A survey on Hamilton cycles in directed graphs

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ABSTRACT

We survey some recent results on long-standing conjectures regarding Hamilton cycles in directed graphs, oriented graphs and tournaments. We also combine some of these to prove the following approximate result towards Kelly's conjecture on Hamilton decompositions of regular tournaments: the edges of every regular tournament can be covered by a set of Hamilton cycles which are 'almost' edge-disjoint. We also highlight the role that the notion of 'robust expansion' plays in several of the proofs. New and old open problems are discussed.

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1. Introduction

The decision problem of whether a graph has a Hamilton cycle is NP-complete and so a satisfactory characterization of Hamiltonian graphs seems unlikely. Thus it makes sense to ask for degree conditions which ensure that a graph has a Hamilton cycle. One such result is Dirac's theorem [21], which states that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. This is strengthened by Ore's theorem [56]: if G is a graph with $n \geq 3$ vertices such that every pair $x \neq y$ of non-adjacent vertices satisfies $d(x) + d(y) \geq n$, then G has a Hamilton cycle. Dirac's theorem can also be strengthened considerably by allowing many of the vertices to have small degree: Pósa's theorem [57] states that a graph on $n \geq 3$ vertices has a Hamilton cycle if its degree sequence $d_1 \leq \cdots \leq d_n$ satisfies $d_i \geq i + 1$ for all $i < (n - 1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd. Again, this is best possible − none of the degree conditions can be relaxed. Chvátal's theorem [14] is a further generalization. It characterizes all those degree sequences which ensure the existence of a Hamilton cycle in a graph: suppose that the degrees of the graph G are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. This condition on the degree sequence is best possible in the sense that for any degree sequence $d_1 \leq \cdots \leq d_n$ violating this condition there is a corresponding graph H with no Hamilton cycle whose degree sequence $d'_1, \ldots, d'_n$

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dominates $d_1, \ldots, d_n$ (i.e. the $d_i' \geq d_i$ for every $i = 1, \ldots, n$). These four results are among the most general and well-known Hamiltonicity conditions. There are many more — often involving additional structural conditions like planarity. The survey [26] gives an extensive overview (which concentrates on undirected graphs).

In this survey, we concentrate on recent progress for directed graphs. Though the problems are equally natural for directed graphs, it is usually much more difficult to obtain satisfactory results. Additional results beyond those discussed here can be found in the corresponding chapter of the monograph [4]. In Section 2, we discuss digraph analogues and generalizations of the above four results. The next section is devoted to oriented graphs — these are obtained from undirected graphs by orienting the edges (and thus are digraphs without 2-cycles). Section 4 is concerned with tournaments. Section 5 is devoted to several generalizations of the notion of a Hamilton cycle, e.g. pancyclicity and k-ordered Hamilton cycles. The final section is devoted to the concept of 'robust expansion'. This has been useful in proving many of the recent results discussed in this survey. We will give a brief sketch of how it can be used. In this paper, we also use this notion (and several results from this survey) to obtain a new result (Theorem 18) which gives further support to Kelly's conjecture on Hamilton decompositions of regular tournaments. In a similar vein, we use a result of [16] to deduce that the edges of every sufficiently dense regular (undirected) graph can be covered by Hamilton cycles which are almost edge-disjoint (Theorem 21).

2. Hamilton cycles in directed graphs

2.1. Minimum degree conditions

For an analogue of Dirac's theorem in directed graphs it is natural to consider the *minimum semidegree* $\delta^0(G)$ of a digraph G, which is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. (Here a directed graph may have two edges between a pair of vertices, but in this case their directions must be opposite.) The corresponding result is a theorem of Ghouila-Houri [25].

Theorem 1 (*Ghouila-Houri [25]*). *Every strongly connected digraph on n vertices with $\delta^+(G)+\delta^-(G) \geq n$ contains a Hamilton cycle. In particular, every digraph with $\delta^0(G) \geq n/2$ contains a Hamilton cycle.*

(When referring to paths and cycles in directed graphs we usually mean that these are directed, without mentioning this explicitly.)

For undirected regular graphs, Jackson [35] showed that one can reduce the degree condition in Dirac's theorem considerably if we also impose a connectivity condition, i.e. every 2-connected d-regular graph on n vertices with $d \geq n/3$ contains a Hamilton cycle. Hilbig [33] and Zhu et al. [71] improved the degree condition to $n/3 - 1$ unless G is the Petersen graph. The example in Fig. 1 shows that the degree condition cannot be reduced any further. Clearly, the connectivity condition is necessary. We believe that a similar result should hold for directed graphs too.

Conjecture 2. *Every strongly 2-connected d-regular digraph on n vertices with $d \geq n/3$ contains a Hamilton cycle.*

Replacing each edge in Fig. 1 with two oppositely oriented edges shows that the degree condition cannot be reduced. Moreover, it is not hard to see that the strong 2-connectivity cannot be replaced by just strong connectivity. For undirected graphs, analogous questions have also been studied for 3-connected graphs.

2.2. Ore-type conditions

Woodall proved the following digraph version of Ore's theorem, which generalizes Ghouila-Houri's theorem. $d^+(x)$ denotes the outdegree of a vertex x, and $d^-(x)$ its indegree.

Theorem 3 (*Woodall [68]*). *Let G be a strongly connected digraph on $n \geq 2$ vertices. If $d^+(x)+d^-(y) \geq n$ for every pair $x \neq y$ of vertices for which there is no edge from x to y, then G has a Hamilton cycle.*
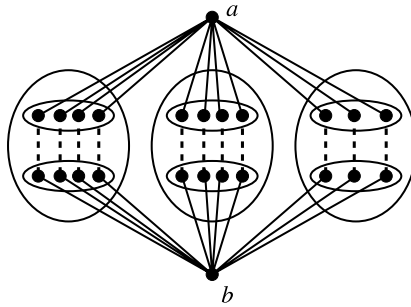
Fig. 1. A $(3s - 1)$-regular 2-connected graph G on $n = 9s + 2$ vertices with no Hamilton cycle. To construct G, start with 3 disjoint cliques on $3s$ vertices each. In the ith clique choose disjoint sets A_i and B_i with $|A_i| = |B_i|$ and $|A_1| = |A_2| = s$ and $|A_3| = s - 1$. Remove a perfect matching between A_i and B_i for each i. Add 2 new vertices a and b, where a is connected to all vertices in the sets A_i and b is connected to all vertices in all the sets B_i.
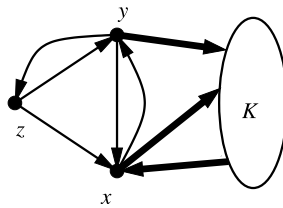
Fig. 2. An extremal example for Conjecture 5: let F be the digraph obtained from the complete digraph K on $n - 3$ vertices and a complete digraph on 3 other vertices x, y, z as follows: remove the edge from x to z, add all edges in both directions between x and K and all edges from y to K.

Woodall's theorem in turn is generalized by Meyniel's theorem, where the degree condition is formulated in terms of the total degree of a vertex. Here the *total degree* $d(x)$ *of* x is defined as $d(x) := d^+(x) + d^-(x)$.

Theorem 4 (*Meyniel [52]*)**.** *Let G be a strongly connected digraph on $n \geq 2$ vertices. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in G, then G has a Hamilton cycle.*

The following conjecture of Bang-Jensen et al. [5] would strengthen Meyniel's theorem by requiring the degree condition only for dominated pairs of vertices (a pair of vertices is *dominated* if there is a vertex which sends an edge to both of them).

Conjecture 5 (*Bang-Jensen et al. [5]*)**.** *Let G be a strongly connected digraph on $n \geq 2$ vertices. If $d(x) + d(y) \geq 2n - 1$ for all dominated pairs of non-adjacent vertices in G, then G has a Hamilton cycle.*

An extremal example F can be constructed as in Fig. 2. To see that F has no Hamilton cycle, note that every Hamilton path in $F - x$ has to start at z. Also, note that the only non-adjacent (dominated) pairs of vertices are z together with a vertex u in K and these satisfy $d(z) + d(u) = 2n - 2$.

Some support for the conjecture is given e.g. by the following result of Bang-Jensen et al. [6]: if we also assume the degree condition for all pairs of non-adjacent vertices which have a common outneighbour, then G has a 1-factor, i.e. a union of vertex-disjoint cycles covering all the vertices of G.

There are also a number of degree conditions which involve triples or 4-sets of vertices, see e.g. the corresponding chapter in [4].

2.3. Degree sequences forcing Hamilton cycles in directed graphs

Nash-Williams [55] raised the question of a digraph analogue of Chvátal's theorem quite soon after the latter was proved: for a digraph G it is natural to consider both its outdegree sequence $d_1^+, \ldots, d_n^+$

and its indegree sequence $d_1^-, \ldots, d_n^-$. Throughout, we take the convention that $d_1^+ \le \cdots \le d_n^+$ and $d_1^- \le \cdots \le d_n^-$ without mentioning this explicitly. Note that the terms d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of G.

Conjecture 6 (*Nash-Williams [55]*). *Suppose that G is a strongly connected digraph on $n \ge 3$ vertices such that for all $i < n/2$*

(i) $d_i^+ \ge i + 1$ *or* $d_{n-i}^- \ge n - i$,

(ii) $d_i^- \ge i + 1$ *or* $d_{n-i}^+ \ge n - i$.

Then G contains a Hamilton cycle.

It is even an open problem whether the conditions imply the existence of a cycle through any pair of given vertices (see [8]). The following example shows that the degree condition in Conjecture 6 would be best possible in the sense that for all $n \ge 3$ and all $k < n/2$ there is a non-Hamiltonian strongly connected digraph G on n vertices which satisfies the degree conditions except that $d_k^+, d_k^- \ge k + 1$ are replaced by $d_k^+, d_k^- \ge k$ in the kth pair of conditions. To see this, take an independent set I of size $k < n/2$ and a complete digraph K of order $n - k$. Pick a set X of k vertices of K and add all possible edges (in both directions) between I and X. The digraph G thus obtained is strongly connected, not Hamiltonian and

$$\underbrace{k, \ldots, k}_{k \text{ times}}, \underbrace{n - 1 - k, \ldots, n - 1 - k}_{n - 2k \text{ times}}, \underbrace{n - 1, \ldots, n - 1}_{k \text{ times}}$$

is both the out- and indegree sequence of G. In contrast to the undirected case there exist examples with a similar degree sequence to the above but whose structure is quite different (see [49,17]). This is one of the reasons which makes the directed case much harder than the undirected one. In [17], the following approximate version of Conjecture 6 for large digraphs was proved.

Theorem 7 (*Christofides et al. [17]*). *For every $\beta > 0$ there exists an integer $n_0 = n_0(\beta)$ such that the following holds. Suppose that G is a digraph on $n \ge n_0$ vertices such that for all $i < n/2$*

(i) $d_i^+ \ge \min\{i + \beta n, n/2\}$ *or* $d_{n-i-\beta n}^- \ge n - i$;

(ii) $d_i^- \ge \min\{i + \beta n, n/2\}$ *or* $d_{n-i-\beta n}^+ \ge n - i$.

Then G contains a Hamilton cycle.

This improved a recent result in [49], where the degrees in the first parts of these conditions were not 'capped' at $n/2$. The earlier result in [49] was derived from a result in [39] on the existence of a Hamilton cycle in an oriented graph satisfying a certain expansion property. Capping the degrees at $n/2$ makes the proof far more difficult: the conditions of Theorem 7 only imply a rather weak expansion property and there are many types of digraphs which almost satisfy the conditions but are not Hamiltonian.

The following weakening of Conjecture 6 was posed earlier by Nash-Williams [54]. It would yield a digraph analogue of Pósa's theorem.

Conjecture 8 (*Nash-Williams [54]*). *Let G be a digraph on $n \ge 3$ vertices such that $d_i^+, d_i^- \ge i + 1$ for all $i < (n - 1)/2$ and such that additionally $d_{\lceil n/2 \rceil}^+, d_{\lceil n/2 \rceil}^- \ge \lceil n/2 \rceil$ when n is odd. Then G contains a Hamilton cycle.*

The previous example shows the degree condition would be best possible in the same sense as described there. The assumption of strong connectivity is not necessary in Conjecture 8, as it follows from the degree conditions. Theorem 7 immediately implies a corresponding approximate version of Conjecture 8. In particular, for half of the vertex degrees (namely those whose value is $n/2$), the result matches the conjectured value.

2.4. Chvátal–Erdős type conditions

Another sufficient condition for Hamiltonicity in undirected graphs which is just as fundamental as those listed in the introduction is the Chvátal–Erdős theorem [15]: suppose that G is an undirected

graph with $n \geq 3$ vertices, for which the vertex-connectivity number $\kappa(G)$ and the independence number $\alpha(G)$ satisfy $\kappa(G) \geq \alpha(G)$, then G has a Hamilton cycle. Currently, there is no digraph analogue of this. Given a digraph G, let $\alpha_0(G)$ denote the size of the largest set S so that S induces no edge and let $\alpha_2(G)$ be the size of the largest set S so that S induces no cycle of length 2. So $\alpha_0(G) \leq \alpha_2(G)$. $\alpha_0(G)$ is probably the more natural extension of the independence number to digraphs. However, even the following basic question (already discussed e.g. in [38]) is still open.

Question 9. *Is there a function $f_0(k)$ so that every digraph with $\kappa(G) \geq f_0(k)$ and $\alpha_0(G) \leq k$ contains a Hamilton cycle?*

Here the connectivity $\kappa(G)$ of a digraph is defined to be the size of the smallest set of vertices S so that $G - S$ is either not strongly connected or consists of a single vertex. The following result shows that the analogous function for $\alpha_2(G)$ does exist.

Theorem 10 (*Jackson [37]*)**.** *If G is a digraph with*

$$\kappa(G) \geq 2^{\alpha_2(G)}(\alpha_2(G) + 2)!,$$

then G has a Hamilton cycle.

The proof involves a 'reduction' of the problem to the undirected case. As observed by Thomassen and Chakroun (see [38] again), there are non-Hamiltonian digraphs with $\kappa(G) = \alpha_2(G) = 2$ and $\kappa(G) = \alpha_2(G) = 3$. But it could well be that every digraph satisfying $\kappa(G) \geq \alpha_2(G) \geq 4$ has a Hamilton cycle. Even the following weaker conjecture is still wide open.

Conjecture 11 (*Jackson and Ordaz [38]*)**.** *If G is a digraph with $\kappa(G) \geq \alpha_2(G) + 1$, then G contains a Hamilton cycle.*

(In fact, they even conjectured that G as above is pancyclic.) Since the problem seems very difficult, even (say) a bound on κ which is polynomial in α_2 in Theorem 10 would be interesting.

3. Hamilton cycles in oriented graphs

Recall that an *oriented graph* is a directed graph with no 2-cycles. Results on oriented graphs seem even more difficult to obtain than results for the digraph case (the Caccetta–Häggkvist conjecture on the girth of oriented graphs of large minimum outdegree is a notorious example of this kind). In particular, most problems regarding Hamiltonicity of such graphs were open until recently and many open questions still remain.

3.1. Minimum degree conditions

Thomassen [64] raised the natural question of determining the minimum semidegree that forces a Hamilton cycle in an oriented graph. Thomassen initially believed that the correct minimum semidegree bound should be $n/3$ (this bound is obtained by considering a 'blow-up' of an oriented triangle). However, Häggkvist [27] later gave a construction which gives a lower bound of $\lceil(3n - 4)/8\rceil - 1$: for n of the form $n = 4m + 3$ where m is odd, we construct G on n vertices as in Fig. 3. Since every path which joins two vertices in D has to pass through B, it follows that every cycle contains at least as many vertices from B as it contains from D. As $|D| > |B|$ this means that one cannot cover all the vertices of G by disjoint cycles. This construction can be extended to arbitrary n (see [39]). The following result exactly matches this bound and improves earlier ones of several authors, e.g. [27,29,65]. In particular, the proof builds on an approximate version which was proved in [42].

Theorem 12 (*Keevash et al. [39]*)**.** *There exists an integer n_0 so that any oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \frac{3n-4}{8}$ contains a Hamilton cycle.*

Jackson conjectured that for regular oriented graphs one can significantly reduce the degree condition.

Conjecture 13 (*Jackson [36]*)**.** *For each $d > 2$, every d-regular oriented graph G on $n \leq 4d + 1$ vertices has a Hamilton cycle.*
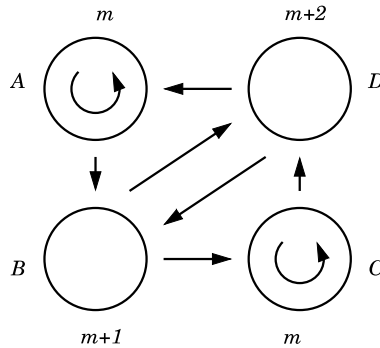
Fig. 3. An extremal example for Theorem 12: partition the vertices into 4 parts A, B, C, D, with $|A| = |C| = m$, $|B| = m + 1$ and $|D| = m + 2$. Each of A and C spans a regular tournament, B and D are joined by a bipartite tournament (i.e. an orientation of the complete bipartite graph) which is as regular as possible. We also add all edges from A to B, from B to C, from C to D and from D to A.

The disjoint union of two regular tournaments on $n/2$ vertices shows that this would be best possible. Note that the degree condition is smaller than the one in Conjecture 2. We believe that it may actually be possible to reduce the degree condition even further if we assume that G is strongly 2-connected: is it true that for each $d > 2$, every d-regular strongly 2-connected oriented graph G on $n \leq 6d$ vertices has a Hamilton cycle? A suitable orientation of the example in Fig. 1 shows that this would be best possible.

3.2. Ore-type conditions

Häggkvist [27] also made the following conjecture which is closely related to Theorem 12. Given an oriented graph G, let $\delta(G)$ denote the minimum degree of G (i.e. the minimum number of edges incident to a vertex) and set $\delta^*(G) := \delta(G) + \delta^+(G) + \delta^-(G)$.

Conjecture 14 (*Häggkvist [27]*)**.** *Every oriented graph G on n vertices with $\delta^*(G) > (3n - 3)/2$ contains a Hamilton cycle.*

(Note that this conjecture does not quite imply Theorem 12 as it results in a marginally greater minimum semidegree condition.) In [42], Conjecture 14 was verified approximately, i.e. if $\delta^*(G) \geq (3/2 + o(1))n$, then G has a Hamilton cycle (note this implies an approximate version of Theorem 12). The same methods also yield an approximate version of Ore's theorem for oriented graphs.

Theorem 15 (*Kelly et al. [42]*)**.** *For every $\alpha > 0$ there exists an integer $n_0 = n_0(\alpha)$ such that every oriented graph G of order $n \geq n_0$ with $d^+(x) + d^-(y) \geq (3/4 + \alpha)n$ whenever G does not contain an edge from x to y contains a Hamilton cycle.*

The construction in Fig. 3 shows that the bound is best possible up to the term αn. It would be interesting to obtain an exact version of this result.

Song [59] proved that every oriented graph on $n \geq 9$ vertices with $\delta(G) \geq n - 2$ and $d^+(x) + d^-(y) \geq n - 3$ whenever G does not contain an edge from x to y is pancyclic (i.e. G contains cycles of all possible lengths). In [59] he also claims (without proof) that the condition is best possible for infinitely many n as G may fail to contain a Hamilton cycle otherwise. Note that Theorem 15 implies that this claim is false.

3.3. Degree sequence conditions and Chvátal–Erdős type conditions

In [49] a construction was described which showed that there is no satisfactory analogue of Pósa's theorem for oriented graphs: as soon as we allow a few vertices to have a degree somewhat below $3n/8$, then one cannot guarantee a Hamilton cycle. The question of exactly determining all those degree sequences which guarantee a Hamilton cycle remains open though.

It is also not clear whether there may be a version of the Chvátal–Erdős theorem for oriented graphs.

4. Tournaments

A *tournament* is an orientation of a complete graph. It has long been known that tournaments enjoy particularly strong Hamiltonicity properties: Camion [12] showed that we only need to assume strong connectivity to ensure that a tournament has a Hamilton cycle. Moon [53] strengthened this by proving that every strongly connected tournament is even pancyclic. It is easy to see that a minimum semidegree of $n/4$ forces a tournament on n vertices to be strongly connected, leading to a better degree condition for Hamiltonicity than that of $(3n - 4)/8$ for the class of all oriented graphs.

4.1. Edge-disjoint Hamilton cycles and decompositions

A *Hamilton decomposition* of a graph or digraph G is a set of edge-disjoint Hamilton cycles which together cover all the edges of G. Not many examples of graphs with such decompositions are known. One can construct a Hamilton decomposition of a complete graph if and only if its order is odd (this was first observed by Walecki in the late 19th century). Tillson [61] proved that a complete digraph G on n vertices has a Hamilton decomposition if and only if $n \neq 4, 6$. The following conjecture of Kelly from 1968 (see [53]) would be a far-reaching generalization of Walecki's result.

Conjecture 16 (*Kelly*)**.** *Every regular tournament on n vertices can be decomposed into $(n - 1)/2$ edge-disjoint Hamilton cycles.*

In [50] we proved an approximate version of Kelly's conjecture. Moreover, the result holds even for oriented graphs G which are not quite regular and whose 'underlying' undirected graph is not quite complete.

Theorem 17 (*Kühn et al. [50]*)**.** *For every $\eta_1 > 0$ there exist $n_0 = n_0(\eta_1)$ and $\eta_2 = \eta_2(\eta_1) > 0$ such that the following holds. Suppose that G is an oriented graph on $n \geq n_0$ vertices such that $\delta^0(G) \geq (1/2 - \eta_2)n$. Then G contains at least $(1/2 - \eta_1)n$ edge-disjoint Hamilton cycles.*

We also proved that the condition on the minimum semidegree can be relaxed to $\delta^0(G) \geq (3/8 + \eta_2)n$. This is asymptotically best possible since the construction described in Fig. 3 is almost regular.

Some earlier support for Kelly's conjecture was provided by Thomassen [66], who showed that the edges of every regular tournament can be covered by at most $12n$ Hamilton cycles. In this paper, we improve this to an asymptotically best possible result. We will give a proof (which relies on Theorem 17) in Section 6.1.

Theorem 18. *For every $\xi > 0$ there exists an integer $n_0 = n_0(\xi)$ so that every regular tournament G on $n \geq n_0$ vertices contains a set of $(1/2 + \xi)n$ Hamilton cycles which together cover all the edges of G.*

Kelly's conjecture has been generalized in several ways, e.g. Bang-Jensen and Yeo [7] conjectured that every k-edge-connected tournament has a decomposition into k spanning strong digraphs. A bipartite version of Kelly's conjecture was also formulated by Jackson [36]. Thomassen made the following conjecture which replaces the assumption of regularity by high connectivity.

Conjecture 19 (*Thomassen [65]*)**.** *For every $k \geq 2$ there is an integer $f(k)$ so that every strongly $f(k)$-connected tournament has k edge-disjoint Hamilton cycles.*

A conjecture of Erdős (see [65]) which is also related to Kelly's conjecture states that almost all tournaments G have at least $\delta^0(G)$ edge-disjoint Hamilton cycles.

Similar techniques as in the proof of the approximate version of Kelly's conjecture were used at the same time in [16] to prove approximate versions of two long-standing conjectures of Nash-Williams on edge-disjoint Hamilton cycles in (undirected) graphs. One of these results states that one can almost decompose any dense regular graph into Hamilton cycles.

Theorem 20 (*Christofides et al. [16]*). *For every $\eta > 0$ there is an integer $n_0 = n_0(\eta)$ so that every d-regular graph on $n \geq n_0$ vertices with $d \geq (1/2 + \eta)n$ contains at least $(d - \eta n)/2$ edge-disjoint Hamilton cycles.*

In Section 6.1 we deduce the following analogue of Theorem 18:

Theorem 21. *For every $\xi > 0$ there is an integer $n_0 = n_0(\xi)$ so that every d-regular graph G on $n \geq n_0$ vertices with $d \geq (1/2 + \xi)n$ contains a set of at most $(d + \xi n)/2$ Hamilton cycles which together cover all the edges of G.*

4.2. Counting Hamilton cycles in tournaments

One of the earliest results on tournaments (and the probabilistic method), was obtained by Szele [60], who showed that the maximum number $P(n)$ of Hamilton paths in a tournament on n vertices satisfies $P(n) = O(n!/2^{3n/4})$ and $P(n) \geq n!/2^{n-1} =: f(n)$. The lower bound is obtained by considering a random tournament. The best upper bound is due to Friedgut and Kahn [23] who showed that $P(n) = O(n^c f(n))$, where c is slightly less than 5/4. The best current lower bound is due to Wormald [69], who showed that $P(n) \geq (2.855 + o(1))f(n)$. So in particular, $P(n)$ is not attained for random tournaments. Also, he conjectured that this bound is very close to the correct value.

Similarly, one can define the maximum number $C(n)$ of Hamilton cycles in a tournament on n vertices. Note that by considering a random tournament again, we obtain $C(n) \geq (n-1)!/2^n =: g(n)$. Unsurprisingly, $C(n)$ and $P(n)$ are very closely related, e.g. we have $P(n) \geq nC(n)$. In particular, the main result in [23] states that $C(n) = O(n^c g(n))$, where c is the same as above. This implies the above bound on $P(n)$, since Alon [2] observed that $P(n) \leq 4C(n + 1)$. Also, Wormald [69] showed that $C(n) \geq (2.855 + o(1))g(n)$. (Note this also follows by combining Alon's observation with the lower bound on $P(n)$ in [69].)

Of course, in general it does not make sense to ask for the minimum number of Hamilton paths or cycles in a tournament. However, the question does make sense for regular tournaments. Friedgut and Kahn [23] asked whether the number of Hamilton cycles in a regular tournament is always at least $\Omega(g(n))$. The best result towards this was recently obtained by Cuckler [20], who showed that every regular tournament on n vertices contains at least $n!/(2+o(1))^n$ Hamilton cycles. This also answers an earlier question of Thomassen. Asking for the minimum number of Hamilton paths in a tournament T also makes sense if we assume that T is strongly connected. Busch [11] determined this number exactly by showing that an earlier construction of Moon is best possible. The related question on the minimum number of Hamilton cycles in a strongly 2-connected tournament is still open (see [11]).

4.3. Sumner's universal tournament conjecture

Sumner's universal tournament conjecture states that every tournament on $2n - 2$ vertices contains every tree on n vertices. In [47] an approximate version of this conjecture was proved and subsequently in [48], the conjecture was proved for all large trees (see e.g. [47] for a discussion of numerous previous results). The proof in [48] builds on several structural results proved in [47].

Theorem 22 (*Kühn et al. [47,48]*). *There is an integer n_0 such that for all $n \geq n_0$ every tournament G on $2n - 2$ vertices contains any directed tree T on n vertices.*

While this result is not directly related to the main topic of the survey (i.e. Hamilton cycles), there are several connections. Firstly, just as with many of the new results in the other sections, the concept of a robust expander is crucial in the proof of Theorem 22. Secondly, the proof of Theorem 22 also makes direct use of the fact that a robust expander contains a Hamilton cycle (Theorem 30). Suitable parts of the tree T are embedded by considering a random walk on (the blow-up of) such a Hamilton cycle.

In [47], we also proved that if T has bounded maximum degree, then it suffices if the tournament G has $(1 + \alpha)n$ vertices. This is best possible in the sense that the 'error term' αn cannot be completely omitted in general. But it seems possible that it can be reduced to a constant which depends only on the maximum degree of T. If T is an orientation of a path, then the error term can be omitted completely:

a result of Thomason [62] implies that every sufficiently large tournament contains every possible orientation of a Hamilton path. Havet and Thomassé [31] showed that this even holds for tournaments whose number of vertices is not 3, 5 or 7 (for arbitrary orientations of Hamilton cycles, see Section 5.2).

5. Generalizations

In this section, we discuss several natural ways of strengthening the notion of a Hamilton cycle.

5.1. Pancyclicity

Recall that a graph (or digraph) is *pancyclic* if it contains a cycle of every possible length. Dirac's theorem implies that a graph on $n \geq 3$ vertices is pancyclic if it has minimum degree greater than $n/2$. (To see this, remove a vertex x and apply Dirac's theorem to the remaining subgraph to obtain a cycle of length $n - 1$. Then consider the neighbourhood of x on this cycle to obtain cycles of all possible lengths through x.) Similarly, one can use Ghouila-Houri's theorem to deduce that every digraph on n vertices with minimum semidegree greater than $n/2$ is pancyclic. In both cases, the complete bipartite (di-)graph whose vertex class sizes are as equal as possible shows that the bound is best possible. More generally, the same trick also works for Meyniel's theorem: let G be a strongly connected digraph on $n \geq 2$ vertices. If $d(x) + d(y) \geq 2n + 1$ for all pairs of non-adjacent vertices $x \neq y$ in G, then G is pancyclic. (Indeed, the conditions imply that either G contains a strongly connected tournament or contains a vertex x with $d(x) > n$, in which case we can proceed as above.) If n is even, the bound $2n+1$ is best possible. For n is odd, it follows from a result of Thomassen [63] that one can improve it to $2n$.

For oriented graphs the minimum semidegree threshold which guarantees pancyclicity turns out to be $(3n - 4)/8$, i.e. the same threshold as for Hamiltonicity (see [43]). The above trick of removing a vertex does not work here. Instead, to obtain 'long' cycles one can modify the proof of Theorem 12. A triangle is guaranteed by results on the Caccetta–Häggkvist conjecture — e.g. a very recent result of Hladký et al. [34] states that every oriented graph on n vertices with minimum semidegree at least $0.347n$ contains a 3-cycle. Short cycles of length $\ell \geq 4$ can be guaranteed by a result in [43] which states that for all $n \geq 10^{10}\ell$ every oriented graph G on n vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ contains an ℓ-cycle. This is best possible for all those $\ell \geq 4$ which are not divisible by 3. Surprisingly, for some other values of ℓ, an ℓ-cycle is forced by a much weaker minimum degree condition. In particular, the following conjecture was made in [43].

Conjecture 23 (*Kelly et al. [43]*). *Let $\ell \geq 4$ be a positive integer and let k be the smallest integer that is greater than 2 and does not divide ℓ. Then there exists an integer $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lfloor n/k \rfloor + 1$ contains an ℓ-cycle.*

The extremal examples for this conjecture are always 'blow-ups' of cycles of length k. Possibly one can even weaken the condition by requiring only the outdegree of G to be large. It is easy to see that the only values of k that can appear in Conjecture 23 are of the form $k = p^s$ with $k \geq 3$, where p is a prime and s a positive integer.

5.2. Arbitrary orientations

As mentioned earlier, the most natural notion of a cycle in a digraph is to have all edges directed consistently. But it also makes sense to ask for Hamilton cycles where the edges are oriented in some prescribed way, e.g. to ask for an 'antidirected' Hamilton cycle where consecutive edges have opposite directions. Surprisingly, it turns out that both for digraphs and oriented graphs the minimum degree threshold which guarantees a 'consistent' Hamilton cycle is approximately the same as that which guarantees an arbitrary orientation of a Hamilton cycle.

Theorem 24 (*Häggkvist and Thomason [28]*). *There exists an n_0 so that every digraph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq n/2 + n^{5/6}$ contains every orientation of a Hamilton cycle.*

In [29], they conjectured an analogue of this for oriented graphs, which was recently proved by Kelly.

Theorem 25 (*Kelly [41]*)**.** *For every $\alpha > 0$ there exists an integer $n_0 = n_0(\alpha)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq (3/8 + \alpha)n$ contains every orientation of a Hamilton cycle.*

The proof of this result uses Theorem 12 as the well as the notion of expanding digraphs. Interestingly, Kelly observed that the thresholds for various orientations do not coincide exactly: for instance, if we modify the example in Fig. 3 so that all classes have the same odd size, then the resulting oriented graph has minimum semidegree $(3n − 4)/8$ but no antidirected Hamilton cycle.

Thomason [62] showed that for large tournaments strong connectivity ensures every possible orientation of a Hamilton cycle. More precisely, he showed that for $n \geq 2^{128}$, every tournament on n vertices contains all possible orientations of a Hamilton cycle, except possibly the 'consistently oriented' one. (Note that this implies that every large tournament contains every orientation of a Hamilton path.) The bound on n was later reduced to 68 by Havet [30]. Thomason conjectured that the correct bound is $n \geq 9$.

5.3. k-ordered Hamilton cycles

Suppose that we require our (Hamilton) cycle to visit several vertices in a specific order. More formally, we say that a graph G is *k-ordered* if for every sequence $s_1, \ldots, s_k$ of distinct vertices of G there is a cycle which encounters $s_1, \ldots, s_k$ in this order. G is *k-ordered Hamiltonian* if it contains a Hamilton cycle with this property. Kierstead et al. [44] determined the minimum degree which forces an (undirected) graph to be k-ordered Hamiltonian.

Theorem 26 (*Kierstead et al. [44]*)**.** *For all $k \geq 2$, every graph on $n \geq 11k − 3$ vertices of minimum degree at least $\lceil n/2 \rceil + \lfloor k/2 \rfloor − 1$ is k-ordered Hamiltonian.*

The extremal example consists of two cliques intersecting in $k − 1$ vertices if k is even and two cliques intersecting in $k − 2$ vertices if k is odd. The case when n is not too large compared to k is still open. The corresponding Ore-type problem was solved in [22]. Here the Ore-type result does not imply the Dirac-type result above. Many variations and stronger notions have been investigated (see e.g. [26] again).

Directed graphs form a particularly natural setting for this kind of question. The following result gives a directed analogue of Theorem 26.

Theorem 27 (*Kühn et al. [51]*)**.** *For every $k \geq 3$ there is an integer $n_0 = n_0(k)$ such that every digraph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lceil (n + k)/2 \rceil − 1$ is k-ordered Hamiltonian.*

Note that if n is even and k is odd the bound on the minimum semidegree is slightly larger than in the undirected case. However, it is best possible in all cases. In fact, if the minimum semidegree is smaller, it turns out that G need not even be k-ordered. Again, the family of extremal examples turns out to be much richer than in the undirected case. Note that every Hamiltonian digraph is 2-ordered Hamiltonian, so the case when $k \leq 2$ in Theorem 27 is covered by Ghouila-Houri's theorem. It would be interesting to obtain an Ore-type or an oriented version of Theorem 27.

5.4. Factors with prescribed cycle lengths

Another natural way of generalizing Dirac's theorem is to ask for a certain set of vertex-disjoint cycles in G which together cover all the vertices of G (note this also generalizes the notion of pancyclicity). For large undirected graphs, Abbasi [1] determined the minimum degree which guarantees k vertex-disjoint cycles in a graph G whose (given) lengths are $n_1, \ldots, n_k$, where the n_i sum up to n and where the order n of G is sufficiently large. As in the case of Hamilton cycles, the corresponding questions for directed and oriented graphs appear much harder than in the undirected case and again much less is known. Keevash and Sudakov [40] recently obtained the following result.

Theorem 28 (*Keevash and Sudakov [40]*)**.** *There exist positive constants c, C and an integer n_0 so that whenever G is an oriented graph on $n \geq n_0$ vertices with minimum semidegree at least $(1/2 − c)n$ and whenever $n_1, \ldots, n_t$ are so that $\sum_{i=1}^{t} n_i \leq n − C$, then G contains vertex-disjoint cycles of length $n_1, \ldots, n_t$.*
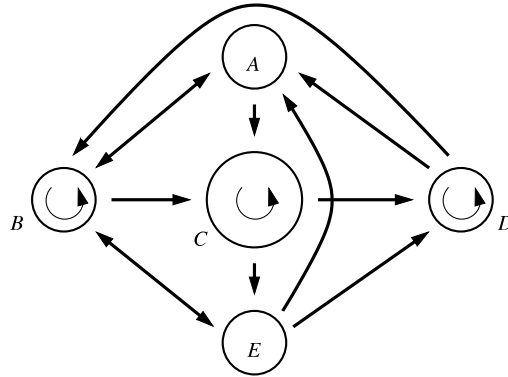
Fig. 4. An extremal example for Conjecture 29: the set sizes are $|A| = m$, $|B| = m - 1$, $|C| = 2m + 1$, $|D| = m - 1$ and $|E| = m + 1$, where m is even. B, C and D induce regular tournaments, while A and E induce independent sets. A single arrow (e.g. from B to C) indicates an orientation of the complete bipartite graph from the initial set towards the final set. A double edge (e.g. between B and E) indicates an orientation of the complete bipartite graph so that within each set, the in- and outdegrees of the vertices differ by at most one.

In general, one cannot take $C = 0$. In the case of triangles (i.e. when all the $n_i = 3$), they show that one can choose $C = 3$. This comes very close to proving a recent conjecture formulated independently by [70], which states that every regular tournament on $n = 6k + 3$ vertices contains vertex-disjoint triangles covering all the vertices of the tournament. Similar questions were also raised earlier by Song [58]. For instance, given t, he asked for the smallest integer $f(t)$ so that all but a finite number of strongly $f(t)$-connected tournaments T satisfy the following: let n be the number of vertices of T and let $\sum_{i=1}^{t} n_i = n$. Then T contains vertex-disjoint cycles of length $n_1, \ldots, n_t$. Chen et al. [13] proved the weaker result that every sufficiently large t-connected tournament G contains t vertex-disjoint cycles which together cover all the vertices of G. This proved a conjecture of Bollobás.

5.5. Powers of Hamilton cycles

Komlós et al. [45] showed that every sufficiently large graph G on n vertices with minimum degree at least $kn/(k+1)$ contains the kth power of a Hamilton cycle. Extremal examples are complete $(k+1)$-partite graphs with classes of almost equal size. It appears likely that the situation for digraphs is similar. However, just as for ordinary Hamilton cycles, it seems that for oriented graphs the picture is rather different. (Both for digraphs and oriented graphs, the most natural definition of the kth power of a cycle is a cyclically ordered set of vertices so that every vertex sends an edge to the next k vertices in the ordering.)

Conjecture 29 (*Treglown [67]*). *For every $\varepsilon > 0$ there is an integer $n_0 = n_0(\varepsilon)$ so that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree at least $(5/12 + \varepsilon)n$ contains the square of a Hamilton cycle.*

A construction which shows that the constant $5/12$ cannot be improved is given in Fig. 4. We claim that the square of any Hamilton cycle would have to visit a vertex of B in between two visits of E. Since $|B| < |E|$, this shows that the graph does not contain the square of a Hamilton cycle. To prove the claim, suppose that F is a squared Hamilton cycle and consider a vertex e of F which lies in E. Then the predecessor of e lies in C or B, so without loss of generality we may assume that it is a vertex $c_1 \in C$. Again, the predecessor of c_1 lies in C or B (since it must lie in the common inneighbourhood of C and E), so without loss of generality we may assume that it is a vertex $c_2 \in C$. The predecessor of c_2 can now lie in A, B or C. If it lies in B we are done again, if it is a vertex $c_3 \in C$, we consider its predecessor, which can again only lie in A, B or C. Since F must visit all vertices, it follows that we eventually arrive at a predecessor $a \in A$ whose successor on F is some vertex $c \in C$. The predecessor of a on F must lie in the common inneighbourhood of a and c, so it must lie in B, as required.

For the case of tournaments, the problem was solved asymptotically by Bollobás and Häggkvist [9]. Given a tournament T of large order n with minimum semidegree at least $n/4 + \varepsilon n$, they proved that (for fixed k) T contains the kth power of a Hamilton cycle. So asymptotically, the semidegree threshold for an ordinary Hamilton cycle in a tournament is the same as that for the kth power of a Hamilton cycle.

6. Robustly expanding digraphs

Roughly speaking, a graph is an expander if for every set S of vertices the neighbourhood $N(S)$ of S is significantly larger than S itself. A number of papers have recently demonstrated that there is a remarkably close connection between Hamiltonicity and expansion (see e.g. [10,32,46]). The following notion of robustly expanding (dense) digraphs was introduced in [49].

Let $0 < \nu \leq \tau < 1$. Given any digraph G on n vertices and $S \subseteq V(G)$, the ν-*robust outneighbourhood* $RN_{\nu,G}^{+}(S)$ *of* S is the set of all those vertices x of G which have at least νn inneighbours in S. G is called a *robust* (ν, τ)-*outexpander* if $|RN_{\nu,G}^{+}(S)| \geq |S| + \nu n$ for all $S \subseteq V(G)$ with $\tau n < |S| < (1 - \tau)n$. As the name suggests, this notion has the advantage that it is preserved even if we delete some vertices and edges from G. We will also use the more traditional (and weaker) notion of a (ν, τ)-*outexpander*, which means $|N^{+}(S)| \geq |S| + \nu n$ for all $S \subseteq V(G)$ with $\tau n < |S| < (1 - \tau)n$.

Theorem 30 (*Kühn et al. [49]*)**.** *Let n_0 be a positive integer and ν, τ, η be positive constants such that $1/n_0 \ll \nu \leq \tau \ll \eta < 1$. Let G be a digraph on $n \geq n_0$ vertices with $\delta^0(G) \geq \eta n$ which is a robust (ν, τ)-outexpander. Then G contains a Hamilton cycle.*

Theorem 30 is used in [49] to give a weaker version of Theorem 7 (i.e. without the degrees capped at $n/2$). In the same paper it is also applied to prove a conjecture of Thomassen regarding a weak version of Conjecture 16 (Kelly's conjecture). One can also use it to prove e.g. Theorem 15 and thus an approximate version of Theorem 12. (Indeed, as proved in [42], the degree conditions of Theorem 15 imply expansion, the proof for robust expansion is similar.) As mentioned earlier, it is also used as a tool in the proof of Theorem 22. Finally, we will also use it in the next subsection to prove Theorem 18.

In [49], Theorem 30 was deduced from a result in [39]. The proof of the result in [39] (and a similar approach in [42]) in turn relied on Szemerédi's regularity lemma and a (rather technical) version of the Blow-up lemma due to Csaba [19]. A (parallel) algorithmic version of Theorem 30 was also proved in [18]. Below, we give a brief sketch of a proof of Theorem 30 which avoids any use of the Blow-up lemma and is based on an approach in [17].

The *density* of a bipartite graph G with vertex classes A and B is defined to be $d(A, B) = \frac{e(A,B)}{|A||B|}$, where $e(A, B)$ denotes the number of edges between A and B. Given $\varepsilon > 0$, we say that G is ε-*regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have that $|d(X, Y) - d(A, B)| < \varepsilon$. We also say that G is (ε, d)-*super-regular* if it is ε-regular and furthermore every vertex $a \in A$ has degree at least $d|B|$ and similarly for every $b \in B$. These definitions generalize naturally to non-bipartite (di-)graphs. In particular, we say that a digraph G on n vertices is ε-*regular* of density d if $\frac{e(X,Y)}{|X||Y|} = d \pm \varepsilon$ for all (not necessarily disjoint) subsets X, Y of $V(G)$ of size at least εn. Here $e(X, Y)$ denotes the number of edges of G from X to Y. We say G is (ε, d)-*super-regular* if it is ε-regular and $\delta^0(G) \geq dn$.

We also need the result that every super-regular digraph contains a Hamilton cycle.

Lemma 31. *Suppose that $1/n_0 \ll \varepsilon \ll d \ll 1$ and G is an (ε, d)-super-regular digraph on $n \geq n_0$ vertices. Then G contains a Hamilton cycle.*

Lemma 31 is a special case e.g. of a result of Frieze and Krivelevich [24], who proved that an (ε, d)-super-regular digraph on n vertices has at least $(d - 4\sqrt{\varepsilon})n$ edge-disjoint Hamilton cycles if n is large and $\varepsilon \ll d$. Here we also give a sketch of a direct proof of Lemma 31.

We first prove that G contains a 1-factor. Consider the auxiliary bipartite graph whose vertex classes A and B are copies of $V(G)$ with an edge between $a \in A$ and $b \in B$ if there is an edge from a to b in G. One can show that this bipartite graph has a perfect matching (by Hall's marriage theorem), which in turn corresponds to a 1-factor in G.

It is now not hard to prove the lemma using the 'rotation–extension' technique: choose a 1-factor of G. Now remove an edge of a cycle in this 1-factor and let P be the resulting path. If the final vertex of P has any outneighbours on another cycle C of the 1-factor, we can extend P into a longer path which includes the vertices of C (and similarly for the initial vertex of P). We repeat this as long as possible (and one can always ensure that the extension step can be carried out at least once). So we may assume that all outneighbours of the final vertex of P lie on P and similarly for the initial vertex of P. Together with the ε-regularity this can be used to find a cycle with the same vertex set as P. Eventually, we arrive at a Hamilton cycle.

Sketch proof of Theorem 30. Choose ε, d to satisfy $1/n_0 \ll \varepsilon \ll d \ll \nu$. The first step is to apply a directed version of Szemerédi's regularity lemma to G (such a version was proved by Alon and Shapira [3]). This gives us a partition of the vertices of G into clusters $V_1, \ldots, V_k$ and an exceptional set V_0 so that $|V_0| \le \varepsilon n$ and all the clusters have size m. Now define a 'reduced' digraph R whose vertices are the clusters $V_1, \ldots, V_k$ and with an edge from V_i to V_j if the bipartite graph spanned by the edges from V_i to V_j is ε-regular and has density at least d. Then one can show (see Lemma 14 in [49]) that R is still a $(\nu/2, 2\tau)$-outexpander (this is the point where we need the robustness of the expansion in G) with minimum semidegree at least $\eta k/2$. This in turn can be used to show that R has a 1-factor $\mathcal{F}$ (using the same auxiliary bipartite graph as in the proof of Lemma 31). By removing a small number of vertices from the clusters, we can also assume that the bipartite subgraphs spanned by successive clusters on each cycle of $\mathcal{F}$ are super-regular, i.e. have high minimum degree. For simplicity, assume that the cluster size is still m.

Moreover, since G is an expander, we can find a short path in G between clusters of different cycles of $\mathcal{F}$ and also between any pair of exceptional vertices. However, we need to choose such paths without affecting any of the useful structures that we have found so far. For this, we will consider paths which 'wind around' cycles in $\mathcal{F}$ before moving to another cycle. More precisely, a *shifted walk* from a cluster A to a cluster B is a walk $W(A, B)$ of the form

$$W(A, B) = X_1 C_1 X_1^- X_2 C_2 X_2^- \cdots X_t C_t X_t^- X_{t+1},$$

where $X_1 = A, X_{t+1} = B, C_i$ is the cycle of $\mathcal{F}$ containing X_i, and for each $1 \le i \le t, X_i^-$ is the predecessor of X_i on C_i and the edge $X_i^- X_{i+1}$ belongs to R. We say that W as above *traverses t cycles* (even if some C_i appears several times in W). We also say that the clusters $X_2, \ldots, X_{t+1}$ are the *entry clusters* (as this is where W 'enters' a cycle C_i) and the clusters $X_1^-, \ldots, X_t^-$ are the *exit clusters* of W. Note that

(i) for any cycle of $\mathcal{F}$, its clusters are visited the same number of times by $W(A, B) - B$.

Using the expansion of R, it is not hard to see that

(ii) for any clusters A and B there is a shifted walk from A to B which does not traverse too many cycles.

Indeed, the expansion property implies that the number of clusters one can reach by traversing t cycles is at least $t\nu k/2$ as long as this is significantly less than the total number k of clusters.

Now we will 'join up' the exceptional vertices using shifted walks. For this, write $V_0 = \{a_1, \ldots, a_\ell\}$. For each exceptional vertex a_i choose a cluster T_i so that a_i has many outneighbours in T_i. Similarly choose a cluster U_i so that a_i has many inneighbours in U_i and so that

(iii) no cluster appears too often as a T_i or a U_i.

Given a cluster X, let X^- be the predecessor of X on the cycle of $\mathcal{F}$ which contains X and let X^+ be its successor. Form a 'walk' W on $V_0 \cup V(R)$ which starts at a_1, then moves to T_1, then follows a shifted walk from T_1 to U_2^+, then it winds around the entire cycle of $\mathcal{F}$ containing U_2^+ until it reaches U_2. Then W moves to a_2, then to a_3 using a shifted walk as above until it has visited all the exceptional vertices (see Fig. 5). Proceeding similarly, we can ensure that W has the following properties:

(a) W is a closed walk which visits all of V_0 and all of $V(R)$.
(b) For any cycle of $\mathcal{F}$, its clusters are visited the same number of times by W.
(c) Every cluster appears at most $m/10$ times as an entry or exit cluster.

(b) follows from (i) and (c) follows from (ii) and (iii). The next step towards a Hamilton cycle would be to find a cycle C in G which corresponds to W (i.e. each occurrence of a cluster in W is replaced by
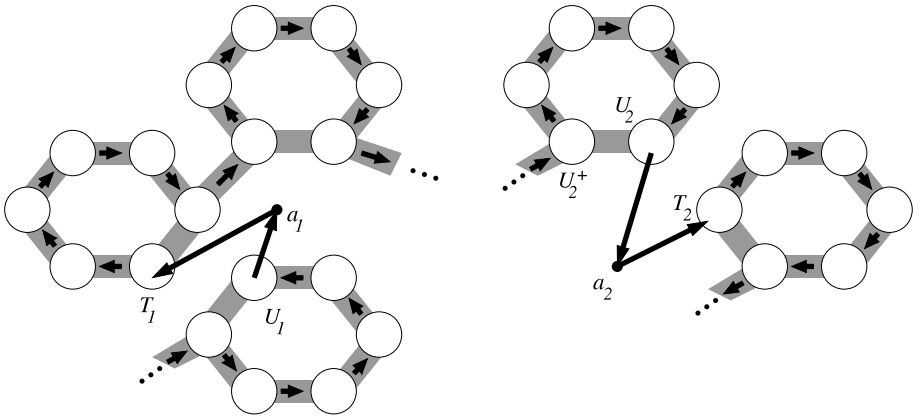
Fig. 5. Constructing the walk W.

a distinct vertex of G lying in this cluster). Unfortunately, the fact that V_0 may be much larger than the cluster size m implies that there may be clusters which are visited more than m times by W, which makes it impossible to find such a C. So we will apply a 'short-cutting' technique to W which avoids 'winding around' the cycles of $\mathcal{F}$ too often.

For this, we now fix edges in G corresponding to all those edges of W that do not lie within a cycle of $\mathcal{F}$. These edges of W are precisely the edges in W at the exceptional vertices as well as all the edges of the form AB where A is used as an exit cluster by W and B is used as an entrance cluster by W. For each edge $a_i T_i$ at an exceptional vertex we choose an edge $a_i x$, where x is an outneighbour of a_i in T_i. We similarly choose an edge $y a_i$ from U_i to a_i for each $U_i a_i$. We do this in such a way that all these edges are disjoint outside V_0. For each occurrence of AB in W, where A is used as an exit cluster by W and B is used as an entrance cluster, we choose an edge ab from A to B in G so that all these edges are disjoint from each other and from the edges chosen for the exceptional vertices (we use (c) here).

Given a cluster A, let A_{entry} be the set of all those vertices in A which are the final vertex of an edge of G fixed so far and let A_{exit} be the set of all those vertices in A which are the initial vertex of an edge of G fixed so far. So $A_{entry} \cap A_{exit} = \emptyset$. Let G_A be the bipartite graph whose vertex classes are $A \setminus A_{exit}$ and $A^+ \setminus A^+_{entry}$ and whose edges are all the edges from $A \setminus A_{exit}$ to $A^+ \setminus A^+_{entry}$ in G. Since W consists of shifted walks, it is easy to see that the vertex classes of G_A have equal size. Moreover, it is possible to carry out the previous steps in such a way that G_A is super-regular (here we use (c) again). This in turn means that G_A has a perfect matching M_A. These perfect matchings (for all clusters A) together with all the edges of G fixed so far form a 1-factor $\mathcal{C}$ of G. It remains to transform $\mathcal{C}$ into a Hamilton cycle.

We claim that for any cluster A, we can find a perfect matching M'_A in G_A so that if we replace M_A in $\mathcal{C}$ with M'_A, then all vertices of G_A will lie on a common cycle in the new 1-factor $\mathcal{C}$.

To prove this claim we proceed as follows. For every $a \in A^+ \setminus A^+_{entry}$, we move along the cycle C_a of $\mathcal{C}$ containing a (starting at a) and let $f(a)$ be the first vertex on C_a in $A \setminus A_{exit}$. Define an auxiliary digraph J on $A^+ \setminus A^+_{entry}$ such that $N_J^+(a) := N_{G_A}^+(f(a))$. So J is obtained by identifying each pair $(a, f(a))$ into one vertex with an edge from $(a, f(a))$ to $(b, f(b))$ if G_A has an edge from $f(a)$ to b (see Fig. 6). Since G_A is super-regular, it follows that J is also super-regular. By Lemma 31, J has a Hamilton cycle, which clearly corresponds to a perfect matching M'_A in G_A with the desired property.

We now repeatedly apply the above claim to every cluster. Since $A_{entry} \cap A_{exit} = \emptyset$ for each cluster A, this ensures that all vertices which lie in clusters on the same cycle of $\mathcal{F}$ will lie on the same cycle of the new 1-factor $\mathcal{C}$. Since by (a) W visits all clusters, this in turn implies that all the non-exceptional vertices will lie in the same cycle of $\mathcal{C}$. Since the exceptional vertices form an independent set in $\mathcal{C}$, it follows that $\mathcal{C}$ is actually a Hamilton cycle. □

6.1. Covering regular graphs and tournaments with Hamilton cycles

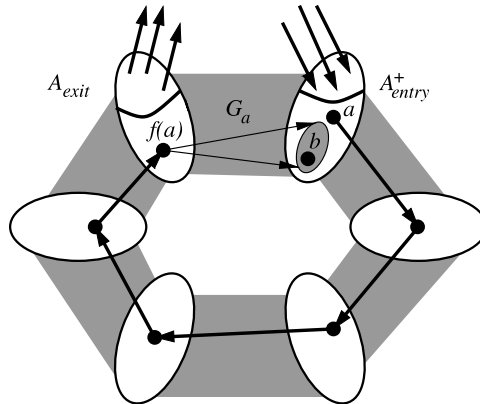Here we give proofs of Theorems 18 and 21. The proof of Theorem 18 uses Theorems 17 and 30.

Fig. 6. An illustration of the auxiliary digraph J, where there is an edge from a to b in J.

Proof of Theorem 18. Choose new constants η_1, ν, τ such that $1/n_0 \ll \eta_1 \ll \nu \leq \tau \ll \xi$. Consider any regular tournament G on $n \geq n_0$ vertices. Apply Theorem 17 to G in order to obtain a collection $\mathcal{C}$ of at least $(1/2 - \eta_1)n$ edge-disjoint Hamilton cycles. Let F be the undirected graph consisting of all those edges of G which are not covered by the Hamilton cycles in $\mathcal{C}$. Note that F is k-regular for some $k \leq 2\eta_1 n$. By Vizing's theorem the edges of F can be coloured with at most $\Delta(F) + 1 \leq 3\eta_1 n$ colours and thus F can be decomposed into at most $3\eta_1 n$ matchings. Split each of these matchings into at most $1/\eta_1^{1/2}$ edge-disjoint matchings, each containing at most $\eta_1^{1/2}n$ edges. So altogether this yields a collection $\mathcal{M}$ of at most $3\eta_1^{1/2}n$ matchings covering all edges of F. It is enough to show that for each $M \in \mathcal{M}$ there exists a Hamilton cycle of G which contains all the edges in M.

So consider any $M \in \mathcal{M}$. As observed in [49] (see the proof of Corollary 16 there), any regular tournament is a robust (ν, τ)-outexpander. Let D be the digraph obtained from G by 'contracting' all the edges in M, i.e. by successively replacing each edge $xy \in M$ with a vertex v_{xy} whose inneighbourhood is the inneighbourhood of x and whose outneighbourhood is the outneighbourhood of y. Using that M consists of at most $\eta_1^{1/2}n$ edges and that $\eta_1 \ll \nu, \tau$, it is not hard to check that D is still a robust $(\nu/2, 2\tau)$-outexpander and $\delta^0(D) \geq (1/2 - 2\eta_1^{1/2})n$. So Theorem 30 implies that D contains a Hamilton cycle, which corresponds to a Hamilton cycle in G containing all edges of M, as required. $\quad\square$

Note that we cannot simply apply Theorem 12 instead of Theorem 30 at the end of the proof, because D may not been an oriented graph. However, instead of using Theorem 30, one can also use the following result of Thomassen [66]: for every set E of $n/24$ independent edges in a regular tournament on n vertices, there is a Hamilton cycle which contains all edges in E.

Theorem 21 can be proved in a similar way, using Ghouila-Houri's theorem instead of Theorem 30.

Proof of Theorem 21. Choose a new constant η such that $1/n_0 \ll \eta \ll \xi$ and apply Theorem 20 to find a collection of at least $(d - \eta n)/2$ edge-disjoint Hamilton cycles. Let F denote the subgraph of G consisting of all edges not lying in these Hamilton cycles. Then F is k-regular for some $k \leq \eta n$. Choose a collection $\mathcal{M}$ of matchings covering all edges of F as in the proof of Theorem 18. So each matching consists of at most $\eta^{1/2}n$ edges. As before, for each $M \in \mathcal{M}$ it suffices to find a Hamilton cycle of G containing all the edges of M. Let D' be the digraph obtained from G by orienting each edge in M and replacing each edge in $E(G) \setminus M$ with two edges, one in each direction. Let D be the digraph obtained from D' by 'contracting' the edges in M as in the proof of Theorem 18. Then D' has minimum semidegree at least $n/2$ and thus contains a Hamilton cycle by Ghouila-Houri's theorem (Theorem 1). This Hamilton cycle corresponds to a Hamilton cycle in G containing all edges of M, as required. $\quad\square$

Acknowledgements

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