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Bounds for the Gini Mean Difference via the Sonin Identity

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Abstract—Some bounds for the Gini mean difference of a continuous distribution by the use of the Sonin identity are given

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1. INTRODUCTION

Let $f : \mathbb{R} \to [0, \infty)$ be a *density function*; this means that f is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(t)\,dt = 1$, and define by

$$F(x) := \int_{-\infty}^{x} f(t)\,dt, \qquad x \in \mathbb{R}, \tag{1.1}$$

its *cumulative function*. We also denote the *expectation of* f by $E(f)$, where

$$E(f) := \int_{-\infty}^{\infty} x f(x)\,dx \tag{1.2}$$

provided that the integral exists and is finite

The *mean difference*

$$R_G(f) := \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\,dF(x)\,dF(y) \tag{1.3}$$

was proposed by Gini in 1912 [1], after whom it is usually named, but was discussed by Helmert and other German writers in the 1870s (cf. [2], see also [3, p 48]). It has a certain theoretical attraction, being dependent on the spread of the variate-values among themselves and not on the deviations from some central value [3, p. 48]. Further, its defining integral (1.3) may converge when that of the *variance* $\sigma^2(f)$,

$$\sigma^2(f) := \int_{-\infty}^{\infty} (x - E(f))^2\,dF(x), \tag{1.4}$$

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does not converge. It is, however, more difficult to compute than the standard deviation. Another useful concept that will be utilised in the following is the *mean deviation* $M_D(f)$, defined by [3, p. 48]

$$M_D(f) := \int_{-\infty}^{\infty} |x - E(f)| \, dF(x). \tag{1.5}$$

As Giorgi noted in [4], some of the many reasons for the success and the relevance of the Gini mean difference or *Gini index* $I_G(f)$,

$$I_G(f) = \frac{R_G(f)}{E(f)}, \tag{1.6}$$

are their simplicity, certain interesting properties and useful decomposition possibilities, and these have been analysed in an earlier work by Giorgi [5]. For a bibliographic portrait of the Gini index, see [4] where numerous references are given.

The main aim of this paper is to establish various general bounds for $R_G(f)$ on utilising specific results and tools from the theory of inequalities. The novel approach at times recaptures either known results; however, further results are obtained which do not seem to exist in the literature.

2. SOME IDENTITIES

We state here some identities that are of interest throughout the paper. For the sake of completeness, we give short proofs, even for identities that have been stated before (see, for instance, the book [3, Exercise 2.9, p. 94]).

The following lemma holds.

LEMMA 1. *We have the identity*

$$R_G(f) = 2\int_{-\infty}^{\infty} x f(x) F(x) \, dx - E(f). \tag{2.1}$$

PROOF. Using the definition of Gini mean difference (1.3) and the integration by parts formulae, we have successively

$$\begin{aligned}
2R_G(f) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} (x-y) f(y) \, dy \right) f(x) \, dx + \int_{-\infty}^{\infty} \left(\int_{x}^{\infty} (y-x) f(y) \, dy \right) f(x) \, dx \\
&= \int_{-\infty}^{\infty} \left[f(x) \left(x \int_{-\infty}^{x} f(y) \, dy - \int_{-\infty}^{x} y f(y) \, dy \right) \right] dx \\
&\quad + \int_{-\infty}^{\infty} \left[f(x) \left(\int_{x}^{\infty} y f(y) \, dy - x \int_{x}^{\infty} f(y) \, dy \right) \right] dx \\
&= \int_{-\infty}^{\infty} x F(x) \, dF(x) - \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} y f(y) \, dy \right) dF(x) \\
&\quad + \int_{-\infty}^{\infty} \left(\int_{x}^{\infty} y f(y) \, dy \right) dF(x) - \int_{x}^{\infty} x \left(1 - F(x)\right) dF(x) \\
&= \int_{-\infty}^{\infty} x F(x) \, dF(x) - \left[\left(\int_{-\infty}^{x} y f(y) \, dy \right) F(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x F(x) f(x) \, dx \right] \\
&\quad + \left(\int_{x}^{\infty} y f(y) \, dy \right) F(x) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} x F(x) f(x) \, dx \\
&\quad - \int_{x}^{\infty} x dF(x) + \int_{-\infty}^{\infty} x F(x) \, dF(x) \\
&= \int_{-\infty}^{\infty} x F(x) \, dF(x) - \left(\int_{-\infty}^{\infty} x f(x) \, dx - \int_{-\infty}^{\infty} x f(x) F(x) \, dx \right)
\end{aligned}$$

$$+\int_{-\infty}^{\infty} xf(x)F(x)\,dx - \int_{x}^{\infty} x\,dF(x) + \int_{-\infty}^{\infty} xF(x)\,dF(x)$$
$$= 4\int_{-\infty}^{\infty} xF(x)\,dF(x) - 2\int_{-\infty}^{\infty} xf(x)\,dx$$
$$= 4\int_{-\infty}^{\infty} xF(x)f(x)\,dx - 2E(f),$$

giving the desired identity (2.1). ∎

The following identity holds.

LEMMA 2. *We have the identity*

$$\int_{-\infty}^{\infty} (1-F(y))\,F(y)\,dy = 2\int_{-\infty}^{\infty} xf(x)F(x)\,dx - E(f). \tag{2.2}$$

PROOF. Denote $R(y) = 1 - F(y) = \int_y^{\infty} f(t)\,dt$, $y \in \mathbb{R}$. Then we have, on integrating by parts, that

$$\int_{-\infty}^{\infty} R(y)F(y)\,dy = yR(y)F(y)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} y\,d\,[R(y)F(y)]\,. \tag{2.3}$$

Since $\lim_{y\to\infty} yR(y) = 0$ and $\lim_{y\to-\infty} yF(y) = 0$, hence $yR(y)F(y)|_{-\infty}^{\infty} = 0$.

Also, we have

$$\int_{-\infty}^{\infty} y\,[R'(y)F(y) + R(y)F'(y)]\,dy = \int_{-\infty}^{\infty} y\,[-f(y)F(y) + (1-F(y))\,f(y)]\,dy$$
$$= \int_{-\infty}^{\infty} y\,[f(y) - 2f(y)F(y)]\,dy$$
$$= -2\int_{-\infty}^{\infty} yf(y)F(y)fy + E(f).$$

Using identity (2.3), we deduce (2.2). ∎

Define the functions $e : \mathbb{R} \to R$, $e(x) = x$ and $F : \mathbb{R} \to R_+$, $F(x) = \int_{-\infty}^{x} f(t)\,dt$. We recall that the covariance of e and F is given by

$$\mathrm{Cov}(e,F) := E\,[(e - E(f))\,(F - E(F))]\,.$$

The following result in terms of covariance also holds

LEMMA 3. *With the above notations, we have*

$$\mathrm{Cov}(e,F) = \int_{-\infty}^{\infty} xF(x)f(x)\,dx - \frac{1}{2}E(f). \tag{2.4}$$

PROOF. Using the definitions involved, we have

$$\mathrm{Cov}(e,F) = \int_{-\infty}^{\infty} (x - E(f))\left(F(x) - \int_{-\infty}^{\infty} f(y)F(y)\,dy\right) f(x)\,dx$$
$$= \int_{-\infty}^{\infty} f(x)\left[xF(x) + E(f)\int_{-\infty}^{\infty} f(y)F(y)\,dy \right.$$
$$\left. - \left(\int_{-\infty}^{\infty} f(y)F(y)\,dy\right) x - E(f)F(x)\right] dx$$
$$= \int_{-\infty}^{\infty} xF(x)f(x)\,dx + E(f)\int_{-\infty}^{\infty} f(y)F(y)\,dy$$
$$- E(f)\int_{-\infty}^{\infty} f(y)F(y)\,dy - E(f)\int_{-\infty}^{\infty} f(x)F(x)\,dx$$
$$= \int_{-\infty}^{\infty} xF(x)f(x)\,dx - E(f)\int_{-\infty}^{\infty} f(x)F(x)\,dx,$$

and since

$$\int_{-\infty}^{\infty} f(x)F(x)\,dx = \int_{-\infty}^{\infty} F(x)\,dF(x) = \frac{1}{2}F^2(x)\Big|_{-\infty}^{\infty} = \frac{1}{2},$$

we deduce the desired identity (2.4). ∎

Using the above lemmas, we may state the following known result (see for instance [3, p. 54]).

THEOREM 1. *With the above assumptions, we have the identities*

$$\begin{aligned} R_G(f) = 2\operatorname{Cov}(e, F) &= \int_{-\infty}^{\infty} (1 - F(y))F(y)\,dy \\ &= 2\int_{-\infty}^{\infty} x f(x)F(x)\,dx - E(f). \end{aligned} \tag{2.5}$$

Before we point out another identity for the *Gini mean difference*, we recall the following identity due to *Sonin* (see for instance [6, p. 246] for the case of univariate real functions):

$$\begin{aligned} &\int_\Omega \rho(s)g(s)h(s)\,d\mu(s) - \int_\Omega \rho(s)g(s)\,d\mu(s) \cdot \int_\Omega \rho(s)h(s)\,d\mu(s) \\ &\quad = \int_\Omega \rho(s)\left(g(s) - \int_\Omega g(\tau)\rho(\tau)\,d\mu(\tau)\right)(h(s) - \gamma)\,d\mu(s) \end{aligned} \tag{2.6}$$

for any $\gamma \in \mathbb{R}$, where $\rho : \Omega \to \mathbb{R}$ is a μ-measurable function with $\int_\Omega \rho(s)\,d\mu(s) = 1$, (Ω, Σ, μ) is a measure space, and $\mu \cdot \Sigma \to [0, \infty)$ is a positive measure on Ω.

The proof for (2.6) may be obtained by direct calculation starting with the right-hand side and we omit the details.

Utilising Sonin's identity (2 6), we may state the following.

THEOREM 2. *With the above assumptions for f and F, we have the identity*

$$\begin{aligned} R_G(f) &= 2\int_{-\infty}^{\infty} (x - E(f))\,(F(x) - \gamma)\,f(x)\,dx \\ &= 2\int_{-\infty}^{\infty} (x - \delta)\left(F(x) - \frac{1}{2}\right) f(x)\,dx, \end{aligned} \tag{2.7}$$

for any $\gamma, \delta \in \mathbb{R}$.

PROOF. The first identity follows by Sonin's identity for $\Omega = \mathbb{R}$, $d\mu(s) = dx$, $\rho(s) = f(s)$, $g(s) = s$, and $h(s) = F(s)$. The second identity follows in a similar manner. ∎

3. SOME GENERAL BOUNDS

The following result comparing the Gini mean difference with the mean absolute deviation may be stated.

THEOREM 3. *With the above assumptions, we have the bounds*

$$\begin{aligned} \frac{1}{2}M_D(f) \le R_G(f) &\le 2\sup_{x\in\mathbb{R}} |F(x) - \gamma|\,M_D(f) \\ &\le M_D(f), \end{aligned} \tag{3.1}$$

for any $\gamma \in [0, 1]$, where $F(\cdot)$ is the cumulative distribution of f and $M_D(f)$ is the mean deviation defined by (1.5).

PROOF. By the definition of $R_G(f)$ and the properties of the integral, we have

$$\begin{aligned} R_G(f) &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f(x) f(y)\,dx\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |x-y| f(x)\,dx\right) f(y)\,dy \\ &\geq \frac{1}{2}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} (x-y) f(x)\,dx\right| f(y)\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} |E(f)-y|\, f(y)\,dy \\ &= \frac{1}{2} M_D(f), \end{aligned}$$

and the first inequality in (3.1) is obtained.

For the second inequality, we use (2.7) to get

$$\begin{aligned} R_G(f) &\leq 2\int_{-\infty}^{\infty} |x-E(f)|\,|F(x)-\gamma|\, f(x)\,dx \\ &\leq 2\sup_{x\in\mathbb{R}} |F(x)-\gamma| \int_{-\infty}^{\infty} |x-E(f)|\, f(x)\,dx \\ &= 2\sup_{x\in\mathbb{R}} |F(x)-\gamma|\, M_D(f), \end{aligned}$$

for any $\gamma \in [0,1]$.

Finally, since, for $\gamma = 1/2$

$$\left|F(x)-\frac{1}{2}\right| \leq \frac{1}{2}, \qquad x\in\mathbb{R},$$

then the last part of (3.1) holds true. ∎

REMARK 1. The inequality

$$\frac{1}{2} M_D(f) \leq R_G(f) \leq M_D(f) \tag{3.2}$$

is known, see for instance the lecture notes online by Wichura located at `http://galton.uchicago.edu/~wichura/Stat304/Handouts/L09.means3.pdf`

If X is a random variable with the density function f and X is nondegenerate, there is equality on the left in (3.2) if and only if the random variable X takes only two values (with probability one). There is no degenerate X for which equality holds on the right; however, given any $\varepsilon > 0$, there exists a nondegenerate X_ε, such that $R_G(f_\varepsilon) \geq (1-\varepsilon) M_D(f)$, which shows that the multiplicative constant 1 in front of $M_D(f)$ in (3.2) cannot be replaced by a smaller quantity.

Now, consider the *absolute moment of order* 1 *about* $\delta \in \mathbb{R}$,

$$M_\delta(f) := \int_{-\infty}^{\infty} |x-\delta| f(x)\,dx.$$

We may state the following upper bound for the Gini difference mean in terms of $M_\delta(f)$ as follows.

THEOREM 4. *With the above assumptions and if $m \in \mathbb{R}$ is a median for f, that is, $F(m) = \int_{-\infty}^{m} f(x) = 1/2$, then we have*

$$R_G(f) \leq M_m(f) \leq M_D(f), \tag{3.3}$$

where $M_m(f) = \int_{-\infty}^{\infty} |x-m| f(x)\,dx$.

PROOF. If we use the second equality in (2.7), then for any $\delta \in \mathbb{R}$, we get

$$\begin{aligned} R_G(f) &\le 2\int_{-\infty}^{\infty} |x-\delta| \left|F(x) - \frac{1}{2}\right| f(x)\,dx \\ &\le \int_{-\infty}^{\infty} |x-\delta|\, f(x)\,dx \\ &= M_\delta(f), \end{aligned}$$

since $|F(x) - 1/2| \le 1/2$ for each $x \in \mathbb{R}$. Taking the infimum over $\delta \in \mathbb{R}$ and taking into account (see for instance Theorem 2 of `http://galton.uchicago.edu/~wichura/Stat304/Handouts/L08.means2.pdf`) that

$$\inf_{\delta\in\mathbb{R}} M_\delta(f) = M_m(f),$$

we deduce the first inequality in (3.3).

The second part is obvious. ∎

Now, consider the *absolute moment of order p about* $E(f)$, namely,

$$M_{E,p}(f) := \int_{-\infty}^{\infty} |x - E(f)|^p\, f(x)\,dx < \infty, \qquad p > 1. \tag{3.4}$$

With this definition, we may state the following result.

THEOREM 5. *Assume that* $p > 1$, $1/p + 1/q = 1$, *and* $f : \mathbb{R} \to [0,\infty)$ *is a distribution with* $M_{E,p}(f) < \infty$. *Then we have the inequality*

$$R_G(f) \le \frac{2}{(q+1)^{1/q}} \left[M_{E,p}(f)\right]^{1/p}. \tag{3.5}$$

PROOF. Using the first identity in (2.7) for $\gamma = 0$ and the weighted Hölder inequality, we have

$$\begin{aligned} R_G(f) &\le 2\int_{-\infty}^{\infty} |x - E(f)|\, F(x) f(x)\,dx \\ &\le 2\left(\int_{-\infty}^{\infty} |x-E(f)|^p\, f(x)\,dx\right)^{1/p} \left(\int_{-\infty}^{\infty} F^q(x) f(x)\,dx\right)^{1/q}. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} F^q(x) f(x)\,dx = \frac{1}{q+1},$$

hence inequality (3.5) is obtained. ∎

REMARK 2. For $p = q = 2$, we have

$$M_{E,2}(f) = \sigma^2(f) = \int_{-\infty}^{\infty} (x - E(f))^2\, f(x)\,dx,$$

and from (3.5), we may state the inequality

$$R_G(f) \le \frac{2\sqrt{3}}{3}\sigma(f). \tag{3.6}$$

Inequality (3.6) is known, see for instance, `http://galton.uchicago.edu/~wichura/Stat304/Handouts/L09.means3.pdf`.

The case of equality holds in (3.6) if f is uniformly distributed on $[0,1]$.

If we denote now by

$$M_{\delta,p}(f) := \int_{-\infty}^{\infty} |x-\delta|^p\, f(x)\,dx, \qquad p > 1$$

the *absolute moment of order p about* δ of f, then we may state the following result.

THEOREM 6 *Let $\delta \in \mathbb{R}$, $p > 1$, $1/p + 1/q = 1$ and assume that $M_{\delta,p}(f) < \infty$. If there exists a unique median m for f, then we have the inequality*

$$R_G(f) \le \frac{1}{(q+1)^{1/q}} \left[M_{\delta,p}(f)\right]^{1/p}. \tag{3.7}$$

PROOF. We use the second identity in (2.7) and Hölder's weighted integral inequality to get

$$\begin{aligned} R_G(f) &\le 2\int_{-\infty}^{\infty} |x-\delta| \left|F(x) - \frac{1}{2}\right| f(x)\,dx \\ &\le 2\left(\int_{-\infty}^{\infty} |x-\delta|^p f(x)\,dx\right)^{1/p} \left(\int_{-\infty}^{\infty} \left|F(x) - \frac{1}{2}\right|^q f(x)\,dx\right)^{1/q}. \end{aligned} \tag{3.8}$$

Now, for m a median,

$$\begin{aligned} \int_{-\infty}^{\infty} \left|F(x) - \frac{1}{2}\right|^q f(x)\,dx &= \int_{-\infty}^{m} \left(\frac{1}{2} - F(x)\right)^q f(x)\,dx + \int_{m}^{\infty} \left(F(x) - \frac{1}{2}\right)^q f(x)\,dx \\ &= -\left.\frac{(1/2 - F(x))^{q+1}}{q+1}\right|_{-\infty}^{m} + \left.\frac{(F(x) - 1/2)^{q+1}}{q+1}\right|_{m}^{\infty} \\ &= \frac{1}{2^{q+1}(q+1)} + \frac{1}{2^{q+1}(q+1)} = \frac{1}{2^q(q+1)}. \end{aligned}$$

Utilising (3.8), we deduce the desired result (3.7). ∎

REMARK 3. If in the above we choose $p = q = 2$, then we have

$$R_G(f) \le \frac{\sqrt{3}}{3} \left[M_{\delta,2}(f)\right]^{1/2}, \tag{3.9}$$

for any $\delta \in \mathbb{R}$. In particular, if we choose $\delta = E(f)$, then from (3.9) we deduce

$$R_G(f) \le \frac{\sqrt{3}}{3}\sigma(f), \tag{3.10}$$

a better result than (3 6), which was available earlier in the literature. Note that this result holds true for distributions for which there exists a unique $m \in \mathbb{R}$, such that $F(m) = 1/2$.

The following result may be stated as well

THEOREM 7. *With the above notations, we have*

$$R_G(f) \le \begin{cases} 2\|f\|_\infty \sigma^2(f), & \text{if } f \in L_\infty(\mathbb{R}), \\ 2\|f\|_p M_{E,1+1/q}(f), & \text{if } f \in L_p(\mathbb{R}), \quad p > 1, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1, \end{cases} \tag{3.11}$$

provided the moments $M_{E,1+1/q}(f)$ defined by (3.4) exists.

PROOF. If we use identity (2.7) with $\gamma = F(E(f))$, then we get

$$\begin{aligned} R_G(f) &\le 2\int_{-\infty}^{\infty} |x - E(f)|\,|F(x) - F(E(f))|\, f(x)\,dx \\ &= 2\int_{-\infty}^{\infty} |x - E(f)| \left|\int_x^{E(f)} f(t)\,dt\right| f(x)\,dx \end{aligned} \tag{3.12}$$

Utilising Hölder's integral inequality, we may state that

$$\left|\int_x^{E(f)} f(t)\,dt\right| \le \begin{cases} |x - E(f)|\,\|f\|_\infty, & \text{if } f \in L_\infty(\mathbb{R}); \\ |x - E(f)|^{1/q}\,\|f\|_p, & \text{if } f \in L_p(\mathbb{R}), \\ & p > 1, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1; \end{cases} \tag{3.13}$$

where

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t\in\mathbb{R}} |f(t)|$$

and

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p\right)^{1/p}, \qquad p \geq 1.$$

Making use of (3.12) and (3.13), we may state that

$$R_G(f) \leq \begin{cases} \|f\|_\infty \int_{-\infty}^{\infty} (x - E(f))^2 f(x)\,dx, & \text{if } f \in L_\infty(\mathbb{R}); \\ \|f\|_p \int_{-\infty}^{\infty} |x - E(f)|^{1+1/q} f(x)\,dx, & \text{if } f \in L_p(\mathbb{R}), \\ & p > 1, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1, \end{cases}$$

and inequality (3.11) is obtained. ∎

Finally, we can state the following result involving the median, m.

THEOREM 8. *Let $m \in \mathbb{R}$ be a median for f. Then we have the inequality*

$$R_G(f) \leq \begin{cases} 2\|f\|_\infty M_{m,2}(f), & \text{if } f \in L_\infty(\mathbb{R}); \\ 2\|f\|_p M_{m,1+1/q}(f), & \text{if } f \in L_p(\mathbb{R}), \quad p > 1, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1, \end{cases} \tag{3.14}$$

provided the moments involved exist. Note that

$$M_{m,s}(f) := \int_{-\infty}^{\infty} |x - m|^s f(x)\,dx, \qquad s > 0.$$

PROOF. By the definition of median, we have

$$\begin{aligned} \left|F(x) - \frac{1}{2}\right| &= \left|\int_m^x f(t)\,dt\right| \\ &\leq \begin{cases} |x - m|\|f\|_\infty, & \text{if } f \in L_\infty(\mathbb{R}); \\ |x - m|^{1/q}\|f\|_p, & \text{if } f \in L_p(\mathbb{R}), \quad p > 1, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1. \end{cases} \end{aligned}$$

Utilising the second identity in (2.7) for $\delta = m$, we have

$$\begin{aligned} R_G(f) &\leq 2\int_{-\infty}^{\infty} |x - m| \left|F(x) - \frac{1}{2}\right| f(x)\,dx \\ &\leq 2\begin{cases} \|f\|_\infty \int_{-\infty}^{\infty} |x - m|^2 f(x)\,dx; \\ \|f\|_p \int_{-\infty}^{\infty} |x - m|^{1+1/q} f(x)\,dx, \end{cases} \end{aligned}$$

giving the desired inequality (3.14). ∎

4. BOUNDS FOR n-TIME DIFFERENTIABLE FUNCTIONS

The following result may be stated.

THEOREM 9. *Assume that the density function $f : \mathbb{R} \to [0,\infty)$ is $(n-1)$-time differentiable $(n \geq 1)$ and the derivative $f^{(n-1)}$ is locally absolutely continuous on $\mathbb{R}$. Then we have the representation*

$$R_G(f) = 2\left[f(E(f))\,\sigma^2(f) + \sum_{k=1}^{n-1} \frac{f^{(k)}(E(f))}{(k+1)!}\tilde{M}_{E,k+2}(f)\right] + S_n(f, E(f)), \tag{4.1}$$

and the remainder $S_n(f, E(f))$ satisfies the bounds

$$|S_n(f,E(f))| \leq \begin{cases} \dfrac{2}{(n+1)!}\left\|f^{(n)}\right\|_\infty M_{E,n+2}(f), & \text{if } f^{(n)} \in L_\infty(\mathbb{R}); \\ \dfrac{2}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p M_{E,n+1+1/q}(f), & \text{if } f^{(n)} \in L_p(\mathbb{R}), \\ & p>1,\quad \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{2}{n!}\left\|f^{(n)}\right\|_1 M_{E,n+1}(f), \end{cases} \tag{4.2}$$

where $\tilde{M}_{E,p}(f) = \int_{-\infty}^{\infty}(t-E(f))^p f(t)\,dt$, $p \in \mathbb{N}$ and $M_{E,\cdot}(f)$ is given by (3.4).

PROOF. We use the Taylor's representation formula with integral remainder for F about a point $t_0 \in \mathbb{R}$ to state that

$$\begin{aligned} F(t) &= F(t_0) + (t-t_0)F^{(1)}(t_0) + \cdots + \frac{(t-t_0)^n}{n!}F^{(n)}(t_0) \\ &\quad + \frac{1}{n!}\int_{t_0}^{t}(t-s)^n F^{(n+1)}(s)\,ds \\ &= F(t_0) + (t-t_0)f(t_0) + \cdots + \frac{(t-t_0)^n}{n!}f^{(n-1)}(t_0) \\ &\quad + \frac{1}{n!}\int_{t_0}^{t}(t-s)^n f^{(n)}(s)\,ds. \end{aligned}$$

Now, if we use identity (2.7), for $\gamma = F(E(f))$, then we have the representation

$$\begin{aligned} R_G(f) &= 2\int_{-\infty}^{\infty}(t-E(f))\,(F(t)-F(E(f)))\,f(t)\,dt \\ &= 2\int_{-\infty}^{\infty}(t-E(f))\left[\frac{(t-E(f))\,f(E(f))}{1!} + \frac{(t-E(f))^2}{2!}f^{(1)}(E(f))\right. \\ &\quad \left. + \cdots + \frac{(t-E(f))^n}{n!}f^{(n-1)}(E(f)) + \frac{1}{n!}\int_{E(f)}^{t}(t-s)^n f^{(n)}(s)\,ds\right] f(t)\,dt \\ &= 2\left[\frac{f(E(f))}{1!}\int_{-\infty}^{\infty}(t-E(f))^2 f(t)\,dt + \frac{f^{(1)}(E(f))}{2!}\int_{-\infty}^{\infty}(t-E(f))^3 f(t)\,dt\right. \\ &\quad \left. + \cdots + \frac{f^{(n-1)}(E(f))}{n!}\int_{-\infty}^{\infty}(t-E(f))^{n+1} f(t)\,dt\right] + S_n(f,E(f)), \end{aligned}$$

where

$$S_n(f,E(f)) := \frac{2}{n!}\int_{-\infty}^{\infty}(t-E(f))\left(\int_{E(f)}^{t}(t-s)^n f^{(n)}(s)\,ds\right) f(t)\,dt.$$

This provides representation (4.1).

To bound the remainder $S_n(f,E(f))$, observe, first, that by Holder's integral inequality

$$\left|\int_{E(f)}^{t}(t-s)^n f^{(n)}(s)\,ds\right| \le \begin{cases} \left\|f^{(n)}\right\|_{\infty}\dfrac{|t-E(f)|^{n+1}}{n+1}, & \text{if } f^{(n)}\in L_{\infty}(\mathbb{R});\\ \left\|f^{(n)}\right\|_{p}\left|\displaystyle\int_{E(f)}^{t}|t-s|^{nq}\,ds\right|^{1/q}, & \text{if } f^{(n)}\in L_{p}(\mathbb{R}),\\ & p>1,\quad \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \left\|f^{(n)}\right\|_{1}|t-E(f)|^{n} \end{cases}$$

$$= \begin{cases} \dfrac{1}{n+1}\left\|f^{(n)}\right\|_{\infty}|t-E(f)|^{n+1}, & \text{if } f^{(n)}\in L_{\infty}(\mathbb{R});\\ \dfrac{1}{(nq+1)^{1/q}}\left\|f^{(n)}\right\|_{p}|t-E(f)|^{n+1/q}, & \text{if } f^{(n)}\in L_{p}(\mathbb{R}),\\ & p>1,\quad \dfrac{1}{p}+\dfrac{1}{q}=1;\\ |t-E(f)|^{n}\left\|f^{(n)}\right\|_{1}. \end{cases} \tag{4.3}$$

Utilising (4.3), we may state that

$$|S_n(f,E(f))| \le \begin{cases} \dfrac{2}{(n+1)!}\left\|f^{(n)}\right\|_{\infty}\displaystyle\int_{-\infty}^{\infty}|t-E(f)|^{n+2}f(t)\,dt, & \text{if } f^{(n)}\in L_{\infty}(\mathbb{R});\\ \dfrac{2}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_{p}\displaystyle\int_{-\infty}^{\infty}|t-E(f)|^{n+1+1/q}f(t)\,dt, & \text{if } f^{(n)}\in L_{p}(\mathbb{R}),\\ & p>1,\quad \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{2}{n!}\left\|f^{(n)}\right\|_{1}\displaystyle\int_{-\infty}^{\infty}|t-E(f)|^{n+1}f(t)\,dt, \end{cases}$$

and inequality (4.3) is proved. ∎

Finally, the following result involving a median may be stated.

THEOREM 10. *Assume that the density $f:\mathbb{R}\to[0,\infty)$ is $(n-1)$-time differentiable $(n\ge 1)$ and the derivative $f^{(n-1)}$ is locally absolutely continuous on $\mathbb{R}$. If $m\in\mathbb{R}$ is a median for f, then we have the representation*

$$R_G(f) = 2\left[\tilde{M}_{m,2}(f)+\sum_{k=1}^{n-1}\frac{f^{(k)}(m)}{(k+1)!}\tilde{M}_{m,k+2}(f)\right]+T_n(f,m), \tag{4.4}$$

where the remainder $T_n(f,m)$ satisfies the bounds

$$|T_n(f,m)| \le \begin{cases} \dfrac{2}{(n+1)!}\left\|f^{(n)}\right\|_{\infty}M_{m,n+2}(f), & \text{if } f^{(n)}\in L_{\infty}(\mathbb{R});\\ \dfrac{2}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_{p}M_{m,n+1+1/q}(f), & \text{if } f^{(n)}\in L_{p}(\mathbb{R}),\\ & p>1,\quad \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{2}{n!}\left\|f^{(n)}\right\|_{1}M_{m,n+1}(f), \end{cases} \tag{4.5}$$

where

$$\tilde{M}_{m,p}(f)=\int_{-\infty}^{\infty}(x-m)^p f(x)\,dx, \qquad p\in\mathbb{N},$$

and

$$M_{m,r}(f) = \int_{-\infty}^{\infty} |x-m|^r f(x)\,dx, \qquad r>0.$$

PROOF. We use the Taylor's representation formula with integral remainder to get

$$F(x) = F(m) + (x-m)f(m) + \cdots + \frac{(x-m)^n}{n!} f^{(n-1)}(m) + \frac{1}{n!}\int_m^x (x-s)^n f^{(n)}(s)\,ds, \qquad x\in\mathbb{R}.$$

Now, if we use identity (2.7) for $\delta = m$, then we get

$$\begin{aligned} R_G(f) &= 2\int_{-\infty}^{\infty} (x-m)\left[(x-m)f(m) + \cdots + \frac{(x-m)^n}{n!} f^{(n-1)}(m)\right. \\ &\quad \left. + \frac{1}{n!}\int_m^x (x-s)^n f^{(n)}(s)\,ds\right] f(x)\,dx \\ &= 2\left[\frac{f(m)}{1!}\int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx + \frac{1}{2!} f'(m)\int_{-\infty}^{\infty} (x-m)^3 f(x)\,dx\right. \\ &\quad \left. + \cdots + \frac{f^{(n-1)}(m)}{n!}\int_{-\infty}^{\infty} (x-m)^{n+1} f(x)\,dx\right] + T_n(f,m), \end{aligned}$$

where, obviously,

$$T_n(f,m) := \frac{2}{n!}\int_{-\infty}^{\infty} (x-m)\left(\int_m^x (t-s)^n f^{(n)}(s)\,ds\right) f(x)\,dx.$$

Since, by Hölder's integral inequality, we have

$$\left|\int_m^x (t-s)^n f^{(n)}(s)\,ds\right| \le \begin{cases} \dfrac{1}{n+1}\left\|f^{(n)}\right\|_\infty |x-m|^{n+1} & \text{if } f^{(n)}\in L_\infty(\mathbb{R}); \\ \dfrac{1}{(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p |x-m|^{n+1/q}, & \text{if } f^{(n)}\in L_p(\mathbb{R}), \\ & p>1, \quad \dfrac{1}{p}+\dfrac{1}{q}=1; \\ |x-m|^n \left\|f^{(n)}\right\|_1, & \end{cases}$$

hence

$$|T_n(f,m)| \le \begin{cases} \dfrac{2}{(n+1)!}\left\|f^{(n)}\right\|_\infty \displaystyle\int_{-\infty}^{\infty} |x-m|^{n+2} f(x)\,dx \\ \dfrac{2}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p \displaystyle\int_{-\infty}^{\infty} |x-m|^{n+1+1/q}, \quad p>1, \quad \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{2}{n!}\left\|f^{(n)}\right\|_1 \displaystyle\int_{-\infty}^{\infty} |x-m|^{n+1} f(x)\,dx, \end{cases}$$

and the theorem is proved. ∎

REMARK 4. Results (4.1),(4.2) and (4.4),(4.5) enable an approximation of $R_G(f)$ in terms of arguably simpler terms with explicit bounds.

REMARK 5. Further results based on Korkine and other recent identities connected with the Grüss inequality are under preparation.

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