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Categorified central extensions, étale Lie 2-groups and Lie's Third Theorem for locally exponential Lie algebras

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Abstract

Lie's Third Theorem, asserting that each finite-dimensional Lie algebra is the Lie algebra of a Lie group, fails in infinite dimensions. The modern account on this phenomenon is the integration problem for central extensions of infinite-dimensional Lie algebras, which in turn is phrased in terms of an integration procedure for Lie algebra cocycles.

This paper remedies the obstructions for integrating cocycles and central extensions from Lie algebras to Lie groups by generalising the integrating objects. Those objects obey the maximal coherence that one can expect. Moreover, we show that they are the universal ones for the integration problem.

The main application of this result is that a Mackey-complete locally exponential Lie algebra (e.g., a Banach–Lie algebra) integrates to a Lie 2-group in the sense that there is a natural Lie functor from certain Lie 2-groups to Lie algebras, sending the integrating Lie 2-group to an isomorphic Lie algebra.

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0. Introduction

This paper sets out to resolve obstructions for integrating Lie algebras and central extensions of them. It is a celebrated theorem that each finite-dimensional Lie algebra is the Lie algebra of a Lie group, which is known as Lie's Third Theorem. It was proven by Lie in a local versions and in full strength by Élie Cartan (cf. [15] and references therein). It has also been Élie Cartan who first remarked in [16] that one may also use the fact that $\pi_2(G)$ vanishes for any finite-dimensional Lie group[1] to prove Lie's Third Theorem. If G is infinite-dimensional, then $\pi_2(G)$ does not vanish any more, for instance for $C^\infty(S^1, \mathrm{SU}(2))$ or $PU(\mathcal{H})$. This was used by van Est and Korthagen in [61] to construct an example of a Banach–Lie algebra which cannot be the Lie algebra of a Lie group (cf. [18] for the corresponding construction for $PU(\mathcal{H})$).

However, there is a large class of infinite-dimensional Lie algebras which integrate to a *local* Lie group, namely locally exponential Lie algebras. In particular, all Banach–Lie algebras belong to this class. The non-existence of a (global) Lie group integrating a locally exponential Lie algebra may thus be regarded as the obstruction against the corresponding local Lie group to enlarge to a (global) Lie group. This is why a Lie algebra, which is the Lie algebra of a (global) Lie group is often called *enlargeable*, whilst a Lie algebra is called *integrable* if it is the Lie algebra of a local Lie group (cf. [41]).

The most sophisticated tool for the analysis of enlargeability of locally exponential Lie algebras is Neeb's machinery for integrating central extensions of infinite-dimensional Lie groups, developed in [40]: if $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a central extension of Lie algebras and G is a 1-connected Lie group with Lie algebra $\mathfrak{g}$, then $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ integrates to a central extension of G if and only if the period group $\mathrm{per}_{\widehat{\mathfrak{g}}}(\pi_2(G)) \subseteq \mathfrak{z}$ is discrete. A variant of this theory (cf. [41, Section VI.1]) applies in particular to a locally exponential Lie algebra $\mathfrak{g}$, since $\mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ad}}$ is a (generalised) central extension and there always exists a 1-connected Lie group G_{ad} with $L(G_{\mathrm{ad}}) = \mathfrak{g}_{\mathrm{ad}}$. Thus the obstruction for $\mathfrak{g}$ to be non-enlargeable is the non-discreetness of $\mathrm{per}_{\mathfrak{g}}(\pi_2(G_{\mathrm{ad}})) \subseteq \mathfrak{z}(\mathfrak{g})$. If $\mathfrak{g}$ is finite-dimensional, then $\pi_2(G_{\mathrm{ad}})$ vanishes and Lie's Third Theorem is immediate. From this point of view the theorem seems to be merely a homotopy-theoretic accident.

Enlarging local groups and integrating central extensions obey a common pattern. The obstruction for enlarging a local (Lie) group to a global one is an associativity constraint, which is coupled to topological properties of the local group (cf. [54,59,60,44,11]). In general, global associativity cannot be achieved as the counterexamples, mentioned above, show. In the integration problem for cocycles the obstruction for $\mathrm{per}_{\widehat{\mathfrak{g}}}(\pi_2(G)) \subseteq \mathfrak{z}$ to be discrete ensures that a cocycle condition holds for a certain universal integrating cocycle.

The upshot of this paper is that one may *relax* global associativity and cocycle conditions at the same time by introducing more generalised *but still coherent* objects, like generalised group cocycles and 2-groups. It is organised as follows. In the first section we line out an integration procedure for Lie algebra cocycles to generalised, locally smooth Lie group cocycles. This is the central idea of this paper, all other results will build on this. The main achievement of this section is the following

Theorem. *If $\mathfrak{z}$ is Mackey-complete and $\mathfrak{g}$ is a Lie algebra with simply connected Lie group G, then each continuous Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ integrates to a generalised cocycle on G.*

[1] Originally, Cartan's condition was that the first two Betti numbers vanish.

Moreover, the generalised cocycle that we construct is universal for generalised cocycles integrating ω. □

The remaining sections describe interpretations of the results of the first section. The second describes an interpretation in the language of loop prolongations. It is discussed which aspects cannot be covered by loop prolongations, which then leads to an interpretation in the language of 2-groups. This is done in sections three and four and the corresponding extension theory is introduced in section five. It is described which rôle étalness plays in this setting, and this section eventually results in the second main result of this paper.

Theorem. *If $\mathfrak{g}$ is the Lie algebra of the simply connected Lie group G, then each topologically split central extension $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ with Mackey-complete $\mathfrak{z}$ integrates to a smooth generalised central extension of étale Lie 2-groups.* □

After having worked this out we apply the previous results to the (generalised) central extension $\mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ad}}$ for $\mathfrak{g}$ locally exponential in order to obtain our version of Lie's Third Theorem in the next section:

Theorem. *If $\mathfrak{g}$ is a Mackey-complete locally exponential Lie algebra, then there exists an étale Lie 2-group $\mathcal{G}$ such that $\mathcal{L}(\mathcal{G})$ is isomorphic to $\mathfrak{g}$.* □

In the end we indicate some directions for further research and give some details on locally convex Lie groups in Appendix A.

There exist many links to neighbouring topics, which we will mention throughout the text. Amongst those are integrability questions for Lie algebroids (Remark 5.7), String group models (Example 4.10), diffeological Lie groups (Remark 7.1) and connections on categorified bundles and n-plectic geometry (Remark 7.2). Since many of them need concepts and notation that we provide in the text we refrain from summarising them here.

Conventions

For us a manifold is a Hausdorff space, which is locally homeomorphic to open subsets of some locally convex space such that the coordinate changes are diffeomorphisms. A Lie group is a group, which also is a manifold such that the group operations are smooth (cf. Definition A.1 for details on this). For M, N smooth manifolds and $f : M \to N$ smooth, $Tf : TM \to TN$ denotes the tangent map of f. If M, N and f are pointed, then $df : T_*M \to T_*N$ denotes the differential in the base point. Moreover, if df vanishes, then one may define $d^2 f : T_*M \times T_*M \to T_*N$ in terms of local coordinates, where the vanishing of df implies the independence of the choice of a chart. For us, a locally smooth map on a pointed manifold is a map which is smooth on some open neighbourhood of the base-point (and *not* on an open neighbourhood of each point).

Unless stated otherwise, G shall always be a 1-connected Lie group with Lie algebra $\mathfrak{g}$, which we usually identify with T_eG. Moreover, $\mathfrak{z}$ shall always denote a Mackey-complete locally convex space (in particular, integrals of smooth functions from standard simplices to $\mathfrak{z}$ always exist, cf. Remark A.6) and Z will denote the abelian Lie group $\mathfrak{z}/\Gamma$ for some discrete subgroup Γ (in some situations we will choose Γ explicitly, but in general any discrete subgroup is fine). Unless stated otherwise, $q : \mathfrak{z} \to Z$ will denote the canonical quotient map.

If A is an abelian Lie group where G acts on (trivially if nothing else is said), then we define

$$C^n(G,A) := \big\{ f : G^n \to A\colon\ f(g_1,\ldots,g_n)=0 \text{ if } g_i = e \text{ for some } i \text{ and}$$
$$f \text{ is smooth on some neighbourhood of } (e,\ldots,e)\in G^n \big\},$$

the group of normalised locally smooth A-valued n-cochains on G. Note that this implies in particular

$$df(e,\ldots,e)(v_1,\ldots,v_n) = \sum_{i=1}^{n} df(e,\ldots,e)(0,\ldots,v_i,\ldots,0)=0.$$

On $C^n(G,A)$ we denote by

$$\mathrm{d}_{\mathrm{gp}} : C^n(G,A)\to C^{n+1}(G,A),$$

$$\mathrm{d}_{\mathrm{gp}} f(g_0,\ldots g_n) = g_0.f(g_1,\ldots,g_n) - \sum_{i=0}^{n-1}(-1)^i f(g_0,\ldots,g_i g_{i+1},\ldots,g_n) - (-1)^n f(g_0,\ldots,g_{n-1})$$

the ordinary group differential (we will also use this formula for $\mathrm{d}_{\mathrm{gp}} f$ in more general situation, where A does not carry a Lie group structure and f does not obey any smoothness condition). If $f\in C^2(G,A)$, $\mathrm{d}_{\mathrm{gp}} f=0$, and G acts trivially, then

$$(a,g)\cdot(b,h) = \big(a+b+f(g,h), gh\big) \tag{1}$$

defines a group structure on $A\times G$, which we denote by $A\times_f G$.

We denote by $\Delta^{(n)}\subseteq\mathbb{R}^n$ the standard n-simplex, which we view as a manifold with corners. For a Hausdorff space X, $C_n(X)=\langle C(\Delta^{(n)},X)\rangle_{\mathbb{Z}}$ denotes the group of singular n-chains in X over $\mathbb{Z}$ and $\partial : C_n(X)\to C_{n-1}(X)$ the corresponding singular differential. Moreover, $Z_n(X)$ and $B_n(X)$ denote the corresponding cycles and boundaries and $H_n(X)$ the singular homology of X. For $\alpha,\alpha'\in C(\Delta^{(n)},G)$, $\alpha+\alpha'$ and $-\alpha$ always refer to the additive structure in $C_n(G)$ whilst $\alpha\cdot\alpha'$ and α^{-1} always refer to the (point-wise) group structures on the Lie group $C(\Delta^{(n)},G)$. Moreover, G acts by left multiplication on $C_n(G)$ and we take this module structure into account when using d_{gp} for $C_n(G)$-valued mappings.

If $\mathcal{C}$ is a small category, then $\mathcal{C}_0$ and $\mathcal{C}_1$ are the sets of objects and morphisms. The structure maps of $\mathcal{C}$ are always denoted by s, t, id and $\circ$. If $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ is a functor, then $\mathcal{F}_0:\mathcal{C}_0\to\mathcal{D}_0$ and $\mathcal{F}_1:\mathcal{C}_1\to\mathcal{D}_1$ are the corresponding maps on the set of objects and morphisms. Likewise, if $\alpha:\mathcal{F}\Rightarrow\mathcal{F}'$ is a natural transformation, then we use the same letter to denote the corresponding map $\alpha:\mathcal{C}_0\to\mathcal{D}_1$. The set of isomorphism classes of $\mathcal{C}$ is denoted by $\pi_0(\mathcal{C})$ and $\pi_0(\mathcal{F})$ is the induced map $\pi_0(\mathcal{C})\to\pi_0(\mathcal{D})$. Almost all categories that we will encounter in this article will be groupoids, i.e., categories in which each morphism is invertible.

1. Integrating cocycles

This section describes the principal construction of this paper. It is an integration procedure for Lie algebra cocycles and generalises the approach from [40]. The main achievement will be to overcome the obstruction from [40] for the aforementioned integration procedure by passing from group cocycles with coefficients in an abelian Lie group to group cocycles with coefficients in a complex of abelian Lie groups. We first recall the setting and the results from [40].

Definition 1.1. Let $\mathfrak{g}$ be a topological Lie algebra and $\mathfrak{z}$ be a topological vector space. A Lie algebra *cocycle* is a continuous bilinear map $\omega: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ satisfying $\omega(x, y) = -\omega(y, x)$ and

$$\omega\big([x, y], z\big) + \omega\big([y, z], x\big) + \omega\big([z, x], y\big) = 0.$$

The cocycle ω is said to be a *coboundary* if there exists a continuous linear map $b: \mathfrak{g} \to \mathfrak{z}$ with $\omega(x, y) = b([x, y])$. The vector space of cocycles is denoted by $Z^2_c(\mathfrak{g}, \mathfrak{z})$ and the space of coboundaries $B^2_c(\mathfrak{g}, \mathfrak{z})$ is a sub space of $Z^2_c(\mathfrak{g}, \mathfrak{z})$. The vector space $H^2_c(\mathfrak{g}, \mathfrak{z}) := Z^2_c(\mathfrak{g}, \mathfrak{z})/B^2_c(\mathfrak{g}, \mathfrak{z})$ is called the (second continuous) *Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in $\mathfrak{z}$. Two cocycles ω and ω' are called *equivalent* if $[\omega] = [\omega']$ in $H^2_c(\mathfrak{g}, \mathfrak{z})$. □

Remark 1.2. Lie algebra cohomology is a concept that unfolds its importance in particular when considering infinite-dimensional Lie algebras. For instance, Whitehead's lemma [29, Theorem III.13] asserts that $H^2_c(\mathfrak{g}, \mathfrak{z})$ vanishes if $\mathfrak{g}$ and $\mathfrak{z}$ are finite-dimensional and $\mathfrak{g}$ is semi-simple.

The importance of $H^2_c(\mathfrak{g}, \mathfrak{z})$ comes from the fact that it classifies (topologically split)[2] *central extensions* of topological Lie algebras, i.e. short exact sequences $\mathfrak{z} \to \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$ for which there exists a continuous and linear right inverse of q [41, Proposition V.2.10]. In infinite dimensions a prominent example for a non-trivial $H^2_c(\mathfrak{g}, \mathfrak{z})$ comes from $\mathfrak{g} = C^\infty(S^1, \mathfrak{k})$, $\mathfrak{z} = \mathbb{R}$ and the Kac–Moody cocycle

$$\omega_{\langle\cdot,\cdot\rangle}: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad (f, g) \mapsto \int_{S^1} \big\langle f(t), g'(t)\big\rangle\, dt, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ is the Killing form of the finite-dimensional simple Lie algebra $\mathfrak{k}$. □

In [40] it is described how Lie algebra cocycles may be integrated to (locally smooth) group cocycles. We shall now introduce slightly more general objects (cf. [12, Section 2]) covering in particular the (locally smooth) group cocycles from [40] (see also [63] or [58] for other occurrences of this concept).

Definition 1.3. Let G be an arbitrary Lie group and $A \xrightarrow{\tau} Z$ be a morphism of abelian Lie groups. Then a *generalised group cocycle* on G with coefficients $A \xrightarrow{\tau} Z$ (shortly called generalised cocycle if the setting is understood) consists of two maps $F \in C^2(G, Z)$ and $\Theta \in C^3(G, A)$ such that

[2] Our central extensions of Lie algebras are always assumed to be topologically split, but different authors follow different conventions for this.

$$\mathrm{d}_{\mathrm{gp}} F = \tau \circ \Theta, \tag{3}$$

$$\mathrm{d}_{\mathrm{gp}} \Theta = 0. \tag{4}$$

A *morphism* of generalised cocycles $(\varphi, \psi): (F, \Theta) \to (F', \Theta')$ consists of two maps $\varphi \in C^1(G, Z)$ and $\psi \in C^2(G, A)$ such that $F = F' + \mathrm{d}_{\mathrm{gp}} \varphi + \tau \circ \psi$ and $\Theta = \Theta' + \mathrm{d}_{\mathrm{gp}} \psi$. Furthermore, a *2-morphism* $\gamma : (\varphi, \psi) \Rightarrow (\varphi', \psi')$ between two morphisms of generalised cocycles is given by a map $\gamma \in C^1(G, A)$ such that $\psi = \psi' + \mathrm{d}_{\mathrm{gp}} \gamma$. □

If we view discrete groups as zero-dimensional Lie groups, then the preceding definition also yields the concept of generalised cocycles without any smoothness assumptions. That is why we do not explicitly distinguish between cocycles with or without smoothness conditions.

In this paper we shall mostly deal with the case that A is a discrete group. This implies that Θ vanishes on some identity neighbourhood for smooth maps are in particular continuous.

Remark 1.4. The previous definition reduces to the case of locally smooth cohomology $H^2(G, Z)$ (cf. [40, Definition 4.4]) if we consider generalised cocycles modulo morphisms with coefficients $0 \to Z$. Generalised cocycles with $(0 \to Z)$-coefficients will sometimes be called *2-cocycles* (or simply cocycles if the dimension is understood) with coefficients (or values in) Z. Similar to the case of Lie algebras, $H^2(G, Z)$ classifies *central extensions of Lie groups*, i.e., short exact sequences $Z \to \widehat{G} \to G$ possessing smooth local sections[3] (see [40, Proposition 4.2] and Example 4.2). Note also that a generalised cocycle (F, Θ) yields a 2-cocycle $q \circ F$ with values in $Z/\operatorname{im}(\tau)$ provided $\operatorname{im}(\tau)$ is discrete. In this case, we call $q \circ F$ the *band* of (F, Θ).

If we take coefficients $A \to 0$ and consider generalised cocycles modulo morphisms, then this yields the corresponding higher locally smooth cohomology $H^3(G, A)$ (cf. [41, Definition V.2.5]). Generalised cocycles with $(A \to 0)$-coefficients will sometimes also be called *3-cocycles* (or simply cocycles if the dimension is understood) with coefficients (or values in) A. □

We are now heading for a description of the integration procedure from [40]. In order to do so, we give a slightly more conceptual construction in the following two lemmata that we will use later on in our generalised construction. They describe the simplicial part of the procedure for enlarging local group cocycles to global ones (cf. [61]). Variants of this construction are implicitly hidden in [27] and [14]. However, none of the above authors relates those cocycles to (locally smooth) group cohomology.

Recall that our assumption is that G is a 1-connected Lie group with Lie algebra $\mathfrak{g}$.

Lemma 1.5. *Assume that there exist maps $\alpha : G \to C^\infty(\Delta^{(1)}, G)$ and $\beta : G^2 \to C^\infty(\Delta^{(2)}, G)$ such that*

$$\alpha_e \equiv e, \qquad \alpha_g(0) = e, \qquad \alpha_g(1) = g, \qquad \beta_{g,g}(s,t) = \alpha_g(s+t) \quad \textit{and} \tag{5}$$

$$\partial \beta_{g,h} = \alpha_g + g.\alpha_h - \alpha_{gh}. \tag{6}$$

[3] This is equivalent to demanding that $Z \to \widehat{G} \to G$ is a locally trivial principal bundle. Our central extensions of Lie groups are always assumed to be locally trivial principal bundles, but as above, different authors follow different conventions for this.

Then $\mathrm{d_{gp}}\,\beta : G^3 \to C_2(G)$ *takes values in* $Z_2(G)$ *and we have* $\mathrm{d_{gp}}\,\Theta_\beta = 0$ *if we set* $\Theta_\beta := q \circ \mathrm{d_{gp}}\,\beta$ *with* $q : Z_2(G) \to H_2(G) \cong \pi_2(G)$ *the canonical quotient map.*

We took β as sole subscript, indicating the dependence of Θ on α and β, for α is completely determined by β.

Proof. From (6) it follows directly that

$$\partial(\mathrm{d_{gp}}\,\beta)(g,h,k) = \partial(g.\beta_{h,k}) - \partial\beta_{gh,k} + \partial\beta_{g,hk} - \partial\beta_{g,h}$$

vanishes and thus Θ_β takes values in $Z_2(G)$. That $\Theta_\beta \in C^3(G, \pi_2(G))$ follows from $\beta_{g,g}(s,t) = \alpha_g(s+t)$, for then $\Theta_\beta(g,h,k)$ is null-homotopic if one of g, h or k equals e. Moreover, $\mathrm{d_{gp}}\,\Theta_\beta = 0$ follows from $\mathrm{d}^2_{\mathrm{gp}} = 0$. □

Lemma 1.6. *If* α', β' *is another pair of maps satisfying* (5) *and* (6)*, then there exists a map* $\gamma : G \to C^\infty(\Delta^{(2)}, G)$ *with* $\partial\gamma_g = \alpha_g + g.\alpha_e - \alpha'_g$*. Moreover,* $b_\gamma(g,h) = \beta_{g,h} - \beta'_{g,h} - \mathrm{d_{gp}}\,\gamma$ *takes values in* $Z_2(G)$ *and satisfies* $\Theta_\beta = \Theta_{\beta'} + q \circ \mathrm{d_{gp}}\, b_\gamma$*.*

Proof. Since G is simply connected, the map γ exists by [40, Proposition 5.6]. Just as above it is checked that b_γ takes values in $Z_2(G)$. Moreover, $\mathrm{d}^2_{\mathrm{gp}} = 0$ yields $q \circ \mathrm{d_{gp}}\, b_\gamma = q \circ \mathrm{d_{gp}}(\beta - \beta') = \Theta_\beta - \Theta_{\beta'}$. □

That Θ_β is a cocycle is actually trivial since we wrote is as a coboundary of the group cochain β. The point here is that it takes values in the much smaller subgroup $Z_2(G)$ and as cocycle with values in this group it is *not* trivial. Its projection to $\pi_2(G)$ is even the other extreme, namely *universal*, at least for discrete abelian groups (see [47] and Example 4.2).

In general, the maps α and β that we are going to choose for our construction are pretty arbitrary. However, when fixing a chart around the identity then there exists a canonical choice for α_g and $\beta_{g,h}$ if g and h are "close" to the identity:

Lemma 1.7. *Let* $\varphi : U \to \widetilde{U} \subseteq \mathfrak{g}$ *be a chart with* $\varphi(e) = 0$*,* $\varphi(U)$ *convex and* $\widetilde{V} \subseteq \widetilde{U}$ *open and convex such that* $e \in V := \varphi^{-1}(\widetilde{V})$ *and* $V \cdot V \subseteq U$*. For* $g \in U$ *we set* $\tilde{g} := \varphi(g)$ *and set* $\tilde{g} * \tilde{h} = \widetilde{gh}$ *for* $g, h \in V$*. If we define*

$$\alpha_g(t) = \varphi^{-1}(t \cdot \tilde{g}), \tag{7}$$

$$\beta_{g,h}(t,s) = \varphi^{-1}\big(t(\tilde{g} * s\tilde{h}) + s\big(\tilde{g} * (1-t)\tilde{h}\big)\big) \tag{8}$$

for $g, h \in V$*, then these assignments can be extended to mappings* α *and* β *satisfying* (5) *and* (6)*. Moreover, if for a different chart* φ' *we set* $\bar{g} := \varphi'(g)$ *and*

$$\gamma_g(s,t) = \varphi^{-1}\left(\frac{s(1-t)}{t+s}\varphi\big(\varphi'^{-1}\big((t+s)\bar{g}\big)\big) + t(1+s)\tilde{g}\right) \tag{9}$$

for $g \in V \cap V'$*, then this assignment can be extended to a map* $\gamma : G \to C^\infty_*(\Delta^{(2)}, G)$*, satisfying* $\partial\gamma_g = \alpha_g - \alpha'_g$*. In addition, if* $W \subseteq V$ *with* $W \cdot W \subseteq V$*, then* $\Theta_\beta|_{W\times W\times W}$ *and* $b_\gamma|_{W\times W}$ *are smooth.*

Proof. It is easily checked that α_g and $\beta_{g,h}$ defined as in (7) and (8) satisfy (5) and (6). Since G is connected, we may choose for each $g \in G\backslash U$ some $\alpha_g \in C^\infty(\Delta^{(1)}, G)$ with $\alpha_g(0) = e$ and $\alpha_g(1) = g$.

For $g, h \in G$ with $g \notin V$ or $h \notin V$, $\alpha_g + g.\alpha_h - \alpha_{gh}$ is in $Z_2(G)$, and thus there exists some $\beta_{g,h} \in C^\infty(\Delta^{(2)}, G)$ with $\partial\beta_{g,h} = \alpha_g + g.\alpha_h - \alpha_{gh}$ because G is simply connected (cf. [40, Proposition 5.6]). Moreover, we may choose $\beta_{g,g}(s,t) = \alpha(s+t)$. It is immediate that for γ_g as defined in (9) we have $\partial\gamma_g = \alpha_g - \alpha'_g$. Since G is simply connected, we may choose for each $g \notin V \cap V'$ some γ_g with $\partial\gamma_g = \alpha_g - \alpha'_g$. The rest is immediate. □

We now come to the description of the integration procedure from [40].[4]

Remark 1.8. Associated to each Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is its *period homomorphism* $\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$. This is given on (piecewise) smooth representatives by $[\sigma] \mapsto \int_\sigma \omega^l$, where ω^l is the left-invariant $\mathfrak{z}$-valued 2-form on G with $\omega^l(e) = \omega$ (cf. [40] or [65] for the fact that each homotopy class contains a smooth representative and [40] for the fact $\int_\sigma \omega^l$ is independent of the choice of a representative). We define $F_{\omega,\beta} : G \times G \to \mathfrak{z}$ by

$$F_{\omega,\beta}(g,h) := \int\limits_{\beta_{g,h}} \omega^l, \tag{10}$$

where $\beta : G^2 \to C^\infty_*(\Delta^{(2)}, G)$ is the map from Lemma 1.7 applied to a chart φ with $d\varphi(e) = \mathrm{id}_\mathfrak{g}$. Since $\beta(g,g)$ and $\beta(e,g)$ are null-homotopic, it follows that $F_{\omega,\beta}(g,e) = F_{\omega,\beta}(e,g) = 0$. That $(F_{\omega,\beta}, \Theta_\beta)$ is a generalised cocycle with coefficients $\pi_2(G) \xrightarrow{\mathrm{per}_\omega} \mathfrak{z}$ follows from $\mathrm{d}_{\mathrm{gp}}\Theta_\beta = 0$, from

$$\begin{aligned} \mathrm{d}_{\mathrm{gp}} F(g,h,k) &= F_{\omega,\beta}(h,k) - F_{\omega,\beta}(gh,k) + F_{\omega,\beta}(g,hk) - F_{\omega,\beta}(g,h) \\ &= \int\limits_{\beta_{h,k}} \omega^l - \int\limits_{\beta_{gh,k}} \omega^l + \int\limits_{\beta_{g,hk}} \omega^l - \int\limits_{\beta_{g,h}} \omega^l \\ &= \int\limits_{g.\beta_{h,k} - \beta_{gh,k} + \beta_{g,hk} - \beta_{g,h}} \omega^l = \mathrm{per}_\omega\big(\Theta_\beta(g,h,k)\big) \end{aligned} \tag{11}$$

and from the fact that the maps $V \times V \ni (g,h) \mapsto \beta_{g,h}$ and $C^\infty(\Delta^{(2)}, G) \ni \beta \mapsto \int_\beta \omega^l$ are smooth.

Since Neeb only considers 2-cocycles (and no generalised cocycles), he is forced[5] to consider Eq. (11) modulo Π_ω and thus obtains a 2-cocycle $f_{\omega,\beta}(g,h) := [F_{\omega,\beta}(g,h)]$ with values in $Z_\omega := \mathfrak{z}/\Pi_\omega$. The drawback is of course that he needs to assume that Π_ω is discrete in order to consider Z_ω as a Lie group (see Remark 7.1 for a proposal on how to use diffeological Lie groups in this context). □

[4] Variants of this procedure for the case of Kac–Moody central extensions can be found, for instance, in [49,37,13,38]. Implicitly, the cocycles that we shall construct here are already apparent in their constructions.

[5] From this discussion it is clear that it is sufficient to divide out $\Pi_\omega := \mathrm{per}_\omega(\pi_2(G))$ in order to ensure the cocycle identity. From Lemma 1.15 is follows that this is also necessary.

We will now make precise in which sense a group cocycle may "integrate" a Lie algebra cocycle. Recall that our standing assumption is that $\mathfrak{g}$ is the Lie algebra of G, that $\mathfrak{z}$ is an arbitrary Mackey-complete locally convex space and that $Z = \mathfrak{z}/\Gamma$ for $\Gamma \leqslant \mathfrak{z}$ discrete.

Lemma 1.9. *Let A be discrete and $A \xrightarrow{\tau} Z$ be a morphism of abelian Lie groups. If $F : G^2 \to Z$ and $\Theta : G^3 \to A$ is a generalised cocycle, then $dF(e,e)$ vanishes and we get a Lie algebra cocycle*

$$L(F) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}, \qquad (x, y) \mapsto d^2F\big((x,0),(0,y)\big) - d^2F\big((y,0),(0,x)\big).$$

Proof. Let $U \subseteq G$ be an identity neighbourhood such that $F|_{U\times U}$ and $\Theta|_{U\times U\times U}$ are smooth maps. From $F(e,g) = F(g,e) = 0$ it follows that $dF(e,e)$ vanishes. Moreover, $\Theta|_{U\times U\times U}$ vanishes since it is in particular continuous and A is discrete. Thus

$$F(g,h) + F(gh,k) - F(g,hk) - F(h,k) = 0$$

for g, h, k in U. Since the computation of $L(F)$ in [40, Lemma 4.6] only depends on the values of F on $U \times U$, the same calculation shows the claim. □

Definition 1.10. A generalised cocycle (F, Θ) as in the previous lemma is said to *integrate* a $\mathfrak{z}$-valued Lie-algebra cocycle ω if $L(F)$ is equivalent to ω. □

Theorem 1.11. *The generalised cocycle $(F_{\omega,\beta}, \Theta_\beta)$ from Remark* 1.8 *integrates ω.*

Proof. Since $F|_{V\times V}$ coincides with the function $f : V \times V \to \mathfrak{z}$ in [40, Lemma 6.2], associated to the cocycle ω and the smooth maps $\sigma_{g,h} : \Delta^{(2)} \to G$ from [40, Lemma 6.2] coincide with $\beta_{g,h}$ as defined in Lemma 1.7, [40, Lemma 6.2] shows

$$L(F)(x,y) = d^2F\big((x,0),(0,y)\big) - d^2F\big((y,0),(0,x)\big) = \omega(x,y). \qquad \square$$

We now argue that the generalised cocycle $(F_{\Omega,\beta}, \Theta_\beta)$ does essentially not depend on the choices that we made.

Remark 1.12. The construction in Remark 1.8 and the preceding proof depends on the actual choice of the map $\beta : G \times G \to C^\infty(\Delta^{(2)}, G)$, which in turn depends on the choice of a chart φ. However, for two different choices the resulting cocycles Θ_β and $\Theta_{\beta'}$ are equivalent by Lemma 1.5 and Lemma 1.7. Moreover, if $\gamma : G \to C^\infty_*(\Delta^{(2)}, G)$ is the corresponding map as defined in Lemma 1.7, then we obtain a morphism $(\varphi, \psi) : (F_{\omega,\beta}, \Theta_\beta) \to (F_{\omega,\beta'}, \Theta_{\beta'})$, given by $\psi(g,h) = [b_\gamma(g,h)]$ and

$$\varphi(g) = \int_{\gamma_g} \omega^l.$$

If two Lie algebra cocycles ω and ω' are equivalent, then $\omega(x,y)=\omega'(x,y)+b([x,y])$ for $b:\mathfrak{g}\to\mathfrak{z}$ linear and continuous. This leads to

$$\int_{\beta_{g,h}} \left(\omega-\omega'\right)^l = \int_{\beta_{g,h}} d\left(b^l\right) = \int_{\partial\beta_{g,h}} b^l = \int_{\alpha_g} b^l + \int_{g.\alpha_h} b^l - \int_{\alpha_{gh}} b^l$$

by Stokes Theorem. We thus obtain a morphism $(\varphi,\psi):(F_{\omega,\beta},\Theta_\beta)\to(F_{\omega',\beta},\Theta_\beta)$ with $\psi\equiv 0$ and

$$\varphi(g)=\int_{\alpha_g} b^l. \qquad \square$$

We conclude this section with showing that the cocycle $(F_{\omega,\beta},\Theta_\beta)$ we constructed here is universal for generalised cocycles that integrate ω. This may be seen as a substitute for the exact sequence [40, Theorem 7.12]

$$0\to \operatorname{Ext}_{\mathrm{Lie}}(G,Z)\xrightarrow{L} H^2_c(\mathfrak{g},\mathfrak{z})\xrightarrow{P}\operatorname{Hom}\left(\pi_2(G),Z\right).$$

The next lemma is the generalisation of the injectivity of L (cf. [40, Proposition 7.4]) for not necessarily discrete subgroups $\Gamma\subseteq\mathfrak{z}$.

Lemma 1.13. *Let $F\in C^2(G,\mathfrak{z})$ be such that $\mathrm{d}_{\mathrm{gp}} F$ vanishes on some identity neighbourhood of G^3, $\mathrm{d}_{\mathrm{gp}} F$ takes values in some subgroup Γ of $\mathfrak{z}$ and $L(F)$ is trivial as a Lie algebra cocycle. Then there exists $\varphi\in C^1(G,\mathfrak{z})$ such that $F-\mathrm{d}_{\mathrm{gp}}\varphi$ vanishes on some identity neighbourhood and takes values in Γ on $G\times G$.*

Proof. First note that $L(F)$ actually defines a Lie algebra cocycle by the same argument as in Lemma 1.9. Since $L(F)$ is trivial, there exists a continuous and linear map $\chi:\mathfrak{g}\to\mathfrak{z}$ such that $L(F)=\chi([\cdot,\cdot])$.

Let $U,V\subseteq G$ be contractible identity neighbourhoods such that $\mathrm{d}_{\mathrm{gp}} F|_{U\times U\times U}$ vanishes, $F|_{U\times U}$ is smooth and $V^2\subseteq U$. Then $(\mathfrak{z}\times U,\mathfrak{z}\times V,\mu_F,(0,e))$ with

$$\mu_F\left((z,g),(w,h)\right):=\left(z+w+F(g,h),gh\right)$$

is a local Lie group with Lie algebra $\mathfrak{z}\oplus_{L(F)}\mathfrak{g}$. Since $L(F)=\chi([\cdot,\cdot])$, we have that

$$\mathfrak{z}\oplus_{L(F)}\mathfrak{g}\ni(z,x)\mapsto z+\chi(x)\in\mathfrak{z}$$

defines a homomorphism of Lie algebras. This we may integrate to a homomorphism of local Lie groups, given by $(z,g)\mapsto z+\varphi(g)$. By shrinking V if necessary we may assume that φ is defined on V. That this map is a homomorphism implies that $F-\mathrm{d}_{\mathrm{gp}}\varphi$ vanishes on $V\times V$.

Since $\mathrm{d}_{\mathrm{gp}} F$ takes values in Γ, we have that $f:=q\circ F:G\times G\to\mathfrak{z}/\Gamma$ is a group cocycle (where $q:\mathfrak{z}\to\mathfrak{z}/\Gamma$ is the canonical projection) and thus $(\mathfrak{z}/\Gamma)\times_f G$ is a group (which actually is topological, but not Hausdorff in general). Now

$$f_\varphi:V\to(\mathfrak{z}/\Gamma)\times_f G,\qquad g\mapsto\left(q\left(\varphi(g)\right),g\right)$$

satisfies $f_\varphi(g)\cdot f_\varphi(h)=f_\varphi(g\cdot h)$ wherever defined. Since G is 1-connected, f_φ extends[6] to a unique group homomorphism. This extension is given by $g\mapsto(\varphi'(g),g)$ for some function $\varphi':G\to\mathfrak{z}/\Gamma$. Moreover, φ' extends $q\circ\varphi$ and satisfies $f-\mathrm{d}_{\mathrm{gp}}\varphi'\equiv 0$. If we choose a lift $s:\mathfrak{z}/\Gamma\to\mathfrak{z}$ with $q(0)\mapsto 0$, then $g\mapsto s(\varphi'(g))$ for $g\notin V$ extends φ to all of G with the desired properties. □

Remark 1.14. The previous proof easily adapts to the case where G is not simply connected. One may construct φ as in the previous proof, but if G is just connected, f_φ does not necessarily extend. However, it determines a homomorphism $\widetilde{G}\to(\mathfrak{z}/\Gamma)\times_f G$, where $\widetilde{G}$ is the 1-connected cover of G. Restricting this homomorphism to $\pi_1(G)$ yields a homomorphism $\pi_1(G)\to\mathfrak{z}/\Gamma$. If this is trivial, f_φ in fact extends to a homomorphism and the argument carries over. □

The following lemma is our version of [40, Theorem 7.9] for non-discrete Γ.

Lemma 1.15. *If $F\in C^2(G,\mathfrak{z})$ is such that $\mathrm{d}_{\mathrm{gp}}F$ vanishes on some identity neighbourhood, $\mathrm{d}_{\mathrm{gp}}F$ takes values in some subgroup Γ of $\mathfrak{z}$ and $L(F)$ is equivalent to ω as a Lie algebra cocycle, then* $\mathrm{per}_\omega(\pi_2(G))\subseteq\Gamma$.

Proof. Since per_ω does only depend on the cohomology class of ω (cf. [40, Remark 5.9]), we may assume that $L(F)=\omega$. Set $\Theta:=\mathrm{d}_{\mathrm{gp}}F$ and let $U,V\subseteq G$ be open and contractible identity neighbourhoods such that $F|_{U\times U}$ is smooth, $\Theta|_{U\times U\times U}$ vanishes and $V\cdot V\subseteq U$. For each $g\in G$ we define $\kappa_g\in\Omega^1(gV,\mathfrak{z})$ by

$$\kappa_g(w_x)=d_2F_{g^{-1}.x}\big(x^{-1}.w_x\big)\quad\text{for } w_x\in T_xgV,$$

where $d_2F_g(w_h):=dF(0_g,w_h)$ for $g,h\in U$ and $w_h\in T_hU$. This is smooth for $F|_{U\times U}$ is smooth and a straight forward computation shows $d\kappa_g=\omega^l|_{gV}$. For $g,h\in G$ with $gV\cap hV\neq\emptyset$ we have $g^{-1}h\in U$. Thus $\mathrm{d}_{\mathrm{gp}}F(g^{-1}h,h^{-1}x,x^{-1}\eta(t))$ vanishes for $\eta(t)\in gV\cap hV$ and this implies

$$(\kappa_g-\kappa_h)(w_x)=d_2F_{g^{-1}.h}\big(h^{-1}.w_x\big).$$

If $\alpha:[0,1]\to gV\cap hV$ is smooth, then this in turn yields

$$\begin{aligned}\int_\alpha\kappa_g-\kappa_h&=\int_0^1 d_2F_{g^{-1}h}\big(h^{-1}.\dot\alpha(t)\big)\,dt\\&=F\big(g^{-1}h,h^{-1}\alpha(1)\big)-F\big(g^{-1}h,h^{-1}\alpha(0)\big)\\&=F\big(h,h^{-1}\alpha(1)\big)-F\big(g,g^{-1}\alpha(1)\big)+\Theta\big(g,g^{-1}h,h^{-1}\alpha(1)\big)\\&\quad-F\big(h,h^{-1}\alpha(0)\big)+F\big(g,g^{-1}\alpha(0)\big)-\Theta\big(g,g^{-1}h,h^{-1}\alpha(0)\big).\end{aligned}$$

[6] The group on the right does not need to be topological for this, cf. [25, Corollary A.2.26].

Now let $[\sigma] \in \pi_2(G)$ be represented by a smooth map $\sigma : [0,1]^2 \to G$ such that σ maps a neighbourhood of $\partial[0,1]^2$ to $\{e\}$. Then there exists some $n \in \mathbb{N}$ such that for $i, j \in \{0, \ldots, n-1\}$

$$\sigma\left(\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]\right) \subseteq g_{ij} V$$

for some $g_{ij} \in G$. We denote by σ_{ij} the restriction of σ to $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$. Then

$$\operatorname{per}_\omega\big([\sigma]\big) = \int_\sigma \omega^l = \sum_{i,j=0}^{n-1} \int_{\sigma_{ij}} \omega^l = \sum_{i,j=0}^{n-1} \int_{\sigma_{ij}} d\kappa_{g_{ij}} = \sum_{i,j=0}^{n-1} \int_{\partial\sigma_{ij}} \kappa_{g_{ij}} \tag{12}$$

by Stokes Theorem. We parametrise the intersection $\sigma_{ij} \cap \sigma_{i+1\,j}$ by $\mu_{ij}(t) := \sigma(\frac{i+1}{n}, \frac{j+t}{n})$ and $\sigma_{ij} \cap \sigma_{ij+1}$ by $\nu_{ij}(t) = \sigma(\frac{i+t}{n}, \frac{j+1}{n})$. In particular, we have the identities

$$\begin{aligned} &\mu_{i0}(0) = \mu_{i\,n-1}(1) = e, \qquad \mu_{ij}(1) = \nu_{ij}(1), \qquad \mu_{ij}(1) = \nu_{i+1\,j}(0), \\ &\nu_{0j}(0) = \nu_{n-1\,j}(1) = e, \qquad \mu_{i\,j+1}(0) = \nu_{ij}(1), \qquad \mu_{i\,j+1}(0) = \nu_{i+1\,j}(0). \end{aligned}$$

Since $\sigma|_{\partial[0,1]^2}$ vanishes, the integrals along $\partial\sigma_{ij} \cap \partial\sigma$ in (12) vanish and we thus have

$$\begin{aligned} \operatorname{per}_\omega\big([\sigma]\big) + \Gamma &= \left(\sum_{i,j=0}^{n-1} \int_{\mu_{ij}} \kappa_{g_{ij}} - \kappa_{g_{i+1\,j}} - \sum_{i,j=0}^{n-1} \int_{\nu_{ij}} \kappa_{g_{ij}} - \kappa_{g_{ij+1}}\right) + \Gamma \\ &= \left(\sum_{i,j=0}^{n-1} F\big(g_{i+1\,j}, g_{i+1\,j}^{-1}\mu_{ij}(1)\big) - F\big(g_{ij}, g_{ij}^{-1}\mu_{ij}(1)\big)\right. \\ &\quad + \sum_{i,j=0}^{n-1} -F\big(g_{i+1\,j}, g_{i+1\,j}^{-1}\mu_{ij}(0)\big) + F\big(g_{ij}, g_{ij}^{-1}\mu_{ij}(0)\big) \\ &\quad + \sum_{i,j=0}^{n-1} -F\big(g_{ij+1}, g_{ij+1}^{-1}\nu_{ij}(1)\big) + F\big(g_{ij}, g_{ij}^{-1}\nu_{ij}(1)\big) \\ &\quad \left. + \sum_{i,j=0}^{n-1} F\big(g_{ij+1}, g_{ij+1}^{-1}\nu_{ij}(0)\big) - F\big(g_{ij}, g_{ij}^{-1}\nu_{ij}(0)\big)\right) + \Gamma. \end{aligned}$$

From the above identities it follows that the correspondingly underlined terms cancel out. Thus $\operatorname{per}_\omega([\sigma])$ is contained in Γ. □

Lemma 1.16. *Let (F', Θ') be a generalised cocycle on G that integrates ω (cf. Definition* 1.10*). Assume that $p \in \operatorname{Hom}(\pi_2(G), A)$ and $\psi \in C^2(G, A)$ are such that $p \circ \Theta_\beta = \Theta' + d_{gp}\psi$. Then the following are equivalent.*

(i) $\tau \circ p = q \circ \mathrm{per}_\omega$.
(ii) $\mathrm{d}_{\mathrm{gp}}(q \circ F_{\omega,\beta} - F' - \tau \circ \psi) = 0$.
(iii) *There exists* $\varphi \in C^1(G, Z)$ *such that* $q \circ F_{\omega,\beta} = F' + \mathrm{d}_{\mathrm{gp}}\varphi + \tau \circ \psi$.

Proof. We fist note that $q : \mathfrak{z} \to Z$ is a covering map and thus there exists a section $s : Z \to \mathfrak{z}$ such that $s(0) = 0$ and s is smooth on some zero neighbourhood.

(i) $\Rightarrow$ (ii): We set $F^\sharp := s \circ (F' + \tau \circ \psi)$. By Lemma 1.13 there exists $\varphi \in C^1(G, \mathfrak{z})$ such that

$$\xi := F_{\omega,\beta} - F^\sharp - \mathrm{d}_{\mathrm{gp}}\varphi$$

vanishes on some identity neighbourhood and takes values in $q^{-1}(A)$. To show the assertion it suffices to show that $\mathrm{d}_{\mathrm{gp}}(q \circ \xi) = 0$. This follows from

$$\begin{aligned} &\tau \circ p \circ \Theta_\beta = q \circ \mathrm{per}_\omega \circ \Theta_\beta \\ &\quad\Rightarrow\quad \tau \circ \left(\Theta' + \mathrm{d}_{\mathrm{gp}}\psi\right) = q \circ \mathrm{d}_{\mathrm{gp}} F_{\omega,\beta} \\ &\quad\Rightarrow\quad \mathrm{d}_{\mathrm{gp}} F' + \tau \circ \mathrm{d}_{\mathrm{gp}}\psi = \mathrm{d}_{\mathrm{gp}}\left(q \circ \xi + F' + \tau \circ \psi\right). \end{aligned}$$

(ii) $\Rightarrow$ (iii): Applying Lemma 1.13 to $s \circ (q \circ F_{\omega,\beta} - F' - \tau \circ \psi)$ gives $\varphi' \in C^1(G, \mathfrak{z})$ such that

$$s \circ \left(q \circ F_{\omega,\beta} - F' - \tau \circ \psi\right) - \mathrm{d}_{\mathrm{gp}}\varphi'$$

has values in Γ. This implies $q \circ F_{\omega,\beta} - F' - \tau \circ \psi - \mathrm{d}_{\mathrm{gp}}\varphi = 0$ if we set $\varphi := q \circ \varphi'$.

(iii) $\Rightarrow$ (i): By [47], $\mathrm{im}(\Theta_\beta)$ generates $\pi_2(G)$. Thus the claim follows from

$$q \circ \mathrm{per}_\omega \circ \Theta_\beta = q \circ \mathrm{d}_{\mathrm{gp}} F_{\omega,\beta} = \mathrm{d}_{\mathrm{gp}}\left(F' + \tau \circ \psi\right) = \tau \circ \left(\Theta' + \mathrm{d}_{\mathrm{gp}}\psi\right) = \tau \circ p \circ \Theta_\beta. \qquad \square$$

Proposition 1.17. *Let* (F', Θ') *be a generalised cocycle on* G *that integrates* ω *(cf. Definition* 1.10*). Then there exists a unique* $p : \pi_2(G) \to A$ *such that* $p \circ \Theta_\beta = \Theta' + \mathrm{d}_{\mathrm{gp}}\psi$ *for some* $\psi \in C^2(G, A)$*. Moreover,* $\tau \circ p = q \circ \mathrm{per}_\omega$*.*

Proof. Since Θ_β is universal for discrete groups, there exists a unique $p_{\Theta'} \in \mathrm{Hom}(\pi_2(G), A)$ such that $[p_{\Theta'} \circ \Theta_\beta] = [\Theta']$ (cf. [47]), which is equivalent to $p \circ \Theta_\beta = \Theta' + \mathrm{d}_{\mathrm{gp}}\psi$ for some $\psi \in C^2(G, A)$. Set $p := p_{\Theta'}$. We will show that $\tau \circ p = q \circ \mathrm{per}_\omega$ and recall the construction of $p_{\Theta'}$ for this sake.

For H an abelian group and an arbitrary cocycle $f : G^3 \to H$, vanishing on some identity neighbourhood $U \times U \times U$, we construct an H-valued Čech-2 cocycle as follows. Take $V \subseteq U$ a symmetric open identity neighbourhood such that $V^2 \subseteq U$. Then the sets $V_g := gV$ for $g \in G$ form an open cover of G and

$$\eta(f, V)_{g,h,k} : V_g \cap V_h \cap V_k \to H, \qquad x \mapsto -f\left(g, g^{-1}h, g^{-1}k\right) - f\left(g^{-1}h, h^{-1}k, k^{-1}x\right)$$

is smooth since $g^{-1}h$, $g^{-1}k$, $h^{-1}k$ are elements of U if $V_g \cap V_h \cap V_k \neq \emptyset$. Moreover, it follows from $\mathrm{d}_{\mathrm{gp}} f(g, g^{-1}h, h^{-1}k, k^{-1}x) = 0$ that

$$\eta(f, V)_{g,h,k}(x) = -f\left(g, g^{-1}h, h^{-1}x\right) - f\left(h, h^{-1}k, k^{-1}x\right) + f\left(g, g^{-1}k, k^{-1}x\right),$$

showing that $\eta(f,V)_{g,h,k}$ constitutes a Čech 2-cocycle on G. The class $[\eta(f)] \in \check{H}^2(G,A)$ of this cocycle only depends on the equivalence class of f in $H^3(G,A)$. Since PG is contractible, this transgresses to an H-valued Čech 1-cocycle on ΩG, giving rise to a covering $H \to \widehat{\Omega G} \to \Omega G$ of ΩG. Choosing a base point in $\widehat{\Omega G}$ turns its connected component $\widehat{\Omega G}_0$ into a central extension $H \to \widehat{\Omega G}_0 \to \Omega G$ of Lie groups (cf. [25, Appendix 2]). Thus there exists a covering morphism $P : \widetilde{\Omega G} \to \widehat{\Omega G}_0$, where $\widetilde{\Omega G}$ is the universal covering of ΩG (note that ΩG is connected since G is assumed to be simply connected). The restriction of P to the subgroup $\pi_2(G) \cong \pi_1(\Omega G) \subseteq \widetilde{\Omega G}$ then gives the homomorphism p_f. In particular, we note that p_f only depends on $[\eta(f)]$.

From this it follows that $\tau \circ p = q \circ \mathrm{per}_\omega$ if and only if $[\eta(\tau \circ p \circ \Theta_\beta)] = [\eta(q \circ \mathrm{per}_\omega \circ \Theta_\beta)]$. In order to show the latter we assume that $\tau \circ p \circ \Theta_\beta$ and $q \circ \mathrm{per}_\omega \circ \Theta_\beta$ are smooth when restricted to $U \times U \times U$ and observe that $[p \circ \Theta_\beta] = [\Theta']$ implies $\eta(\tau \circ p \circ \Theta_\beta, V) \sim \eta(\tau \circ \Theta', V)$. Now $\tau \circ \Theta' = \mathrm{d}_{\mathrm{gp}} F'$ implies

$$\begin{aligned}
\eta\big(\tau \circ \Theta', V\big)_{g,h,k}(x) &= -\tau\big(\Theta'\big(g, g^{-1}h, g^{-1}k\big) - \Theta'\big(g^{-1}h, h^{-1}k, k^{-1}x\big)\big) \\
&= -\mathrm{d}_{\mathrm{gp}} F'\big(g, g^{-1}h, h^{-1}k\big) - \mathrm{d}_{\mathrm{gp}} F'\big(g^{-1}h, h^{-1}k, k^{-1}x\big) \\
&= F'\big(g, g^{-1}h\big) + F'\big(h, h^{-1}k\big) - F'\big(g, g^{-1}k\big) \\
&\quad - \big(F'\big(g^{-1}h, h^{-1}x\big) + F'\big(h^{-1}k, k^{-1}x\big) - F'\big(g^{-1}k, k^{-1}x\big)\big).
\end{aligned}$$

This is equivalent to the cocycle

$$V_g \cap V_h \cap V_k \ni x \mapsto -\big(F'\big(g^{-1}h, h^{-1}x\big) + F'\big(h^{-1}k, k^{-1}x\big) - F'\big(g^{-1}k, k^{-1}x\big)\big) \tag{13}$$

since $V_g \cap V_h \ni x \mapsto F'(g, g^{-1}h) \in Z$ is a 1-cochain. Similarly, $\mathrm{per}_\omega \circ \Theta_\beta = \mathrm{d}_{\mathrm{gp}} F_{\omega,\beta}$ implies that $\eta(q \circ \mathrm{per}_\omega \circ \Theta_\beta, V)$ is equivalent to the cocycle

$$V_g \cap V_h \cap V_k \ni x \mapsto -q\big(F_{\omega,\beta}\big(g^{-1}h, h^{-1}x\big) + F_{\omega,\beta}\big(h^{-1}k, k^{-1}x\big) - F_{\omega,\beta}\big(g^{-1}k, k^{-1}x\big)\big). \tag{14}$$

Since $F_{\omega,\beta}$ and F' both integrate ω, their restrictions to some open neighbourhood (which we may still assume to be U) are equivalent as local cocycles. Any local group coboundary between them gives rise to a coboundary between the Čech cocycles (13) and (14). □

Corollary 1.18. *The generalised cocycle* $(F_{\omega,\beta}, \Theta_\beta)$ *from Remark* 1.8 *is universal for generalised cocycles integrating* ω*. This means that if* (F', Θ') *integrates* ω *(cf. Definition* 1.10*), then there exists some* $p \in \mathrm{Hom}(\pi_2(G), A)$ *and a morphism* $(\varphi, \psi) : (q \circ F_{\omega,\beta}, p \circ \Theta_\beta) \to (F', \Theta')$ *of generalised cocycles. Moreover, the existence of* (φ, ψ) *determines* p *uniquely.*

2. Loop prolongations

In this section we provide the minimal algebraic structure that the generalised cocycles from the previous section yield. However, it will turn out that these algebraic structures do not mix very well with the underlying smooth structures, so we will go to slightly advanced algebraic structures in the next section to treat smoothness issues appropriately.

Definition 2.1. A *loop* is a set X together with a map $\mu : X \times X \to X$, $(x,y) \mapsto \mu(x,y) =: x \cdot y$ and a distinguished element $e \in X$ such that $x \cdot e = e \cdot x = x$ for all $x \in X$ and such that the maps $\lambda_x, \rho_x : X \to X$, $\lambda_x(y) = x \cdot y$ and $\rho_x(y) = y \cdot x$ are bijective. A morphism between two loops X and Y is a map $\varphi : X \to Y$ satisfying $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$. □

Remark 2.2. (Cf. [19].) Let X be a loop. Then in general we do not have that $(x \cdot y) \cdot z$ equals $x \cdot (y \cdot z)$, but since $\rho_{(x \cdot y) \cdot z}$ is bijective, there exists a unique element $A(x,y,z)$ such that

$$x \cdot (y \cdot z) = A(x,y,z) \cdot \bigl((x \cdot y) \cdot z\bigr).$$

We call the map $A : X \times X \times X \to X$ the *associator* of X and sometimes refer to $A(x,y,z)$ as the associator of (x,y,z).

We have to add the following data and assumptions in order to come from general loops to group cohomology. Suppose that we have a homomorphism $\varphi : X \to H$ for H an arbitrary group. Then the kernel of φ is a normal subloop[7] of X and since H is a group, all associators are contained in $\ker(\varphi)$. If we choose a sub*group* A of $\ker(\varphi)$ and assume that

$$\varphi \text{ is surjective,} \tag{15}$$

$$A(k,x,y) = A(x,k,y) = e \quad \text{for all } x,y \in X \text{ and } k \in \ker(\varphi), \tag{16}$$

$$k \cdot A(x,y,z) = A(x,y,z) \cdot k \quad \text{for all } x,y,z \in X \text{ and } k \in \ker(\varphi), \tag{17}$$

$$x \cdot a = a \cdot x \quad \text{for all } x \in X \text{ and } a \in A, \tag{18}$$

then we call $(A, \varphi : X \to H)$ a (general) *loop prolongation* of H by A (cf. [19, Section 4]). In this case, one can actually show that $\ker(\varphi)$ is a sub*group* of X [19, (4.6)] and that A and each associator is contained in $Z(\ker(\varphi))$. It has been shown in [19] that if we assume, in addition, that all associators are contained in A and that $A(x,y,k) = e$ for each $k \in K$, then A factors through a map from $H \times H \times H \to A$ which is in fact a cocycle. We will from now on assume that this is the case (and drop the adjective "general" to indicate this in the notation).

Note that the assignment of a 3-cocycles to a loop prolongation gives rise to a group isomorphism between (equivalence classes of) loop prolongations and $H^3(G,A)$ (with respect to a suitably defined product, see [19]). □

Lemma 2.3. *Let (F,Θ) be a generalised cocycle on the discrete group H with coefficients $A \subseteq Z$, also discrete. Then $\mu((z,g),(w,h)) = (z + w + F(g,h), gh)$ endows $Z \times H$ with the structure of a loop, which we denote by $Z \times_F H$. Moreover, if $q : Z \to Z/A$ is the canonical quotient homomorphism, then $\varphi : Z \times_F H \to (Z/A) \times_{(q \circ F)} H$, $(z,h) \mapsto (q(z),h)$ defines a loop prolongation of $(Z/A) \times_{(q \circ F)} H$ by A. The group 3-cocycle associated to this loop prolongation then equals Θ.*

Proof. That $Z \times_F H$ defines a loop is directly checked from the definition, as well as conditions (15), (17) and (18). From (1) it follows immediately that φ defines a homomorphism. Since

[7] One should not get confused by the different notions of normal subloops, for instance in [39,45,2] or the one used in [19]. One easily checks that they are all equivalent, the probably easiest one is presented, for instance, in [30]. In particular, the usual kernel-epimorphism correspondence goes through, see [2] or [39, p. 14] and references given there.

$$\begin{aligned}\big((z,g)(w,h)\big)(v,k) &= \big(z+w+v+F(g,h)+F(gh,k), ghk\big)\\ &= \big(\Theta(g,h,k),e\big)\big(\big(z+w+v+F(h,k)+F(g,hk), ghk\big)\big)\\ &= \big(\Theta(g,h,k),e\big)\big((z,g)\big((w,h)(v,k)\big)\big)\end{aligned}$$

follows from $\mathrm{d}_{\mathrm{gp}} F = \Theta$, we have $A((z,g),(w,h),(v,k)) = (\Theta(g,h,k),e)$. Thus (16) follows from the normalisation conditions of (F,Θ). From the above equation it is also clear that Θ is the group 3-cocycles associated to this loop prolongation. □

The preceding lemma shows in particular that the integrating cocycle $(F_{\beta,\omega},\Theta_\beta)$ from Remark 1.8 gives rise to a loop prolongation and one might be tempted to incorporate smoothness assumptions into the game. But we shall see now that in general one does not have a smooth structure on $\mathfrak{z}\times_{F_{\omega,\beta}} G$ which has $\mathfrak{z}\times V$ (the subset on which μ already is smooth) as an open subset.

Example 2.4. Let $G = C^\infty(S^1,\mathrm{SU}_2)$, $\mathfrak{g} = C^\infty(S^1,\mathfrak{su}_2)$ and $\omega_{\langle\cdot,\cdot\rangle}$ be the Kac–Moody cocycle from (2). If we normalise $\langle\cdot,\cdot\rangle$ such that the left-invariant extension of $\langle[\cdot,\cdot],\cdot\rangle$ is a generator of $H^3_{\mathrm{dR}}(\mathrm{SU}_2,\mathbb{Z})$, then it follows from the calculation in the proof of [36, Theorem III.9] that $\mathrm{per}_{\omega_{\langle\cdot,\cdot\rangle}} = \mathbb{Z}$. Thus $f := q\circ F_{\omega_{\langle\cdot,\cdot\rangle},\beta}$ is a 2-cocycle (cf. Remark 1.8). We thus obtain a central extension of Lie groups $U(1)\to\widehat{G}\to G$ with $\widehat{G} := U(1)\times_f G$. From [40, Proposition 5.11] it follows that the connecting homomorphism $\pi_2(G)\to\pi_1(U(1))$ in the long exact homotopy sequence of this fibration is (up to the choice of a sign) the identity on $\mathbb{Z}$. This implies that $\pi_2(\widehat{G}) = 0$.

Now consider the loop $\mathbb{R}\times_F G$ with $F := F_{\omega_{\langle\cdot,\cdot\rangle},\beta}$ from the previous lemma. If there existed a topology on $\mathbb{R}\times G$ having $\mathbb{R}\times V$ as an open subset (for V as in Remark 1.8) and turning μ into a globally smooth map, then the exact sequence $\mathbb{Z}\to\mathbb{R}\times_F G\to\widehat{G}$, induced by φ as in the previous lemma, would define a locally trivial principal bundle over $\widehat{G}$. In fact, the maps

$$g\cdot V\ni x\mapsto (0,g)\cdot\big(0,g^{-1}\cdot x\big)\in\mathbb{R}\times G$$

would provide smooth local sections (it is here that we use the global smoothness of μ). Since this bundle is in fact a covering and $\widehat{G}$ is simply connected, this covering would be trivial and $\widehat{G}$ would be diffeomorphic to $\mathbb{Z}\times G$. But $\pi_2(G)=\mathbb{Z}$, a contradiction to the existence of a globally smooth extension of the locally smooth multiplication. □

Remark 2.5. It is always the case that a group which restricts to a local (analytic) Lie group possesses a compatible (global) smooth (analytic) structure, at least if the group is generated by the local group (see Theorem 4.1 for the precise statement).

It is a well-known fact that an analogous statement does not hold for loops. In fact, each finite-dimensional almost smooth loop restricts to a smooth local loop on some neighbourhood of e. If there existed a compatible smooth structure on this loop, then the identity would yield a diffeomorphism between this smooth structure and the almost smooth structure. This is due to the fact that a morphism between almost smooth loops is smooth if and only if it is so on an open neighbourhood of e. So each almost smooth loop which is not a smooth loop yields a counterexample to the generalisation of the introductory statement of this remark to loops. See [39, Section 1.3] for details. In fact, the situation is even worse, there exist one-dimensional analytic loops which may not be extended to a global analytic loop [26, lines before Remark IX.6.8]. □

As the previous discussion shows, we are forced to consider as integrating objects of ω loop prolongations, which are compatible with the smooth structure only in an identity neighbourhood:

Definition 2.6. A loop prolongation $(A, \varphi : L \to H)$ is called *locally smooth* if L is endowed with the structure of a smooth local loop. With this we mean that there exists a subset W containing the identity, which is endowed with some manifold structure such that $L|_W := (\mu^{-1}(W) \cap (W \times W), W, \mu, 0)$ is a local loop and all structure maps are smooth (cf. [39, Section 1.1]). In particular, the manifold structure is part of the data. If, in addition, the associator of $(A, \varphi : L \to H)$ vanishes on some identity neighbourhood (in $W \times W \times W$), then we call the loop prolongation *locally associative*. □

Of course, local smoothness and local associativity make sense for loops in general, but we shall use this concepts only for loops that are parts of a loop prolongation.

Lemma 2.7. *If $(A, \varphi : L \to H)$ is a locally smooth loop prolongation and L is generated by some open $V \subseteq W$ with $V \cap A = \{e\}$, then L/A carries a Lie group structure such that the quotient map $q : L \to L/A$ restricts to a local diffeomorphism on some open neighbourhood of e.*

Proof. Since $V \cap A = \{e\}$, q is injective on V and we use it to endow $q(V) \subseteq L/A$ with a manifold structure. Clearly, the group multiplication and inversion are locally smooth on A/L with respect to this smooth structure. Since V generates L we have that $q(V)$ generates L/A and the assertion follows from Theorem 4.1. □

Remark 2.8. If $(A, \varphi : L \to H)$ is a locally smooth and locally associative loop prolongation, then $L|_V$ is a local Lie group for some open identity neighbourhood $V \subseteq W$. This then gives rise to a Lie algebra $\mathcal{L}(L)$, which is independent of the choice of V. □

Proposition 2.9. *If $\mathfrak{g}$ is a Mackey-complete locally exponential Lie algebra, then there exists a locally smooth and locally associative loop prolongation such that the associated Lie algebra is isomorphic to $\mathfrak{g}$.*

Since the upcoming sections are independent of this result, there is no harm in postponing the proof until the end of Section 6.

3. 2-Groups

We now introduce 2-groups in oder to overcome the discrepancy between the globally defined algebraic structure and the locally given smoothness in the next section (cf. Remark 4.9).

Definition 3.1. A (unital) *2-group* is a small groupoid $\mathcal{G}$, together with a *multiplication functor* $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, an *inversion functor* $^{-} : \mathcal{G} \to \mathcal{G}$ and an object $\mathbb{1}$ (frequently identified with its identity morphism $\mathrm{id}_{\mathbb{1}}$), together with natural isomorphisms

$$\alpha_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$$

for g, h, k objects of $\mathcal{G}$, called *associators*. We require $\mathbb{1} = \bar{\mathbb{1}}$, $g \otimes \mathbb{1} = g = \mathbb{1} \otimes g$ and $g \otimes \bar{g} = \mathbb{1} = \bar{g} \otimes g$ on objects and morphisms and we require that

$$\alpha_{g,h,k\otimes l} \circ \alpha_{g\otimes h,k,l} = (\mathrm{id}_g \otimes \alpha_{h,k,l}) \circ \alpha_{g,h\otimes k,l} \circ (\alpha_{g,h,k} \otimes \mathrm{id}_l)$$

for all g, h, k, l. The last requirement is frequently referred to as *pentagon identity*. Moreover, we require that $\alpha_{g,h,k}$ is an identity if one of g, h or k is $\mathbb{1}$ and that $\alpha_{g,\bar{g},g} = \mathrm{id}_g$ and $\alpha_{\bar{g},g,\bar{g}} = \mathrm{id}_{\bar{g}}$.

A *morphism* of (unital) 2-groups is a functor $\mathcal{F} : \mathcal{G} \to \mathcal{G}'$, together with natural isomorphisms $\beta_{g,h} : \mathcal{F}(g) \otimes' \mathcal{F}(h) \to \mathcal{F}(g \otimes h)$, satisfying $\mathcal{F}(\mathbb{1}) = \mathbb{1}'$, $\mathcal{F}(\bar{g}) = \overline{\mathcal{F}(g)}$, $\beta_{g,\mathbb{1}} = \mathrm{id}_{\mathcal{F}(g)} = \beta_{\mathbb{1},g}$ and $\beta_{g,\bar{g}} = \mathbb{1}' = \beta_{\bar{g},g}$ for all objects g of $\mathcal{G}$. Here, we require

$$\mathcal{F}\big(\alpha(g,h,k)\big) \circ \beta_{g\otimes h,k} \circ \big(\beta_{g,h} \otimes' \mathrm{id}_k\big) = \beta_{g,h\otimes k} \circ \big(\mathrm{id}_{\mathcal{F}(g)} \otimes' \beta_{h,k}\big) \circ \alpha'\big(\mathcal{F}(g), \mathcal{F}(h), \mathcal{F}(k)\big)$$

for all g, h, k. Finally, a *2-morphism* between morphisms $\mathcal{F}$ and $\mathcal{F}'$ consists of natural isomorphisms $\gamma_g : \mathcal{F}(g) \to \mathcal{F}'(g)$ such that

$$\gamma_{g\otimes h} \circ \beta_{g,h} = \beta'_{g,h} \circ \big(\gamma_g \otimes' \gamma_h\big).$$

The resulting 2-category is denoted by **2-Grp**. □

We took the clumsy notation for the inversion functor to distinguish it explicitly from the functor $\mathcal{G} \to \mathcal{G}^{\mathrm{op}}$ that maps each morphism to its inverse morphism. Non-unital 2-groups involve additional natural isomorphisms, replacing the identities $g \otimes \mathbb{1} = g = \mathbb{1} \otimes g$ and $g \otimes \bar{g} = \mathbb{1} = \bar{g} \otimes g$, which are themselves required to obey certain coherence conditions. However, in our constructions these isomorphisms shall always be identities which is why we excluded them from our definition. This is also why we drop the adjective "unital" in the sequel.

Remark 3.2. 2-Groups are special kinds of monoidal categories (cf. [7]), just as groups are special kinds of monoids. However, observe that a group is a monoid with a special *property* (existence of inverses), while a 2-group is a monoidal category with an additional *structure* (an inversion functor). Our 2-groups are also examples of coherent 2-groups, as considered in [7]. □

Example 3.3. A particular important class of examples form the so called *strict 2-groups*. They can be characterised to be 2-groups, for which all natural isomorphisms in Definition 3.1 are the identities.

Besides this description, strict 2-groups can be described by crossed modules as follows (in fact, the 2-category of strict 2-groups is 2-equivalent to the 2-category of crossed modules, cf. [46,20] or [34]). A *crossed module* is a morphism of groups $\tau : H \to G$ (for G now an arbitrary group), together with an automorphic action of G on H such that

$$\tau(g.h) = g \cdot \tau(h) \cdot g^{-1}, \tag{19}$$

$$\tau(h).h' = h \cdot h' \cdot h^{-1}. \tag{20}$$

Note that these two equations force $\ker(\tau)$ to be central in H and $\mathrm{im}(\tau)$ to be normal in G. From this one can build up a 2-group $\mathcal{G}$ as follows. The objects are G, the morphisms are $H \rtimes G$ and the structure maps are given by $s(h,g) = g$, $t(h,g) = \tau(h) \cdot g$, $\mathrm{id}_g = (e,g)$, $(h', \tau(h) \cdot g) \circ (h,g) = (h'h, g)$. The identity object is e and the multiplication and inversion functor are given by multiplication and inversion on the groups G and $H \rtimes G$. Then (19) and (20) ensure that this defines a functor with the desired properties if we set the associator from Definition 3.1 to be the identity. □

Example 3.4. We obtain a slightly weaker version of the previous example if we are given instead of a crossed module a loop prolongation $(A, \varphi: L \to G)$. From this we construct an (in general non-strict) 2-group as follows. The set of objects is L, the set of morphisms is $A \times L$ and the structure maps are given by $s(a,x) = x$, $t(a,x) = a \cdot x$ and $(a, b \cdot y) \circ (b, y) = (a + b, y)$. Then we set $x \otimes y = x \cdot y$ and $(a,x) \otimes (b,y) = (a \cdot b, x \cdot y)$, which clearly defines a functor. Since the loop L may fail to be associative, this only defines a 2-group if we introduce the associators $\alpha_{x,y,z} = (A(x,y,z), (x \cdot y) \cdot z)$. One readily checks that this defines a 2-group with the aid of the axioms of a loop prolongation from Remark 2.2. □

We finish this section with a few constructions that will be important later on.

Remark 3.5. Each 2-group comes along with a couple of natural groups associated to it.

- The set of isomorphism classes $\pi_0(\mathcal{G})$ of $\mathcal{G}$. Since $\otimes$ is a functor, it induces a map $\pi_0(\mathcal{G}) \times \pi_0(\mathcal{G}) \to \pi_0(\mathcal{G})$. This clearly defines a group multiplication for isomorphic objects in $\mathcal{G}$ become equal in $\pi_0(\mathcal{G})$.
- The set $\mathcal{G}_{\mathbb{1}}$ of morphisms in the full subcategory of $\mathcal{G}$, generated by $\mathbb{1}$. On $\mathcal{G}_{\mathbb{1}}$, we define a map

$$\mathcal{G}_{\mathbb{1}} \times \mathcal{G}_{\mathbb{1}} \to \mathcal{G}_{\mathbb{1}}, \qquad (g,h) \mapsto g \otimes h.$$

  If we assume that $\alpha_{g,h,k}$ is an identity if one of g, h or k is isomorphic to $\mathbb{1}$,[8] then this defines an associative multiplication on $\mathcal{G}_{\mathbb{1}}$. Since $f \in \mathcal{G} \Leftrightarrow \overline{f} \in \mathcal{G}_{\mathbb{1}}$, this is in fact a group.
- The source and target fibres $s^{-1}(\mathbb{1})$ and $t^{-1}(\mathbb{1})$ are a subgroup of $\mathcal{G}_{\mathbb{1}}$.
- The endomorphisms $\pi_1(\mathcal{G}) := \mathrm{End}(\mathbb{1}) = s^{-1}(\mathbb{1}) \cap t^{-1}(\mathbb{1})$ of $\mathbb{1}$ form a subgroup of $\mathcal{G}_{\mathbb{1}}$. □

4. Lie 2-groups

In this section we shall elaborate on our concept of smoothness on 2-groups. Note the subtlety that we call our objects of main interest *Lie* 2-groups, so we emphasise their Lie-theoretic interpretation. Other authors calls their corresponding objects *smooth* 2-groups to put an emphasis on their properties as generalisations of smooth manifolds.

Since our 2-groups are internal to sets (for they are assumed to be small categories), it seems to be natural to work internal to manifolds (i.e., require sets to be manifolds and maps to be smooth), but this turns out to be too restrictive. The perspective to Lie groups that we will follow for our notion of Lie 2-groups is that a Lie group is a group with a locally smooth group multiplication. We make this precise in the following theorem. Note that Lie 2-groups make sense for smooth spaces in a more general setting than just locally convex manifolds, but to stay clear and brief we will stick to the manifold case.

[8] This should follow from coherence, but we were not able to find a reference for it. However, all 2-groups that we encounter in this article obey this condition.

Theorem 4.1. *Let G be a group, $U \subseteq G$ be a subset containing e and let U be endowed with a manifold structure. Moreover, assume that there exists $V \subseteq U$ open such that*

(i) $e \in V$, $V = V^{-1}$ *and* $V \cdot V \subseteq U$,
(ii) *the maps* $V \times V \ni (g,h) \mapsto gh \in U$ *and* $V \ni g \mapsto g^{-1} \in V$ *are smooth,*
(iii) V *generates* G *(as a monoid or, equivalently, as a group).*

Then there exists a unique Lie group structure on G such that the inclusion $U \hookrightarrow G$ is a diffeomorphism on some open identity neighbourhood. In particular, $V \hookrightarrow G$ is a diffeomorphism onto its open image and any other choice of V satisfying the above conditions gives the same smooth structure on G.

Proof. The proof is standard, see for instance [10, Proposition III.1.9.18]. However, we shall repeat the essential parts to illustrate the general idea.

Let $W \subseteq V$ be open such that $e \in W$, $W \cdot W \subseteq V$ and $W = W^{-1}$. Then we transport the smooth structure from W to gW by left translation $\lambda_g : W \to gW$ (i.e. we *define* λ_g to be a diffeomorphism). This is well-defined since for $gW \cap hW \neq \emptyset$ we have $h^{-1}g \in V$ so that the coordinate change

$$\lambda_g^{-1}(gW \cap hW) \ni x \mapsto \lambda_{h^{-1}g}(x) \in \lambda_h^{-1}(gW \cap hW)$$

is smooth by (ii). In particular, $V \subseteq G$ is open and $V \hookrightarrow G$ is a diffeomorphism onto its image.

To verify that the group multiplication is smooth, we first observe that for each $h \in G$ there exists $W_h \subseteq V$ open with $e \in W_h$ such that $h^{-1}W_h h \subseteq V$ and $x \mapsto h^{-1}xh$ is smooth. In fact, the set of all $h \in G$ such that W_h exists forms a sub monoid containing V, which equals G by (iii). Thus,

$$gW_h \times hW \ni (x,y) \mapsto xy = \lambda_{gh}\big(\big(h^{-1} \cdot \lambda_g^{-1}(x) \cdot h\big) \cdot \lambda_h^{-1}(y)\big) \in ghV$$

is smooth. A similar argument shows that inversion is also smooth.

If G is endowed with a Lie group structure such that $U \hookrightarrow G$ restricts to a diffeomorphism on V', then the restriction of id_G is smooth on $V' \cap V$. Since a homomorphism between Lie groups is smooth if and only if it so on an identity neighbourhood, this shows that id_G is in fact a diffeomorphism. This applies in particular to a possibly different choice of V. □

Note that the previous theorem tells us that the group structure determines the global topology, as soon as the local topology is fixed. It also says that a Lie group may equally well be defined as a group G, together with the smooth structure on U such that $V \subseteq U$ open with the corresponding properties exist. This is a very familiar pattern in Lie theory and we shall take this perspective when defining Lie 2-groups below. We have already seen that a statement corresponding to the preceding theorem is not valid for loops and we shall see below that non-strict 2-groups yet have a different behaviour.

The following example illustrates an important application of the preceding theorem.

Example 4.2. Let G be an arbitrary connected Lie group and let $f : G \times G \to Z$ be a 2-cocycle, which is smooth on some identity neighbourhood $U \times U$. This gives a group $Z \times_f G$ as in (1). Taking $V \subseteq U$ open with $e \in V$, $V^{-1} = V$ and $V \cdot V \subseteq U$ shows that this multiplication is smooth when restricted to $(V \times G)^2$ (a similar argument works for the inversion, since $(a, g)^{-1} = (-a - f(g, g^{-1}), g^{-1})$). Thus Theorem 4.1 yields a Lie group structure on $Z \times_f G$. This turns $Z \to Z \times_f G \to G$ into a central extension of Lie groups, possessing $U \ni x \mapsto (0, x) \in Z \times_f G$ as smooth local section. This construction applies in particular to the 2-cocycle $q \circ F_{\omega_{\langle \cdot, \cdot \rangle}, \beta}$ from Example 2.4.

Another important application of this construction is the following. Let PG be the space of continuous pointed paths in G and $G \to PG$, $g \mapsto \alpha_g$ be a section of the map that evaluates in 1 (the compact-open topology defines in fact a Lie group topology on PG with Lie algebra $P\mathfrak{g}$). Moreover, assume that α is smooth on some identity neighbourhood. Then

$$\Theta_\alpha(g, h) = [\alpha_g + g.\alpha_h - \alpha_{gh}] \in H_1(G) \cong \pi_1(G)$$

is a group cocycle for the universal covering group $\widetilde{G} \cong PG/(\Omega G)_0$. In fact, reconstructing cocycles from the central extension $\pi_1(G) \to \widetilde{G} \to G$ as in [40, Proposition 4.2] shows that Θ_α is equivalent to each of those cocycles. □

We now turn to the development of our notion of Lie 2-group. At first, we need the concept of a smooth 2-space.

Definition 4.3. A *smooth 2-space* is a (possibly infinite-dimensional) Lie groupoid. This means that it is a groupoid $\mathcal{C}$ such that $\mathcal{C}_0$ and $\mathcal{C}_1$ are endowed with smooth manifold structures, source and target maps are smooth surjective submersions[9] and the other structure maps are smooth.

A *smooth functor* between smooth 2-spaces is a functor whose respective maps on objects and morphisms are smooth. A *smooth natural transformation* between smooth functors is a natural transformation such that the corresponding map from objects to morphisms is smooth. The resulting 2-category is denoted by **2-Man**. Two smooth 2-spaces $\mathcal{C}$ and $\mathcal{D}$ are called *isomorphic* if there exist smooth functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$ (on the nose). □

The correct notion of equivalence of smooth 2-spaces is *Morita equivalence*, a more involved notion than the naive one. We shall not need this notion in this article. The previous definition takes Lie groupoids as internal categories in the category of locally convex manifolds. For more general purposes as we are aiming for here this definition is insufficient. The category of locally convex manifolds has bad categorical properties: it lacks pull-backs, quotients and internal homs, also when restricting to finite-dimensional ones. This can be remedied by introducing smooth 2-spaces as categories internal to smooth spaces (also called Chen or diffeological spaces), for which we refer to [5].

The following proposition is the equivalent statement to the previous theorem for *strict* 2-groups. In order to state it we fist have to introduce the following notation.

Remark 4.4. Let $\mathcal{C}$ be a small monoidal category (e.g., a 2-group) and $\mathcal{V} \subseteq \mathcal{C}$ be a subcategory. Then the *monoidal subcategory generated by* $\mathcal{V}$ is the smallest monoidal subcategory containing $\mathcal{V}$. Since intersections of monoidal subcategories are in turn monoidal subcategories, there is a unique smallest monoidal subcategory, which we denote by $\langle \mathcal{V} \rangle$. □

[9] Surjective submersion in the strong sense that it is a projection in local coordinates. This ensures in particular that the space of composable morphisms $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ is a smooth manifold (cf. [42, Appendix A]).

Proposition 4.5. *Let $\mathcal{G}$ be a strict 2-group, $\mathcal{U}$ be a full subcategory containing $\mathbb{1}$ and let $\mathcal{U}$ be endowed with the structure of a smooth 2-space. Moreover, assume that there exists a full subcategory $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}_0$ is open in $\mathcal{U}_0$,*

(i) $\mathbb{1} \in \mathcal{V}$, $\overline{\mathcal{V}} = \mathcal{V}$ *and* $\mathcal{V} \otimes \mathcal{V} \subseteq \mathcal{U}$,
(ii) *the functors* $\otimes|_{\mathcal{V}\times\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathcal{U}$ *and* $\bar{}\,|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ *are smooth,*
(iii) $\langle \mathcal{V} \rangle = \mathcal{G}$.

Then there exists on $\mathcal{G}$ the structure of a smooth 2-space such that $\bar{}$ and $\otimes$ are smooth functors and the inclusion $\mathcal{U} \hookrightarrow \mathcal{G}$ restricts to an isomorphism on some full subcategory $\mathcal{V}'$ with $\mathcal{V}'_0 \subseteq \mathcal{U}$ open. Moreover, the smooth structure on $\mathcal{G}$ is unique with respect to these properties.

The following proof relies heavily on the fact that strict 2-groups are actually category objects internal to the category of groups, i.e., spaces of objects, morphisms and composable morphisms are groups and all structure maps are group homomorphisms (cf. [46,7,20]).

Proof of Proposition 4.5. It is clear from the assumptions that $\mathcal{U}_0 \subseteq \mathcal{G}_0$ is a subset containing the identity which is endowed with a manifold structure and $\mathcal{V}_0 \subseteq \mathcal{U}_0$ is an open subset satisfying the assumptions from Theorem 4.1. This yields a smooth structure on $\mathcal{G}_0$. On morphisms, we have the smooth manifold $\mathcal{U}_1 = s^{-1}(\mathcal{U}_0) \cap t^{-1}(\mathcal{U}_0)$ containing $\mathrm{id}_{\mathbb{1}}$ and the open subset $\mathcal{V}_1 = s^{-1}(\mathcal{V}_0) \cap t^{-1}(\mathcal{V}_0)$. Now $\mathcal{V}_1$ generates $\mathcal{G}_1$ by assumption, so Theorem 4.1 also yields a smooth structure on $\mathcal{G}_1$. Moreover, s and t are submersions since they are so on $\mathcal{U}_1$ and the smoothness of $\bar{}$ and $\otimes$ is part of the conclusion of Theorem 4.1. The uniqueness assertion also follows immediately from the one in Theorem 4.1. □

It might look quite promising to expect a similar construction of globally smooth 2-group structures from locally ones also in the case of non-strict 2-groups, but this expectation is too optimistic. In fact, the following lemmata show that the topology of $\mathcal{G}_1$ splits as a product in this case into the part that comes from the identities and the arrow part.

Lemma 4.6. *Let $\mathcal{G}$ be a 2-group which is also a smooth 2-space such that the functors $\bar{}$, $\otimes$ and the associator are smooth. Then $s^{-1}(\mathbb{1})$ is a submanifold of $\mathcal{G}_1$ and in particular a Lie group. Moreover,*

$$s^{-1}(\mathbb{1}) \times \mathcal{G}_1 \to \mathcal{G}_1, \qquad (a, f) \mapsto a \otimes f$$

defines a smooth action. Moreover, this action is free, $\mathcal{G}_1/s^{-1}(\mathbb{1}) \cong \mathcal{G}_0$ as manifolds and $\mathcal{G}_1$ is a trivial smooth principal $s^{-1}(\mathbb{1})$-bundle.

Proof. Since inverse images of points under submersions are submanifolds, $s^{-1}(\mathbb{1})$ is a Lie group. That the action is free follows from

$$a \otimes f = b \otimes f \quad \Rightarrow \quad a \otimes \underbrace{\left(f \otimes f^{-1}\right)}_{=\mathbb{1}} = b \otimes \underbrace{\left(f \otimes f^{-1}\right)}_{=\mathbb{1}} \quad \Rightarrow \quad a = b.$$

The source map $\mathcal{G}_1 \to \mathcal{G}_0$ is $s^{-1}(\mathbb{1})$-invariant and thus induces a smooth map $\mathcal{G}_1/s^{-1}(\mathbb{1}) \to \mathcal{G}_0$. The identity map $\mathcal{G}_0 \to \mathcal{G}_1$ provides a smooth global section, proving the claim. □

Lemma 4.7. *Let $\mathcal{G}$ be a 2-group which is also a smooth 2-space such that the functors $^{-}$, $\otimes$ and the associator are smooth. If $s^{-1}(\mathbb{1})$ is discrete in the induced topology then the arrow part*

$$(\mathcal{G}_0)^3 \ni (g,h,k) \mapsto \alpha(g,h,k) \otimes \mathrm{id}_{\overline{(g\otimes h)\otimes k}} \in s^{-1}(\mathbb{1}) \subseteq \mathcal{G}_1$$

of α is locally constant.

Proof. This is due to the fact that smooth maps between locally convex manifolds are in particular continuous. □

The importance of the previous lemma is that we are forced to work with 2-groups with $s^{-1}(\mathbb{1})$ discrete if we want a reasonable interpretation of a Lie 2-group integrating an ordinary Lie algebra (cf. Section 5). Thus it illustrates the limitation on building 2-groups with too many smoothness conditions. However, locally smoothness of the group multiplication is essential for passing from Lie 2-groups to Lie 2-algebras. In view of Theorem 4.1 and Proposition 4.5, the following definition seems to be natural.

Definition 4.8. A *Lie 2-group* is a tuple $(\mathcal{G},\mathcal{U})$ such that $\mathcal{G}$ is a 2-group and $\mathcal{U}$ is a full subcategory containing $\mathbb{1}$, which is endowed with the structure of a smooth 2-space. Moreover, there has to exist a full subcategory $\mathcal{V} \subseteq \mathcal{U}$ with $\mathcal{V}_0 \subseteq \mathcal{U}_0$ open such that

(i) $\mathbb{1} \in \mathcal{V}$, $\overline{\mathcal{V}} = \mathcal{V}$ and $\mathcal{V} \otimes \mathcal{V} \subseteq \mathcal{U}$,
(ii) the functors $\otimes|_{\mathcal{V}\times\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathcal{U}$ and $^{-}|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ are smooth,
(iii) $\langle\mathcal{V}\rangle = \mathcal{G}$.

When working with Lie 2-groups we will sometimes not mention $\mathcal{U}$ explicitly if it is understood. A *morphism* of Lie 2-groups is a morphism of the underlying 2-groups such that the constituting functors and natural transformations restrict (and co-restrict) to smooth functors and natural transformations on some of the subcategories $\mathcal{V}$ from above. Likewise, *2-morphisms* between morphisms of Lie 2-groups are defined. □

For the case of a non-strict 2-group our notion of a Lie 2-group does *not* fit with the notion used in [7, Definition 27], where the functors and natural transformation defining the 2-group structure are required to be globally smooth. For the reasons explained above we find our notion more natural in the non-strict case (see also [7, Theorem 59]). The observation that the concept introduced in [7] is sometimes inadequate has also been made by Henriques in [24, Section 9] (the latter notion of smooth 2-groups has also been used in [50]). However, the previous definition covers strict Lie 2-groups by Proposition 4.5 (cf. [3,7,66]). Moreover, it leads to a locally smooth group structure on the group $\pi_0(\mathcal{G})$ (in the appropriate category where $\pi_0(\mathcal{G})$ is a smooth space) and thus a globally smooth group structure thereon by Theorem 4.1. We thus may interpret our Lie 2-groups as a categorified version of a Lie group, much like Lie groupoids are categorified manifolds.

Remark 4.9. It is the fact that 2-groups form a 2-category which makes them more interesting then loop prolongations. Thus 2-groups allow for the notion of equivalence, which is weaker than isomorphism. Moreover, the category of smooth 2-spaces also allows for a weaker notion of morphisms, the so-called Hilsum–Skandalis morphisms (also called spans or weak morphisms or bibundles) between Lie groupoids. Making use of this concept, one can show that certain Lie 2-groups in the above sense are in fact equivalent to weak group objects in the weak 2-category of Lie groupoids with morphisms the aforementioned Hilsum–Skandalis morphisms (the latter are called *stacky Lie groups* in [9]). This applies in particular to the 2-groups that we will construct in Theorem 5.16 (cf. Remark 7.2 and [67]) and to the one from the following example (cf. [50]). □

Although it does not play a rôle in the main theme of the paper, we present the following example for it illustrates the use and simplicity of our concept of Lie 2-groups.

Example 4.10. Let G be compact, simple and simply connected. Then $\langle[\cdot,\cdot],\cdot\rangle$ is a Lie-algebra 3-cocycle on $\mathfrak{g}$, where $\langle\cdot,\cdot\rangle$ denotes the Killing form of $\mathfrak{g}$. Under this assumptions the left-invariant extension $\langle[\cdot,\cdot],\cdot\rangle^l$ is a generator of $H^3_{\mathrm{dR}}(G,\mathbb{Z})\cong\mathbb{Z}$. Consequently, the corresponding period homomorphism

$$\mathrm{per}_{\langle[\cdot,\cdot],\cdot\rangle}:\pi_3(G)\to\mathbb{R},\qquad [\sigma]\mapsto\int_\sigma\big\langle[\cdot,\cdot],\cdot\big\rangle^l$$

(cf. [41, Definition V.2.12]) has image $\mathbb{Z}$. Now the maps

$$\alpha:G\to C^\infty\big(\Delta^{(1)},G\big)\quad\text{and}\quad\beta:G^2\to C^\infty\big(\Delta^{(2)},G\big)$$

from Lemma 1.7 are accompanied by an additional map $\gamma:G^3\to C^\infty(\Delta^{(3)},G)$ satisfying

$$\partial\gamma_{g,h,k}=g.\beta_{h,k}-\beta_{gh,k}+\beta_{g,hk}-\beta_{g,h}\tag{21}$$

for the above assumptions on G imply that it is 2-connected. Moreover, one can choose $\gamma_{g,h,k}$ to depend smoothly on (g,h,k) in some neighbourhood of (e,e,e), similar to α and β in Eqs. (7) and (8). Then we set

$$\varphi_\gamma:G^3\to U(1)=\mathbb{R}/\mathbb{Z},\qquad (g,h,k)\mapsto\exp\Big(\int_{\gamma_{g,h,k}}\big\langle[\cdot,\cdot],\cdot\big\rangle^l\Big)$$

where $\exp:\mathbb{R}\to\mathbb{R}/\mathbb{Z}$ is the canonical quotient map. This defines a 3-cocycle since

$$\mathrm{d}_{\mathrm{gp}}(\varphi_\gamma)(g,h,k,l)=\int_{(\mathrm{d}_{\mathrm{gp}}\gamma)(g,h,k,l)}\big\langle[\cdot,\cdot],\cdot\big\rangle^l\in\mathbb{Z},$$

which in turn follows from $(\mathrm{d}_{\mathrm{gp}}\gamma)(g,h,k,l)\in Z_3(G)$, similar to Remark 1.8.

This is in fact a locally smooth 3-cocycle and by [41, Theorem V.2.6] we may differentiate this cocycle to get back the Lie algebra 3-cocycle $\langle[\cdot,\cdot],\cdot\rangle$. Similar to the argument from Remark 1.12 we see that the cohomology class of φ_γ does not depend on the choice of γ, as long as (21) is fulfilled (that is why we drop the subscript from now on). From this cocycle we get a 2-group $\mathcal{G}_G$ by setting $(\mathcal{G}_G)_0$ to be G and $(\mathcal{G}_G)_1$ to be $U(1)\times G$ with source and target map equal to the projection to G and composition of morphism induced by the group structure on $U(1)$. The monoidal structure is given by the group multiplication in G (on objects) and in $U(1)\times G$ (on morphisms) and the associator is given by $\alpha_{g,h,k}=(\varphi(g,h,k),ghk)$.

Now φ is smooth on $U \times U \times U$ for $U \subseteq G$ some open identity neighbourhood. Choosing some $V \subseteq U$ open with $e \in V$, $V = V^{-1}$ and $V^2 \subseteq U$ one directly checks that all requirements from Definition 4.8 are satisfied. This turns $\mathcal{G}_G$ into a Lie 2-group.

The natural generalisation of the differentiation process described in the next section enables one to differentiate $\mathcal{G}_G$ to a Lie 2-algebra. Since the differentiation of φ is the 3-cocycle $\langle[\cdot,\cdot],\cdot\rangle$, this Lie 2-algebra is the non-strict Lie 2-algebra determined by the 3-cocycle $\langle[\cdot,\cdot],\cdot\rangle$ (cf. [3]). This is (one model for) the string Lie 2-algebra, and the Lie 2-group $\mathcal{G}_G$ would thus be another model for the string 2-group (cf. [55,56,4,24,50] or [42]). There is certainly much more to say about this Lie 2-group (cf. Remark 7.2), but this lies beyond the scope of the present paper. □

5. Categorified central extensions and étale Lie 2-groups

In this section we define central extensions of (Lie) 2-groups and show how they arise from generalised cocycles (cf. Remark 5.3), for the more general setting see [12]. In particular, we seek for an interpretation of the integrating cocycle from Theorem 1.11 in terms of central extensions.

In the first part of the section we shall describe the route from generalised cocycles to generalised central extensions. The second part elaborates on the basic notions of Lie theory for Lie 2-groups and central extensions. Our perspective will be that central extensions are described by group cohomology, see [35, Section IV.3] for ordinary groups, [1] for generalisations and [40] for the specialisation to topological and Lie groups. The approach to Schreier-like invariants for extensions of groupoids in [8] does not fit into our situation, for our sequences of groupoids shall not be bijectively on objects.

Remark 5.1. In order to match the following definition with the situation of extensions of groups recall that a short exact sequence $A \xrightarrow{i} B \xrightarrow{j} C$ is a sequence of order two such that the diagram

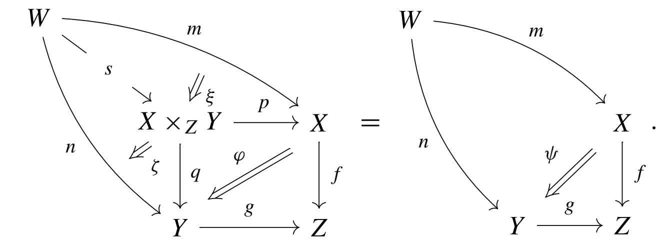
$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B \\ \downarrow & & \downarrow{\scriptstyle j} \\ * & \longrightarrow & C \end{array} \tag{22}$$

is at the same time a pullback (i injective and $\mathrm{im}(i) \subseteq \ker(j)$) and a pushout ($j$ surjective and $\ker(j) \subseteq \mathrm{im}(i)$).

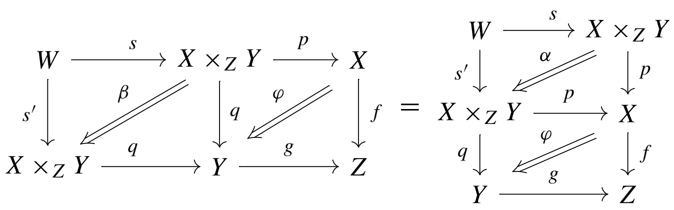
In the case that we are working with a strict 2-category we have to replace the (ordinary) pullback by a 2-pullback and likewise replace a pushout by a 2-pushout. If $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ are morphisms in a 2-category, then a *2-pullback* consists of an object, denoted $X \times_Z Y$, 1-morphisms $X \times_Z Y \xrightarrow{p} X$ and $X \times_Z Y \xrightarrow{q} Y$ and a 2-isomorphism $\varphi : f \circ p \Rightarrow g \circ q$ such that the diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\;p\;} & X \\ {\scriptstyle q}\downarrow & \overset{\varphi}{\Swarrow} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\;g\;} & Z \end{array}$$

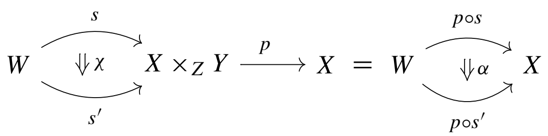
has the following universal property: For any object W that comes equipped with morphisms $W \xrightarrow{m} X$ and $W \xrightarrow{n} Y$ and a 2-isomorphism $\psi : f \circ m \Rightarrow g \circ n$ there exists a morphism $s : W \to X \times_Z Y$ and 2-isomorphisms $\xi : m \Rightarrow p \circ s$ and $\zeta : q \circ s \Rightarrow n$ such that



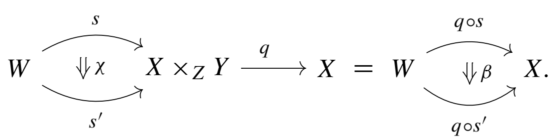
Moreover, given another morphism $W \xrightarrow{s'} X \times_Z Y$ and 2-isomorphisms $\alpha : p \circ s \Rightarrow p \circ s'$ and $\beta : q \circ s \Rightarrow q \circ s'$ such that



there has to be a unique 2-isomorphism $\chi : s \Rightarrow s'$ such that



and



Along the same lines, one defines *2-pushouts*. □

Definition 5.2. (Cf. [50, Definition 66].) If $\tau : A \to Z$ is a morphism of abelian groups (viewed as a crossed module for the trivial action of Z on A), then we denote the associated 2-group from Example 3.3 by $\mathcal{Z}_\tau$, which we also call a *strict abelian 2-group*.

For an arbitrary group G denote by $\underline{G}$ the 2-group with objects $g \in G$, only identity morphisms and the 2-group structure induced by multiplication in G. Then we define an *abelian* *extension* of $\underline{G}$ by $\mathcal{Z}_\tau$ to be a sequence of 2-groups $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$ such that $q \circ i$ is the constant functor $\mathbb{1}$ and that the diagram

$$\begin{array}{ccc} \mathcal{Z}_\tau & \xrightarrow{\ i\ } & \widehat{\mathcal{G}} \\ \downarrow & \swarrow\!\!\!\!\swarrow \ \mathrm{id}_{\mathbb{1}} & \downarrow q \\ * & \longrightarrow & \underline{G} \end{array} \tag{23}$$

is a 2-pullback and a 2-pushout in the 2-category **2-Grp**. Such an extension is called *central* if the two functors

$$\mathcal{Z}_\tau \times \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}, \qquad (z,g) \mapsto g \otimes \bigl(i(z) \otimes \overline{g}\bigr) \quad \text{and} \quad \mathcal{Z}_\tau \times \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}, \qquad (z,g) \mapsto i(z) \tag{24}$$

are naturally isomorphic. □

Note that the fact that $\underline{G}$ has only identity morphisms enforces us to put $\mathrm{id}_{\mathbb{1}}$ into the 2-cell of the above diagram.

Remark 5.3. Let G be a discrete group and A, Z be discrete abelian groups. For (F,Θ) a generalised group cocycle with coefficients in $\tau : A \to Z$, given by $\Theta \in C^3(G,A)$ and $F \in C^2(G,A)$ satisfying (3) and (4), the following assignment defines a 2-group $\widehat{\mathcal{G}}_{(F,\Theta)}$. The category $\widehat{\mathcal{G}}_{(F,\Theta)}$ is given by

$$\begin{aligned} \mathrm{Obj}(\widehat{\mathcal{G}}_{(F,\Theta)}) &= Z \times G, \qquad s(a,x,g) = (x,g), \qquad t(a,x,g) = \bigl(\tau(a)+x,g\bigr), \\ \mathrm{Mor}(\widehat{\mathcal{G}}_{(F,\Theta)}) &= A \times Z \times G, \qquad \mathrm{id}_{(x,g)} = (0,x,g), \qquad (a,x,g)\circ(b,y,g) = (a+b,y,g) \end{aligned}$$

and the multiplication functor by

$$(a,x,g) \otimes (b,y,h) = \bigl(a+b, x+y+F(g,h), gh\bigr).$$

Since F satisfies the cocycle identity only up to correction by Θ, this assignment defines a monoidal category if we define $\mathbb{1} = (0,e)$ and

$$\alpha_{(x,g),(y,h),(z,k)} = \bigl(\Theta(g,h,k), x+y+z+F(g,h)+F(gh,k), ghk\bigr).$$

We clearly have $1 \otimes g = g = g \otimes \mathbb{1}$, the source-target matching condition of α is equivalent to $\mathrm{d}_{\mathrm{gp}} F = \tau \circ \Theta$ and the pentagon identity is equivalent to $\mathrm{d}_{\mathrm{gp}} \Theta = 0$. Moreover,

$$\overline{(a,x,g)} = \bigl(-a, -x-F\bigl(g,g^{-1}\bigr), g^{-1}\bigr)$$

defines an inversion functor on $\widehat{\mathcal{G}}_{(F,\Theta)}$, turning it into a 2-group. In addition to the 2-group structure on $\widehat{\mathcal{G}}_{(F,\Theta)}$, we have canonical functors $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}}_{(F,\Theta)}$ and $\widehat{\mathcal{G}}_{(F,\Theta)} \xrightarrow{q} \underline{G}$. □

Lemma 5.4. *In the setting of the previous remark,* $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}}_{(F,\Theta)} \xrightarrow{q} \underline{G}$ *is a central extension of* $\underline{G}$ *by* $\mathcal{Z}_\tau$.

Proof. We first notice that the functors from (24) are actually equal in this case, so the extension will be central.

We abbreviate $\mathcal{Z} := \mathcal{Z}_\tau$ and $\mathcal{G} = \widehat{\mathcal{G}}_{(F,\Theta)}$. Assume that $m : W \to \mathcal{G}$ is given such that $q \circ m = \mathbb{1}$. Then on objects we have that $m_0(w) \in q_0^{-1}(\mathbb{1}) = Z \times e_G \cong \mathcal{Z}_0$ and on morphisms we have $m_1(v) \in q_1^{-1}(\mathbb{1}) = A \times Z \times \{e_G\} \cong \mathcal{Z}_0$. So m factors (on the nose) through a morphism $s : W \to \mathcal{Z}$, i.e., we may choose $\xi : i \circ s \Rightarrow m$ (and of course also $\zeta : * \circ s \Rightarrow *$) to be the identity natural transformations. Moreover, if we have $s' : W \to \mathcal{Z}$ and 2-isomorphisms $\alpha : i \circ s \Rightarrow i \circ s'$ such that $q_1(\alpha(w)) = \mathrm{id}_{\mathbb{1}}$, then $\alpha(w) \in q_1^{-1}(\mathbb{1}) = A \times Z \times \{e_G\} \cong \mathcal{Z}_1$ so that α factors through a 2-isomorphism $\chi : s \Rightarrow s'$ which obviously satisfies the requirements. This shows that $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}}_{(F,\Theta)} \xrightarrow{q} \underline{G}$ is a 2-pull back. Along similar lines one shows that it also is a 2-pushout. □

We will now consider extensions of Lie 2-groups. Note that in the case of Lie groups (in [40]) or in the setting of smooth 2-groups (in [50]) there is an additional requirement on a sequence $A \xrightarrow{i} B \xrightarrow{q} C$ besides that the diagram from (22), respectively (23), is a (2-)pullback and a (2-)pushout. For Lie group extensions one requires the existence of a smooth local section (this then implies that $B \to C$ is a locally trivial principal A-bundle) and in [50] it is required that $A \xrightarrow{i} B \xrightarrow{q} C$ is an A-gerbe over C.

In our treatment we restrict from now on to étale Lie 2-groups, a concept that we are heading for now. This concept will be tailored to fit our Lie theoretic needs.

Remark 5.5. Similarly to the concept of a smooth 2-space (and smooth functors and natural transformations, cf. Definition 4.3), one defines (topological) *vector 2-spaces* to be internal categories in locally convex vector spaces, i.e., small categories such that all sets occurring in the definition of a small category are locally convex spaces, all structure maps are continuous linear maps and source and target are projections. Likewise, linear functors and natural transformations are defined internally, defining the 2-category **2-Vect**.

There is a natural functor T from the category $\mathbf{Man_{pt}}$ (of pointed manifolds with smooth base-point preserving maps) to the category **Vect** (of topological vector spaces with continuous linear maps), sending manifolds to the tangent spaces at the base-point and smooth maps to their differentials at the base-point. Since this functor preserves pull-backs, it maps categories, functors and natural transformation in $\mathbf{Man_{pt}}$ to ones in **Vect** and thus defines a 2-functor $\mathcal{T} : \mathbf{2\text{-}Man_{pt}} \to \mathbf{2\text{-}Vect}$. □

If we want to enforce $\mathcal{T}$ to take values in **Vect** instead of **2-Vect**, then we need a canonical identification of $\mathcal{T}(\mathcal{M})_0$ and $\mathcal{T}(\mathcal{M})_1$. This is the case if $\mathcal{M}$ is étale, as defined below.

Definition 5.6. A smooth 2-space is called *étale* if all structure maps are local diffeomorphisms. A Lie 2-group $(\mathcal{G}, \mathcal{U})$ is called étale if $\mathcal{U}$ is an étale 2-space. Morphisms and 2-morphisms for étale Lie 2-groups are defined to be morphisms and 2-morphisms of Lie 2-groups. The corresponding 2-category is denoted by $\mathbf{Lie\ 2\text{-}Grp_{\acute{e}t}}$. □

Most of the Lie 2-groups that we shall encounter in this article are étale. Note that the differentials of local diffeomorphisms give canonical identifications of the tangent spaces at the base-points. Thus $\mathcal{T}(\mathcal{M})$ is in fact a vector space for étale $\mathcal{M}$. We shall make this precise for Lie 2-groups below. Note also that $s^{-1}(\mathbb{1})$ is discrete in an étale Lie 2-group. In particular, Lemma 4.7 applies to étale Lie 2-groups with globally smooth group operations.

Remark 5.7. The passage from general Lie 2-groups to étale ones will have an effect that has also been used in [57] for the solution of the integration problem of (finite-dimensional) Lie algebroids (cf. [17]). It is a long-standing observation that Lie algebroids integrate to local Lie group [48], but in general the integrating Lie groupoid may not be enlarged to a global Lie groupoid. The reasons for this failure is essentially the non-discreteness of the image of a period map (cf. [17, Section 3.2 and Theorem 4.1]) for which [17, Example 3.7] and [57, Example 1] give examples, very close to the integration problem that this article deals with. In [57] the integrating objects are *Weinstein groupoids*, which are also étale and categorified replacements of Lie groupoids. The result from [57] can also be seen as integrating Lie algebroids to locally defined Lie groupoids and then solve the associativity-constraint by passing to Weinstein groupoids. In the same spirit, étale Lie 2-groups will be the integrating objects for integration locally exponential Lie algebras. □

Definition 5.8. Let G be an arbitrary Lie group and $\tau : A \to Z$ be a morphism of abelian Lie groups with *discrete* A. Then a *smooth generalised central extension* (s.g.c.e.) is a sequence $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$ of étale Lie 2-groups such that $p \circ i = \mathbb{1}$ and that the diagram (23) is a 2-pullback and a 2-pushout in **Lie 2-Grp**$_{\text{ét}}$. Moreover, we demand that there exists a smooth functor $q : \underline{U} \to \widehat{\mathcal{G}}$ satisfying $q \circ s = \mathrm{id}_{\underline{U}}$, where $U \subseteq G$ is some open identity neighbourhood and that the functors

$$\mathcal{Z}_\tau \times \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}, \qquad (z,g) \mapsto g \otimes \big(i(z) \otimes \overline{g}\big) \quad \text{and} \quad \mathcal{Z}_\tau \times \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}, \qquad (z,g) \mapsto i(z)$$

are smoothly isomorphic when restricted to some neighbourhood of $(0, e_G) \in (\mathcal{Z}_\tau \times \mathcal{G})_0$. □

The requirement on a s.g.c.e. to be a sequence in *étale* Lie 2-group will enable us to take an easy way to central extensions of Lie algebras (cf. Proposition 5.14). The étalness is not crucial for the definition to make sense, the same definition of course also works in **Lie 2-Grp**. The following lemma is immediate from the definitions.

Lemma 5.9. *If (F, Θ) is a generalised cocycle on G with coefficients $\tau : A \to Z$ and A is discrete, then the 2-group $\widehat{\mathcal{G}}_{(F,\Theta)}$ from Remark 5.3 is canonically an étale Lie 2-group and $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}}_{(F,\Theta)} \xrightarrow{q} \underline{G}$ is a s.g.c.e.*

The following proposition describes the way back from generalised central extensions to ordinary ones. It is the categorical version of the discreteness condition for $\mathrm{per}_\omega(\pi_2(G))$ from [40].

Proposition 5.10. *Let $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$ be a s.g.c.e. such that $\tau(A) \subseteq Z$ is discrete. Then $\pi_0(\mathcal{Z}_\tau)$ and $\pi_0(\widehat{\mathcal{G}})$ carry Lie group structures with modelling space $\mathfrak{z}$ and $\mathfrak{z} \times \mathfrak{g}$ (respectively), turning*

$$\pi_0(\mathcal{Z}_\tau) \xrightarrow{\pi_0(i)} \pi_0(\widehat{\mathcal{G}}) \xrightarrow{\pi_0(q)} G \tag{25}$$

into a central extension of Lie groups.

Proof. First we note that $\pi_0(\mathcal{Z}_\tau) \cong Z/\tau(A)$ has a natural Lie group structure with Lie algebra $\mathfrak{z}$. Let $s : \underline{U} \to \widehat{\mathcal{G}}$ be a smooth section of q. Then $(\pi_0(q))^{-1}(U) \cong \pi_0(\mathcal{Z}_\tau) \times U$ as a set and we endow $(\pi_0(q))^{-1}(U)$ with the smooth structure making this identification a diffeomorphism. Since the group multiplication on $\widehat{\mathcal{G}}$ is smooth on an open subcategory containing $\mathbb{1}$, the group multiplication in $\pi_0(\widehat{\mathcal{G}})$ is locally smooth. Since U generates G, $\pi_0(q)^{-1}(U)$ generates $\pi_0(\widehat{\mathcal{G}})$ and the assertion follows from Theorem 4.1. □

Definition 5.11. The induced central extension (25) is called the *band* of $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$. □

Corollary 5.12. *If $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is a Lie algebra cocycle and (F, Θ) is a generalised cocycle which integrates ω (cf. Definition 5.15) and if $\tau(A) \subseteq Z$ is discrete, then the band of $\widehat{\mathcal{G}}_{(F,\Theta)}$ is a central extension $Z/\tau(A) \to \widehat{G} \to G$ integrating $\mathfrak{z} \to \mathfrak{z} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$.*

Proof. To see that the band of $\widehat{\mathcal{G}}_{(F,\Theta)}$ integrates $\mathfrak{z} \to \mathfrak{z} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$ we first observe that for $q : Z \to Z/\tau(A)$ the canonical quotient map $Tq(e) : \mathfrak{z} = T_e Z \to T_e(Z/\tau(A))$ is an isomorphism for $\tau(A)$ is discrete. Using this to identify $\mathfrak{z}$ with $T_e(Z/\tau(A))$ the claim follows from $L(q \circ F) = Tq(e) \circ L(F)$. □

Remark 5.13. We now derive a Lie algebra canonically associated to each étale Lie group $\mathcal{G}$. We first show that the associator α is trivial on some neighbourhood of $\mathbb{1}$. Since $\mathcal{G}$ is étale, the identity map $\mathcal{G}_0 \to \mathcal{G}_1$ is a local inverse around $\mathbb{1}$ for both, s and t. Since $\alpha_{\mathbb{1},\mathbb{1},\mathbb{1}} = \mathbb{1}$, we thus have $\mathrm{id} \circ t \circ \alpha = \mathrm{id} \circ s \circ \alpha$, which implies $s \circ \alpha = t \circ \alpha$ on some neighbourhood of $\mathbb{1}$. Now multiplying $\alpha(g,h,k)$ with $\mathrm{id}_{\overline{(g\otimes h)\otimes k}}$ defines a map with values in $s^{-1}(\mathbb{1})$, which is continuous on some identity neighbourhood and thus constantly $\mathbb{1}$. Since $\alpha_{x,\bar{x},x}$ is an identity for each x and α is natural, all of this implies

$$\mathrm{id}_{(g\otimes h)\otimes k} = (\alpha_{g,h,k} \otimes \mathrm{id}_{\overline{(g\otimes h)\otimes k}}) \otimes \mathrm{id}_{(g\otimes h)\otimes k} = \alpha_{g,h,k}$$

on some identity neighbourhood, which yields

$$(g \otimes h) \otimes k = g \otimes (h \otimes k)$$

for g, h, k from some neighbourhood of $\mathbb{1}$. Thus the multiplication functor defines on $\mathcal{G}_0$ the structure of a local Lie group and induces on $T_{\mathbb{1}}\mathcal{G}_0$ a Lie bracket. We denote this Lie algebra by $\mathcal{L}(\mathcal{G})$.

A similar argument as above shows that for a morphism $\mathcal{F} : \mathcal{G} \to \mathcal{G}'$ of Lie 2-groups, where $\mathcal{G}$ and $\mathcal{G}'$ are étale, we have $\mathcal{F}(g) \otimes' \mathcal{F}(h) = \mathcal{F}(g \otimes h)$ for g, h from some identity neighbourhood. Thus $\mathcal{F}_0$ induces a morphism of local Lie groups and thus of Lie algebras $\mathcal{L}(\mathcal{F}) : \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\mathcal{G}')$. Likewise, a 2-morphisms θ between two such morphisms has to be the identity on some identity neighbourhood. Summarising,

$$\mathcal{L} : \textbf{Lie 2-Grp}_{\textbf{ét}} \to \textbf{Lie Alg}$$

defines a 2-functor to the category of Lie algebras, considered as a 2-category with only identity 2-morphisms. □

Proposition 5.14. *Let $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$ be a s.g.c.e. Then*

$$\mathcal{L}(\mathcal{Z}_\tau) \xrightarrow{\mathcal{L}(i)} \mathcal{L}(\widehat{\mathcal{G}}) \xrightarrow{\mathcal{L}(q)} \mathcal{L}(\underline{G}) \tag{26}$$

is a central extension of Lie algebras.

Proof. That (26) is short exact follows the fact that the 2-functor $\mathcal{L}$ preserves 2-limits, which turn into ordinary limits in **LieAlg**. The differential of a section (on objects) of $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} G$ provides a linear and continuous section of (26). □

Definition 5.15. For a s.g.c.e. $\mathcal{Z}_\tau \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{q} \underline{G}$ its *derived* central extension is the central extension (26). If $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a topologically split central extension, then it is said to *integrate* to a smooth generalised central extension if there exists a s.g.c.e. such that its derived central extension is equivalent to $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$. □

Theorem 5.16. *If $\mathfrak{g}$ is the Lie algebra of the simply connected Lie group G, then each topologically split central extension $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ integrates to a smooth generalised central extension of étale Lie 2-groups.*

Proof. We may assume that $\widehat{\mathfrak{g}}$ is equivalently given by a Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$, which we integrate to a $\mathcal{Z}_{\mathrm{per}_\omega}$-valued cocycle $(F_{\omega,\beta}, \Theta_\beta)$ by Theorem 1.11 for some appropriate choice of β. Then Lemma 5.9 yields a s.g.c.e. $\mathcal{Z}_{\mathrm{per}_\omega} \to \widehat{\mathcal{G}}_{(F_{\omega,\beta},\Theta_\beta)} \to \underline{G}$.

Let $U, V \subseteq G$ be open identity neighbourhoods such that $F|_{U\times U}$ and $\Theta|_{U\times U\times U}$ are smooth and $V \cdot V \subseteq U$. To calculate the derived central extension we consider the restriction of the multiplication functor m to the full subcategory with objects in $\mathfrak{z} \times U$, where it is given by

$$m_0\big((z,g),(w,h)\big) = \big(z + w + F_{\omega,\beta}(g,h), gh\big)$$

on objects. By the definition of the Lie bracket of a local Lie group, the Lie bracket on $\mathcal{L}(\widehat{\mathcal{G}}_{(F_{\omega,\beta},\Theta_\beta)})$ is given by

$$\big((z,x),(w,y)\big) \mapsto \big(L(F_{\omega,\beta})(x,y),[x,y]\big)$$

and $L(F_{\omega,\beta}) = \omega$ shows the claim. □

We thus recover the classical case of central extensions by passing from a generalised central extension to its band in the case that $\mathrm{per}_\omega(\pi_2(G)) \subseteq \mathfrak{z}$ is discrete. Moreover, we can interpret the proof of the previous theorem as first passing to a 2-connected cover of G and then solve a trivial integration problem in the following sense.

Remark 5.17. Let $\beta : G^2 \to C^\infty_*(\Delta^{(2)}, G)$ be the map from Lemma 1.7, applied to a chart φ with $d\varphi(e) = \mathrm{id}_\mathfrak{g}$ and $\Theta_\beta : G^3 \to \pi_2(G)$ be the corresponding group 3-cocycle from Lemma 1.5. Then Θ_β determines an (in general non-strict) Lie 2-group $\mathcal{G} := \widetilde{\mathcal{G}}_{(0,\Theta)}$ (cf. Lemma 5.9), which we interpret as an appropriate version of a 2-connected cover of G (cf. [47]). In particular, we have a smooth generalised central extension

$$B\pi_2(G) \to \mathcal{G} \to \underline{G},$$

where $B\pi_2(G)$ is the strict Lie 2-group, associated to the crossed module $\pi_2(G) \to \{*\}$. Now $\widehat{\mathcal{G}} := \widehat{\mathcal{G}}_{(F_{\omega,\beta},\Theta_\beta)}$ can be seen as a central extension $\underline{\mathfrak{z}} \to \widehat{\mathcal{G}} \to \mathcal{G}$ (when generalising central extensions of Lie 2-groups to non-étale ones in the obvious way). Summarising, we have the commutative diagram

$$\begin{array}{ccccc}
\underline{\mathfrak{z}} & \longrightarrow & \underline{\mathfrak{z}} & \longrightarrow & \{*\} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{Z}_{\mathrm{per}_\omega} & \longrightarrow & \widehat{\mathcal{G}} & \longrightarrow & \underline{G} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
B\pi_2(G) & \longrightarrow & \mathcal{G} & \longrightarrow & \underline{G}
\end{array}$$

with exact rows and columns. Since $\mathcal{G}$ is an étale Lie 2-group with Lie algebra $\mathfrak{g}$, one may interpret $\underline{\mathfrak{z}} \to \widehat{\mathcal{G}} \to \mathcal{G}$ also as a central extension of étale 2-groups integrating $\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$. □

6. Lie's Third Theorem

We conclude this paper with the following generalisation of Lie's Third Theorem. We briefly recall definitions and some basic facts.

Definition 6.1. A locally convex Lie algebra $\mathfrak{g}$ is said to be locally exponential if there exists a circular convex open zero neighbourhood $U \subseteq \mathfrak{g}$ and an open subset $D \subseteq U \times U$ on which there exists a smooth map

$$m_U : D \to U, \qquad (x, y) \mapsto x * y$$

such that $(D, U, m_U, 0)$ is a local Lie group and such that the following hold.

(i) For $x \in U$ and $|t|, |s|, |t+s| \leqslant 1$, we have $(tx, sx) \in D$ with $tx * sx = (t+s)x$.
(ii) The second order term in the Taylor expansion of m_U in 0 is $b(x, y) = \frac{1}{2}[x, y]$. □

Remark 6.2. (Cf. [41, Example IV.2.4].) All Banach–Lie algebras are locally exponential, as well as all Lie algebras of locally exponential Lie groups. □

Theorem 6.3. *(See [41, Theorem IV.3.8].) Let* $\mathfrak{g}$ *be a locally exponential Lie algebra. Then the adjoint group* $G_{\mathrm{ad}} \leqslant \mathrm{Aut}(\mathfrak{g})$ *carries the structure of a locally exponential Lie group whose Lie algebra is* $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. □

The route to Lie's Third Theorem seems to be clear, simply integrate $\mathfrak{z}(\mathfrak{g}) \xrightarrow{\mathrm{incl}} \mathfrak{g} \xrightarrow{q} \mathfrak{g}_{\mathrm{ad}}$. But the latter need not be topologically split, as the following example shows.

Example 6.4. Let $F \leqslant E := \ell^p(\mathbb{N})$ for some $1 < p < 2$ be a non-complemented, in particular infinite-dimensional subspace, i.e., there exists no continuous projection $E \to F$. We choose a linearly independent sequence $(e_n)_{n\in\mathbb{N}}$ in F. Moreover, we choose a linearly independent sequence $(a_n)_{n\in\mathbb{N}}$ in $F^\perp$ such that $\operatorname{span}\{a_n\}$ is dense in $F^\perp$ and for each a_n another linearly independent $b_n \in F^\perp$. Having fixed this we set

$$[x, y] := \sum_{n=1}^{\infty} \frac{1}{2^n}\big(a_n(x)b_n(y) - a_n(y)b_n(x)\big)e_n.$$

Since $a_n(e_m) = b_n(e_m) = 0$ we have that $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ and thus $[\cdot,\cdot]$ defines a Lie bracket on E. An element $x \in E$ is in the centre precisely if the map $[x, \cdot]$ is trivial. This is the case if $x \in F$. On the other hand, if $x \notin F$, then $a_n(x) \neq 0$ for at least one $n \in \mathbb{N}$. For each $0 \neq y \in \ker(a_n)$ we have $y \notin \ker(b_n)$ and thus $[x, y] \neq 0$. This shows $x \notin F \Rightarrow [x, \cdot] \neq 0$ and thus F is the centre of E. □

A procedure similar to considering generalised central extensions as in [41, Section VI.1] now remedies this failure.

Theorem 6.5. *If $\mathfrak{g}$ is a Mackey-complete locally exponential Lie algebra, then there exists an étale Lie 2-group $\mathcal{G}$ such that $\mathcal{L}(\mathcal{G})$ is isomorphic to $\mathfrak{g}$.*

Proof. We consider $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and the map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$. This map vanishes if $x \in \mathfrak{z}(\mathfrak{g})$ or $y \in \mathfrak{z}(\mathfrak{g})$ and thus induces a continuous cocycle $\omega_{\mathfrak{g}} : \mathfrak{g}_{\mathrm{ad}} \times \mathfrak{g}_{\mathrm{ad}} \to |\mathfrak{g}|$, where $|\mathfrak{g}|$ denotes the Mackey-complete locally convex space underlying $\mathfrak{g}$. This integrates by Theorem 1.11 to a generalised cocycle $(F_{\omega_{\mathfrak{g}}}, \Theta)$, which in turn gives rise to an étale Lie 2-group $\widehat{\mathcal{G}}_{(F_{\omega_{\mathfrak{g}}},\Theta)}$ with set of objects $|\mathfrak{g}| \times G_{\mathrm{ad}}$. Moreover, we may assume that $F_{\omega_{\mathfrak{g}}}$ is smooth on $V \times V$ and $V = V^{-1}$.

The exponential function $\exp_{\mathfrak{g}_{\mathrm{ad}}} : \mathfrak{g}_{\mathrm{ad}} \to G_{\mathrm{ad}}$ restricts to a diffeomorphism on some open zero neighbourhood $U \subseteq \mathfrak{g}_{\mathrm{ad}}$ and we may assume that $\exp(U) \subseteq V$. We now want to construct a local exponential function for $|\mathfrak{g}| \oplus_{\omega_{\mathfrak{g}}} \mathfrak{g}_{\mathrm{ad}}$ and for this first define $\gamma_x(t) := \exp_{\mathfrak{g}_{\mathrm{ad}}}(tx)$ for $x \in \mathfrak{g}_{\mathrm{ad}}$ and for $x \in U$ and $t \in [0, 1]$ we set

$$z_x(t) := -\int_0^t TF_{\omega_{\mathfrak{g}}}\left(0_{\gamma_x(s)^{-1}}, \left.\frac{d}{du}\right|_{u=s} \gamma_x(u)\right) ds,$$

where $0_{\gamma_x(s)^{-1}}$ denotes the zero element in $T_{\gamma_x(s)^{-1}}G_{\mathrm{ad}}$. Note that the integral exists since $|\mathfrak{g}|$ is Mackey-complete. With this we set

$$\eta_{(z,x)} : [0, 1] \to |\mathfrak{g}| \times G_{\mathrm{ad}}, \qquad \eta_{(z,x)}(t) := \big(tz + z_x(t), \gamma_x(t)\big)$$

and observe that $\dot{z}_x(t) = -TF_{\omega_{\mathfrak{g}}}(0_{\gamma_x(t)^{-1}}, \dot{\gamma}_x(t))$ implies

$$\left.\frac{d}{dt}\right|_{t=0} \eta_{(z,x)}(t_0)^{-1}\eta_{(z,x)}(t_0 + t) = (z, x)$$

for $t_0 \in (0, 1)$. Thus

$$\exp : |\mathfrak{g}| \times U \to |\mathfrak{g}| \times \exp_{\mathfrak{g}_{\mathrm{ad}}}(U) \subseteq |\mathfrak{g}| \times G_{\mathrm{ad}}, \qquad (z, x) \mapsto \big(z + z_x(1), \exp_{\mathfrak{g}_{\mathrm{ad}}}(x)\big)$$

defines a local exponential function, which is a diffeomorphism since $(z, x) \mapsto (z, \exp_{\mathfrak{g}_{\mathrm{ad}}}(x))$ is one.

Now $\mathfrak{g}$ is isomorphic to the closed ideal $\{(x, q(x))\colon x \in \mathfrak{g}\}$ of $|\mathfrak{g}| \oplus_{\omega_{\mathfrak{g}}} \mathfrak{g}_{\mathrm{ad}}$, and thus exp restricts to a diffeomorphism of $(|\mathfrak{g}| \times U) \cap \mathfrak{g}$ onto $W := (|\mathfrak{g}| \times \exp_{\mathfrak{g}_{\mathrm{ad}}}(U)) \cap \exp(\mathfrak{g})$. Note that we have in particular $\mathfrak{z}(\mathfrak{g}) \times \exp_{\mathfrak{g}_{\mathrm{ad}}}(U) \subseteq W$. We define $\mathcal{G}$ to be the monoidal subcategory of $\widehat{\mathcal{G}}_{(F_{\omega_{\mathfrak{g}}}, \Theta)}$, generated by the full subcategory $\mathcal{W}$ determined by W.

It remains to check that $\mathcal{G}$ defined this way actually is an étale Lie 2-group with Lie algebra $\mathfrak{g}$. We have $\mathcal{W}_0 = W$ (by definition) and $\mathcal{W}_1 = \pi_2(G_{\mathrm{ad}}) \times W$, since $\mathrm{per}_{\omega_{\mathfrak{g}}}\colon \pi_2(G_{\mathrm{ad}}) \to |\mathfrak{g}|$ takes values in $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}$ by [41, Theorem VI.1.6] and $\mathfrak{z}(\mathfrak{g}) \times \exp_{\mathfrak{g}_{\mathrm{ad}}}(U) \subseteq W$. With the restricted structure maps this clearly is an étale 2-space and thus $(\mathcal{G}, \mathcal{W})$ is an étale Lie 2-group. Since exp is a local exponential function it is also clear that the Lie algebra, associated to the local group $(\mu^{-1}(W), W, \mu, (0, e))$ (with $\mu((z, x), (w, y)) = (z + w + F_{\omega_{\mathfrak{g}}}(x, y), xy)$) is isomorphic to $\mathfrak{g}$. Thus $\mathcal{L}(\mathcal{G}) \cong \mathfrak{g}$. □

Proof of Proposition 2.9. The set of objects of the étale Lie 2-group $\mathcal{G}$ constructed in the previous theorem give rise to a loop, which restricts to a locally smooth loop on some open neighbourhood of $\mathbb{1}$. Since the Lie 2-group is étale, this locally smooth loop is also locally associative. Moreover, the Lie algebra associated to this locally smooth and locally associative loop coincides with $\mathcal{L}(\mathcal{G})$ and thus is isomorphic to $\mathfrak{g}$. □

7. Prospects

We tried to develop a completed account on the integration of infinite-dimensional Lie algebras to Lie 2-groups. In order to do so we dropped some topics that may be at hand which we shortly line out in this section. Most of them deserve to be worked out seriously.

Remark 7.1 *(Diffeological Lie groups).* The problem that one encounters when trying to integrate central extensions of infinite-dimensional Lie algebras to Lie groups is that one has to factor out subgroups from locally convex spaces that may be not discrete. This has to be done to ensure that the cocycle condition for a certain universal cocycle holds.

However, one may resolve this problem by enlarging the category of smooth manifolds to a category in which this quotient exists. For instance, the category of diffeological spaces (or more general smooth spaces, cf. [5]) has this property. From our cocycle $(F_{\omega,\beta}, \Theta_\beta)$, integrating a given Lie algebra cocycle ω, one obtains an ordinary group cocycle $q \circ F_{\omega,\beta}$, which is in general (locally) smooth as a map between diffeological spaces, because the quotient map $q\colon \mathfrak{z} \to \mathfrak{z}/\Pi_\omega$ is smooth, no matter whether $\Pi_\omega := \mathrm{per}_\omega(\pi_2(G))$ is discrete or not. With the corresponding version of Theorem 4.1 for diffeological spaces one thus constructs a diffeological group $\widehat{G}_\omega$ and

$$\mathfrak{z}/\Pi_\omega \to \widehat{G}_\omega \to G$$

is a candidate for a central extension of diffeological groups, integrating

$$\mathfrak{z} \to \mathfrak{z} \oplus_\omega \mathfrak{g} \to \mathfrak{g}.$$

The crucial point here would be to set up the notion of a Lie functor from diffeological spaces to vector spaces such that it takes $\mathfrak{z}/\Pi_\omega$ to $\mathfrak{z}$, even if Π_ω is not discrete (such a thing should exist according to [51]).

In [27], a similar construction has been done in order to obtain a prequantisation for an arbitrary symplectic manifold (M,ω), with not necessarily integral $[\omega]\in H^2_{\mathrm{dR}}(M)$. In particular, prequantisation can be performed by directly passing to the dual of the Lie algebra, without constructing a Lie algebra at first[10] [28]. □

Remark 7.2 *(Differential geometry of generalised extensions).* One perspective to the integration procedure for central extensions of Lie algebras is to find a Lie group extension as a principal bundle with a prescribed curvature. It should be possible to develop such a point of view also for smooth generalised central extensions, a similar perspective has been taken, for instance, by Schommer-Pries [50].

On the level of cocycles, the passage is quite clear. For a cocycle $f:G\times G\to Z$, smooth on $U\times U$, the central extension $Z\to Z\times_f G\to G$ is a principal bundle, described by the transgressed Čech cocycle

$$\gamma_{g,h}:gV\cap hV\to Z,\qquad x\mapsto f\big(g,g^{-1}x\big)-f\big(h,h^{-1}x\big)$$

where $V\subseteq U$ is an open identity neighbourhood with $V\cdot V\subseteq U$. That $\gamma_{g,h}$ is smooth follows from

$$f\big(g,g^{-1}x\big)-f\big(h,h^{-1}x\big)=f\big(g^{-1}h,h^{-1}x\big)-f\big(g,g^{-1}h\big)$$

and from $g^{-1}h\in U$ if $gV\cap hV\neq\emptyset$. For a generalised cocycle (F,Θ) the transgressed non-abelian Čech cocycle is accordingly given by

$$\gamma_{g,h}:gV\cap hV\to Z,\qquad x\mapsto F\big(g,g^{-1}x\big)-F\big(h,h^{-1}x\big)-\tau\big(\Theta\big(g,g^{-1}h,h^{-1}x\big)\big)$$

and

$$\eta_{g,h,k}:gV\cap hV\cap kV\to Z,$$
$$x\mapsto -\Theta\big(g,g^{-1}h,h^{-1}x\big)-\Theta\big(h,h^{-1}k,k^{-1}x\big)+\Theta\big(g,g^{-1}k,k^{-1}x\big).$$

This yields a principal $\mathcal{Z}$-2-bundle $\mathcal{P}$ (over $\underline{G}$) [66], which is as a groupoid (without any additional structure) equivalent to $\mathcal{G}_{F,\Theta}$. Applied to the string cocycle φ from Example 4.10 this 2-bundle is the prequantisation for the 2-plectic manifold $(G,\langle[\cdot,\cdot],\cdot\rangle)$ [13,6]. In general, the interpretation of $\mathcal{P}$ as a bundle with connection is be a bit more tricky since the principal bundle $\pi_0(\mathcal{P})$ should admit curvature (in fancy terms, we want the fake curvature *not* to vanish). The theory of higher bundles with connection is being developed at the moment (cf. [53,43,52], references therein and [62] for the case of group extensions). □

Remark 7.3 *(Non-locally exponential Lie algebras).* One may wonder whether a similar theorem as our version for Lie's Third Theorem is also in reach for non-locally exponential Lie algebras. To our best knowledge it would be unlikely to expect a similar result in this direction, for the algebraic properties of non-locally exponential Lie algebras couple very hardly to their

[10] Even if there is a Lie algebra around, there is a priori no canonical dual space, associated to it, for the usual topologies on dual spaces are not good enough (cf. [41]). So it is more natural to pass directly to the dual.

local Lie groups (if they exist at all). For instance, Lempert proved that $\mathcal{V}(M)_{\mathbb{C}}$ is even not *integrable* for any compact manifold M (cf. [33]), which relies on more involved arguments as the counterexample of van Est and Korthagen in [61]. □

Remark 7.4 *(Higher Lie algebras and Lie algebroids).* In a sense, we performed a similar integration procedure as Henriques in [24]. It thus seems to be promising to carry this analogy further to integrate even infinite-dimensional Lie 2-algebras or to enlarge Henriques' procedure beyond the Banach case. Since the obstructions for integrating locally exponential Lie algebras and finite-dimensional Lie algebroids [17] seem to be the same, an integration procedure for special classes of infinite-dimensional Lie algebroids (e.g. Banach–Lie algebroids) as in [57] is quite likely. □

Remark 7.5 *(Stacky Lie groups).* Our definition of a Lie 2-group is somewhat weaker than one would expect at first. However, if one leaves the world of manifolds and considers Lie groupoids as presentations of differentiable stacks, then we expect that Lie 2-groups as defined above lead to stacky Lie groups in the sense of [9].

Problem. If $\mathcal{G}$ is a Lie 2-group (in the sense of Definition 4.8), does there exist a stacky Lie group (in the sense of [9]) or alternatively a smooth group stack $\mathcal{H}$ such that the underlying 2-groups are equivalent and the smooth stacks are equivalent in a "neighbourhood" of the identity? If this is the case, can this correspondence be promoted to an equivalence of the corresponding 2-categories?

One possible way to obtain this would be to follow the usage of the associativity of the group multiplication through Theorem 4.1. The coordinates of the Lie group structure on G would yield a Lie groupoid, the multiplication on G a Hilsum–Skandalis morphism describing the multiplication morphism between the stacky Lie groups and the usage of the associativity, finally, the associator 2-morphism.

The above problem seems to be solvable since in the cases know to the author an ad-hoc construction yields stacky Lie groups from Lie 2-groups. For the String 2-group from Example 4.10 this is the construction of Schommer-Pries in [50] and for the 2-groups $\widehat{\mathcal{G}}_{(F_{\omega,\beta},\Theta_\beta)}$ and $\widetilde{\mathcal{G}}_{(0,\Theta_\beta)}$ from Theorem 5.16 and Remark 5.17 this is carried out in [67]. Moreover, it would be desirable to work out a Lie theory of stacky Lie groups directly in the correct categorical setup. □

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Appendix A. Differential calculus on locally convex spaces

We provide some background material on locally convex Lie groups and their Lie algebras in this appendix.

Definition A.1. Let X and Y be a locally convex spaces and $U \subseteq X$ be open. Then $f : U \to Y$ is *differentiable* or C^1 if it is continuous, for each $v \in X$ the differential quotient

$$df(x).v := \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}$$

exists and if the map $df : U \times X \to Y$ is continuous. If $n > 1$ we inductively define f to be C^n if it is C^1 and df is C^{n-1} and to be C^∞ or *smooth* if it is C^n. We say that f is C^∞ or *smooth* if f is C^n for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U,Y)$ and $C^\infty(U,Y)$.

A (locally convex) *Lie group* is a group which is a smooth manifold modelled on a locally convex space such that the group operations are smooth. A locally convex Lie algebra is a Lie algebra, whose underlying vector space is locally convex and whose Lie bracket is continuous. □

Remark A.2. We have the chain rule

$$d(g\circ f)(x).v = dg\big(f(x)\big).\big(df(x).v\big)$$

and the identities $d^2f(x)(v,w) = \mathrm{pr}_2(d(Tf)(x,v).(w,0))$ (more precisely

$$d(Tf)(x,v)(w,0) = \big(df(x).w, d^2f(x)(v,w)\big)\big)$$

and

$$d(Tf)(x,v)\big(w,w'\big) = d(Tf)\big((w,0) + \big(0,w'\big)\big) = \big(df(x).w, d^2f(x)(v,w)\big) + \big(0, df(x).w'\big).$$

This implies the "chain rule" for d^2f:

$$d^2(g\circ f)(x).(v,w) = d^2g\big(f(x)\big)\big(df(x).v, df(x).w\big) + dg\big(f(x)\big).d^2f(x)(v,w). \quad (27)$$

If M is a manifold and we take the definition of the tangent bundle

$$TM := \left(\bigcup_{i\in I}\{i\}\times\varphi_i(U_i)\times X\right)\Big/\sim$$

with $(i,\varphi_i(x),v) \sim (i',\varphi_{i'}(x), d(\varphi_{i'}\circ\varphi_i^{-1})(\varphi_i(x)).v)$ if $x \in U_i \cap U_{i'}$, then the map

$$d^2f : (T_xM)^2 \to T_{f(x)}N,$$
$$\big[i,\varphi_i(x),v\big],\big[i,\varphi_i(x),w\big] \mapsto \big[j,\psi_j\big(f(x)\big), d^2\big(\psi_j\circ f\circ\varphi_i^{-1}\big)\big(\varphi_i(x)\big)(v,w)\big]$$

is well-defined according to (27). □

Definition A.3. Let G be a locally convex Lie group. The group G is said to have an *exponential function* if for each $x \in \mathfrak{g}$ the initial value problem

$$\gamma(0) = e, \qquad \gamma'(t) = T\lambda_{\gamma(t)}(e).x$$

has a solution $\gamma_x \in C^\infty(\mathbb{R}, G)$ and the function

$$\exp_G : \mathfrak{g} \to G, \qquad x \mapsto \gamma_x(1)$$

is smooth. Furthermore, if there exists a zero neighbourhood $W \subseteq \mathfrak{g}$ such that $\exp_G|_W$ is a diffeomorphism onto some open identity neighbourhood of G, then G is said to be *locally exponential*. □

Lemma A.4. *If G and G' are locally convex Lie groups with exponential function, then for each morphism $\alpha : G \to G'$ of Lie groups and the induced morphism $d\alpha(e) : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras, the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ \uparrow{\scriptstyle \exp_G} & & \uparrow{\scriptstyle \exp_{G'}} \\ \mathfrak{g} & \xrightarrow{d\alpha(e)} & \mathfrak{g}' \end{array}$$

commutes.

Remark A.5. The Fundamental Theorem of Calculus for locally convex spaces (cf. [21, Theorem 1.5]) yields that a locally convex Lie group G can have at most one exponential function (cf. [41, Lemma II.3.5]).

Typical examples of locally exponential Lie groups are Banach–Lie groups (by the existence of solutions of differential equations and the inverse mapping theorem, cf. [32]) and groups of smooth and continuous mappings from compact manifolds into locally exponential groups ([22, Section 3.2], [64]). However, diffeomorphism groups of compact manifolds are never locally exponential (cf. [41, Example II.5.13]) and direct limit Lie groups not always (cf. [23, Remark 4.7]). For a detailed treatment of locally exponential Lie groups and their structure theory we refer to [41, Section IV]. □

Remark A.6. Let X be a locally convex space. Then X is said to be Mackey-complete if each Mackey–Cauchy sequence converges in X (cf. [31, Section I.2]). In particular, sequentially complete spaces are Mackey-complete. The main reason for working with this weaker concept of completeness is that it ensures the existence of (weak) integrals of smooth curves (cf. [31, Theorem I.2.14]), even for non-complete spaces. Moreover, it implies the existence of integrals for smooth functions on cubes and standard simplices (cf. [41, Remark I.4.4]). □

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