \otimes_R S).$$

We interpret this equation as follows: *cutting out of a twisted-linear ideal the twisted-linear part of given degree is functorial with respect to base change.*

Proof of Lemma 5.4. Since $R[\mathfrak{X}]_1/I_1$ is R-flat and of finite presentation, and since our claim is local on R, we can assume that $R[\mathfrak{X}]_1/I_1$ is free. Since invertible linear transformations of the degree 1-variables do not affect twisted-linearity nor flatness, we can assume without loss of generality that the f_j of degree 1 are in fact in $\mathfrak{X}$, i.e. variables of degree 1. If all the f_j have degree 1, we are done. Otherwise, we apply induction on n, where p^n is the maximum degree of a generator f_j. By our assumption on I_1, we have $R[\mathfrak{X}]/I \simeq R[\mathfrak{X}']/I'$, where $\mathfrak{X}'$ is $\mathfrak{X}$ minus the variables generating I_1, and I' is the image of I under the quotient map $R[\mathfrak{X}] \to R[\mathfrak{X}']$. Hence, I' is twisted-linear and generated by those $g_j = \mathrm{image}(f_j) \in R[\mathfrak{X}']$ which have degree $\geqslant p$.

Set $\mathfrak{X}'' = \{x \in \mathfrak{X}';\ \deg x > 1\} \cup \{x^p \in \mathfrak{X}';\ \deg x = 1\}$. Then the g_j already lie in $R[\mathfrak{X}'']$ and generate an ideal $I'' \subset R[\mathfrak{X}'']$. We obtain $R[\mathfrak{X}']/I' \simeq (R[\mathfrak{X}'']/I'')[p\text{-th roots of some variables in } \mathfrak{X}'']$. By faithful flatness of extension by p-th roots, $R[\mathfrak{X}'']/I''$ is flat over R. Now we only deal with variables of degree

$\geqslant p$, and I'' is still twisted-linear, so for the moment we can think of all degrees divided by p and apply the induction hypotheses in order to obtain the claim of the lemma for $I'' \subset R[\mathfrak{X}'']$:

$$\sum_{1 \leqslant e_j \leqslant N} R g_j^{p^{N-e_j}} = F_{p^N}(R)'' \cap I''.$$

(Here, by abuse of notation, we denote by g_j also the (unique) preimages of the g_j in $R[\mathfrak{X}'']$. As for $R[\mathfrak{X}]$, we denote by $F_{p^N}(R)''$ and $F_{p^N}(R)'$ the modules of twisted-linear monomials of degree p^N in the algebras $R[\mathfrak{X}'']$ and $R[\mathfrak{X}']$, respectively.) Since

$$R\big[\mathfrak{X}'\big]/I' \simeq \big(R\big[\mathfrak{X}''\big]/I''\big)[p\text{-th roots}],$$

we have $I'' = I' \cap R[\mathfrak{X}'']$, $F_{p^N}(R)' = F_{p^N}(R)'' \subset R[\mathfrak{X}'']$ and thus obtain

$$\sum_{1 \leqslant e_j \leqslant N} R g_j^{p^{N-e_j}} = F_{p^N}(R)' \cap I'.$$

Now let $\varphi : R[\mathfrak{X}'] \to R[\mathfrak{X}]$ be the canonical splitting of the quotient map $R[\mathfrak{X}] \to R[\mathfrak{X}']$. Then $I \subset \varphi(I') \oplus \sum_{m \neq 1} mR[\mathfrak{X}]$, where the sum runs over all monomials m in variables in $\mathfrak{X} - \mathfrak{X}'$ not equal to 1. Since similarly $F_{p^N}(R) = \varphi(F_{p^N}(R)') \oplus \bigoplus_{x \in \mathfrak{X} - \mathfrak{X}'} Rx^{p^N}$, we have

$$F_{p^N}(R) \cap I \subset \varphi\big(F_{p^N}(R)' \cap I'\big) \oplus \bigoplus_{x \in \mathfrak{X} - \mathfrak{X}'} Rx^{p^N} = \sum_{e_j \leqslant N} R f_j^{p^{N-e_j}}.$$

The opposite inclusion is trivial. □

By a similar induction argument we prove

Lemma 5.7. *Assume that the twisted-linear ideal $I \subset R[\mathfrak{X}]$ is generated in degrees $d \leqslant p^n$ and that the graded components $(R[\mathfrak{X}]/I)_d$ are flat over R for $d \leqslant p^n$. Then $R[\mathfrak{X}]/I$ is R-flat.*

Proof. We proceed by induction on n. Since the graded components of $R[\mathfrak{X}]/I$ are of finite presentation, $(R[\mathfrak{X}]/I)_d$ is even locally free for $d \leqslant p^n$. Thus, since our claim is local on R, we can assume that $R[\mathfrak{X}]/\langle I_1 \rangle$ is isomorphic to a polynomial ring $R[\mathfrak{X}']$, with $\mathfrak{X}' \subset \mathfrak{X}$. In case $n = 0$ we are already done. Otherwise, the image of $I/\langle I_1 \rangle$ in $R[\mathfrak{X}']$ is a twisted-linear ideal $I' \subset R[\mathfrak{X}']$, generated by the images g_j of those generators of I which have degree $\geqslant p$. Set $\mathfrak{X}'' = \{x \in \mathfrak{X}';\ \deg x > 1\} \cup \{x^p \in \mathfrak{X}';\ \deg x = 1\}$. Then the g_j of degree > 1 lie in $R[\mathfrak{X}'']$ and generate an ideal $I'' \subset R[\mathfrak{X}'']$. We have

$$R[\mathfrak{X}]/I \simeq R\big[\mathfrak{X}'\big]/I' \simeq \big(R\big[\mathfrak{X}''\big]/I''\big)\big[p\text{-th roots of some variables in } \mathfrak{X}''\big].$$

Still, I'' is twisted-linear and $R[\mathfrak{X}'']/I''$ is R-flat in degrees $\leqslant p^n$, since adjoining p-th roots is faithfully flat. In $\mathfrak{X}''$ there only occur variables of degrees p^e with $e \geqslant 1$, so we can divide all degrees by p and arrive at a situation where we can use the induction hypotheses: $R[\mathfrak{X}'']/I''$ is R-flat in *all* degrees. But then so is $R[\mathfrak{X}']/I' \simeq R[\mathfrak{X}]/I$. □

Remark 5.8.

1. For any ideal $I \subset R[\mathfrak{X}]$, we denote by $I^{\leqslant n}$ the ideal generated by the graded components of I which have degree $\leqslant n$. Then, if $R[\mathfrak{X}]/I$ is R-flat and I is twisted-linear, Lemma 5.7 tells us that the same properties hold for $R[\mathfrak{X}]/I^{\leqslant p^n}$. In particular, we obtain from Corollary 5.6 that $R[\mathfrak{X}]_{p^n}/(F_{p^n}(R) + (I^{\leqslant p^m})_{p^n})$ is R-flat for all m and n.

2. Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal generated in degrees $\leqslant p^n$. For the quotient $R[\mathfrak{X}]/I$ to be R-flat it is sufficient that $(R[\mathfrak{X}]/I)_{p^i}$ is R-flat for $i = 0, \dots, n$. This follows from Lemma 5.7 by induction on n.

Combining these two remarks, we obtain a stronger version of the preceding lemma:

Corollary 5.9. *Assume that the twisted-linear ideal $I \subset R[\mathfrak{X}]$ is generated in degrees $d \leqslant p^n$. Then $R[\mathfrak{X}]/I$ is R-flat if and only if the R-modules*

$$F_{p^m}(R)/\big(F_{p^m}(R) \cap I\big)$$

are flat for $m = 0, \dots, n$.

Proof. The 'only if'-part was proved in Corollary 5.6. For the 'if'-part, we proceed by induction on n, the case $n = 0$ being a consequence of Lemma 5.7, since $F_{p^0}(R)/(F_{p^0}(R) \cap I) = (R[\mathfrak{X}]/I)_1$. To verify the statement for $n > 0$ assume $R[\mathfrak{X}]/I^{\leqslant p^{n-1}}$ is R-flat, and consider the exact sequence

$$0 \to F_{p^n}(R)/\big(F_{p^n}(R) \cap I\big) \to R[\mathfrak{X}]_{p^n}/I_{p^n} \to R[\mathfrak{X}]_{p^n}/\big(F_{p^n}(R) + \big(I^{\leqslant p^{n-1}}\big)_{p^n}\big) \to 0.$$

Since the right-hand module is R-flat by Corollary 5.6, and the left-hand module is R-flat by hypotheses, the same holds for the middle module. Now use (2) of Remark 5.8. □

We are now prepared to show that twisted-linearity is a closed condition on the Hilbert scheme H_k^h. In order to give this statement a precise meaning and a neat formulation, we generalize the notion of twisted-linearity to sheaves of ideals:

Definition 5.10. Let X be any k-scheme, and let $\mathcal{I} \subset \mathcal{O}_X[\mathfrak{X}]$ be a sheaf of ideals. We say $\mathcal{I}$ is twisted-linear, if for every open affine subscheme $Y = \operatorname{Spec} R$ of X, $\Gamma(Y, \mathcal{I}) \subset R[\mathfrak{X}]$ is a twisted-linear ideal.

Remark 5.11. Twisted linearity can be tested on an affine open covering: a sheaf of ideals $\mathcal{I}$ as in the previous definition is twisted-linear, if and only if there exists a covering by open affines, $X = \bigcup Y_i$, $Y_i = \operatorname{Spec} R_i$, such that every $I_i = \Gamma(Y_i, \mathcal{I}) \subset R_i[\mathfrak{X}]$ is a twisted-linear ideal. The 'only if'-part is of course trivial. To verify the 'if'-part, we can assume that $X = \operatorname{Spec} R$ is affine. Then $I = \Gamma(X, \mathcal{I}) \subset R[\mathfrak{X}]$ is twisted-linear if and only if

$$\big\langle I \cap F_{p^n}(R),\ n \in \mathbb{N}\big\rangle = I.$$

(That the property of being finitely generated, which we require for twisted-linear ideals, is local for the Zariski-topology on $\operatorname{Spec} R$, even for the faithfully flat topology, is a well-known fact.) This equation holds, by flatness of $R \to R_i$, if and only if it holds after localizing in R_i for every i. But, using Corollary 5.5, this is twisted-linearity on each Y_i, which was our assumption.

Proposition 5.12. *Let $p : X \to H_k^h$ be a morphism of k-schemes such that the corresponding sheaf of ideals on X is twisted-linear. Assume that $p_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{H_k^h}$-module and let Y be the scheme-theoretic image of p. Then the corresponding sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y[\mathfrak{X}]$ is twisted-linear.*

Proof. The defining ideal-sheaf of $Y \subset H_k^h$ is equal to $\ker(\mathcal{O}_H \to p_*\mathcal{O}_X)$. Hence to the map $f : X \to Y$ corresponds an *injective* map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and, in particular, on an affine open subset $\operatorname{Spec} R \subset Y$ we get $R \hookrightarrow \mathcal{O}_X(f^{-1}(\operatorname{Spec} R))$. Covering $f^{-1}(\operatorname{Spec} R) = \bigcup \operatorname{Spec} S_i$ by open affines, we obtain an injective map of rings

$$R \hookrightarrow \mathcal{O}_X\big(f^{-1}(\operatorname{Spec} R)\big) \hookrightarrow \prod S_i =: S.$$

By assumption, the ideal J_i corresponding to $\operatorname{Spec} S_i \hookrightarrow f^{-1}(\operatorname{Spec} R) \to X \to H_k^h$ is twisted-linear. We have to verify that the same holds for the ideal I corresponding to $\operatorname{Spec} R \hookrightarrow Y \to H_k^h$. Then the claim follows from the remark, as Y can be covered by open affines like $\operatorname{Spec} R$.

We inductively construct a generating system for I consisting of twisted-linear elements. First note that any element of degree 1 is trivially twisted-linear, i.e. $I^{\leqslant 1}$ is twisted linear. For the inductive step, assume that $I^{\leqslant p^{n-1}}$ is twisted linear. Then, by the remarks following Lemma 5.7, $M := R[\mathfrak{X}]_{p^n}/(F_{p^n}(R) + (I^{\leqslant p^{n-1}})_{p^n})$ is projective over R, whence, in the line below, the middle map is injective:

$$I_{p^n} \to M \hookrightarrow M \otimes_R S \xrightarrow{\simeq} \prod (M \otimes_R S_i). \tag{5.1}$$

To justify that the right-hand map is an isomorphism, note that, if M is even free, it is a finite product of copies of R, and the above map is indeed an isomorphism since arbitrary products commute. In general, M is a direct summand (and hence also a direct factor) of a finitely generated free R-module $M \oplus M'$. This shows that $(M \otimes_R S) \oplus (M' \otimes_R S) \simeq (\prod M \otimes_R S_i) \oplus (\prod M' \otimes_R S_i)$. Since the composition of the maps in (5.1) is zero ($I \otimes_R S_i = J_i$ being twisted-linear), so is the left-hand map. This shows that, in order to obtain a generating system of $I^{\leqslant p^n}$, we can extend a generating system of $I^{\leqslant p^{n-1}}$ by twisted-linear elements living in $F_{p^n}(R)$. This proves the inductive step. □

Corollary 5.13. *The sheaf of ideals corresponding to $\mathcal{D}(\lambda) \subset H_k^h$ is twisted-linear.*

Proof. The ideal $I(\lambda)$ (whose orbit is by definition $\mathcal{O}(\lambda)$) is twisted-linear. By definition of the ring operations in $R[\![z]\!]^F$, the operation of $\mathrm{L}^{F,\geqslant 0}\mathrm{Sl}_n$ on H_k^h preserves twisted-linearity, whence the sheaf of ideals corresponding to the inclusion $p : \mathcal{O}(\lambda) \subset H_k^h$ is twisted-linear. Now the claim follows from Proposition 5.12, since $\mathcal{D}(\lambda)$ is the closure (= scheme-theoretic image) of $\mathcal{O}(\lambda)$ in H_k^h. □

6. The structure of $\mathcal{D}(\lambda)$

6.1. An iterated bundle of ordinary Grassmannians

From now on, we write $\mathfrak{X} = \{x_{i,j};\ i = 1, \ldots, n;\ j = 0, \ldots, N-1\}$, i.e. $R[\mathfrak{X}]$ is the affine coordinate ring of $\mathbb{A}_R^{nN}$ with the grading $\deg x_{i,j} = p^j$, introduced in Section 3. Moreover, we denote by $z^\#$ the comorphism of multiplication by z on $\mathbb{A}_R^{nN}$, i.e. the ringhomomorphism

$$z^\# : R[\mathfrak{X}] \to R[\mathfrak{X}]; \qquad x_{i,j} \mapsto x_{i,j-1}^p.$$

In the previous section we introduced the ideals $I^{\leqslant p^i}$, $i = 0, 1, \ldots$, associated with a twisted linear admissible ideal $I \subset R[\mathfrak{X}]$. If we consider ideals I with the property that $z^\# I^{\leqslant p^i} \subset I^{\leqslant p^{i-1}}$ (which is the case for R-valued points of $\mathcal{D}(\lambda)$), the assignments $I \to I^{\leqslant p^i}$ have a nice geometric interpretation, which we are going to describe in the sequel.

Denote by h_m the Hilbert function of $k[\mathfrak{X}]/I(\lambda)^{\leqslant p^{m-1}}$ for $m = 0, \ldots, N$. In particular, h_0 is the Hilbert function of $k[\mathfrak{X}]$ (namely, $I(\lambda)$ contains no elements of degree $\leqslant p^{-1}$ whence $I^{\leqslant p^{-1}} = 0$), and $h_N = h$, the Hilbert function of $k[\mathfrak{X}]/I(\lambda)$. For $m = 0, \ldots, N$ denote by $\mathcal{T}_m(\lambda)$ the following (set)-valued functor on the category of k-algebras:

$$\begin{aligned}\mathcal{T}_m(\lambda)(R) = \big\{&\text{admissible twisted linear ideals } I \subset R[\mathfrak{X}] \\ &\text{with Hilbert function } h_m \text{ and such that } z^\# I^{\leqslant i} \subset I^{\leqslant i-1}\big\}.\end{aligned} \tag{6.1}$$

Note that these functors are sheaves for the Zariski-topology by Remark 5.11. Note further that $\mathcal{T}_0(\lambda) = \operatorname{Spec} k$ and $\mathcal{D}(\lambda) \hookrightarrow \mathcal{T}_N(\lambda)$ as functors. Moreover, we have morphisms of functors

$$\mathcal{T}_{m+1}(\lambda) \to \mathcal{T}_m(\lambda); \quad I \mapsto I^{\leqslant p^{m-1}},$$

once we know that the Hilbert function of $R[\mathfrak{X}]/I$, for I admissible, twisted linear and generated in degrees $\leqslant p^{m-1}$, is completely determined by its values on $\{0, \dots, p^{m-1}\}$. But this follows from

Proposition 6.1. *The Hilbert function of an admissible twisted linear ideal $I \subset R[\mathfrak{X}]$, which is generated in degrees $\leqslant p^m$, determines the ranks of the R-modules $F_{p^0}(R) \cap I, \dots, F_{p^m}(R) \cap I$ and vice versa.*

Proof. By Corollary 5.6 the R-modules $F_{p^i}(R)/F_{p^i}(R) \cap I$ are flat, whence the statement of the proposition makes sense. Consider first ideals $I \subset k[\mathfrak{X}]$ having a *monomial* twisted linear generating system that is, a generating system consisting of monomials $\xi_1, \dots, \xi_l$, each a power of a variable $x_{i,j}$ and of degree $\leqslant p^m$. Hence, if the ξ_i form a minimal generating system of I (i.e. none of them can be omitted), they also form a regular sequence in I. Moreover, the number of elements in $\{\xi_i\}$ having a given degree is determined by the dimensions of the vector spaces $F_{p^i}(k) \cap I, i = 0, \dots, m$. On the other hand Hilbert functions are additive with respect to short exact sequences as

$$0 \to k[\mathfrak{X}]/\langle \xi_1, \dots, \xi_{i-1} \rangle \xrightarrow{\cdot \xi_i} k[\mathfrak{X}]/\langle \xi_1, \dots, \xi_{i-1} \rangle \to k[\mathfrak{X}]/\langle \xi_1, \dots, \xi_i \rangle \to 0.$$

Thus the Hilbert function of $R[\mathfrak{X}]/I$ is determined by the dimensions of $F_{p^i}(k) \cap I$, $i = 0, \dots, m$, and vice versa, as claimed. Now we show that the general case reduces to the case just studied: by flatness of $R[\mathfrak{X}]/I$ and by Corollaries 5.6 and 5.5 we may assume $R = k$. Consider the action on $H_k^{h'}$ (where h' denotes the Hilbert function of I) of the 1-parameter subgroup

$$\left\{ D(t) = \operatorname{diag}\left(t^{-n+1}, t^{-n+3}, \dots, t^{n-1}\right);\ t \in k \right\} \subset T \subset \operatorname{Sl}_n(k).$$

By properness of $H_k^{h'}$ the orbit of I extends to a closed curve $C \subset H_k^{h'}$, and by Proposition 5.12 the ideal I' corresponding to $t = 0$ is generated by twisted linear homogeneous elements of degrees $\leqslant p^m$. Let $\sum_{i=1}^n P_i(x_{i,0}, \dots, x_{i,N-1})$ be such an element, the $P_i(x_{i,0}, \dots, x_{i,N-1})$ denoting polynomials of the same degree d over k in N variables, or the zero polynomial. Then, by construction, the multiplicative group acts (via $D(t)$) on P_i with weight $(-n-1+2i)d$. Since I' is a fixed point under the action of $D(t)$, this shows that each of the summands $P_i(x_{i,0}, \dots, x_{i,N-1})$ is itself contained in I'. But from this it follows easily that I' is even a monomial ideal generated by elements of the form $x_{i,j}^{p^{e_{i,j}}}$ for some non-negative integers $e_{i,j}$. Namely, if $P(x_{i,0}, \dots, x_{i,N-1}) = a_e x_{i,0}^{p^{d+e}} + \dots + a_0 x_{i,e}^{p^d}$ with $a_0 \neq 0$, then we have $x_{i,j}^{p^{d+e-j}} \in I'$ for every $0 \leqslant j \leqslant e$, since I' is stable under the map $z^{\#} : x_{i,j} \mapsto x_{i,j-1}^p$. Of course, the Hilbert functions of $k[\mathfrak{X}]/I$ and $k[\mathfrak{X}]/I'$ coincide, and the application of Corollary 5.5 to the specializations $\operatorname{Spec} k \to C$ given by $t = 0$ and $t = 1$, respectively, shows that also the dimensions of $F_{p^i}(k) \cap I$ and $F_{p^i}(k) \cap I'$ coincide for $i = 0, \dots, m$. Thus we are indeed reduced to the special case of monomial ideals. □

For the following recall the standard decomposition of λ defined in Section 4.2.2.

Theorem 6.2. *For each $m = 0, \dots, N$ the morphism $\mathcal{T}_m(\lambda) \to \mathcal{T}_{m-1}(\lambda)$ is relatively representable by a bundle of ordinary Grassmannians* $\operatorname{Grass}(|\mu_m|, n)$. *In particular, it is smooth of relative dimension $|\mu_m|(n - |\mu_m|)$.*

Corollary 6.3. *The functor $\mathcal{T}_N(\lambda)$ is representable by a smooth, connected k-scheme of dimension $\sum_{m=1}^N |\mu_m|(n - |\mu_m|) = \dim \tilde{\Sigma}(\mu_1, \dots, \mu_N)$. The functorial map $\mathcal{D}(\lambda) \hookrightarrow \mathcal{T}_N(\lambda)$ is thus a closed immersion of k-schemes.*

Proof. Recall that $\mathcal{T}_0(\lambda) = \operatorname{Spec} k$ and use induction on N. □

Proof of Theorem 6.2. Let I be any R-valued point of $\mathcal{T}_{m-1}(\lambda)$ and consider the exact sequence of flat R-modules

$$\begin{aligned} 0 \;\to\; K := \ker\bigl(z^{\#}\bigr) &\to F_{p^{m-1}}(R)/\bigl(F_{p^{m-1}}(R)\cap I\bigr) \\ &\xrightarrow{z^{\#}} F_{p^{m-1}}(R)/\bigl(F_{p^{m-1}}(R)\cap I\bigr) \to \sum_{i=1}^{n} Rx_{i,m-1} \to 0. \end{aligned}$$

Then K is flat over R, and hence locally free since it is of finite presentation, and of rank n. By Proposition 6.1 and Corollary 5.9 the fiber of $\mathcal{T}_m(\lambda) \to \mathcal{T}_{m-1}(\lambda)$ is in functorial bijection with the set of R-submodules $L \subset K$ such that K/L is flat over R and of constant rank

$$\operatorname{rk}\bigl(I(\lambda)^{\leqslant p^{m-1}}/I(\lambda)^{\leqslant p^{m-2}}\bigr)_{p^{m-1}} = \operatorname{rk} I(\lambda)_{p^{m-1}} - \operatorname{rk} I(\lambda)_{p^{m-2}} = |\mu_m|. \tag{6.2}$$

But these are exactly the R-valued points of a $\mathrm{Grass}(|\mu_m|, n)$-bundle over $\operatorname{Spec} R$. In particular, the relative dimension of $\mathcal{T}_m(\lambda) \to \mathcal{T}_{m-1}(\lambda)$ is equal to $(|\mu_m|)(n - |\mu_m|)$. $\square$

6.2. *The relation with Demazure varieties*

Recall the ideal $I(\lambda)$ of the standard lattice scheme associated to λ and its orbit closure $\mathcal{D}(\lambda)$. Denote by $h : \mathbb{N} \to \mathbb{N}$ the Hilbert function of $I(\lambda)$. Furthermore, let N be as in 4.2.1 and, as in the previous section, let $\mathfrak{X} = \{x_{i,j};\ i = 1, \dots, n;\ j = 0, \dots, N-1\}$.

Denote by $\mathrm{Frob}_R : R[\mathfrak{X}] \to R[\mathfrak{X}]$ the relative Frobenius morphism over R. Let F^* be the pullback-functor on the category of R-modules along the absolute Frobenius morphism $R \to R$, whose effect on a submodule of $R[\mathfrak{X}]$ is raising the coefficients of its elements to the p-th power. Note that applying Frob_R to a submodule of $R[\mathfrak{X}]$ and then pulling the image back via F^* yields the image of the original submodule under the *absolute* Frobenius. The pullback functor F^* induces an $\mathbb{F}_p$-morphism $\varphi : \mathcal{G}^{(N)}_{\mathrm{Gl}_n} \to \mathcal{G}^{(N)}_{\mathrm{Gl}_n}$; $\mathcal{L} \mapsto F^*(\mathcal{L})$. We set

$$\Phi := \varphi^{N-1} \times \cdots \times \varphi \times \mathrm{id} : \prod_{i=1}^{N} \mathcal{G}^{(N)}_{\mathrm{Gl}_n} \to \prod_{i=1}^{N} \mathcal{G}^{(N)}_{\mathrm{Gl}_n}.$$

Furthermore set $M := F_{p^{N-1}}(R)$ and note that on M we have an endomorphism $z^{\#}$ which is defined by restriction of scalar multiplication with z on $R[\mathfrak{X}]$: it sends $x_{i,j}^{p^{N-1-j}}$ to $x_{i,j-1}^{p^{N-j}}$ if $j > 0$ and to 0 otherwise. Let $\{e_{i,j}\}$ be the dual basis of $\{x_{i,j}^{p^{N-1-j}}\}$ and consider the dual map of $z^{\#}$, which we again denote by z:

$$z : V := \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, R) \simeq V; \qquad e_{i,j-1} \mapsto e_{i,j}.$$

Note that $V \simeq R^{nN}$, as well as its R-submodules with projective quotient which are stable under z, are lattices in the sense of Section 2 (they correspond to precisely those lattices $\mathcal{L} \subset R[\![z]\!]^n$ which contain $z^N R[\![z]\!]^n$).

We make an additional remark before stating the main result of this section: on $\mathcal{D}(\lambda)$ there is an action of the positive loop group $\mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n$ given as the pullback of the natural $\mathrm{L}^{F,\geqslant 0}\,\mathrm{Sl}_n$-action via the morphism in Eq. (4.2). On the other hand, the morphism $\mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n \times_k k \to \mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n$, where the fiber product is taken over $k \to k; x \mapsto x^{p^{N-1}}$, defines by composition an action of $\mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n \simeq \mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n \times_k k$ on the affine Grassmannian.

Theorem 6.4. *Let $\Sigma(\lambda) = \tilde{\Sigma}(\mu_1, \ldots, \mu_N)$ denote the variety of lattice chains defined in Section 2, with $\mu_1, \ldots, \mu_N$ the standard decomposition of λ. Then there is a closed immersion $\iota : \mathcal{D}(\lambda) \to \prod_{i=1}^N \mathcal{G}_{\mathrm{Gl}_n}^{(N)}$, such that the following diagram commutes*:

$$\begin{tikzcd} \mathcal{D}(\lambda) \arrow[r, "\iota"] \arrow[d, "\sigma"] & \prod_{i=1}^N \mathcal{G}_{\mathrm{Gl}_n}^{(N)} \arrow[d, "\Phi"] \\ \Sigma(\lambda) \arrow[r, hook] & \prod_{i=1}^N \mathcal{G}_{\mathrm{Gl}_n}^{(N)}. \end{tikzcd}$$

The map σ is a universal homeomorphism and the map from $\mathcal{D}(\lambda)$ to the fiber product of $\Sigma(\lambda)$ with $\prod_{i=1}^N \mathcal{G}_{\mathrm{Gl}_n}^{(N)}$ is a nil-immersion. Moreover, σ is equivariant for the above defined actions of $\mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n$.

Proof. Recall that every R-valued point of $\mathcal{D}(\lambda)$ is a twisted-linear ideal I by Corollary 5.13. For $l = 1, \ldots, N$ we set $L_l := \mathrm{Frob}_R^{N-l}(F_{p^{l-1}}(R) \cap I)$ and $\mathcal{L}_l := \mathrm{Hom}_R(M/L_l, R)$ (Compare Example 5.1 at the beginning of the previous section!). Then by Corollary 5.6, the modules L_l are projective. By Lemma 5.4 this assignment is functorial, and finally, since any twisted-linear ideal I is generated by the sets $F_{p^l}(R) \cap I$, we have a functorial *injection*. Since split exact sequences are preserved by both Frob_R and $\mathrm{Hom}_R(-, R)$, $V/\mathcal{L}_l$ is again projective, whence the map $\iota : I \mapsto (\mathcal{L}_1, \ldots, \mathcal{L}_N)$ is well defined. Note furthermore the equation $F^* L_{l-1} = \mathrm{Frob}_R^{N-l}\,\mathrm{Frob}(F_{p^{l-2}}(R) \cap I) \subset L_l$. It implies

$$F^* \mathcal{L}_{l-1} = F^* \mathrm{Hom}_R(M/L_{l-1}, R) = \mathrm{Hom}_R\big(M/F^* L_{l-1}, R\big) \supset \mathcal{L}_l, \tag{6.3}$$

which proves that the image of $\Phi \circ \iota$ indeed consists of descending lattice chains. Again by projectivity, the rank of successive quotients $F^* \mathcal{L}_{l-1}/\mathcal{L}_l$ is constant on $\operatorname{Spec} R$, i.e. equal to $|\mu_l|$ by Eq. (6.2). Hence the map σ. It is easily seen that σ is equivariant for the above-mentioned actions.

It remains to check that the immersion $\alpha : \mathcal{D}(\lambda) \to S := \Sigma(\lambda) \times_{\prod_{i=1}^N \mathcal{G}_{Gl_n}^{(N)}} \prod_{i=1}^N \mathcal{G}_{Gl_n}^{(N)}$ of k-varieties is indeed a nil-immersion, i.e. that it is surjective. (Since $S \to \Sigma(\lambda)$ is a universal homeomorphism, this will imply that σ is a universal homeomorphism as well.) But $\mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n$-equivariance shows that the image of $\mathcal{O}(\lambda)$ is dense in $\Sigma(\lambda)$, and in particular both have the same dimension. By finiteness of Φ we see that α has dense image as well, and is therefore surjective. □

Let us recall the iterated bundle of Grassmannians $\mathcal{T}_N(\lambda)$ defined in Section 6.1. In Corollary 6.3 we saw that there is a natural closed immersion $\mathcal{D}(\lambda) \hookrightarrow \mathcal{T}_N(\lambda)$, where the dimension of the latter equals the dimension of $\Sigma(\lambda)$. But in Theorem 6.4 we have now seen that this is also the dimension of $\mathcal{D}(\lambda)$. Thus we have proved

Corollary 6.5. *The varieties $\mathcal{D}(\lambda)$ and $\mathcal{T}_N(\lambda)$ are equal. In particular, $\mathcal{D}(\lambda)$ is an iterated bundle of ordinary Grassmannians.*

Example 6.6. We illustrate the theorem by calculating explicitly the smallest non-trivial example: let $n = 2$ and $N = 2$. We are thus dealing with ideals in the polynomial ring $R[x_0, x_1, y_0, y_1]$ for some k-algebra R. Let $\lambda = (1, -1) \in \check{\mathrm{X}}_+(T)$ whence $\tilde{\lambda} = (2, 0)$, $\mu_1 = \mu_2 = (1, 0)$ and $I(\lambda) = (x_0, x_1)$.

Proposition 6.7. *In the above situation the variety $\mathcal{D}(\lambda)$ is isomorphic to the projective space bundle $X = \mathrm{Proj}_{\mathbb{P}_k^1}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(-2p))$. The boundary of the big cell is the divisor $\mathbb{P}_k^1 \times_k \{\infty\}$.*

Note that X can be explicitly constructed by gluing two copies of $U = \mathbb{A}^1_k \times \mathbb{P}^1_k$ via the self-inverse isomorphism

$$\chi : \big(\mathbb{A}^1_k - \{0\}\big) \times \mathbb{P}^1_k \xrightarrow{\simeq} \big(\mathbb{A}^1_k - \{0\}\big) \times \mathbb{P}^1_k; \big(a, (c:d)\big) \mapsto \big(1/a, \big(c : a^{2p} d\big)\big).$$

We will use this description in the

Proof of Proposition 6.7. Define two maps on R-valued points.

$$\varphi : U(R) \to \mathcal{D}(\lambda)(R),$$
$$\big(a, (c:d)\big) \mapsto \big(ax_0 + y_0, cx_0^p + da^p x_1 + dy_1\big),$$
$$\psi : U(R) \to \mathcal{D}(\lambda)(R),$$
$$\big(a, (c:d)\big) \mapsto \big(x_0 + ay_0, -cy_0^p + da^p y_1 + dx_1\big).$$

The maps $\varphi(R)$ and $\psi(R)$ are clearly functorial in R, so they constitute morphisms of k-schemes. Furthermore, observe that $\varphi \circ \chi = \psi$ on $(\mathbb{A}^1_k - \{0\}) \times \mathbb{P}^1_k$. This shows that φ and ψ give a morphism $f : X \to \mathcal{D}(\lambda)$. To show that this is an isomorphism, we have to find an inverse $f(R)^{-1}$ which is functorial in R. But by Proposition 5.13 every R-valued point of $\mathcal{D}(\lambda)$ is twisted-linear, i.e. locally of the form $(ax_0 + by_0, cx_0^p + da^p x_1 + db^p y_1)$ or $(ax_0 + by_0, -cy_0^p + da^p x_1 + db^p y_1)$ (depending on whether $b \neq 0$ or $a \neq 0$) with a or b a unit in R and c or d a unit in R. Such an ideal defines in a functorial way an R-valued point of X. Obviously, the divisor $\{d = 0\} = \mathbb{P}^1_k \times_k \{\infty\}$ maps bijectively to the boundary of the big cell: it parametrizes the ideals of the form $(ax_0 + y_0, x_0^p)$ or $(x_0 + ay_0, y_0^p)$. $\square$

Similarly, we compute the Demazure-variety $\Sigma(\lambda)$. Its k-valued points are descending chains of subspaces $k^{2\times 2}_{x_0,x_1,y_0,y_1} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2$ which are stable under multiplication by z: $x_0 \mapsto x_1, y_0 \mapsto y_1$ and of codimension 1 and 2, respectively. Hence one sees like in the proposition above that $\Sigma(\lambda) \simeq \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2))$, using two charts on R-valued points

$$\varphi' : U(R) \mapsto \mathcal{D}_\mu(R),$$
$$\big(a, (c:d)\big) \mapsto (ax_0 + y_0, cx_0 + dax_1 + dy_1),$$
$$\psi' : U(R) \mapsto \mathcal{D}_\mu(R),$$
$$\big(a, (c:d)\big) \mapsto (x_0 + ay_0, -cy_0 + day_1 + dx_1),$$

and gluing them via $(a, (c:d)) \mapsto (1/a, (c : a^2 d))$.

Since the Frobenius morphism commutes with dualization, to compute the map $\sigma : \mathcal{D}(\lambda) \to \Sigma(\lambda)$ of Theorem 6.4 explicitly, we may compute in terms of defining relations of lattices, instead of generators of lattices. Hence, using the charts from above, we immediately obtain the picture

$$\begin{tikzcd}
\mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2p)) \arrow[r, "\sigma"] \arrow[d] & \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2)) \arrow[d] \\
\mathbb{P}^1_k \arrow[r, "\mathrm{Frob}"] \arrow[d] & \mathbb{P}^1_k \arrow[d] \\
\mathrm{Spec}\, k \arrow[r, "\mathrm{Frob}"] & \mathrm{Spec}\, k,
\end{tikzcd}$$

where the map σ is given on the respective charts by

$$\big(a, (c:d)\big) \mapsto \big(a^p, (c:d)\big).$$

It is elementary to check that the upper square is cartesian, by looking at the transition functions defining the respective line bundles. This is in agreement with Theorem 6.4: namely, a diagram relating this with the square of Theorem 6.4 has the form

$$\begin{tikzcd}
\mathcal{D}(\lambda) \arrow[r, "\iota"] \arrow[d, "\sigma"] \arrow[rr, bend left] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \times \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \arrow[r, "\pi_1"] \arrow[d, "\varphi\times\mathrm{id}"] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \arrow[d, "\varphi"] & \mathbb{P}^1 \arrow[l] \arrow[d, "\mathrm{Frob}"] \\
\Sigma(\lambda) \arrow[r, hook] \arrow[rr, bend right] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \times \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \arrow[r, "\pi_1"] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} & \mathbb{P}^1. \arrow[l]
\end{tikzcd}$$

Here the horizontal maps on the right are given by identifying $\mathbb{P}^1_k$ with the space of lattices $\mathcal{L}_1$ with $z\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_0$. Note that the middle square is trivially cartesian, while the right-hand square is not. Thus also the left-hand square fails to be cartesian and the map α of Theorem 6.4 is indeed a non-trivial nil-immersion.

6.3. Demazure varieties as schemes of lattices with infinitesimal structure

The k-valued points of $\mathcal{D}(\lambda)$ are lattice schemes in $\mathbb{A}^{nN}_k$ (see Definition 4.2), which, in general, will not be reduced. For instance, in the situation of Example 6.6, the set of points in $\mathcal{D}(\lambda) \simeq \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2p))$ which have non-reduced fibers in the universal family, is exactly the divisor $d = 0$ (the boundary of the big cell $\mathcal{O}(\lambda)$). The corresponding lattice schemes have the form $(ax_0 + by_0, x_0^p, y_0^p)$ with a and b not both zero. In general, we obtain

Corollary 6.8. *If k is a perfect field of characteristic p, then the map $\sigma : \mathcal{D}(\lambda) \to \Sigma(\lambda)$ defines a bijection of k-valued points. In particular, for any lattice $\mathcal{L} \in \mathcal{S}(\lambda)$, the fiber of $\mathcal{L}$ in $\Sigma(\lambda)$ can be interpreted as a variety of infinitesimal structures on $\mathcal{L}$. The lattice schemes in $\mathcal{D}(\lambda)$ which have non-trivial infinitesimal structure are exactly those lying in the boundary of the big cell $\mathcal{O}(\lambda)$.*

Proof. Only the very last assertion requires a proof: by construction, the lattices in $\mathcal{O}(\lambda)$ are reduced (since $I(\lambda)$ is). On the other hand, take any lattice $\mathcal{L}$ in the boundary of $\mathcal{O}(\lambda)$. Such a lattice maps to the boundary of $\mathcal{S}(\lambda)$ under $\mathcal{D}(\lambda) \to \Sigma(\lambda) \to \mathcal{S}(\lambda)$ by the $\mathrm{L}^{\geqslant 0}\,\mathrm{Sl}_n$-equivariance of σ (see proof of Theorem 6.4). Thus its reduced structure corresponds to a point in $\mathcal{O}(\lambda')$ for some $\lambda' < \lambda$ (Bruhat-order), whence it has a Hilbert function different from that of $I(\lambda)$. Thus, by constancy of the Hilbert function on $\mathcal{D}(\lambda)$, the reduced structure of $\mathcal{L}$ cannot belong to $\mathcal{D}(\lambda)$. $\square$

Finally, let us briefly study the invariants of lattices in $\mathcal{M}^{(N),h}_n$. (If $d_1, \ldots, d_n$ denote the elementary divisors of a lattice $\mathcal{L} \subset k((z))^n$, then by the *invariants* of $\mathcal{L}$ we mean the vector $(\mathrm{val}_z\, d_1, \ldots, \mathrm{val}_z\, d_n)$ with entries ordered by decreasing size.) We will need the following purely combinatorial lemma about partitions of a given positive integer. For an n-tuple $\sigma \in \mathbb{N}^n$ we write $|\sigma| = \sum_{i=1}^n \sigma_i$.

Lemma 6.9. *Let $\sigma, \sigma' \in \mathbb{N}^n$ with $|\sigma| = |\sigma'|$ and such that $\sigma_i \geqslant \sigma_{i+1}$ and $\sigma'_i \geqslant \sigma'_{i+1}$ for all $1 \leqslant i \leqslant n$. Let N be the maximum of all the σ_i and σ'_i and set*

$$\tau_i := \textit{number of entries in } \sigma \textit{ which are } \geqslant i,$$

$$\tau'_i := \textit{number of entries in } \sigma' \textit{ which are } \geqslant i,$$

for $i = 1, \dots, N$ (the 'dual partitions' for σ and σ'). Then, with respect to the Bruhat order, $\sigma \geqslant \sigma'$ if and only if $\tau \leqslant \tau'$.

Proof. By definition, $\tau \leqslant \tau'$ if and only if for all $1 \leqslant i \leqslant N$, $0 \leqslant \sum_{j=1}^{i}(\tau'_j - \tau_j)$. Since $|\tau| = |\tau'|$, this is equivalent to $0 \leqslant \sum_{j=i}^{N}(\tau_j - \tau'_j) = \sum_{j=i}^{N}(\#\{l \mid \sigma_l \geqslant j\} - \#\{l \mid \sigma'_l \geqslant j\}) = \sum_{l=1}^{n}(\max\{\sigma_l, i\} - \max\{\sigma'_l, i\})$. We show that this holds if $\sigma \geqslant \sigma'$: let $1 \leqslant r \leqslant n$ be the largest index such that both $\sigma_r \geqslant i$ and $\sigma'_r \geqslant i$, and let $1 \leqslant s \leqslant n$ be the largest index such that σ_s or σ'_s is $\geqslant i$. If $\sigma_{r+1} < i$, then we have

$$\sum_{j=i}^{N}(\tau_j - \tau'_j) = \sum_{l=1}^{r}(\sigma_l - \sigma'_l) + \sum_{l=r+1}^{s}(i - \sigma'_l) \geqslant \sum_{l=1}^{s}(\sigma_l - \sigma'_l).$$

On the other hand, if $\sigma'_{r+1} < i$ we obtain

$$\sum_{j=i}^{N}(\tau_j - \tau'_j) = \sum_{l=1}^{r}(\sigma_l - \sigma'_l) + \sum_{l=r+1}^{s}(\sigma_l - i) \geqslant \sum_{l=1}^{r}(\sigma_l - \sigma'_l).$$

Both expressions are thus non-negative if $\sigma \geqslant \sigma'$, which proves one direction. The other implication holds by duality: τ (resp. τ') can be regarded as the dual partition for σ (resp. σ'). Thus all the arguments remain valid if we interchange σ and τ' (resp. σ' and τ). □

Let $h : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of $I(\lambda)$ and let $V \subset \mathbb{A}_k^{nN}$ be a lattice scheme corresponding to a k-valued point in $\mathcal{M}_n^{(N),h}$. Since k is assumed to be perfect, the map in (3.2) for $\bar{R} = k$ is bijective, whence $k[\![z]\!]^F$ is by definition isomorphic to $k[\![z]\!]$. Via this isomorphism we can regard the set $V(k)$ of k-valued points as a lattice $z^N k[\![z]\!]^n \subset \mathcal{L} \subset z^{-N} k[\![z]\!]^n$ (after multiplication by a suitable power of z). From Lemma 6.9 we obtain

Corollary 6.10. *Assume that the defining ideal of V is twisted-linear, and let λ' denote the invariants (ordered by decreasing size) of the corresponding lattice $\mathcal{L} \subset k((z))^n$. Then $\lambda' \leqslant \lambda$.*

Proof. Let $I(V)$ denote the defining ideal of V. Multiplying with a suitable matrix in $\mathrm{Sl}_n(k[\![z]\!]^F)$ we may assume that V_{red} is given by the ideal $I(V)_{red} = \langle x_{1,0}, \dots, x_{1,\tilde{\lambda}'_1 - 1}, \dots, x_{n,0}, \dots, x_{n,\tilde{\lambda}'_n - 1} \rangle$ for some $\tilde{\lambda}' \in \mathbb{N}^n$ with $\tilde{\lambda}'_i \geqslant \tilde{\lambda}'_{i+1}$. Since $I(V)$ is twisted linear, its Hilbert function h determines the dimension of the k-vector space $I(V) \cap F_{p^N}(k) = I(V)_{red} \cap F_{p^N}(k)$: it is therefore equal to the dimension of $I(\lambda) \cap F_{p^N}(k)$, whence the determinants of the respective lattices coincide. In other words: $|\tilde{\lambda}| = |\tilde{\lambda}'|$. Now we have to show that $\tilde{\lambda} \geqslant \tilde{\lambda}'$, or equivalently (by Lemma 6.9) $\mu \leqslant \mu'$, where μ and μ' are the respective duals in the sense of Lemma 6.9. Since $\mu_i = \dim_k I(V)_{p^{i-1}} \cap F_{p^{i-1}}(k)$, while $\mu'_i = \dim_k (I(V)_{red})_{p^{i-1}} \cap F_{p^{i-1}}(k)$, the claim follows. □

Thus the k-valued points in $\mathcal{M}_n^{(N),h}$ which are given by twisted linear ideals correspond to lattices with invariants $\leqslant \lambda$. (I don't know, if every k-valued point in $\mathcal{M}_n^{(N),h}$ is twisted-linear.) Thinking of the analogous situation for Schubert varieties in the affine Grassmannian (where the Schubert variety $\mathcal{S}(\lambda)$ parametrizes exactly those lattices with invariants $\leqslant \lambda$) one could be tempted to think that $\mathcal{M}_n^{(N),h}$ and $\mathcal{D}(\lambda)$ coincide. However, looking once more at the simple situation of Example 6.6, we see that this is in general not the case:

Let $P = k[x_0, x_1, y_0, y_1]$, $\lambda = (2, 0)$ and set $I = (y_0, x_0^p) \in \mathcal{T} := \mathcal{T}_2$, $\mathcal{M} := \mathcal{M}_2^{(1),h}$. Note that a deformation $\tilde{I} \in \mathcal{M}(k[\epsilon])$ of I is given by 2 generators with indeterminate coefficients $a, b, c \in k$:

$$g_1 := y_0 + \epsilon a x_0,$$
$$g_2 := x_0^p + \epsilon(b y_1 + c x_1).$$

Hence $\dim \mathrm{T}_I \mathcal{M} = 3$, while the dimension of $\mathrm{T}_I \mathcal{T}$ is 2, e.g. by Theorem 6.2. (In a more elementary way, we could argue that on $\mathrm{T}_I \mathcal{T}$ we have the additional condition that $z^\# g_2 \subset (g_1)$, which means that $\epsilon(b y_0^p + c x_0^p)$ must be a multiple of $g_1^p = y_0^p$. This forces $c = 0$, whence again $\dim \mathrm{T}_I \mathcal{T} = 2$).

Of course, this exhibits only a difference in the infinitesimal structures at the point $I(\lambda)$. But also the topological spaces of $\mathcal{M}_n^{(N),h}$ and $\mathcal{D}(\lambda)$ differ in general: let $n = 4$, $\lambda = (1, 1, -1, -1)$, whence $N = 2$ and $\tilde{\lambda} = (2, 2, 0, 0)$. Then $I(\lambda) = \langle x_0, x_1, y_0, y_1 \rangle \subset k[x_i, y_i, z_i, w_i;\ i = 0, 1]$. On the other hand, consider the twisted-linear ideal $I = \langle x_0, y_0, z_0^p, z_1 \rangle$ in the same polynomial ring. Certainly, it has the same Hilbert function as $I(\lambda)$, and it defines a lattice scheme with invariants $(1, 0, 0, -1)$. Thus it is a k-valued point of $\mathcal{M}_n^{(N),h}$. However, it violates the condition $z^\# I = z^\# I^p \subset I^{\leqslant 1}$ which, by (6.1), is satisfied by points of $\mathcal{T}_2(\lambda) = \mathcal{D}(\lambda)$.

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