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Example of a weighted algebra $\mathcal{L}_p^w(G)$ on an uncountable discrete group ☆

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ABSTRACT

We construct examples of weighted algebras $\mathcal{L}_p^w(G)$ with $1 < p \leqslant 2$ on uncountable free groups. For $p > 2$ no weighted algebras exist on these groups. On the other hand, we prove that for any amenable locally compact group G, if $\mathcal{L}_p^w(G)$ is an algebra for some weight w and $p > 1$, then G is σ-compact.

Throughout the paper, let G be a locally compact group with a fixed Haar measure μ. Let $w > 0$ be a measurable function. We consider weighted spaces $\mathcal{L}_p^w(G)$ with $p \geqslant 1$, which are by definition equal to

$$\mathcal{L}_p^w(G) = \left\{ f\colon \int_G |fw|^p < \infty \right\}.$$

Norm of a function f in this space is $\|f\|_{p,w} = \|fw\|_p = (\int_G |fw|^p)^{1/p}$.

Sufficient conditions are well-known, under which a weighted space becomes an algebra with respect to usual convolution,

$$(f * g)(t) = \int_G f(s)g\left(s^{-1}t\right) ds.$$

For $p = 1$ it is submultiplicativity [3]:

$$w(st) \leqslant w(s)w(t), \tag{1}$$

and for $p > 1$ the following inequality [10] (pointwise locally almost everywhere):

$$w^{-q} * w^{-q} \leqslant w^{-q}. \tag{2}$$

Here q is the conjugate exponent to p, so that $1/p + 1/q = 1$. Commutativity of such an algebra is equivalent to that of the group.

Natural example of a weighted algebra is given by the usual algebra of summable functions $\mathcal{L}_1(G)$ with the trivial weight $w \equiv 1$. But if $p > 1$, existence of weighted algebras on a group G is a nontrivial question. For example, the usual spaces $\mathcal{L}_p(G)$ are closed under convolution on compact groups only; this is the positive solution of renowned L_p-conjecture, see final result in Saeki [9].

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We are interested in possibly full description of the class $\mathcal{WL}_p$ of groups on which weighted algebras $\mathcal{L}_p^w(G)$ with $p > 1$ exist. It has long been known that the real line belongs to this class [10] (one may take, e.g., $w(t) = 1 + t^2$ for any $p > 1$), as well as $\mathbb{Z}$ and $\mathbb{R}^n$. In [7] it is proved that this class contains all σ-compact groups (i.e. representable as a countable union of compact sets). It is also proved [7] that for abelian groups σ-compactness is necessary for containment in this class.

In the present paper we show that $\mathcal{WL}_p$ for $1 < p \leqslant 2$ contains some non-σ-compact groups. Theorem 1.1 in presents a construction of weighted algebras on an uncountable discrete group; note that in discrete case σ-compactness is the same as countability. For $p > 2$ there are no weighted algebras on these groups, so that the classes $\mathcal{WL}_p$ are different for different p.

Section 2 treats weighted algebras with involution. The results of this section allow us to extend to the class of amenable groups Theorem 1.1 in [7], claiming that σ-compactness is necessary for the existence of weighted algebras. This extension is carried out in Section 3. Of course, groups in Theorem 1.1 are not amenable—they are free groups with uncountable set of generators.

The examples of Theorem 1.1 give also solution to the following problem. It was known before [6] that inequality (2) may not hold for a weight of an algebra with $p > 1$. But in the examples of [6] algebras are not translation invariant, and weights are not locally bounded. Theorem 1.1 shows that even these additional assumptions do not imply (2).

Weighted algebras on locally compact groups with $p = 1$, which are also called Beurling algebras, have been studied recently by Dales and Lau in [1]. In particular, this memoir contains close results on symmetric algebras and algebras on free groups.

1. Example of a weighted algebra on a free group

Theorem 1.1. *On a free group F of any infinite cardinality there is a weight w such that the weighted space $\mathcal{L}_p^w(F)$ for all p, $1 < p \leqslant 2$ is a Banach algebra with respect to convolution.*

Proof. Denote by A the set of generators of F. We consider elements of F as reduced words in the alphabet $A \cup A^{-1}$. Let $|\alpha|$ denote the length of a word $\alpha \in F$, and put $A_n = \{\alpha\colon\ |\alpha| = n\}$, $n \geqslant 0$. We define

$$w|_{A_n} = (n+1)^3$$

(any number > 2 may be taken instead of 3).

In this example we denote $\ell_p = \mathcal{L}_p(F)$, $\ell_p^w = \mathcal{L}_p^w(F)$. It is convenient to fix the Haar measure so that the measure of each point is equal to 1. Since $\ell_p^w = \ell_p/w$, it is sufficient to take $f, g \in \ell_p$ and show that

$$h = w \cdot \left(\frac{f}{w} * \frac{g}{w}\right) \in \ell_p.$$

We may assume that f, g are nonnegative. For any α

$$h(\alpha) = w(\alpha) \cdot \left(\frac{f}{w} * \frac{g}{w}\right)(\alpha) = w(\alpha) \sum_\beta \frac{f_\beta g_{\beta^{-1}\alpha}}{w(\beta) w(\beta^{-1}\alpha)}.$$

Let $\alpha \in A_n$, $\beta \in A_k$.

(1) If $k \geqslant n/2$, then $w(\beta) \geqslant (n/2+1)^3 \geqslant (n+1)^3/8 = w(\alpha)/8$, and $w(\alpha)/w(\beta) \leqslant 8$.
(2) If $k < n/2$, then $\beta^{-1}\alpha$ contains no less than $n-k$ letters, so that $w(\beta^{-1}\alpha) \geqslant (n-k+1)^3 > (n/2+1)^3 \geqslant w(\alpha)/8$, and then $w(\alpha)/w(\beta^{-1}\alpha) \leqslant 8$.

Thus, we may write an estimate $h(\alpha) \leqslant 8\varphi(\alpha) + 8\psi(\alpha)$, where

$$\varphi(\alpha) = \sum_\beta \frac{f_\beta g_{\beta^{-1}\alpha}}{w(\beta)},$$

$$\psi(\alpha) = \sum_\beta \frac{f_\beta g_{\beta^{-1}\alpha}}{w(\beta^{-1}\alpha)} = \sum_\beta \frac{f_{\alpha\gamma^{-1}} g_\gamma}{w(\gamma)}.$$

One may see later that φ and ψ are estimated similarly, and it will suffice to estimate φ alone.

$$\varphi(\alpha) = \sum_\beta \frac{f_\beta g_{\beta^{-1}\alpha}}{w(\beta)} = \sum_{k=0}^{\infty} \sum_{\beta \in A_k} \frac{f_\beta g_{\beta^{-1}\alpha}}{(k+1)^3}.$$

Each set A_k may be split into disjoint sets $A_{kj}(\alpha)$, $0 \leqslant j \leqslant k$, by number of common letters in α and β:

$$A_{kj}(\alpha) = \{\beta \in A_k\colon\ \beta_1 = \alpha_1,\ \ldots,\ \beta_j = \alpha_j,\ \beta_{j+1} \neq \alpha_{j+1}\}.$$

If $j > n = |\alpha|$, we assume that $A_{kj}(\alpha) = \emptyset$. Thus, $A_k = \bigcup_{j=0}^{k} A_{kj}(\alpha)$, so that

$$\varphi(\alpha) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{\beta\in A_{kj}(\alpha)} \frac{f_\beta g_{\beta^{-1}\alpha}}{(k+1)^3} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\varphi_{kj}(\alpha)}{(k+1)^3},$$

where

$$\varphi_{kj}(\alpha) = \sum_{\beta\in A_{kj}(\alpha)} f_\beta g_{\beta^{-1}\alpha}.$$

In other words,

$$\varphi = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\varphi_{kj}}{(k+1)^3}.$$

Estimate now the norms $\|\varphi_{kj}\|_p$, taking into account that $\varphi_{kj}(\alpha) = 0$ when $n = |\alpha| < j$. Now, and only now, we use the fact that $p \leqslant 2$: it implies that $\|x\|_q \leqslant \|x\|_p$ for the conjugate exponent q and all $x \in \ell_p$

$$\|\varphi_{kj}\|_p^p = \sum_{n=0}^{\infty}\sum_{\alpha\in A_n} \varphi_{kj}(\alpha)^p = \sum_{n=j}^{\infty}\sum_{\alpha\in A_n} \varphi_{kj}(\alpha)^p = \sum_{n=j}^{\infty}\sum_{\alpha\in A_n}\left(\sum_{\beta\in A_{kj}(\alpha)} f_\beta g_{\beta^{-1}\alpha}\right)^p \leqslant \sum_{n=j}^{\infty}\sum_{\alpha\in A_n}\sum_{\beta\in A_{kj}(\alpha)} f_\beta^p \sum_{\gamma\in A_{kj}(\alpha)} g_{\gamma^{-1}\alpha}^p.$$

A word $\alpha \in A_n$ may be represented in the form $\alpha = \hat\alpha\check\alpha$, where $\hat\alpha = \alpha_1\ldots\alpha_j \in A_j$, $\check\alpha = \alpha_{j+1}\ldots\alpha_n \in A_{n-j}$ and $\alpha_{j+1} \neq \alpha_j^{-1}$. Then every $\beta \in A_{kj}(\alpha)$ has form $\beta = \hat\alpha\hat\beta$, where $\hat\beta = \beta_{j+1}\ldots\beta_k \in A_{k-j}$ and $\beta_{j+1} \neq \alpha_j^{-1}$, $\beta_{j+1} \neq \alpha_{j+1}$. Similar representation holds for $\gamma = \hat\alpha\hat\gamma$, so that $\gamma^{-1}\alpha = \hat\gamma^{-1}\check\alpha$. Thus,

$$\begin{aligned}
\|\varphi_{kj}\|_p^p &\leqslant \sum_{n=j}^{\infty}\sum_{\hat\alpha\in A_j}\sum_{\substack{\check\alpha\in A_{n-j}\\ \alpha_{j+1}\neq\alpha_j^{-1}}}\ \sum_{\substack{\hat\beta\in A_{k-j}\\ \beta_{j+1}\neq\alpha_j^{-1},\, \beta_{j+1}\neq\alpha_{j+1}}} f_{\hat\alpha\hat\beta}^p \sum_{\substack{\hat\gamma\in A_{k-j}\\ \gamma_{j+1}\neq\alpha_{j+1},\, \gamma_{j+1}\neq\alpha_j^{-1}}} g_{\hat\gamma^{-1}\check\alpha}^p \\
&\leqslant \sum_{n=j}^{\infty}\sum_{\hat\alpha\in A_j}\sum_{\substack{\hat\beta\in A_{k-j}\\ \beta_{j+1}\neq\alpha_j^{-1}}} f_{\hat\alpha\hat\beta}^p \sum_{\check\alpha\in A_{n-j}}\sum_{\substack{\hat\gamma\in A_{k-j}\\ \gamma_{j+1}\neq\alpha_{j+1}}} g_{\hat\gamma^{-1}\check\alpha}^p \\
&= \sum_{n=j}^{\infty}\sum_{\xi=\hat\alpha\hat\beta\in A_k} f_\xi^p \sum_{\eta=\hat\gamma^{-1}\check\alpha\in A_{n+k-2j}} g_\eta^p \\
&\leqslant \|f\|_p^p \sum_{n=j}^{\infty}\sum_{\eta\in A_{n+k-2j}} g_\eta^p \leqslant \|f\|_p^p\cdot\|g\|_p^p.
\end{aligned}$$

That is $\|\varphi_{kj}\|_p \leqslant \|f\|_p\cdot\|g\|_p$. Hence

$$\|\varphi\|_p \leqslant \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{\|\varphi_{kj}\|_p}{(k+1)^3} \leqslant \sum_{k=0}^{\infty}(k+1)\frac{\|f\|_p\cdot\|g\|_p}{(k+1)^3} = \sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\|f\|_p\cdot\|g\|_p \equiv C\|f\|_p\cdot\|g\|_p.$$

The norm $\|\psi\|_p$ is estimated similarly, replacing sum over β with that over $\gamma = \beta^{-1}\alpha$. Thus, $\|h\|_p \leqslant 16C\|f\|_p\cdot\|g\|_p$, what was to show. □

It is easy to show that examples of this sort are impossible for $p > 2$:

Proposition 1.2. *If $\mathcal{L}_p^w(G)$ is an algebra and G is a discrete uncountable group, then $p \leqslant 2$.*

Proof. Since G is uncountable, for some $C > 0$ the set $A = \{x\colon\ \max(w(x), w(x^{-1}) \leqslant C\}$ is also uncountable. Note that $A = A^{-1}$. Now, if $f \in \ell_p(A)$, then

$$\|f\|_{p,w}^p = \sum_{x\in A}\bigl|f(x)w(x)\bigr|^p \leqslant C\sum_{x\in A}\|f\|_p^p,$$

so that $\ell_p(A) \subset \mathcal{L}_p^w(G)$. In particular,

$$\sum_{x\in A} f(x)g\bigl(x^{-1}\bigr) = (f*g)(e) < \infty$$

for all nonnegative $f, g \in \ell_p(A)$. We have now $\ell_p(A)\cdot\ell_p(A) \subset \ell_1(A)$, what is possible for $p \leqslant 2$ only. □

2. Symmetric weighted algebras

Let G be a locally compact group. Then natural involution is not defined in all algebras $\mathcal{L}_p^w(G)$, but in some of them which we call symmetric. After a series of lemmas we prove that symmetric algebras possess an injective involutive representation in $\mathcal{L}_2(G)$, and are as a consequence semisimple. The fact that commutative, not necessarily symmetric algebras are semisimple, was proved in [6, Theorem 4].

In this section, Δ will denote the modular function of G. We use also notation $^\triangledown$: $f^\triangledown(t) = f(t^{-1})$.

Definition 2.1. Define

$$f^*(t) = \bar{f}\left(t^{-1}\right)\Delta\left(t^{-1}\right). \tag{3}$$

A space $\mathcal{L}_p^w(G)$ is called *symmetric*, if $f^* \in \mathcal{L}_p^w(G)$ and $\|f\|_{p,w} = \|f^*\|_{p,w}$ for all $f \in \mathcal{L}_p^w(G)$.

The set of all functions of the type f^*, where $f \in \mathcal{L}_p^w(G)$, constitutes a weighted space $\mathcal{L}_p^{\widetilde{w}}(G)$ with the weight $\widetilde{w} = \Delta^{(p-1)/p} w^\triangledown$, whereas $\|f^*\|_{p,w} = \|f\|_{p,\widetilde{w}}$:

$$\|f^*\|_{p,w}^p = \int \left|\bar{f}\left(t^{-1}\right)\Delta\left(t^{-1}\right)w(t)\right|^p dt = \int \left|f(t)w\left(t^{-1}\right)\right|^p \Delta(t)^{p-1}\,dt = \|f\|_{p,\widetilde{w}}^p.$$

Sufficient condition for symmetry is equality $w = \widetilde{w}$, which turns to usual evenness in the case of $p = 1$ or a unimodular group.

Lemma 2.2. *If the space $\mathcal{L}_p^w(G)$ is an algebra, then the space $\mathcal{L}_p^{\widetilde{w}}(G)$ with the weight $\widetilde{w} = \Delta^{(p-1)/p} w^\triangledown$ is also an algebra.*

Proof. This follows from the fact that $f^* * g^* = (g * f)^*$ for all $f, g \in \mathcal{L}_p^w(G)$. Moreover, obviously, $\|\varphi * \psi\|_{p,\widetilde{w}} \leqslant \|\varphi\|_{p,\widetilde{w}} \|\psi\|_{p,\widetilde{w}}$ for all $\varphi, \psi \in \mathcal{L}_p^{\widetilde{w}}(G)$. $\square$

Lemma 2.3. *If the spaces $\mathcal{L}_p^w(G)$, $\mathcal{L}_p^v(G)$ are both algebras, then the space $\mathcal{L}_p^u(G)$ with the weight $u = \max\{w, v\}$ is also an algebra.*

Proof. This follows from the fact that $\mathcal{L}_p^u(G) = \mathcal{L}_p^w(G) \cap \mathcal{L}_p^v(G)$. $\square$

Lemma 2.4. *If the space $\mathcal{L}_p^w(G)$ is an algebra, there is a weight v on G such that the space $\mathcal{L}_p^v(G)$ is a symmetric algebra.*

Proof. According to Lemmas 2.2, 2.3, the proper weight is

$$v = \max\left\{w, \Delta^{-(p-1)/p} w^\triangledown\right\}. \qquad \square$$

Lemma 2.5. *If $\mathcal{L}_p^w(G)$ is a symmetric algebra, then $\|f * g\|_{q,w^{-1}} \leqslant \|f\|_{p,w} \cdot \|g\|_{q,w^{-1}}$ for all $f \in \mathcal{L}_p^w(G)$, $g \in \mathcal{L}_q^{w^{-1}}(G)$.*

Proof. Take arbitrary $f, \varphi \in \mathcal{L}_p^w(G)$, $g \in \mathcal{L}_q^{w^{-1}}(G)$ and transform the following expression:

$$\begin{aligned}
\langle \varphi, f * g\rangle &= \int_G \int_G \varphi(t) f(s) g\left(s^{-1}t\right) ds\,dt = \int_G \int_G \varphi(su) f(s) g(u)\,ds\,du \\
&= \int_G \int_G \varphi\left(s^{-1}u\right) f\left(s^{-1}\right)\Delta\left(s^{-1}\right) g(u)\,ds\,du \\
&= \int_G \int_G \varphi\left(s^{-1}u\right) \bar{f}^*(s) g(u)\,ds\,du = \left\langle \bar{f}^* * \varphi, g\right\rangle.
\end{aligned}$$

Since the spaces $\mathcal{L}_q^{w^{-1}}(G)$ are $\mathcal{L}_p^w(G)$ conjugate to each other, and their natural norms coincide with those in the sense of conjugate spaces

$$\|f * g\|_{q,w^{-1}} = \sup_{\varphi \neq 0} \frac{|\langle \varphi, f * g\rangle|}{\|\varphi\|_{p,w}} = \sup_{\varphi \neq 0} \frac{|\langle \bar{f}^* * \varphi, g\rangle|}{\|\varphi\|_{p,w}} \leqslant \sup_{\varphi \neq 0} \frac{\|\bar{f}^* * \varphi\|_{p,w} \|g\|_{q,w^{-1}}}{\|\varphi\|_{p,w}}.$$

Using the inequality $\|\bar{f}^* * \varphi\|_{p,w} \leqslant \|\bar{f}^*\|_{p,w} \cdot \|\varphi\|_{p,w}$, and symmetricity of $\mathcal{L}_p^w(G)$, we get:

$$\|f * g\|_{q,w^{-1}} \leqslant \|\bar{f}^*\|_{p,w} \|g\|_{q,w^{-1}} = \|f\|_{p,w} \|g\|_{q,w^{-1}},$$

what was to show. $\square$

Operation $f \mapsto f^*$ is an involution on a symmetric algebra. Similarly to the case of usual group algebra, this yields a representation in $\mathcal{L}_2(G)$. In the proof of the next theorem ideas of Kerman and Sawyer [5] are used.

Theorem 2.6. *If $\mathcal{L}_p^w(G)$ is a symmetric algebra, then*:

(1) *Convolution $f * g$ is defined and belongs to $\mathcal{L}_2(G)$ for all $f \in \mathcal{L}_p^w(G)$, $g \in \mathcal{L}_2(G)$, and $\|f * g\|_2 \leqslant \|f\|_{p,w}\|g\|_2$;*
(2) *Convolution operator $T_f(g) = f * g$ on $\mathcal{L}_2(G)$ is bounded, and $\|T_f\| \leqslant \|f\|_{p,w}$;*
(3) *The map $f \mapsto T_f$ is an injective involutive representation of $\mathcal{L}_p^w(G)$.*

Proof. It is enough to prove (1) for nonnegative f and g. As any summable function, g^2 may be represented in the form $g^2 = g_1 g_2$, where $g_1 = g^{2/p} \in \mathcal{L}_p(G)$, $g_2 = g^{2/q} \in \mathcal{L}_q(G)$. Put $\varphi = g_1/w \in \mathcal{L}_p^w(G)$, $\psi = g_2 w \in \mathcal{L}_q^{w^{-1}}(G)$. Then $g^2 = \varphi\psi$ and $\|\varphi\|_{p,w} = \|\psi\|_{q,w^{-1}} = \|g\|_2$. Now in every point $s \in G$, using Cauchy–Bunyakovskii inequality, we have:

$$(f * g)(s) = \int_G f(t)\sqrt{\varphi(t^{-1}s)\psi(t^{-1}s)}\,dt \leqslant \left(\int_G f(t)\varphi(t^{-1}s)\,dt \int_G f(r)\psi(r^{-1}s)\,dr\right)^{1/2} = \sqrt{(f * \varphi)(s)(f * \psi)(s)},$$

and, with account of Lemma 2.5,

$$\begin{aligned}\|f * g\|_2^2 &= \int_G \left|(f * g)(s)\right|^2 ds \leqslant \int_G (f * \varphi)(s)(f * \psi)(s)\,ds \leqslant \|f * \varphi\|_{p,w}\|f * \psi\|_{q,w^{-1}}\\ &\leqslant \|f\|_{p,w}^2\|\varphi\|_{p,w}\|\psi\|_{q,w^{-1}} = \|f\|_{p,w}^2\|g\|_2^2.\end{aligned}$$

Thus (1) is proved, and (2) follow trivially.
Proof of (3) is identical to the case of $\mathcal{L}_1(G)$, see, e.g., [8, §28]. □

Corollary 2.7. *Every symmetric algebra $\mathcal{L}_p^w(G)$ is semisimple.*

3. Corollaries for amenable groups

We prove first a weighted analog of the theorem of Żelazko [11].

Lemma 3.1. *Let $\mathcal{L}_p^w(G)$ be a symmetric algebra. Then for any sets A, B of positive finite measure*

$$\mu(AB) \geqslant \mu(B)\|I_A\|_{q,w^{-1}}^2.$$

Proof. Pick any A and B (of finite measure). Using simple properties of convolution, Cauchy–Bunyakovskii inequality and Theorem 2.6, we get:

$$\begin{aligned}\|I_A\|_{q,w^{-1}}^q\|I_B\|_2^2 &= \int_A w^{-q} \cdot \int_B 1 = \int_G I_A w^{-q} \cdot \int_G I_B = \int_G (I_A w^{-q}) * I_B = \int_{AB} (I_A w^{-q}) * I_B\\ &\leqslant \|I_{AB}\|_2 \left\|(I_A w^{-q}) * I_B\right\|_2 \leqslant \|I_{AB}\|_2 \left\|I_A w^{-q}\right\|_{p,w}\|I_B\|_2\\ &= \|I_{AB}\|_2\left(\int_A w^{-pq+p}\right)^{1/p}\|I_B\|_2 = \|I_{AB}\|_2\left(\int_A w^{-q}\right)^{1/p}\|I_B\|_2\\ &= \|I_{AB}\|_2\|I_A\|_{q,w^{-1}}^{q/p}\|I_B\|_2\end{aligned}$$

(as $-pq + p = -q$). Hence, as $q - q/p = 1$,

$$\|I_A\|_{q,w^{-1}}\|I_B\|_2 \leqslant \|I_{AB}\|_2.$$

Passing to squares, we get the statement of the lemma. □

In the abelian case algebra with an even weight, $w(x) = w(x^{-1})$, is contained in $\mathcal{L}_1(G)$ (this was proved for $G = \mathbb{R}^n$ by Kerman and Sawyer [5]). The next theorem generalizes this result to amenable groups, but the condition of evenness must be replaced by symmetricity of algebra. Theorem 1.1 shows that for arbitrary groups the same result does not hold.

Theorem 3.2. *Let G be an amenable group and let $\mathcal{L}_p^w(G)$, $p > 1$, be a symmetric algebra. Then $w^{-q} \in \mathcal{L}_1(G)$ and $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$.*

Proof. We use the uniform Følner condition [2]: for any compact set $A \subset G$ containing identity, and any $\varepsilon > 0$ there is a compact set $B \subset G$ such that $\mu(AB\Delta B) < \varepsilon\mu(B)$. Hence it follows that $\mu(AB) < (1+\varepsilon)\mu(B)$. But by Lemma 3.1 $\mu(AB) \geqslant \mu(B)\|I_A\|^2_{q,w^{-1}}$. Thus, $\|I_A\|_{q,w^{-1}} \leqslant 1$, i.e. $\int_A w^{-q} \leqslant 1$. As A was chosen arbitrary, we conclude that $\int_G w^{-q} \leqslant 1$, i.e. $w^{-q} \in \mathcal{L}_1(G)$. Hence it follows easily [6, Proposition 2] that $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$. □

Finally, we show that among amenable groups, weighted algebras with $p > 1$ exist on σ-compact groups only.

Corollary 3.3. *If G is an amenable group and with some w and $p > 1$ the space $\mathcal{L}_p^w(G)$ is an algebra, then the group G is σ-compact.*

Proof. By Lemma 2.4 we may assume $\mathcal{L}_p^w(G)$ is symmetric. Then by Theorem 3.2 $w^{-q} \in \mathcal{L}_1(G)$. As w^{-q} is strictly positive, and the support of any summable function may be chosen to be σ-compact [4, 11.40], we conclude that the entire group G is σ-compact. □

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