On right pure semisimple hereditary rings and an Artin problem

Daniel Simson

Faculty of Mathematics and Informatics, N. Copernicus University,
ul. Chopina 12/18, 87-100 Toruń, Poland

Communicated by C.A. Weibel; received 3 November 1993; revised 8 February 1994

Abstract

The pure semisimplicity conjecture (pss\(_R\)) stated below is studied in the paper mainly for hereditary rings \(R\). One of our main results is Theorem 3.6 containing various conditions which are equivalent to the conjecture (pss\(_R\)) for hereditary rings \(R\). It follows from our main results together with recent results of Herzog [16] that in order to prove (pss\(_R\)) for any \(R\) it is sufficient (and necessary) to construct an indecomposable module of infinite length over any hereditary ring \(R\) of the form \(\left( \begin{array}{c} F \otimes M_G \\ 0 \\ G \end{array} \right)\), where \(F, G\) are division rings and \(F M_G\) is a simple \(F\)-\(G\)-bimodule such that \(\dim M_G\) is finite and \(\dim_F M\) is infinite (see Corollary 5.1). Moreover, the existence of a counterexample \(R\) to the pure semisimplicity conjecture is equivalent to a generalized Artin problem for division rings (see 4.3-4.6), which is much more difficult than the Artin problem for division ring extensions solved by Cohn in [5] and by Schofield in [20]. It may frighten people of finding an easy solution to the pure semisimplicity problem. On the other hand, it is concluded in Section 5 that studying generalized Artin problems can help solve the pure semisimplicity conjecture.

1. Introduction

Throughout this paper \(R\) is a ring with an identity element. We denote by \(\text{Mod}(R)\) the category of all unitary right \(R\)-modules, and by \(\text{mod}(R)\) the full subcategory of \(\text{Mod}(R)\) consisting of finitely generated modules.

We recall from [21-23,25,26] that a ring \(R\) with an identity element is said to be right pure semisimple if every right \(R\)-module is a direct sum of finitely generated modules, or equivalently, if every right \(R\)-module is algebraically compact. Various characterizations of right pure semisimple rings can be found in [3,4,11,14,15,18,21-23,25,26,31,32].

\(^1\) Partially supported by Polish KBN Grant 1221/2/91. Email: simson@mat.uni.torun.pl.
A ring $R$ is of finite representation type if $R$ is artinian and there is a finite number of the isomorphism classes of finitely generated indecomposable right (and left) $R$-modules. It is well known that a ring $R$ is of finite representation type if and only if $R$ is right pure semisimple and $R$ is left pure semisimple. However the following pure semisimplicity conjecture remains an open problem (see [3,21–23,25,26]).

The conjecture (pss$_R$) was proved by Auslander [3] for any artin algebra $R$, and by the author for a PI-ring $R$ which is local [27], or hereditary, or the square of the Jacobson radical $J(R)$ of $R$ is zero, or $R$ is a selfinjective and $J(R)^3 = 0$ [26]. In [17] the conjecture (pss$_R$) is proved for any piecewise prime PI-ring $R$ in the sense that given primitive orthogonal idempotents $e$, $f$, $g$ in $R$ the equality $eJ(R)fJ(R)g = 0$ implies $eJ(R)f = 0$ or $fJ(R)g = 0$. The structure of such rings is described in [17]. A generalization of this result is given in [13]. After the paper has been submitted I received the preprint [16] of Herzog, who proves the conjecture for arbitrary PI-rings $R$ (see Section 5).

Here by a PI-ring $R$ we mean a ring $R$ satisfying a polynomial identity. It is well-known that a basic artinian ring $R$ is a PI-ring if and only if $R/J(R)$ is a product of division rings each of which is finite-dimensional over its center.

One of the aims of this paper is to give necessary and sufficient conditions for an arbitrary hereditary ring $R$ to have the property (pss$_R$).

We know from Theorem 3.3 in [26] that the problem for hereditary rings reduces to the case when $R$ has the form

$$R_M = \begin{pmatrix} F & F M_G \\ 0 & G \end{pmatrix},$$

where $F$ and $G$ are division rings, and $F M_G$ is a nonzero simple $F$-$G$-bimodule.

In Sections 3 and 4 the conjecture (pss$_R$) is studied in detail for hereditary rings of the form $R_M$ in a connection with the generalized Artin conjecture discussed in [8] and [20]. One of our main results is Theorem 3.6 which asserts, among others, that (pss$_R$) holds for any hereditary ring $R$ if and only if for any pair of division rings $F$ and $G$, and for any simple bimodule $F M_G$ with $\dim F M = \infty$ there exists an indecomposable right $R_M$-module of infinite length (see Corollary 3.7), or equivalently, if there exists an indecomposable preprojective non-injective module $X$ in $\text{mod}(R_M)$ for which there is no almost split sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in $\text{mod}(R_M)$.

We show in Section 4 that the existence of a counterexample to the conjecture (pss$_R$) for hereditary rings $R$ or for local rings $R$ of length two is equivalent to a generalized Artin problem for division rings (see 4.3–4.6), which is much more difficult than the Artin problem for division ring extensions solved by Cohn in [5,6] and by Schofield in [20].

We finish the paper by Section 5 (added to the revision version of the paper in January 1994), where some consequences of our main results and recent results of
Herzog [16] are collected. In particular we point out there that studying generalized
Artin problems should help in solving the pure semisimplicity conjecture or in finding
a counterexample.

Main results of this paper were presented on the algebra seminar in University of Mur-
cia in May 1992, and on the representation theory seminar in University of Sherbrooke
in August 1992. The author would like to thank these institutions for their hospitality
and financial support.

2. Preliminaries

We recall from [2,3,11,14,21–23,32] the following characterization of right pure
semisimple rings.

**Theorem 2.1.** Let $R$ be a ring. The following statements are equivalent.

(a) $R$ is right pure semisimple.

(b) $R$ is right artinian and every indecomposable right $R$-module is of finite length.

(c) $R$ is right artinian and if

$$X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_m \xrightarrow{f_m} X_{m+1} \rightarrow \cdots$$

is a sequence of non-isomorphisms between indecomposable modules $X_1, X_2, \ldots$
in mod$(R)$ then $f_m f_{m-1} \cdots f_2 f_1 = 0$ for some $m > 1$.

(d) The ring $R$ is right artinian and every covariant additive functor $H : \text{mod}(R) \rightarrow \text{Ab}$ has a simple subfunctor.

(e) The ring $R$ is right artinian and every covariant additive functor $H : \text{mod}(R) \rightarrow \text{Ab}$ satisfies the descending chain conditions for finitely generated subfunctors.

It is well known (see [2] and Section 11.3 in [30]) that a non-split exact sequence

$$e : 0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$$
in mod$(R)$ with indecomposable modules $X$ and $Z$ is called an *almost split sequence*
if $e$ has one of the following equivalent properties:

(AR$_1$) For any map $f : Z' \rightarrow Z$ in mod$(R)$ which is not a splittable
epimorphism, there exists $f' : Z' \rightarrow Y$ such that $f = v f'$.

(AR$_1'$) For any map $g : X \rightarrow X'$ in mod$(R)$ which is not a splittable
monomorphism, there exists $g' : Y \rightarrow X'$ such that $g = g' u$.

We say that mod$(R)$ has almost split sequences if for any non-injective module $X$
there is an almost split sequence $e$ in mod$(R)$ and for any non-projective module $Z$
there is an almost split sequence $e$ in mod$(R)$. A map $f : X \rightarrow Y$ in mod$(R)$ is called
*irreducible* if $f$ is neither a splittable monomorphism nor a splittable epimorphism and
for any factorization \( f = hg \) of \( f \) in \( \text{mod}(R) \) either \( g \) is a splittable monomorphism or \( h \) is a splittable epimorphism.

**Proposition 2.2.** Let \( R \) be a right pure semisimple ring. Then for any non-projective module \( Z \) there is an almost split sequence \( \mathbf{e} \) in \( \text{mod}(R) \). If \( \text{mod}(R) \) has almost split sequences, then \( R \) is of finite representation type.

**Proof.** The first statement is proved in Proposition 2.4 of [26]. The second statement follows from Theorem 2.1 and [2, Proposition 1.11 and Theorem 3.1], because the existence of almost split sequences in \( \text{mod}(R) \) implies that every simple functor \( H : \text{mod}(R) \to \text{Ab} \) is finitely presented [2]. □

Following Gabriel [12] we associate to any ring \( R \) with \( J(R)^2 = 0 \) the hereditary ring

\[
A(R) = \begin{pmatrix} R/J & (R/J)^2 \cr 0 & R/J \end{pmatrix},
\]

where \( J = J(R) \) is viewed as an \((R/J)-(R/J)\)-bimodule in a natural way, and a reduction functor

\[
\mathbb{F} : \text{mod}(R) \to \text{mod}(A(R))
\]

defined by attaching to any module \( X \) in \( \text{mod}(R) \) the triple \( \mathbb{F}(X) = (X', X'', t) \), where \( X' = X/XJ \), \( X'' = XJ \) are viewed as right \( R/J \)-modules and \( t : X' \otimes_{R/J} J_{R/J} \to X''/J \) is an \( R/J \)-homomorphism defined by formula \( t(\bar{x} \otimes r) = x \cdot r \) for \( \bar{x} = x + J \) and \( r \in J \).

The triple \( \mathbb{F}(X) \) is viewed as a right \( A(R) \)-module in a natural way. If \( f : X \to Y \) is an \( R \)-homomorphism we set \( \mathbb{F}(f) = (f', f'') \), where \( f'' : X'' \to Y'' \) is the restriction of \( f \) to \( X'' = XJ \) and \( f' : X' \to Y' \) is the \( R/J \)-homomorphism induced by \( f \).

**Lemma 2.3.** Suppose that \( R \) is an artinian ring and \( J(R)^2 = 0 \).

(a) The ring \( A(R) \) is hereditary artinian. The functor \( \mathbb{F} \) is full and establishes a representation equivalence between \( \text{mod}(R) \) and the category \( \text{Im} \mathbb{F} \). A right \( A(R) \)-module \( Z \) belongs to \( \text{Im} \mathbb{F} \) if and only if \( Z \) has no nonzero summand isomorphic to a simple projective right \( A(R) \)-module.

(b) The ring \( R \) is right pure semisimple (resp. of finite representation type) if and only if \( A(R) \) is right pure semisimple (resp. of finite representation type).

**Proof.** The statement (a) follows from 9.1 in [12]. Since it is easy to check that the property (c) in Theorem 2.1 is preserved and respected by the functor \( \mathbb{F} \) then (b) follows from Theorem 2.1. □

**Remark 2.4.** Since any right pure semisimple ring \( R \) is right artinian then the length \( l_R \) of \( R_R \) viewed as a right \( R \)-module is finite. Following an idea suggested in [29, 7.3] one can try to prove the conjecture \((\text{pss}_R)\) by induction on \( l_R \).
Without loss of generality we can suppose that \( R \) is indecomposable in a direct product of rings.

(a) In case \( l_R = 1 \) the ring \( R \) is a division ring and there is nothing to prove.

(b) If \( l_R = 2 \) then the ring \( R \) is right serial local with \( J(R)^2 = 0 \) and in view of Lemma 2.3 the problem reduces to the case \( R \) is of the form \( R_M \).

(c) If \( l_R = 3 \) then either \( R \) is local, or else \( R \) is hereditary and \( R \) or \( R^{op} \) is of the form \( R_M \). If \( R \) is local and \( l_R = 3 \) then, by Theorem 2.2 in [27], either \( J(R)^2 \neq 0 \) and \( R \) is both left and right serial, or \( J(R)^2 = 0 \) and Lemma 2.3 applies.

Consequently, in view of Theorem 2.2 in [27], it follows that the lower induction steps require the solution of the conjecture (pss) for hereditary rings \( R \) of the form \( R_M \), which are studied in details in the following section.

3. Pure semisimple hereditary rings

Throughout this section we suppose that \( F \) and \( G \) are division rings, and \( FMG \) is an \( F-G \)-bimodule. Our main aim is to give necessary and sufficient conditions for the right pure semisimple hereditary ring

\[
R_M = \begin{pmatrix} F & FMG \\ 0 & G \end{pmatrix}
\]

to be of finite representation type. Throughout we identify modules \( X \) in \( \text{mod}(R_M) \) with triples \( X = (X'_F, X''_G, t) \), where \( X'_F, X''_G \) are finite-dimensional vector spaces over \( F \) and \( G \), respectively, and \( t : X'_F \otimes_F F N_G \to X''_G \) is a \( G \)-linear map. The vector

\[
\dim X = (\dim X'_F, \dim X''_G) \in \mathbb{Z}^2
\]

is called the \textit{dimension-vector} of \( X \).

Suppose that \( FN_G \) is an \( F-G \)-bimodule and \( G N'_F \) is a \( G-F \)-bimodule such that \( \dim N_G < \infty \), \( \dim N'_F < \infty \) and there exists a \( G-F \)-bimodule isomorphism \( G N'_F \cong \text{Hom}_G(F N_G, G) \). Following [19] and [26, pp. 199–200] we define the pair of reflection functors

\[
\begin{array}{c}
\text{mod}(R_{N'}) \\
\xrightarrow{S^+}
\text{S}^{-}
\end{array}
\text{mod}(R_N)
\]

as follows. To any module \( Y = (Y'_F, Y''_G, t : Y'_F \otimes_G N'_F \to Y''_G) \) in \( \text{mod}(R_{N'}) \) we associate the right \( R_{N'} \)-module \( S^+(Y) = (X'_F, X''_G, s) \), where \( X'_F = \text{Ker} t \), \( X''_G = Y'_F \) and \( s : X'_F \otimes_F N_G \to X''_G \) is the \( G \)-linear map corresponding to the inclusion map \( X'_F = \text{Ker} t \leftarrow Y'_F \otimes_G N'_F \) via the composed natural isomorphism

\[
\text{Hom}_F(X'_F, Y'_F \otimes_G N'_F) \cong \text{Hom}_F(X'_F, \text{Hom}_G(F N_G, X''_G)) \cong \text{Hom}_G(X'_F \otimes_F N_G, X''_G).
\]

The functor \( S^- \) is defined analogously by taking the cokernel.
Given an $F$-$G$-bimodule $\mathcal{F}N_G$ we set
\[
\text{l.dim}(N) = \dim_F N \quad \text{and} \quad \text{r.dim}(N) = \dim N_G
\]
and we consider two dual $G$-$F$-bimodules (see [19])
\[
N^* = \text{Hom}_F(\mathcal{F}N_G, F) \quad \text{and} \quad N^{**} = \text{Hom}_G(\mathcal{F}N_G, G).
\]
We associate to our bimodule $FM_G$ a sequence of iterated dual bimodules of $FM_G$ by setting
\[
M'(0) = M,
M'(j) = (M'(j-1))^* \quad \text{for } j \geq 1,
M'(j) = (M'(j+1))^* \quad \text{for } j \leq -1.
\]
We also set
\[
R_{2j} = \begin{pmatrix} F & M^{(2j)} \\ 0 & G \end{pmatrix},
R_{2j+1} = \begin{pmatrix} G & M^{(2j+1)} \\ 0 & F \end{pmatrix},
d_j^M = \text{r.dim}(M'(j))
\]
for $j \in \mathbb{Z}$.

The following technical result is playing a crucial role in the study of hereditary rings.

**Lemma 3.1.** Suppose that $FM_G$ is an $F$-$G$-bimodule such that $d_{j-1}^M$ and $d_j^M$ are finite numbers, and let
\[
s_{j-1}^+, s_{j-1}^- : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2
\]
be group homomorphisms defined by the formulas
\[
s_{j-1}^+(x, y) = (d_j^M x - y, x), \quad s_{j-1}^-(x, y) = (y, d_{j-1}^M y - x).
\]
Then $d_j^M = \text{l.dim}(M'(j-1))$ and the following statements hold.

(a) There exists a pair of reflection functors
\[
\text{mod}(R_{j-1}) \overset{S_{j-1}^+}{\longrightarrow} \text{mod}(R_j) \overset{S_{j-1}^-}{\longrightarrow} \text{mod}(R_{j-1})
\]
satisfying the following conditions:

(c) The functor $S_{j-1}^-$ is left adjoint to $S_{j-1}^+$. If $X$ is an indecomposable module in $\text{mod}(R_{j-1})$ then $S_{j-1}^+ X = 0$ if and only if $X$ is isomorphic to a unique simple projective $R_{j-1}$-module $P_0^{(j-1)}$ with $\text{dim} P_0^{(j-1)} = (0, 1)$. If $S_{j-1}^+ X \neq 0$ then $S_{j-1}^+ S_{j-1}^- X \cong X$ and $\text{dim} S_{j-1}^- X = s_{j-1}^-(\text{dim} X)$. 
If $X$ and $Y$ are indecomposable modules in $\text{mod}(R_{j-1})$ such that $S_{j-1}^{+}X \neq 0$ and $S_{j-1}^{+}Y \neq 0$, then the reflection functor $S_{j-1}^{+}$ induces an isomorphism

$$\text{Hom}_{R_{j-1}}(X,Y) \cong \text{Hom}_{R_{j}}(S_{j-1}^{+}X,S_{j-1}^{+}Y).$$

If $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\text{mod}(R_{j-1})$ and the $R_{j}$-module $S_{j-1}^{+}X$ is nonzero and non-injective then the induced sequence $0 \to S_{j-1}^{+}X \to S_{j-1}^{+}Y \to S_{j-1}^{+}Z \to 0$ in $\text{mod}(R_{j})$ is almost split.

The conditions (c)$^-$ analogous to (c)$^+$ with $S_{j-1}^{+}$, $S_{j-1}^{-}$, simple projective $R_{j}$-module $P_{0}^{(j-1)}$ with $\dim P_{0}^{(j-1)} = (0,1)$ and $S_{j-1}^{+}$, $S_{j-1}^{-}$, simple injective $R_{j}$-module $Q_{0}^{(j)}$ with $\dim Q_{0}^{(j)} = (1,0)$ interchanged.

(b) The ring $R_{j-1}$ is right pure semisimple (resp. of finite representation type) if and only if $R_{j}$ is right pure semisimple (resp. of finite representation type).

**Proof.** The equality $d_{j}^{M} = 1.\dim(M^{(j-1)})$ is immediate. Then the statement (a) follows from the discussion in [19], [10] and in [26, pp. 199-200]. We take for $S_{j}^{+}$ and $S_{j}^{-}$ the corresponding reflection functors $S_{j}^{+}$ and $S_{j}^{-}$ constructed by applying (3.1) to $N = N^{(j)}$ and $N' = N^{(j-1)}$.

(b) By our assumption the rings $R_{j}$, $R_{j-1}$ are right artinian. Then (b) follows, because in view of (a) the functors $S_{j-1}^{+}$ and $S_{j-1}^{-}$ preserve the finite representation type and they preserve the property (c) in Theorem 2.1.

We recall from [7] that a module $X$ over a hereditary ring $R$ is said to be proprojective (resp. preinjective) if the number of non-isomorphic indecomposable modules $Y$ satisfying $\text{Hom}_{R}(Y,X) \neq 0$ (resp. $\text{Hom}_{R}(X,Y) \neq 0$) is finite.

**Proposition 3.2.** Assume that $F$ and $G$ are division rings and the ring $R_{M} = (F \otimes_{G} M_{G})$ is right pure semisimple. Then the following statements hold.

(a) For any $j \geq 0$, the number $d_{j}^{M}$ is finite and the $(-j)$th iterated dual bimodule $M^{(-j)}$ of $F_{M}G$ is simple as a bimodule.

(b) There exists a sequence of reflection functors

$$\cdots \to \text{mod}(R_{-m}) \xrightarrow{S_{m}^{+}} \text{mod}(R_{-m+1}) \xrightarrow{S_{m}^{-}} \cdots \to \text{mod}(R_{-1}) \xrightarrow{S_{1}^{+}} \text{mod}(R_{1}) \xrightarrow{S_{1}^{-}} \text{mod}(R_{M}).$$

(3.4)

For any $j \geq 1$, the ring $R_{-j}$ is right pure semisimple and left artinian. There is a Morita duality $(\text{mod}(R_{M_{R}}^{(m)}))^{\text{op}} \cong \text{mod}(R_{-m})$ for any $m \geq 0$, where $R_{0} = R_{M}$.

(c) An indecomposable module $X$ in $\text{mod}(R_{M})$ is preinjective if and only if there exists $m \geq 0$ such that $X$ is isomorphic to

$$Q_{m}^{(0)} := S_{-1}^{+}S_{-2}^{+}\cdots S_{-m}^{+}Q_{0}^{(m)}$$

(3.5)
where $Q_0^{(-m)}$ is the unique (up to isomorphism) simple injective module in
$\text{mod}(R_{-m})$ such that $\dim Q_0^{(-m)} = (1, 0)$. Any preinjective $R_M$-module $X$ is
uniquely determined by $\dim X$ up to isomorphism.

(d) The preinjective modules in $\text{mod}(R_M)$ form a connected component of the form

$$\ldots \rightarrow Q_{2n+2}^{(0)} \rightarrow Q_{2n}^{(0)} \rightarrow Q_{2n+1}^{(0)} \rightarrow \ldots \rightarrow Q_4^{(0)} \rightarrow Q_2^{(0)} \rightarrow Q_0^{(0)} \rightarrow \ldots$$

in the Auslander-Reiten quiver of $\text{mod}(R_M)$. For any $j \geq 1$ there are bimodule
isomorphisms $\text{Irr}(Q_j^{(0)}, Q_{j-1}^{(0)}) \cong \text{Hom}_{R_M}(Q_j^{(0)}, Q_{j-1}^{(0)}) \cong M^{(-j-2)}$, where
$\text{Irr}(X, Y)$ means the bimodule of irreducible maps from $X$ to $Y$ (see [30, 11.4]).

**Proof.** Since $R_M$ is right pure semisimple, $R_M$ is right artinian and the injective envelope
$E(P_0)$ of the unique simple projective right ideal $P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is of finite length.
It follows that $d_0^M = \dim M_G$ is finite and according to [28, Proposition 2.5] the module
$E(P_0)$ has the form $(\text{Hom}_G(FM_G, G), G, \psi)$. Consequently, the number $d_1^M = \dim \text{Hom}_G(FM_G, G)_F$ is also finite and Lemma 3.1 applies to $i = 0$. It follows that
there exist the pair $S^+_1, S^-_1$ in the required sequence of reflection functors. Moreover,
according to Lemma 3.1(b), the ring $R_1$ is right pure semisimple. Since

$$\dim(M^{(-1)}) = \dim \text{Hom}_G(FM_G, G) = \dim M_G = d_0^M$$

is finite, the ring $R_1$ is left artinian. Now statement (b) and the first part of (a) follow
by an easy induction. Applying the arguments in the proof of Lemma 1.7 in [26] one can show that there is a Morita duality (see [1]) $(\text{mod}(R_{-m-2}))^{op} \cong \text{mod}(R_{-m})$ for
any $m \geq 0$.

In order to finish the proof of (a) it remains to show that $M^{(-j)}$ is simple as a bimodule, for any $j \geq 0$. Assume to the contrary that $M^{(-j)}$ is not simple. It follows that $dd' \geq 4$, where $d = \dim(M^{(-j)})$ and $d' = r\dim(M^{(-1)}) = d_1^M$.

First suppose that $dd' = 4$. By Theorem 4 in [19] there exist a field $K$ and a full
exact embedding of the category $\text{fin}(K[X, e, \delta])$ of finite-dimensional right $K[X, e, \delta]$-
modules into $\text{mod}(R_{-j})$, where $K[X, e, \delta]$ is a skew polynomial ring in one variable
$X, e$ is an automorphism of $K$ and $\delta$ is a $(1, e)$-derivation of $K$. Since the condition
(c) of Theorem 2.1 is not satisfied in $\text{fin}(K[X, e, \delta])$ (see Theorem 4 in [19]), it is
not satisfied in $\text{mod}(R_{-j})$ and therefore $R_{-j}$ is not right pure semisimple contrary to the
statement (b) proved above.

Next suppose that $dd' \geq 5$. It follows from Theorem 5 in [19] that there exist a field
$K$ and a full exact embedding $\text{Mod}(K\langle X, Y \rangle) \rightarrow \text{Mod}(R_{-j})$, where $K\langle X, Y \rangle$ is the
free polynomial algebra in two non-commuting variables $X$ and $Y$. The condition (c) of
Theorem 2.1 is not satisfied in $\text{fin}(K\langle X, Y \rangle)$. Hence we conclude as above that $R_{-j}$ is
not right pure semisimple which is a contradiction. This proves that $M^{(-j)}$ is a simple
bimodule.

In order to prove the statements (c) and (d) of the proposition we note first that by
the properties of the reflection functors in Lemma 3.1 the modules of the form $Q_m^{(0)}$
are preinjective. Conversely, suppose that $X$ is an indecomposable preinjective module
in \text{mod}(R_M)$. Since $R_M$ is right artinian and there is only finitely many non-isomorphic indecomposable modules $Y$ with $\text{Hom}_R(X,Y) \neq 0$ then there exists a sequence of irreducible maps

$$X = X_1 \rightarrow \cdots \rightarrow X_s \rightarrow Q_0^0,$$

where $X_1, \ldots, X_s$ are indecomposable preinjective. This fact follows by applying Theorem 11.9 in [30] to the full subcategory $\mathcal{A}$ of \text{mod}(R_M) consisting of direct sums of indecomposable modules $Y$ such that $\text{Hom}_{R_M}(Y,X) \neq 0$.

If $s = 1$ a simple consideration shows that $X$ is injective and $S_{-1}^{-1}X \cong Q_0^{(-1)}$. Consequently, $X \cong S_{-1}^{-1}Q_0^{(-1)}$ and we are done. If $s \geq 2$, the module $X_s$ is injective and in view of $(c_i^{-})$ there is a sequence of irreducible maps

$$S_{-1}^{-1}X = S_{-1}^{-1}X_1 \rightarrow \cdots \rightarrow S_{-1}^{-1}X_s \cong Q_0^{(-1)}.$$

By an inductive hypothesis the module $S_{-1}^{-1}X$ in \text{mod}(R_{-1}) has the required form and therefore $X$ has, because of the properties $(c_i^{-})-(c_i^{+})$ and $(c_i^{-})-(c_i^{+})$ of the reflection functors given in Lemma 3.1.

For the irreducible map formula we note that $Q_1^{(0)} = E_{R_M}(P_0)$ has the form $(M^{(-1)}, G)$ and therefore for $j = 1$ we get $\text{Irr}(Q_1^{(0)}, Q_0^{(0)}) \cong \text{Hom}_{R_M}(Q_1^{(0)}, Q_0^{(0)}) \cong \text{Hom}_F(M^{(-1)}, F) \cong M^{(-2)}$. The case $j > 1$ reduces to the above one by applying the reflection functors (3.4).

\textbf{Remark 3.3.} In the proof above we use [19, Theorem 51], which is proved in [19] under the hypothesis that the dimension of the hereditary ring $(F, M_0)$ over its center is less than $\kappa_i$, the first strongly inaccessible cardinal number. The author was informed by C.M. Ringel that [19, Theorem 51] remains valid without the above dimension hypothesis.

Following [8] we define inductively the set

$$D = D_2 \cup D_3 \cup \cdots \cup D_m \cup \cdots$$

of \textit{dimension-sequences} $(d_1, \ldots, d_m)$, $m \geq 1$, to be the minimal set satisfying the following two conditions.

(i) $D_2 = \{(0,0)\}$ and $D_3 = \{(1,1,1)\}$.

(ii) If the set $D_m$ is defined we define $D_{m+1}$ to be the set of all sequences of the form $(d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1} + 1, d_{i+2}, \ldots, d_m)$, where $(d_1, \ldots, d_m) \in D_m$ and $i = 1, \ldots, m-1$.

We note that the set $D_m$ is closed under the action of cyclic permutations.

Let us present the list of all dimension-sequences of length $\leq 7$ up to cyclic permutations and reversions:

$(0,0), \quad (1,1,1), \quad (1,2,1,2), \quad (1,2,2,1,3), \quad (1,2,2,2,1,4),$

$(1,2,3,1,2,3), \quad (1,3,1,3,1,3), \quad (1,2,2,2,2,1,5),$

$(1,2,2,3,1,2,4), \quad (1,2,3,2,1,3,3), \quad (1,4,1,2,3,1,3).$
Let us denote by $D^\vee$ the set of all sequences obtained form sequences in $D$ by omitting the last coordinate. Any sequence from $D^\vee$ will be called a simple restriction of a dimension-sequence.

Note that $D^\vee$ contains the following sequences and their reversions: $(0)$, $(1,1)$, $(1,2,1)$, $(2,1,2)$, $(1,2,2,1)$, $(2,2,1,3)$, $(2,1,3,1)$.

**Theorem 3.4.** Suppose that $F$ and $G$ are division rings, and $FM_G$ is a nonzero $F$-$G$-bimodule such that the hereditary ring $R_M = \left(\begin{array}{c} F \\ \_ \_ \\ \_ \_ \_ \_ \_ G \end{array}\right)$ is right pure semisimple. The following conditions are equivalent.

(a) The ring $R_M$ is of finite representation type.

(b) For any indecomposable preprojective non-injective module $X$ in $\text{mod}(R_M)$ there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\text{mod}(R_M)$, where $Y$ and $Z$ are preprojective.

(c) There exists an almost split sequence

$$0 \to P^{(j)}_0 \to Y \to Z \to 0$$

in $\text{mod}(R_j)$ for any $j \geq 0$, where $P^{(j)}_0$ is the unique simple projective right ideal in $R_j$. In this case the modules $Y$ and $Z$ are preprojective.

(d) The numbers $d_0^M, d_1^M, d_2^M, \ldots, d_j^M, \ldots$ are finite.

(e) There exists a unique natural number $m \geq 2$ such that the sequence

$$d_{m-1}(M) := (d_0^M, \ldots, d_{m-2}^M)$$

is a simple restriction of a dimension-sequence (i.e. it belongs to $D^\vee$) and $d_{i+2m}^M = d_i^M$ for $i = 0, \ldots, m - 1$ and all $t \in \mathbb{N}$.

(f) Each indecomposable summand of any direct product of preinjective right $R_M$-modules is a module of finite length.

**Proof.** The equivalence (a)$\Leftrightarrow$(f) follows from Corollary B in [31]. The implication (a)$\Rightarrow$(e) was proved in [8], and (e)$\Rightarrow$(d) is obvious, because it follows from [9, Corollary 1] that if $d_{m-1}(M) \in D^\vee$ then $(d_0^M, \ldots, d_{m-1}^M) \in D$.

(d)$\Leftrightarrow$(c) It follows from Lemma 1.3 in [26] that there exists an almost split sequence $0 \to P^{(j)}_0 \to Y \to Z \to 0$ in $\text{mod}(R_j)$ if and only if the number $\text{l.dim}(M^{(j)})$ is finite. Hence (d)$\Leftrightarrow$(c) follows, because obviously $\text{l.dim}(M^{(j)}) = d_j^M$ for all $j \geq 0$.

(c)$\Rightarrow$(b) In view of the implication (c)$\Rightarrow$(d) and of Lemma 3.1 there exists a sequence of reflection functors

$$\text{mod}(R_M) \xrightarrow{S_0^+} \text{mod}(R_1) \xrightarrow{S_1^-} \cdots \xrightarrow{S_{m-1}^+} \text{mod}(R_{m-1}) \xrightarrow{S_{m-1}^-} \text{mod}(R_m) \xrightarrow{S_m^-} \cdots$$

and the rings $R_0 := R_M, R_1, R_2, \ldots, R_m, \ldots$ are right artinian. It follows from (c)$^-$ in Lemma 3.1 that the right $R_j$-module

$$P^{(j)}_i = S_{j+1}^- S_{j-1}^- \cdots S_{j+i-2}^- S_{j+i-1}^- P^{(j+i)}_0$$

(3.9)
is indecomposable preprojective for all \( j \geq 0 \) and \( i \geq 0 \).

We shall show that for each \( j \geq 0 \) any indecomposable preprojective non-injective module \( X \) in \( \text{mod}(R_j) \) is isomorphic to one of the modules

\[
P_0^{(j)}, P_1^{(j)}, \ldots, P_m^{(j)}, \ldots.
\]

Suppose for simplicity that \( j = 0 \) and let \( X \) be an indecomposable preprojective non-injective module in \( \text{mod}(R_M) \). Since \( \text{Hom}_{R_M}(P_0^{(j)}, X) \neq 0 \) then from Theorem 11.9 in \([30]\) applied to the full subcategory \( A \) of \( \text{mod}(R_M) \) consisting of direct sums of indecomposable modules \( Y \) such that \( \text{Hom}_{R_M}(Y, X) \neq 0 \) it follows that there is a sequence

\[
P_0^{(0)} = Y_0 \to Y_1 \to \cdots \to Y_s = X
\]

of irreducible maps between indecomposable modules in \( \text{mod}(R_0) \).

If \( s = 0 \) we are done. Assume that \( s \geq 2 \). Then by Lemma 1.3(2) in \([26]\) the module \( Y_1 \) is non-simple projective and therefore the module \( S_0^\perp Y_1 \neq 0 \) is simple projective, because \( \text{dim} Y_1 = (0,1) \) and it follows from \( (c_2^+) \) in Lemma 3.1 that \( s_0^\perp (\text{dim} Y_1) = (0,1) \). According to the property \( (c_2^-) \), by applying the functor \( S_0^+ \) we derive the sequence

\[
P_0^{(1)} \cong S_0^\perp Y_1 \to S_0^\perp Y_2 \to \cdots \to S_0^\perp Y_s = S_0^\perp X
\]

of irreducible maps. Then, by the inductive hypothesis, \( S_0^\perp X \cong P_s^{(1)} \) and therefore \( X \cong S_0^\perp P_s^{(1)} \cong P_s^{(0)} \) as required. Now applying \( (c_2^-) \) we easily conclude the statement (b).

(b) \( \Rightarrow \) (a) We apply the functor category method due to Auslander \([2]\). It follows from \([2]\) that every simple additive functor \( S : \text{mod}(R_M) \to \text{Ab} \) has the form

\[
S_X = h^X/Jh^X,
\]

where \( X \) is an indecomposable module in \( \text{mod}(R_M) \), \( h^X = \text{Hom}_{R_M}(X,-) \) and \( Jh^X \) is the Jacobson radical of \( h^X \), that is, \( Jh^X(N) \) is the subgroup of \( h^X(N) \) consisting of all non-isomorphisms from \( X \) to \( N \) for any indecomposable module \( N \) in \( \text{mod}(R_M) \) (see \([2,21-23,25]\)). From the definition of almost split sequences it follows that if there exists an almost split sequence \( 0 \to X \to Y \to Z \to 0 \) in \( \text{mod}(R_M) \) then the functor \( S_X \) is finitely presented and the induced exact sequence

\[
0 \to h^Z \to h^Y \to Jh^X \to 0
\]

of functors is a minimal projective resolution of \( Jh^X \). In view of (b) this implies that if there exists an epimorphism of functors \( h^P \to S \), where \( S \) is semisimple and \( P \) is preprojective in \( \text{mod}(R_M) \), then \( S \) admits a minimal projective resolution

\[
h^V \to h^U \to S \to 0,
\]

where \( U \) and \( V \) are preprojective.

Let \( H = h^{P_0^{(0)}} \), where \( P_0^{(0)} = (\begin{smallmatrix} 0 & 0 \\ 0 & G \end{smallmatrix}) \) is the unique projective simple right ideal of \( R_M \). Given \( m \geq 2 \) we set \( J^mH = J^{m-1}H \) (see \([21-23,25,26]\)). We shall show by induction
on $m \geq 1$ that there is an epimorphism $h^W \rightarrow J^m H$, where $W_m$ is a preprojective module in mod$(R_M)$.

The discussion above proves our claim for $m = 1$. Assume that $m \geq 2$. Since, by the inductive hypothesis, there exists an epimorphism $h^{W_{m-1}} \rightarrow J^{m-1} H$, where $W_{m-1}$ is preprojective, then by the observation above the semisimple functor $S = J^{m-1} H/J^m H$ has a minimal projective resolution (*), where $U$ and $V$ are preprojective. Hence we conclude that there is an epimorphism $h^V \rightarrow J^m H$ as we required. This proves our claim. Consequently, all terms in the radical sequence

$$H \supseteq JH \supseteq \ldots \supseteq J^m H \supseteq J^{m+1} H \supseteq \ldots$$

are finitely generated functors. Since the ring $R_M$ is right pure semisimple then according to Theorem 2.1(e) applied to $H$ the radical sequence terminates, that is, there exists an $m$ such that $J^m H = J^{m+1} H = JJ^m H$. Since $J^m H$ is finitely generated then the Nakayama's lemma yields $J^m H = 0$. It follows that the functor $H$ is of finite length, because the semisimple functor $J^i H/J^{i+1} H$ is finitely generated for $i = 0, 1, \ldots, m - 1$. Then according to Proposition 2.2 in [2] there is only finitely many pairwise non-isomorphic indecomposable modules $X$ in mod$(R_M)$ such that

$$H(X) = \text{Hom}_{R_M}(P^{(0)}_0, X) \neq 0.$$ 

It follows that the ring $R_M$ is of finite representation type, because $P^{(0)}_0$ is a unique simple projective $R_M$-module and any non-injective nonzero $R_M$-module $Y$ admits a monomorphism $P^{(0)}_0 \rightarrow Y$. This finishes the proof of the theorem. 

The results of Section 3 in [8] (see also [9] and [10]) together with Theorem 3.4 yield the following.

**Corollary 3.5.** Let $F$ and $G$ be division rings and let $F M_G$ be an $F$-$G$-bimodule. The following statements are equivalent.

(a) The ring $R_M = (F^{F M_G})$ has precisely $m$ pairwise non-isomorphic indecomposable right modules of finite length.

(b) $(d^M_0, \ldots, d^M_{m-1})$ is a dimension-sequence.

(c) $(d^M_0, \ldots, d^M_{m-2})$ is a simple restriction of a dimension-sequence.

(d) $s^m_{-m+1}s^m_{-m+2} \cdots s^m_{-2}s^m_{-1}(0, 1) = (1, 0)$ (see (3.3)).

Now we are able to prove the following extension of Theorem 3.3 in [26].

**Theorem 3.6.** The following conditions are equivalent.

(a) Every right pure semisimple hereditary ring is of finite representation type.

(b) Every right pure semisimple hereditary ring is left artinian.

(c) If $F$, $G$ are division ring and $F M_G$ is a nonzero simple $F$-$G$-bimodule such that the ring $R_M = (F^{F M_G})$ is right pure semisimple, then dim$_F M$ is finite.

(d) Given a pair of division rings $F$ and $G$ and a nonzero simple $F$-$G$-bimodule $F M_G$ such that dim $M_G$ is finite and dim$_F M = \infty$ there exists an indecomposable right
module of infinite length over the hereditary ring $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$.

(c) For every right pure semisimple hereditary ring of the form $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$, where $F$ and $G$ are division rings and $F_M G$ is a nonzero simple $F$-$G$-bimodule, there exists a Morita duality for left $R_M$-modules such that the left Morita dual ring (see [1]) is two-sided unitarian.

(f) If $F$, $G$ are division rings and $F_M G$ is a nonzero simple $F$-$G$-bimodule such that the ring $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$ is right pure semisimple, then the iterated dual bimodules $M^{(j)}$, $j \geq 0$, are finite-dimensional as right (and left) spaces.

(g) If $F$, $G$ are division ring and $F_M G$ is a nonzero simple $F$-$G$-bimodule such that the ring $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$ is right pure semisimple, then there exists an almost split sequence

$$0 \rightarrow P_0 \rightarrow Y \rightarrow Z \rightarrow 0$$

in $\text{mod}(R_M)$, where $P_0 = (\begin{smallmatrix} 0 \\ 0 \\ G \end{smallmatrix})$ is the unique simple right ideal of $R_M$ up to isomorphism.

**Proof.** The implications (a)$\iff$(b)$\implies$(c) are proved in [26, Theorem 3.3].

(c)$\implies$(b) It follows from Proposition 3.2 that any $F$-$G$-bimodule $F_M G$ is simple if the ring $R_M$ is right pure semisimple. Then (c)$\implies$(b) is a consequence of Theorem 3.3 in [26].

(c)$\implies$(d) Suppose to the contrary that (d) does not hold, that is, there exist division rings $F$ and $G$, and a nonzero simple $F$-$G$-bimodule $F_M G$ such that $\dim M_G$ is finite, $\dim F M = \infty$ and every indecomposable right module over the ring $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$ is finitely generated. It follows from Theorem 2.1 that $R_M$ is right pure semisimple. Since $\dim F M = \infty$ we get a contradiction with (c).

(d)$\implies$(c) Assume to the contrary that (c) does not hold. Then there exists a right pure semisimple ring $R_M$ such that the bimodule $F_M G$ is simple and $\dim F M = \infty$. This contradicts (d) of Theorem 2.1.

(c)$\implies$(f) Assume that $F$ and $G$ are division rings, $F_M G$ is a nonzero simple $F$-$G$-bimodule and the ring $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$ is right pure semisimple. It follows from (c) that the number $d_1^M = \dim F M$ is finite and Lemma 3.1 applies. Hence the ring $R_M^{(1)}$ is right pure semisimple and obviously $M^{(1)}$ is a simple bimodule, because $F_M G$ is simple. Then (c) yields $d_2^M = \text{l.dim}(M^{(1)}) < \infty$. Continuing this procedure we show that the numbers $d_3^M, d_4^M, \ldots$ are finite. Recall that $\text{l.dim}(M^{(j)}) = d_{j+1}^M$.

The implication (f)$\implies$(c) follows from the equality $\dim F M = d_1^M$, and the equivalence (g)$\implies$(c) is a consequence of Corollary 1.4 in [26].

The implication (a)$\implies$(e) easily follows from [2], because minimal injective cogenerators in $\text{Mod}(T)$ and in $\text{Mod}(T^{\text{op}})$ are of finite length if $T$ is a ring of finite representation type.

It remains to show that (e) implies (c). Let $R_M = (\begin{smallmatrix} F & M_G \\ 0 & G \end{smallmatrix})$ be a right pure semisimple ring, where $F$ and $G$ are division ring and $F_M G$ is a nonzero simple $F$-$G$-bimodule. It follows from [23, Note Added in Proof] and from [26, Proposition 2.4(a)] that there is a Morita duality $\text{mod}(R_M) \cong (\text{mod}(T^{\text{op}}))^{\text{op}}$ (see [1]), where $T$ is right pure
semisimple and left artinian. Since $R_M$ is hereditary, the ring $T$ is hereditary. It follows from Proposition 2.5 in [28] that $T$ has the form $R_N = \left( \begin{array}{cc} F & N_G \\ 0 & G \end{array} \right)$, where $N_G$ is a $F$-$G$-bimodule. Since $T$ is right pure semisimple then by Proposition 3.2 the bimodule $N_G$ is simple. Then (e) applied to the ring $T$ implies that $R_M$ is left artinian, because $R_M$ is left Morita dual to $T$. Hence $\dim_F M$ is finite and (c) follows. This finishes the proof of the theorem. \hfill \Box

Corollary 3.7.

(a) The conjecture (pss$R$) has a positive solution for any hereditary ring $R$ if and only if it has a positive solution for any ring of the form $R_M = \left( \begin{array}{cc} F & M_G \\ 0 & G \end{array} \right)$, where $F$ and $G$ are division rings.

(b) The conjecture (pss$R$) has a positive solution for all hereditary rings $R$ if and only if for pair of division rings $F$, $G$ and any simple $F$-$G$-bimodule $M_G$ such that $\dim M_G$ is finite and $\dim_F M = \infty$ one can construct an indecomposable right module of infinite length over the hereditary ring $R_M = \left( \begin{array}{cc} F & M_G \\ 0 & G \end{array} \right)$, or equivalently (see Theorem 2.1), one can construct a sequence

$$X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_m \xrightarrow{f_m} X_{m+1} \longrightarrow \cdots,$$

where $X_1, X_2, \ldots$ are indecomposable right $R_M$-modules of finite length and $f_1, f_2, \ldots$ are non-isomorphisms such that $f_m f_{m-1} \cdots f_2 f_1 \neq 0$ for any $m > 1$. \hfill \Box

4. A generalized Artin problem on division rings

For any pair of numbers $n, m \in \mathbb{N} \cup \{\infty\}$ Schofield constructs in [20] a pair of division rings $G \subseteq F$ in such a way that

$$\dim F_G = n \quad \text{and} \quad \dim_G F = m$$

where $F_G$ and $G F$ mean the $G$-vector space $F$ viewed as a right and as a left $G$-module, respectively. This is a solution of an old Artin problem for division ring extensions.

In [8] a generalized Artin problem was formulated in a connection with a study of hereditary rings $R_M = \left( \begin{array}{cc} F & M_G \\ 0 & G \end{array} \right)$ of finite representation type, where $F$ and $G$ are division rings and $M_G$ is an $F$-$G$-bimodule. It was proved there that the ring $R_M$ is of finite representation type if and only if the sequence $d_m(M)$ (3.7) is a dimension-sequence for some $m$ (i.e. $d_m(M) \in \mathcal{D}$) and $d_{i+2mt} = d_i^m$ for $i = 0, \ldots, m - 1$ and all $t \in \mathbb{N}$. It follows from Proposition 1 in [8] that if $d_m(M) \in \mathcal{D}$ then there exists $j \leq m - 1$ such that $d_i^m = 1$. This means that there exists a ring embedding $\sigma : G \to F$ or a ring embedding $\tau : F \to G$ such that $M(j)$ is isomorphic to the $F$-$G$-bimodule $g F_F$ or to the $G$-$F$-bimodule $F G_G$. Since there exists a cyclic permutation $\sigma$ such that $d_m(M) = \sigma \ast d_m(M(j))$ then without loss of generality we may suppose that $j = 1$ and there exists a ring embedding $G \subseteq F$ such that $M(1) \cong G F_F$. Note that in this case we have
\[ d_0^M = \dim F_G, \quad d_1^M = \dim F_F = 1, \]
\[ d_2^M = \dim G_F, \quad d_3^M = \dim G \Hom_G(F_F, G) \]

and therefore the existence of a bimodule \( FM_G \) such that \( d_m(M) \) is a dimension-sequence \( d = (d_1, \ldots, d_m) \in \mathcal{D} \) is equivalent to a generalized Artin problem. We call it an Artin problem of dimension-type \( d \in \mathcal{D} \), or a bimodule Artin problem.

The solution of the problem is trivial for the dimension-sequences \((0, 0)\), \((1, 1, 1)\), \((1, 2, 1, 2)\), \((1, 3, 1, 3, 1, 3)\). The problem was solved by A. Schofield in [20] for the dimension-sequence \( d_5 = (2, 1, 3, 1, 2) \). The bimodule Artin problem remains an open question for the dimension-sequences \( d \in \mathcal{D} \) which are different from the above ones up to a cyclic permutation.

The discussion above together with Corollary 3.5 yields (see [8] and [9, Corollary 2]).

**Corollary 4.1.** Assume that \( F \subseteq G \) is a pair of division rings such that \( \dim G_F \leq \dim F_G \), and let \( R_G = \left( \begin{array}{c} F G \\ G 0 \end{array} \right) \).

(a) The category \( \text{mod}(R_G) \) has exactly 3 indecomposable modules up to isomorphism if and only if \( F = G \).

(b) The category \( \text{mod}(R_G) \) has exactly 4 indecomposable modules up to isomorphism if and only if \( \dim G_F = \dim F_G = 2 \).

(c) The category \( \text{mod}(R_G) \) has exactly 5 indecomposable modules up to isomorphism if and only if \( \dim G_F = 2, \dim F_G = 3 \) and \( \dim \Hom_F(F_G, F) = 1 \). This is the case if and only if the dimension-sequence \( d_5(F_G, F) \) is equal to \((1, 3, 1, 2, 2)\). □

It follows from the result of Schofield [20] mentioned above that there exists a pair of division rings \( F \subseteq G \) such that the category \( \text{mod}(F_G, G) \) has exactly 5 indecomposable modules up to isomorphism.

Since we are not able to solve the conjecture \( \text{pss}_R \) even for hereditary rings \( R \), we consider the question how a counterexample to the conjecture could be constructed. We shall show below that this problem restricted to rings of low length leads to a kind of Artin problem which is more general than the Artin problems of dimension-type \( d \in \mathcal{D} \), and depends on infinitely many conditions for the dimensions of the iterated dual bimodules \( M^{(-1)}; M^{(-2)}; M^{(-3)}; \ldots \) of a bimodule \( FM_G \). This is given by the following result.

**Proposition 4.2.** The conjecture \( \text{pss}_R \) does not hold for a hereditary ring \( R \) if and only if there exist division rings \( F \) and \( G \) and a nonzero \( F \)-\( G \)-bimodule \( FM_G \) satisfying the following conditions:

(i) \( d_1^M = \dim F M = \infty \) and \( d_0^M = \dim M_G < \infty \).

(ii) The coordinates \( d_{-j}^M, j \geq 0 \), of the infinite dimension-sequence

\[ d_{-\infty}^M(M) = (\ldots, d_{-j}^M, \ldots, d_{-2}^M, d_{-1}^M, d_0^M, \infty) \quad (4.1) \]
associated to the bimodule \( F M_G \) are finite.

(iii) For any \( m \geq 0 \) and \( n \geq 1 \) the sequence \( (d_{m-n}, d_{m-n+1}, \ldots, d_{m-1}, d_m) \) is not a simple restriction of a dimension-sequence.

(iv) The bimodules \( F M_G = M(0), M(-1), \ldots, M(-j), \ldots \) are simple and pairwise non-isomorphic as bimodules.

(v) The ring \( R_M = \left( \begin{array}{c} F \\ G \end{array} \right) \) is right pure semisimple.

**Proof.** If the conjecture \((pss_R)\) does not hold for a hereditary ring \( R \) then according to the equivalence \((a) \iff (d)\) in Theorem 3.6 there exist a pair of division rings \( F \) and \( G \), and a nonzero \( F \)-\( G \)-bimodule \( F M_G \) such that (i) holds and every indecomposable right \( R_M \)-module is of finite length. It follows from Theorem 2.1 that (v) holds and therefore Proposition 3.2 yields (ii) and (iv), because the existence of a bimodule isomorphism \( M \cong M(-s) \) with \( s > 0 \) yields \( M_1 \cong M(-s+1) \), which implies \( \infty = d_1^M = d_{-s+1}^M < \infty \); a contradiction.

Since \( \dim_F M = \infty \) the ring \( R_M \) is not left artinian and therefore \( R_M \) is not of finite representation type. In order to prove the condition (iii) assume to the contrary that there exist \( m \geq 0 \) and \( n \geq 1 \) such that

\[
(d_{m-n}, d_{m-n+2}, \ldots, d_{m-1}, d_m) = d \in D^N.
\]

It follows from Corollary 3.5 that the ring \( R_N \) with \( N = N(-m-n) \) is of finite representation type, because \( d_{n+1}(N) = d \in D^N \). Since \( R_M \) is right pure semisimple then by Proposition 3.2 there exists a sequence of reflection functors shown there. Since \( R_N \) is of finite representation type then according to Lemma 3.1 the ring \( R_M \) is of finite representation type and we get a contradiction. This finishes the proof of the "only if" part. Since the "if" part of the proof is easy, the proposition is proved. \( \square \)

**Corollary 4.3.** The conjecture \((pss_R)\) does not hold for a ring \( R \) of right length \( l_R = 2 \) if and only if there exist division ring embedding \( G \cong F \subseteq G \) and a nonzero \( F \)-\( G \)-bimodule \( F M_G \) such that the conditions (i)-(v) in Proposition 4.2 are satisfied and there exists a bimodule isomorphism \( M(-j) \cong F G_G \) for some \( j \geq 0 \) (i.e. \( d^M_j = 1 \)).

**Proof.** Suppose that \( R \) is indecomposable, \( J = J(R) \) and \( l_R = 2 \). Then \( R \) is a right serial local ring, \( J' = 0 \) and \( \dim J_G = 1 \), where \( G = R/J \). Given a nonzero \( x \in G \) we define a ring embedding \( \sigma : G \rightarrow G \) by formula \( g \cdot x = x \sigma (g) \). It follows that the division ring \( F = \text{Im } \sigma \) is isomorphic to \( G \) and there is a bimodule isomorphism \( \sigma J_G \cong F G_G \) along \( \sigma \).

It follows from Lemma 2.3 that \((pss_R)\) does not hold for \( R \) if and only if the hereditary ring \( R_J = \left( \begin{array}{c} G \\ J_G \end{array} \right) \) is right pure semisimple and is not of finite representation type. In view of Theorem 3.4 this happens if and only if there exists \( m \geq 0 \) such that \( l \dim(J^m) = \infty \) and \( l \dim(J^j) < \infty \) for all \( j < m \). It follows from Proposition 4.2 and the discussion above that \((pss_R)\) does not hold for \( R \) if and only if the \( m \)-iterated dual bimodule \( M = J^m \) of \( J \) has the required properties. \( \square \)
Remark 4.4. Assume that $FM_G$ is a bimodule such that $\dim M_G = d^M_0 = 1$ and the conditions (i)-(v) in Proposition 4.2 are satisfied. Then there exists a division ring embedding $F \subseteq G$ such that

$$FM_G \cong rG_G, \quad M'(-1) = \text{Hom}_G(rG_G, G) \cong \text{G}_G, \quad M'(1) = \text{Hom}_F(rG_G, F),$$

$$d^M_{-1} = \dim G_G < \infty, \quad d^M_1 = \dim F_G = \infty \quad \text{and} \quad d^M_{-j} < \infty \text{ for all } j \geq 2.$$

Then the existence of such a bimodule $FM_G$ is an infinite version of an Artin problem for division ring extensions (see [5], [6] and [20]).

It seems to us that if $R_M$ is right pure semisimple then the sequence $\{d^M_j\}_{j \leq -1}$ is bounded, and $d^M_j = 1$ for some $j \in \mathbb{Z}$.

Definition 4.5. A generalized Artin problem of the infinite dimension-type $d_{-\infty} = (\ldots, d_{-j}, \ldots, d_{-1}, d_0, \infty)$ is the existence of a bimodule $FM_G$ satisfying the conditions (i)-(v) in Proposition 4.2, and such that $d_{-\infty}(FM_G) = d_{-\infty}$.

Remark 4.6. (a) It follows from our discussion above that a construction of a counterexample to the conjecture (pss$_R$) for hereditary rings $R$ (see Proposition 4.2), or for local rings $R$ of length two (see Corollary 4.3) contains a solution of a generalized Artin problem of an infinite dimension-type $d_{-\infty} = (\ldots, d_{-j}, \ldots, d_{-1}, d_0, \infty)$.

This shows that the existence of a counterexample as well as the affirmative solution of the conjecture (see Corollary 3.7) depends strongly on a generalized Artin problem of an infinite dimension-type, and shows that the solution of the conjecture (pss$_R$) leads to deep non-commutative algebra problems. We believe that the conjecture has a positive solution in general, or at least for PI-rings.

(b) It seems to us that if (pss$_R$) does not hold then there exists a counterexample of the form $R_M = (F, FM_G)$, where $FM_G$ is a bimodule such that $d_{-\infty}(FM_G) = (\ldots, 2, 2, 2, 2, 1, \infty)$. This means that there exists a division ring embedding $F \subseteq G$ such that $\dim F_G = \infty, \dim G_G = 2, \quad FM_G \cong rG_G$ and $d^M_{-j} = \text{r.dim } M'(-j) = 2$ for all $j \geq 2$. Unfortunately we are not able to construct such a division ring embedding $F \subseteq G$.

5. Consequences of recent results of I. Herzog

After the paper has been submitted I received the preprint [16] of Herzog, where the conjecture (pss$_R$) is proved for any PI-ring $R$ by reducing the problem to the case when $R$ is hereditary (studied in [26] and in the present paper), and then by applying the results of [26, Section 3]. Furthermore, Herzog has proved in [16] that (pss$_R$) holds for an arbitrary ring $R$ if and only if it holds for any hereditary ring $R$ of the form $R_M = (F, FM_G)$, where $F$ and $G$ are division rings, and $FM_G$ is an $F$-$G$-bimodule.

$^2$This section was added to the revised version of the paper in January 1994.
In other words, the conjecture \((\text{pss}_R)\) is reduced in [16] just to the case when \(R\) is a hereditary ring of the form \(R_M\) studied in details in the present paper.

As a consequence of Herzog's results together with our main results of this paper we get the following important fact.

**Corollary 5.1.** The following conditions are equivalent.

(a) Every right pure semisimple ring is of finite representation type.

(b) For any pair of division rings \(F\) and \(G\), and for any simple \(F\)-\(G\)-bimodule \(FM_G\) such that \(\dim M_G = 0\) one can construct an indecomposable right module of infinite length over the hereditary ring \(R_M = \begin{pmatrix} F & M_G \\ 0 & G \end{pmatrix}\).

(c) For any pair of division rings \(F\) and \(G\), and for any simple \(F\)-\(G\)-bimodule \(FM_G\) such that \(\dim M_G = 0\) and \(\dim F = \infty\) one can construct a sequence

\[X_1 \overset{f_1}{\rightarrow} X_2 \rightarrow \cdots \rightarrow X_m \overset{f_m}{\rightarrow} X_{m+1} \rightarrow \cdots\]

of indecomposable right \(R\)-modules \(X_1, X_2, \ldots\) of finite length connected by non-isomorphisms \(f_1, f_2, \ldots\) such that \(f_m f_{m-1} \cdots f_2 f_1 \neq 0\) for any \(m > 1\).

(d) For any infinite dimension-sequence \(d_{-\infty}\), all generalized Artin problems of dimension-type \(d_{-\infty}\) have no solution. □

**Remark 5.2.** (a) Although the pure semisimplicity conjecture remains still an open problem, the above corollary substantially "localize" the crucial difficulty, which remains to be solved, as one of the equivalent conditions \((b)-(d)\) above. The corollary should essentially help solve the pure semisimplicity conjecture, because the conditions \((b)-(d)\) are formulated in linear algebra terms on bimodules over division rings, and therefore they are much more easy to handle than the original conjecture \((\text{pss}_R)\).

(b) It follows from the above discussion and our results of Section 4 that a counterexample \(R\) to the conjecture \((\text{pss}_R)\) can be constructed if and only if a generalized Artin problem of dimension-type \(d_{-\infty}\) has a solution for some infinite dimension-sequence \(d_{-\infty}\). This observation should be very useful in producing a counterexample to the conjecture. On the other hand this warns the reader that constructions of counterexamples to the conjecture are much more difficult than the solution of the Artin problem for division ring extensions given by Schofield in [20]. It may frighten people of finding an easy solution to the pure semisimplicity problem.

(c) The above discussion shows that studying generalized Artin problems of infinite dimension-types should help in solving the pure semisimplicity conjecture or in finding a counterexample.

**Note added in proof.** Let us present a useful addition to Remark 4.6(b). Assume that there exists a division ring embedding \(F \subseteq G\) such that \(\dim FG = \infty\), \(\dim GF = 2\), and \(\text{r.dim}(F_G)^{(j)} = 2\) for all \(j \geq 2\) as in Remark 4.6(b). This means that \(d_{-\infty}(FG_G) = (\ldots, 2, 2, 1, \infty)\).

It follows that the hereditary ring \(R_G = \begin{pmatrix} F & G \\ 0 & G \end{pmatrix}\) is of infinite representation type and by applying Lemma 3.1 and Proposition 3.2 one can prove the following.
(a) Every indecomposable preprojective module in \( \text{mod}(R_G) \) is isomorphic to one of the projective modules \( P_0 = (0, G) \) and \( P_1 = (G_F, G) \).

(b) Every indecomposable preinjective module in \( \text{mod}(R_G) \) is isomorphic to one of the modules in the sequence

\[ \cdots \rightarrow Q_m^{(0)} \rightarrow Q_{m-1}^{(0)} \rightarrow \cdots \rightarrow Q_2^{(0)} \rightarrow Q_1^{(0)} \rightarrow Q_0^{(0)} \]

of irreducible epimorphisms, where \( Q_0^{(0)} = (F, 0) \), \( Q_1^{(0)} = E(P_0) = (G_F, G) \) (see Proposition 3.2).

(c) \( \dim Q_m^{(0)} = (m + 1, m) \) for any \( m \geq 0 \).

(d) For any \( m \geq 0 \) there exists an exact sequence \( 0 \rightarrow P_0 \xrightarrow{\mu_m} P^m \rightarrow Q_m^{(0)} \rightarrow 0 \), where \( \mu_m \) is an irreducible map.

(d) Every indecomposable module in \( \text{mod}(R_G) \) is isomorphic with one of the modules \( P_0, P_1, Q_0^{(0)}, Q_1^{(0)}, \ldots, Q_m^{(0)}, \ldots \).

It follows that \( R_G \) is right pure semisimple and not of finite representation type.

Proofs of the above statements will be published in our forthcoming paper “An Artin problem for division ring extensions and the pure semisimplicity conjecture” in *Archiv der Mathematik*.

### References


[27] D. Simson, Indecomposable modules over one-sided serial local rings and right pure semisimple rings, Tsukuba J. Math. 7 (1983) 87–103.


