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An almost quadratic bound on vertex Folkman numbers

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ABSTRACT

The vertex Folkman number $F(r,n,m)$, $n<m$, is the smallest integer t such that there exists a K_m-free graph of order t with the property that every r-coloring of its vertices yields a monochromatic copy of K_n. The problem of bounding the Folkman numbers has been studied by several authors. However, in the most restrictive case, when $m=n+1$, no polynomial bound has been known for such numbers. In this paper we show that the vertex Folkman numbers $F(r,n,n+1)$ are bounded from above by $O(n^2\log^4 n)$. Furthermore, for any fixed r and any small $\varepsilon>0$ we derive the linear upper bound when the cliques bigger than $(2+\varepsilon)n$ are forbidden.

1. Introduction

For a given number of colors r and integers n,m, $n<m$, the *vertex Folkman number* $F(r,n,m)$ is the smallest integer t such that there exists a K_m-free graph of order t with the property that every r-coloring of its vertices yields a monochromatic copy of K_n. J. Folkman [11] proved that for any r,n,m, $n<m$, the vertex Folkman number $F(r,n,m)$ is well-defined. Determining the precise value of $F(r,n,m)$ is not an easy problem in general. Only a few of these numbers are known and mostly they were found with the aid of computers (see, *e.g.*, [5]). Some special cases were considered for example in [14,17,18]. Obviously, since $F(r,n,m)\leqslant F(r,n,n+1)$ for any $n<m$, the most restrictive and challenging case is to determine (or more realistically to estimate) the exact value of $F(r,n,n+1)$. The upper bound on $F(r,n,n+1)$, based on Folkman's proof [11], is an iterated power function (see also Theorem 2 in [19]). Nenov [18] improved this bound and showed that for instance for 2 colors $F(2,n,n+1)=O(n!)$. This result was also proved independently by Łuczak, Ruciński and Urbański [17]. In this paper we show the following theorem.

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Theorem 1. *For a given natural number r there exists a constant $C = C(r)$ such that for every n the vertex Folkman number satisfies*

$$F(r, n, n+1) \leqslant Cn^2 \log^4 n.$$

Perhaps Theorem 1 can still be considerably improved. We show that this is the case when cliques of size bigger than $(2 + o(1))n$ are forbidden.

Theorem 2. *For a given natural number r and an arbitrarily small $\varepsilon > 0$ there exists a constant $C = C(r, \varepsilon)$ such that for every n the vertex Folkman number satisfies*

$$F\big(r, n, \lceil (2+\varepsilon)n \rceil\big) \leqslant Cn.$$

Theorem 2 complements the results of Łuczak, Ruciński and Urbański [17] and Kolev and Nenov [14] who proved that $F(r, n, r(n-1))$ is bounded from above by $r(n-1) + n^2 + 1$ and $r(n-1) + 3n + 1$, respectively,

In this paper we also investigate a more general problem. We write $H \to (G)^v_r$ if for every r-coloring of the vertices of H, there exists a monochromatic copy of G. If it is required that such a monochromatic copy should be also induced, then we write $H \underset{ind}{\longrightarrow} (G)^v_r$. Let $\omega(G)$ be the *clique number* of G, *i.e.*, the order of a maximal clique in G. J. Folkman [11] also proved that for every graph G there exists a graph H such that $H \to (G)^v_r$ and $\omega(H) = \omega(G)$. Clearly, $\omega(H) \geqslant \omega(G)$ for any graph with $H \to (G)^v_r$, and thus, Folkman's theorem is in this sense the best possible. However, Folkman's proof gives no good bound on the order of graph H. We give an alternative proof of a stronger version of Folkman's theorem (replacing $H \to (G)^v_r$ by $H \underset{ind}{\longrightarrow} (G)^v_r$) with the relatively small order of H. A related result (without controlling the clique number) was obtained by J. Brown and the second author [4]. They proved that for every natural number r there are constants c and C such that

$$cn^2 \leqslant \max_G \left\{ \min_H \left\{ \big|V(H)\big| : H \underset{ind}{\longrightarrow} (G)^v_r \right\} \right\} \leqslant Cn^2 \log^2 n, \tag{1}$$

where the maximum is taken over all graphs G of order n. With the added requirement that $\omega(H) = \omega(G)$ we were able to show the cubic upper bound only.

Theorem 3. *For a given natural number r there exists a constant $C = C(r)$ such that for every graph G of order n the following inequality holds*

$$\min\left\{ \big|V(H)\big| : H \underset{ind}{\longrightarrow} (G)^v_r \text{ and } \omega(H) = \omega(G) \right\} \leqslant Cn^3 \log^3 n.$$

The base of all logarithms in this paper is e.

2. Incidence structures

In this short section we describe some basic properties of projective planes and generalized quadrangles (for more information see [13,16,22]), which we use to prove Theorems 1 and 3.

A *projective plane* $PG(2, q)$ is an incidence structure of a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines such that:

(P1) any two points lie in a unique line,
(P2) any two lines meet in a unique point,
(P3) every line contains $q + 1$ points, and every point lies on $q + 1$ lines.

Slightly changing the first two conditions one can define a *generalized quadrangle* $Q(4, q)$ as an incidence structure of a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines such that:

(Q1) any two points lie in at most one line,
(Q2) if u is a point not on a line ℓ, then there is a unique point $w \in \ell$ collinear with u, and hence, no three lines form a triangle,
(Q3) every line contains $q+1$ points, and every point lies on $q+1$ lines.

It is known that for every prime power q such incidence structures $PG(2,q)$ and $Q(4,q)$ exist with $|\mathcal{P}| = |\mathcal{L}|$ equals q^2+q+1 and q^3+q^2+q+1, respectively.

3. Proof of Theorem 1

Fix a natural number r. Let α be a real number satisfying $0 < \alpha < 1$. We will show that for a given n there exists a graph H of order $Cn^2 \log^4 n$, $C = C(\alpha)$, such that $\omega(H) = n$ and any subgraph of H induced by a set of cardinality $\lfloor \alpha |V(H)| \rfloor$ contains a copy of K_n. Clearly, setting $\alpha = \frac{1}{r}$ will imply the statement of Theorem 1.

By Bertrand's postulate (see, *e.g.*, [10]) there is a prime number q such that

$$n \log^2 n \leqslant q+1 \leqslant 2n \log^2 n. \tag{2}$$

Let $PG(2,q)$ be a projective plane with a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines. We construct a "random graph" H with the vertex set $\mathcal{P}$. Clearly, $|V(H)| = q^2+q+1 = \Theta(n^2 \log^4 n)$. First, we partition every line $\ell \in \mathcal{L}$ into $n+1$ sets, where n of them have size

$$x = \left\lfloor \frac{2}{\alpha} \log n \right\rfloor.$$

More precisely, for each line ℓ we choose one ordered partition $\ell_0, \ell_1, \dots, \ell_n$, $|\ell_0| = q+1-nx$, $|\ell_1| = \dots = |\ell_n| = x$, randomly and uniformly from the set of all such partitions (note that since $|\ell_0| = \Theta(n \log^2 n)$, ℓ_0 is much bigger than the other parts of the partitions). Second, we join every $u \in \ell_i$ and $w \in \ell_j$, $1 \leqslant i < j \leqslant n$, by an edge obtaining a complete n-partite graph of order nx. Note that since every two points from $\mathcal{P}$ lie in a unique line the graph H is well-defined. We show that a graph H randomly chosen from the space $\mathcal{H}(n,q)$ of all such graphs has almost surely the following properties:

(i) $\omega(H) = n$, and
(ii) every set $U \subseteq V(H)$, $|U| = \lfloor \alpha |\mathcal{P}| \rfloor$, induces in H a subgraph with a copy of K_n.

First we prove (i).

3.1. *Almost all graphs $H \in \mathcal{H}(n,q)$ have the clique number n*

First we provide an auxiliary result. For a given set $S \subseteq \mathcal{P}$ let $\mathcal{L}(S) = \{\ell\colon |S \cap \ell| \geqslant 2\}$. Moreover, for a given set $T \subseteq \mathcal{P}$ let $B(T)$ be the event that the subgraph of H induced on T is a clique of size $|T|$, *i.e.*, $H[T] = K_{|T|}$.

Fact 4. *Let $S \subseteq \mathcal{P}$. Then,*

$$\Pr\bigl(B(S)\bigr) \leqslant \left(\frac{2}{\alpha \log n}\right)^{\sum_{\ell \in \mathcal{L}(S)} |S \cap \ell|}.$$

Proof. Fix a line $\ell \in \mathcal{L}(S)$. Note that if $B(S)$ happens, then we have also $B(S \cap \ell)$ for every $\ell \in \mathcal{L}(S)$. Also if $B(S \cap \ell)$ happens, then $H[S \cap \ell] = K_{|S \cap \ell|}$. That means that all elements of $S \cap \ell$ must belong to different ℓ_i, $1 \leqslant i \leqslant n$, where $\ell = \bigcup_{i=0}^n \ell_i$ is a partition of ℓ. Consequently, $\Pr(B(S \cap \ell))$ can be counted by fixing a partition $\ell_0, \ell_1, \dots, \ell_n$, and randomly selecting a subset of size $t = |S \cap \ell|$ such that $|S \cap \ell_0| = 0$ and $|S \cap \ell_i| \leqslant 1$, $1 \leqslant i \leqslant n$. Thus,

$$\Pr\bigl(B(S\cap\ell)\bigr)=\frac{\binom{n}{t}}{\binom{q+1}{t}}x^t=\frac{n-t+1}{(q+1)-t+1}\cdots\frac{n}{q+1}x^t\leqslant\left(\frac{n}{q+1}\right)^t x^t,$$

and consequently by (2),

$$\Pr\bigl(B(S\cap\ell)\bigr)\leqslant\left(\frac{x}{\log^2 n}\right)^{|S\cap\ell|}\leqslant\left(\frac{2}{\alpha\log n}\right)^{|S\cap\ell|}.$$

Clearly, $B(S)=\bigcap_{\ell\in\mathcal{L}(S)}B(S\cap\ell)$ and since all events $B(S\cap\ell)$ are independent (for different ℓ) the last inequality implies

$$\Pr\bigl(B(S)\bigr)=\prod_{\ell\in\mathcal{L}(S)}\Pr\bigl(B(S\cap\ell)\bigr)\leqslant\left(\frac{2}{\alpha\log n}\right)^{\sum_{\ell\in\mathcal{L}(S)}|S\cap\ell|}. \qquad \Box$$

Let $\gamma=\frac{2\log n}{\log\log n}$. Denote by $A_>$ and $A_\leqslant$ the events that the randomly chosen graph $H\in\mathcal{H}(n,q)$ contains a clique $[S]^2=K_{n+1}$ with the property $\sum_{\ell\in\mathcal{L}(S)}|S\cap\ell|>\gamma(n+1)$ and $\sum_{\ell\in\mathcal{L}(S)}|S\cap\ell|\leqslant\gamma(n+1)$, respectively. Clearly, the probability of containing a copy of K_{n+1} in H is bounded from above by $\Pr(A_>)+\Pr(A_\leqslant)$. If $A_>$ holds, then Fact 4 will directly imply that $\Pr(A_>)=o(1)$. But we also need to prove that $\Pr(A_\leqslant)=o(1)$. If $A_\leqslant$ occurs, then we will observe that there is a "large" line ℓ, *i.e.*, a line satisfying $|S\cap\ell|\geqslant\frac{n}{\gamma}$. Consequently, the edges of K_{n+1} belong to some large line and at least one point from a different line (which, as we will prove, is also unlikely since such a point is not typically connected to many vertices from the same line). This will imply that $\Pr(A_>)+\Pr(A_\leqslant)=o(1)$, and consequently, that almost surely $\omega(H)=n$ holds.

Proposition 5. $\Pr(A_>)=o(1)$.

Proof. Let $S\subseteq\mathcal{P}$ be a set of size $n+1$. In view of Fact 4 the probability that $H[S]=K_{n+1}$ satisfying $\sum_{\ell\in\mathcal{L}(S)}|S\cap\ell|>\gamma(n+1)$ is less than $(\frac{2}{\alpha\log n})^{\gamma(n+1)}$. Hence,

$$\Pr(A_>)<\binom{|\mathcal{P}|}{n+1}\left(\frac{2}{\alpha\log n}\right)^{\gamma(n+1)}\leqslant\left(\frac{e|\mathcal{P}|}{n+1}\right)^{n+1}\left(\frac{2}{\alpha\log n}\right)^{\gamma(n+1)}.$$

Recall that $\gamma=\frac{2\log n}{\log\log n}$. Moreover, since $|\mathcal{P}|=q^2+q+1<(q+1)^2\leqslant 4n^2\log^4 n$ (*cf.* (2)) we get

$$\begin{aligned}\Pr(A_>)&\leqslant\left(4en\log^4 n\left(\frac{2}{\alpha}\log^{-1}n\right)^{\gamma}\right)^{n+1}\\&=\exp\bigl((n+1)\bigl(\log n-\gamma\log\log n+o(\log n)\bigr)\bigr)\\&=\exp\bigl((n+1)\bigl(-\log n+o(\log n)\bigr)\bigr),\end{aligned}$$

which tends to zero as n tends to infinity. $\Box$

Proposition 6. $\Pr(A_\leqslant)=o(1)$.

Proof. Let $S\subseteq\mathcal{P}$ be a set of size $n+1$ such that $H[S]=K_{n+1}$ and $\sum_{\ell\in\mathcal{L}(S)}|S\cap\ell|\leqslant\gamma(n+1)$. Since $\sum_{\ell\in\mathcal{L}(S)}|S\cap\ell|=\sum_{u\in S}deg(u)$, where $deg(u)=|\{\ell\colon\ \ell\in\mathcal{L}(S)\text{ and }u\in\ell\}|$, there exists $u\in S$ with $deg(u)\leqslant\gamma$. Consequently, since every pair of points of S belongs to some $\ell\in\mathcal{L}(S)$, there is $\ell\in\mathcal{L}(S)$ with $u\in\ell$ such that $|S\cap\ell|\geqslant\frac{n}{\gamma}$. Moreover, since ℓ may induce only a clique of size n, there is a point $w\in S$ which does not lie in ℓ.

Set $m=\lceil\frac{n}{\gamma}\rceil\geqslant\frac{n\log\log n}{2\log n}$. Let A_{m+1} be the event that there is a set T of size $m+1$ inducing a clique K_{m+1} in which precisely m points lie in some line ℓ. Clearly, $\Pr(A_\leqslant)\leqslant\Pr(A_{m+1})$. In order to prove Proposition 6, it is enough to show that $\Pr(A_{m+1})=o(1)$. Since every such set T is associated with a line ℓ, $|T \cap \ell| = m$, and one point $w \in \mathcal{P}$, $w \notin \ell$, there are at most $|\mathcal{L}|\binom{q+1}{m}|\mathcal{P}|$ choices for T. Moreover, since $|T \cap \ell| = 2$ for precisely m lines in $\mathcal{L}(T)$ and there is one line $\ell \in \mathcal{L}(T)$ with $|T \cap \ell| = m$ we infer $\sum_{\ell \in \mathcal{L}(T)} |T \cap \ell| = 3m$. Hence, Fact 4 implies $\Pr(B(T)) \leqslant (\frac{2}{\alpha \log n})^{3m}$. Recalling (2), $q + 1 \leqslant 2n \log^2 n$, and consequently, $|\mathcal{P}| = |\mathcal{L}| \leqslant 4n^2 \log^4 n$, we obtain,

$$\begin{aligned}\Pr(A_{m+1}) &\leqslant |\mathcal{L}|\binom{q+1}{m}|\mathcal{P}| \Pr\bigl(B(T)\bigr) \\ &\leqslant 16n^4 \log^8 n \left(\frac{\mathrm{e}(q+1)}{m}\right)^m \left(\frac{2}{\alpha \log n}\right)^{3m} \\ &= 16n^4 \log^8 n \left(\frac{16\mathrm{e}n}{\alpha^3 m \log n}\right)^m \\ &\leqslant 16n^4 \log^8 n \left(\frac{32\mathrm{e}}{\alpha^3 \log\log n}\right)^{\frac{n \log\log n}{2 \log n}},\end{aligned}$$

which tends to zero as n tends to infinity. □

Now we prove property (ii).

3.2. *For almost all graphs $H \in \mathcal{H}(n, q)$ every large set induces K_n*

The proof of this statement goes along the lines of the proof of Theorem 2.2 from [4].

For $U \subseteq V(H)$ with cardinality $|U| = \lfloor \frac{1}{r}|\mathcal{P}| \rfloor = \lfloor \alpha|\mathcal{P}| \rfloor$ let $C(U)$ be the event that K_n is not a subgraph of $H[U]$. Clearly, $C(U)$ implies $C(\ell \cap U)$ for each $\ell \in \mathcal{L}$. Consequently,

$$C(U) \subseteq \bigcap_{\ell \in \mathcal{L}} C(\ell \cap U),$$

and since all events $C(\ell \cap U)$ are independent,

$$\Pr\bigl(C(U)\bigr) \leqslant \prod_{\ell \in \mathcal{L}} \Pr\bigl(C(\ell \cap U)\bigr). \tag{3}$$

For a fixed line $\ell \in \mathcal{L}$ we bound from above the probability that $C(\ell \cap U)$ occurs. Note that if $C(\ell \cap U)$ occurs, then for some $i = i(\ell)$, $1 \leqslant i \leqslant n$, in partition $\ell = \bigcup_{j=0}^n \ell_j$, $\ell_i \cap U = \emptyset$, *i.e.*, ℓ_i and U are disjoint. Let $|U \cap \ell| = u_\ell$. The probability that for a fixed i, $1 \leqslant i \leqslant n$, $U \cap \ell_i = \emptyset$ equals to the probability that for a fixed partition $\ell = \bigcup_{i=0}^n \ell_i$ randomly chosen subset T with $|T| = u_\ell$ satisfies $T \cap \ell_i = \emptyset$. Hence,

$$\Pr\bigl(C(\ell \cap U)\bigr) \leqslant n \frac{\binom{q+1-x}{u_\ell}}{\binom{q+1}{u_\ell}} \leqslant n \exp\left(-\frac{x u_\ell}{q+1}\right),$$

since for any natural numbers a, b, c satisfying $a - b \geqslant c$ the following is true

$$\frac{\binom{a-b}{c}}{\binom{a}{c}} = \frac{(a-b)-(c-1)}{a-(c-1)} \cdots \frac{a-b}{a} \leqslant \left(\frac{a-b}{a}\right)^c \leqslant \exp\left(-\frac{bc}{a}\right).$$

Consequently, (3) yields,

$$\Pr\bigl(C(U)\bigr) \leqslant n^{|\mathcal{L}|} \exp\left(-\frac{x}{q+1} \sum_{\ell \in \mathcal{L}} u_\ell\right).$$

Moreover, since every point in U belongs to exactly $q + 1$ lines

$$\sum_{\ell \in \mathcal{L}} u_\ell = \sum_{\ell \in \mathcal{L}} |U \cap \ell| = |U|(q+1).$$

Hence,

$$\Pr\big(C(U)\big) \leqslant n^{|\mathcal{L}|} \exp\big(-x|U|\big) = n^{|\mathcal{P}|} \exp\big(-x|U|\big).$$

This implies that

$$\Pr\Big(\bigcup_U C(U)\Big) \leqslant \binom{|\mathcal{P}|}{\lfloor \alpha|\mathcal{P}|\rfloor} n^{|\mathcal{P}|} \exp\big(-x|U|\big) \leqslant \left(\frac{\mathrm{e}}{\alpha}\right)^{\alpha|\mathcal{P}|} n^{|\mathcal{P}|} \exp\big(-x|U|\big),$$

where the union is taken over all subsets $U \subseteq V(H)$ with cardinality $\lfloor \alpha|\mathcal{P}|\rfloor$. Finally, since $x|U| = \lfloor \frac{2}{\alpha}\log n\rfloor \lfloor \alpha|\mathcal{P}|\rfloor \geqslant \frac{3}{2}|\mathcal{P}|\log n$, we get

$$\Pr\Big(\bigcup_U C(U)\Big) \leqslant \exp\Big(|\mathcal{P}|\Big(-\frac{1}{2}\log n + o(\log n)\Big)\Big),$$

which tends to zero as n tends to infinity.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let r and $\varepsilon > 0$ be given. Set $\alpha = \frac{1}{r}$. In order to prove Theorem 2, we show that with a positive probability a binomial random graph $G(m, 1-p)$, $p = \frac{c}{m}$ and $c = c(r,\varepsilon)$, does not contain a clique of size bigger or equal than $\frac{2\log c}{c}m$ and every set of size $\lceil \alpha m\rceil$ induces a clique of size at least $\frac{2\log c}{(2+\varepsilon)c}m$. This will imply that for m sufficiently large

$$F\left(r, \left\lceil \frac{2\log c}{(2+\varepsilon)c}m\right\rceil, \left\lceil \frac{2\log c}{c}m\right\rceil\right) \leqslant m,$$

and consequently, setting $n = \lceil \frac{2\log c}{(2+\varepsilon)c}m\rceil$ yields that

$$F\big(r, n, \lceil (2+\varepsilon)n\rceil\big) \leqslant \frac{(2+\varepsilon)c}{2\log c}n$$

holds for sufficiently large n. Indeed, since

$$n = \left\lceil \frac{2\log c}{(2+\varepsilon)c}m\right\rceil \geqslant \frac{2\log c}{(2+\varepsilon)c}m,$$

we get

$$\frac{(2+\varepsilon)c}{2\log c}n \geqslant m,$$

and

$$(2+\varepsilon)n \geqslant \frac{2\log c}{c}m, \quad \text{and hence,} \quad \lceil (2+\varepsilon)n\rceil \geqslant \left\lceil \frac{2\log c}{c}m\right\rceil.$$

Thus,

$$F\big(r, n, \lceil (2+\varepsilon)n\rceil\big) \leqslant F\left(r, n, \left\lceil \frac{2\log c}{c}m\right\rceil\right) \leqslant m \leqslant \frac{(2+\varepsilon)c}{2\log c}n.$$

First, we are going to observe that with a positive probability a random graph $G = G(m, \frac{c}{m})$, where c is a sufficiently large constant, satisfies the following three properties:

(i) for every subset $U \subseteq V(G)$, $|U| = \lceil \alpha m\rceil$, the number of edges induced by U is bounded by $||E(G[U])| - p\binom{|U|}{2}| \leqslant \frac{\varepsilon\alpha}{3}p\binom{|U|}{2}$,

(ii) the independence number of G is less than $\frac{2\log c}{c}m$,
(iii) G is a triangle-free graph.

The first two properties hold with the probability tending to one. Part (i) follows from the standard application of Chernoff's inequality (see, *e.g.*, Section II.3 in [2]).

Part (ii) is a special case of a result of Frieze [12] about the asymptotic concentration of the independence number in a random graph.

Lemma 7. *(See [12].) For every $\varepsilon > 0$ there is a constant c_ε such that for $c = pm$ if $c_\varepsilon \leqslant c = o(m)$ then*

$$\left|\alpha\big(G(m,p)\big) - \frac{2m}{c}(\log c - \log\log c - \log 2 + 1)\right| \leqslant \frac{\varepsilon m}{c}$$

asymptotically almost surely.

Finally, note that the property (iii) holds with the probability $\exp(-c^3/6)$ as m tends to infinity. To see it, recall that the number of triangles in G has a Poisson distribution with mean $\frac{c^3}{6}$ (see, *e.g.*, Section 10.1 in [1]). Hence, there exists a graph G of order m which satisfies simultaneously (i)–(iii). To complete the proof of Theorem 2, it remains to show that every set of size $\lceil \alpha m\rceil$ induces in G a graph with the independence number at least $\frac{2\log c}{(2+\varepsilon)c}m$.

Now we are going to show that for any subset $U \subseteq V(G)$, $|U| = \lceil \alpha m\rceil$, the graph $G[U]$ contains a (large) independent set of size $\frac{2\log c}{(2+\varepsilon)c}m$.

Property (i) yields that the average degree of $G[U]$, say d, satisfies

$$\left(\alpha - \frac{\varepsilon\alpha}{3}\right)c \leqslant d \leqslant \left(\alpha + \frac{\varepsilon\alpha}{3}\right)c.$$

Moreover, property (iii) implies that G, and hence also $G[U]$, is triangle-free. Now we apply the following Shearer's result for triangle-free graphs.

Lemma 8. *(See [20].) Let $G = (V, E)$ be a triangle-free graph with the average degree d. Moreover, let*

$$f(d) = \frac{d\log d - d + 1}{(d-1)^2},$$

where $f(0) = 1$ and $f(1) = \frac{1}{2}$. Then

$$\alpha(G) \geqslant f(d)\big|V(G)\big|.$$

Lemma 8 implies that the independence number of $G[U]$ is at least

$$\frac{d\log d - d + 1}{(d-1)^2}\alpha m \geqslant \frac{\log d - 1}{d-1}\alpha m. \tag{4}$$

Note that for c large enough

$$\frac{\log d - 1}{d-1}\alpha \geqslant \frac{\log((\alpha - \frac{\varepsilon\alpha}{3})c) - 1}{(\alpha + \frac{\varepsilon\alpha}{3})c - 1}\alpha \geqslant \frac{2\log c}{(2+\varepsilon)c}. \tag{5}$$

Hence, by Lemma 8, (4) and (5) there exists an independent set in $G[U]$ of size at least

$$\frac{d\log d - d + 1}{(d-1)^2}\alpha m \geqslant \frac{2\log c}{(2+\varepsilon)c}m.$$

This completes the proof of Theorem 2.

5. Proof of Theorem 3

The proof of Theorem 3 was already given by the authors in an extended abstract [6]. Here for completeness we sketch this proof once more.

Fix a natural number r and set $\alpha = \frac{1}{r}$. We will show that there exists a graph H of order $Cn^3 \log^3 n$, $C = C(r)$, such that $\omega(H) = \omega(G)$ and any subgraph of H induced by a set of cardinality $\lfloor \alpha |V(H)| \rfloor$ contains an induced copy of G.

Bertrand's postulate yields the existence of a prime number q for which

$$\frac{3}{\alpha} n \log n \leqslant q + 1 \leqslant \frac{6}{\alpha} n \log n.$$

Let x be an integer satisfying

$$q + 1 = xn + m, \tag{6}$$

where $0 \leqslant m < n$. Consequently,

$$\frac{2}{\alpha} \log n \leqslant x \leqslant \frac{6}{\alpha} \log n.$$

Let $Q(4, q)$ be a generalized quadrangle with a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines. We construct a "random graph" H with the vertex set $\mathcal{P}$. Hence, $|V(H)| = q^3 + q^2 + q + 1 = \Theta(n^3 \log^3 n)$. In view of (6) one can partition each line into n sets of size x and $x + 1$, respectively. For each line ℓ we choose one ordered partition $\ell_1, \dots, \ell_n$, $x \leqslant |\ell_i| \leqslant x + 1$, $1 \leqslant i \leqslant n$, randomly and uniformly from the set of all such partitions (this time there is no ℓ_0). Let $V(G) = \{v_1, \dots, v_n\}$. For each $u \in \ell_i$ and $w \in \ell_j$ we join $\{u, w\}$ by an edge if and only if $\{v_i, v_j\} \in E(G)$. Note that H is well-defined because of condition (Q1). Moreover, condition (Q2) yields that $\omega(H) = \omega(G)$. In fact, more is true: every triangle of H is contained entirely within some ℓ. In other words, all edges are inside a line and there is no triangle of lines.

The rest of the proof goes along the lines of Section 3.2 with a new setup introduced above.

6. Concluding remarks

With some additional work (see the approach taken in [8]) one can reduce the factor $n^2 \log^4 n$ from Theorem 1 to $n^2 \log^2 n$ and similarly the factor $n^3 \log^3 n$ from Theorem 3 to $n^3 \log n$. However, since we were unable to find any nontrivial lower bound on $F(r, n, n + 1)$, we chose to include a somewhat simpler argument presented in this paper. It would be interesting to decide if the ratio $\frac{F(r,n,n+1)}{n}$ tends to infinity together with n.

One can also ask about the value of $F(r, s, s + 1)$ for a fixed value of $s \geqslant 3$ as r tends to infinity. This question was already considered by several researchers in a different setting. For fixed integers $2 \leqslant t < u$ let

$$f_{t,u}(n) = \min\bigl\{\max\bigl\{T\colon\ T \subseteq V(G) \text{ and } T \text{ spans no } K_t\bigr\}\bigr\},$$

where the minimum is taken over all K_u-free graphs G of order n. Function $f_{t,u}$ was introduced by Erdős and Rogers [9], and further examined by Bollobás and Hind [3], Krivelevich [15], and Sudakov [21]. It is known that for a fixed s we have

$$\Omega\bigl(n^{\frac{1}{2}+o(1)}\bigr) \leqslant f_{s,s+1}(n) \leqslant O\bigl(n^{\frac{s}{s+2}+o(1)}\bigr) \tag{7}$$

(see, *e.g.*, [3,15,21]). The lower bound states that every K_{s+1}-free graph G of order n contains an induced subgraph with $n^{\frac{1}{2}+o(1)}$ vertices with no K_s. We remove the vertex set of such a K_s-free subgraph from $V(G)$ and apply the lower bound of (7) again. Repeating this argument we eventually obtain a partition of $V(G)$ into $r = n^{\frac{1}{2}+o(1)}$ parts (with each part inducing K_s-free subgraph). Hence,

$$\Omega\bigl(r^{2+o(1)}\bigr) \leqslant F(r, s, s + 1).$$

The upper bound of (7) implies that there is a K_{s+1}-free graph G of order n such that any subset of vertices of size $O(n^{\frac{s}{s+2}+o(1)})$ contains a copy of K_s. Consequently, for any partition of G into

$$r = \frac{n}{O(n^{\frac{s}{s+2}+o(1)})} = O\left(n^{\frac{2}{s+2}+o(1)}\right)$$

parts, one of the parts contains K_s. This implies that

$$F(r, s, s+1) \leqslant O\left(r^{\frac{s+2}{2}+o(1)}\right) \tag{8}$$

holds.[2]

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[2] Recently we improved the upper bound (8). With the proof essentially same as in Theorem 3 one can show that $F(r, s, s+1) \leqslant O(r^3)$ and $f_{s,s+1} \leqslant O(r^{2/3})$. Some further refinement of the above bounds are discussed in [7].