View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

ADVANCES IN MATHEMATICS 31, 110–130 (1979)

Sacks Forcing Does Not Always Produce a Minimal Upper Bound

FRED G. ABRAMSON*

University of Wisconsin–Milwaukee, Milwaukee, Wisconsin 53201

THEOREM. There is a countable admissible set, $\mathfrak{A}$, with ordinal ω_1^{CK} such that if S is Sacks generic over $\mathfrak{A}$ then $\omega_1{}^S > \omega_1^{CK}$ and S is a nonminimal upper bound for the hyperdegrees in $\mathfrak{A}$. (The same holds over $\mathfrak{A}$ for any upper bound produced by any forcing which can be construed so that the forcing relation for $\mathbf{\Sigma}_1$ formulas is $\mathbf{\Sigma}_1$.) A notion of forcing, the "delayed collapse" of ω_1^{CK}, is defined. The construction hinges upon the symmetries inherent in how this forcing interacts with $\mathbf{\Sigma}_1$ formulas. It also uses Steel trees to make a certain part of the generic object $\mathbf{\Sigma}_1$ over the final inner model, $\mathfrak{A}$, and, indeed, over many generic extensions of $\mathfrak{A}$.

INTRODUCTION

One of the very first uses of forcing was to produce a standard (atomless) model of set theory which violates the Axiom of Choice (Cohen [3]). Since then forcing has been a major technique in choiceless set theory. It has been used to compare the strengths of various weakened forms of AC, to measure just how much choice is needed for certain classical results of mathematics, and to provide relative consistency proofs for some of the consequences of axioms which contradict AC, particularly the Axiom of Determinacy. (The publications in this area are too numerous to attempt listing them here.)

The main theorem of this paper can be regarded as a result in the area of "choiceless admissibility theory." Given the wide use of admissible sets in many branches of logic, it is not surprising that there are naturally arising questions, such as the minimal upper bound problem for sets of hyperdegrees, which lead to work in this area. Proofs in this area, some of which are referred to below, tend not to be simply a matter of pushing through the set theoretical arguments in a weaker context. Instead, they require arguments particularly designed for their interaction with Σ_1-admissibility.

Sacks [7] shows that if $\mathfrak{A}$ is any countable admissible set satisfying $\mathbf{\Sigma}_1$-dependent choice, then the set of hyperdegrees contained in $\mathfrak{A}$ has a minimal upper bound. This result partially lifts the earlier result of Sacks [6] that *any* countable set of Turing degrees has a minimal upper bound. Hyperdegrees differ from

* Research for this paper was supported, in part, by a grant from the Graduate School of the University of Wisconsin–Milwaukee, and by NSF Grant MCS76-07025.

0001-8708/79/010110–21$05.00/0
Copyright © 1979 by Academic Press, Inc.
All rights of reproduction in any form reserved.

Turing degrees largely because of one fact: The upper bound on the length of Turing computations from a real x is independent of x, i.e., is always ω, while the corresponding upper bound for hyperarithmetical computations from x is ω_1^x, which does depend on x. A major obstacle in lifting theorems to hyperdegrees is to find a way to control ω_1^x. Sacks' construction of a minimal upper bound for certain sets of hyperdegrees appears to make a crucial use of $\mathbf{\Sigma}_1$-DC in surmounting this obstacle. The main result of this paper is that $\mathbf{\Sigma}_1$-DC actually is crucial to Sacks' argument. Moreover, we rule out as a technique for producing a minimal upper bound any type of forcing which possesses what seems to be an essential property for success in forcing over an admissible set.

Friedman [4] has previously constructed an admissible set violating Σ_1-DC by using "proof theoretic" techniques. Here we have a forcing extension of $L_{\omega_1^{CK}}$ which violates $\mathbf{\Sigma}_1$-DC. (It can be shown that $L_{\omega_1^{CK}}$ is a $\mathbf{\Sigma}_1$-elementary substructure of our model, so the parameter is required in the violation.)

The very way in which Sacks uses $\mathbf{\Sigma}_1$-DC suggests the approach which produces the model $\mathfrak{A}$ mentioned in the abstract. Note that if x is an upper bound for $\mathfrak{A}$'s hyperdegrees then x imposes a wellordering on $\mathfrak{A}$'s reals, i.e., the canonical wellordering of $L_{\omega_1^x}(x)$. Now, as we see in Section 1, to ensure that a real which is Sacks-generic over $\mathfrak{A}$ is a nonminimal upper bound, it really suffices, in this context, to violate the main goal of Sacks' construction, namely, the preservation of the admissibility of ω_1^{CK}. So the obvious approach is to make $\mathfrak{A}$ have a failure of $\mathbf{\Sigma}_1$-DC such that any imposition of a wellordering on $\mathfrak{A}$ results in collapsing ω_1^{CK} to ω. Of course, a certain delicate balance must be achieved in order to do this without collapsing ω_1^{CK} within $\mathfrak{A}$ itself.

Now, this program cannot succeed exactly as stated. If $\mathfrak{A}$ is any admissible set with ordinal ω_1^{CK} then a Barwise-compactness with type-omitting argument as in Section 3 of Sacks [8] always produces an upper bound, S, on $\mathfrak{A}$'s hyperdegrees with $\omega_1^S = \omega_1^{CK}$. Nonetheless, this program can be carried out for certain generically produced upper bounds, S, in particular for S Sacks generic over $\mathfrak{A}$. The next paragraph outlines how this is accomplished. But first note in passing that there seems to be little point in trying to have the ordinal of $\mathfrak{A}$ be other than ω_1^{CK}, for by [8, Corollary 3.18], Sacks forcing could only fail to produce a minimal upper bound if the ordinal of $\mathfrak{A}$ is ω_1^x for some $x \in \mathfrak{A}$.

Here is how the construction goes: Start with $L_{\omega_1^{CK}}$. Generically add for each $n < \omega$ a Steel tree T_n (as in Steel [10]) and a single distinguished path P_n through T_n. For the moment, view $M = L_{\omega_1^{CK}}[W]$ as the ground model, where W is (a real coding) the generic object above. Now do a certain forcing, the "delayed collapse" of ω_1^{CK}, over M to produce a tree T such that T is a subtree of $(\omega_1^{CK})^{<\omega}$ (i.e., T's nodes are finite sequences of recursive ordinals, ordered by extension), such that any path through T has ordinals unbounded in ω_1^{CK}. Now pass to an inner model, $\mathfrak{A}$, obtained by adding to $L\omega_1^{CK}$ a real coding all the Steel trees, and adding (very carefully) P_n for exactly those n such that

$n = r(\sigma)$ for some node $\sigma \in T$ (where r is a fixed ω_1^{CK}-recursive map: $(\omega_1^{CK})^{<\omega} \to^{1-1} \omega$.) (Note that $M \not\subseteq \mathfrak{A}$.) What we first prove is that Sacks forcing over $\mathfrak{A}$ produces no paths through any T_n except those P_n purposely put into $\mathfrak{A}$. Thus T is $\mathbf{\Sigma}_1$ over $L_{\omega_1^{CK}}(S)$, and now it is easy to see that $\omega_1^S > \omega_1^{CK}$, which, as is shown in Section 1, is enough, provided that $\mathfrak{A}$ itself is admissible. Showing this fact is, indeed, the main difficulty and relies entirely upon the properties of the delayed collapse. Note one thing—the counterexample to Sacks' technique hinges on exactly the same point as the technique itself! One must preserve the admissibility of the ordinal one starts with.

Organization of This Paper

In Section 1 we define the forcings to be used, construct the model $\mathfrak{A}$, and use a Steel-type symmetry argument to show that Sacks forcing adds no paths other than those purposely put into $\mathfrak{A}$. The Main Lemma, i.e., that $\mathfrak{A}$ is admissible, and a Little Lemma are stated, and the results given in the abstract are proved assuming these lemmas. (In this section, the phrase "can be construed so that ..." is made precise.)

In Section 2, the main technical work takes place. Here the properties of the delayed collapse are developed to prove the lemmas required in Section 1.

Section 3 concludes with some remarks and open questions.

Conventions

When forcing over $L_{\omega_1^{CK}}$ we use the ramified language of Cohen [3]. However, we always use "$p \mid\vdash_{\mathscr{P},\mathscr{B}} \mathscr{F}$" for p in the partial ordering $\mathscr{P}$ and $\mathscr{B}$ any structure to mean the standard weak forcing, i.e., that $\mathscr{F}$ holds in any extension of $\mathscr{B}$ made by a $\mathscr{B}$-generic subset of $\mathscr{P}$ containing p. Of course, by the usual syntactic analysis, one can prove, even in the context of $L_{\omega_1^{CK}}$ rather than a model of ZF, that for a generic G, $L_{\omega_1^{CK}}[G] \models \mathscr{F}$ iff $\exists\, p \in G\, (p \mid\vdash \mathscr{F})$. Partial orderings used in this context are often proper classes over the ground model, but in the case of one actually an element of $L_{\omega_1^{CK}}$, it is routine to check that $L_{\omega_1^{CK}}[G]$ is admissible. For W a real, we use the obvious relativized notions over $L_{\omega_1^{CK}}[W]$. (Forcing over sufficiently strange models, such as the $\mathfrak{A}$ constructed here, requires more care. This issue is discussed further at the end of Section 1 in the definition of $\mathbf{\Sigma}_1$-honesty.) "Sacks forcing over $\mathfrak{A}$" means forcing with hyperarithmetically-pointed perfect sets coded in $\mathfrak{A}$. We assume familiarity with some of the results in Sections 1–3 of Sacks [8].

1. The Model $\mathfrak{A}$

$\mathfrak{A}$ will be an inner model of $L_{\omega_1^{CK}}[W \times G]$, where $W \times G$ is a generic subset of $\mathscr{S} \times \mathscr{D}$. $\mathscr{S}$ is the direct limit of ω-many copies of the Steel forcing which produces a subtree of $\omega^{<\omega}$ with one distinguished path. $\mathscr{D}$ is the "delayed collapse." We will use the usual fact about product forcing, namely, that $W \times G$ is $L_{\omega_1^{CK}}$-generic on $\mathscr{S} \times \mathscr{D}$ iff W is $L_{\omega_1^{CK}}$-generic on $\mathscr{S}$ and G is $L_{\omega_1^{CK}}[W]$-generic on $\mathscr{D}$. It will often be convenient to work over $M \equiv L_{\omega_1^{CK}}[W]$, which will be an admissible set.

The Partial Ordering $\mathscr{S}$ and the Model M

We first define one-path Steel forcing, $(Q, \leqslant_Q)$. (*Notational convention*: We drop subscripts on "$\leqslant$" when the context makes the meaning clear.) Let $<_R$ be a recursive linear ordering on ω with wellfounded initial part of order type ω_1^{CK}. We take $<_R$ to have a largest element, b, and adjoin a new symbol ∞, with the convention $\infty >_R \infty >_R b$. For $\alpha < \omega_1^{CK}$, $i(\alpha)$ denotes the integer whose ordertype in $<_R$ is α. A typical condition (u, v) of Q is a nonempty finite subtree, u, of $\omega^{<\omega}$, together with a tagging $v: u \to \omega \cup \{\infty\}$ such that

(i) $\sigma \subsetneq \tau \in u \to v(\sigma) >_R v(\tau)$,

(ii) $v(\varnothing) = \infty$, and each level of u has at most one σ with $v(\sigma) = \infty$.

$$(u', v') \leqslant_Q (u, v) \qquad \text{iff} \quad u' \supseteq u \text{ and } v' \supseteq v.$$

Note that Q has a greatest element, which we denote by $\mathbb{1}_Q$, with $u = \{\varnothing\}$ and $v(\varnothing) = \infty$. Any generic subset H of Q builds a subtree of $\omega^{<\omega}$ with a distinguished path given by $\{\sigma \mid \exists (u, v) \in H \ (\sigma \in u \text{ and } v(\sigma) = \infty)\}$. (This subtree, of course, has many paths in the real world other than the distinguished one.)

$\mathscr{S}$ is just $\{f: \omega \to Q \mid f(n) = \mathbb{1}_Q$ for all but finitely many $n\}$ and $g \leqslant_{\mathscr{S}} f$ iff $\forall n[g(n) \leqslant_Q f(n)]$.

Fix W an $L_{\omega_1^{CK}}$-generic subset of $\mathscr{S}$.

We take $W \in L_{\omega_1^{CK}+17}$. For each n, $W_n = \{f(n) \mid f \in W\}$ is $L_{\omega_1^{CK}}$-generic on Q. Let T_n denote the tree built by W_n, and P_n its distinguished path. Let $\hat{W} = \{(n, \sigma) \mid \sigma \in T_n\}$. We identify W with a real coding W in some standard way. As $(\mathscr{S}, \leqslant_{\mathscr{S}})$ is an element of $L_{\omega_1^{CK}}$, $M \equiv L_{\omega_1^{CK}}(W)$ is an admissible set, and clearly $\hat{W}$ and $\{(n, P_n) \mid n < \omega\}$ are $\in M$.

The Partial Ordering $\mathscr{D}$

Notation. For X any ordered set, $[X]^{<\omega} \equiv \{f: n \to X \mid n < \omega$ and f is strictly increasing$\}$. This set forms a subtree of $X^{<\omega}$. For any finite sequence σ of length >0, $\sigma\cdot$ denotes the last element of σ. For any finite sequence σ and any x, $\sigma \frown x$ denotes the extension of σ of length one greater than that of σ such that $(\sigma \frown x)\cdot = x$.

A typical condition of $\mathscr{D}$ is a quadruple, (c, t, u, d), where

(i) $c \in [\omega_1^{CK} \cap \mathrm{Lim}]^{<\omega}$,

(ii) t is a finite nonempty subtree of $[\omega_1^{CK}]^{<\omega}$,

(iii) $u: t \to 2$ such that $u(\varnothing) = 1$ and whenever $\sigma \in t$ and $u(\sigma) = 0$ then for no $\tau \supsetneq \sigma$ is $\tau \in t$.

(iv) $d: u - \{\varnothing\} \to \omega_1^{CK}$ in such a way that if $\sigma \neq \varnothing$ and $u(\sigma) = u(\sigma ^\frown \delta) = 1$ then

$$\textit{either } d(\sigma ^\frown \delta) < d(\sigma)$$
$$\textit{or } \exists\, j \in \operatorname{dom}(c)\ [\sigma^{\cdot} < c(j) \leqslant \delta].$$

$(c', t', u', d') \leqslant_{\mathscr{D}} (c, t, u, d)$ just in case $c' \supseteq c$, $t' \supseteq t$, $u' \supseteq u$, and $d' \supseteq d$.

c is thought of as a condition in a Cohen collapse of ω_1^{CK} to ω. Clearly, if G is M-generic on $\mathscr{D}$, the c's from the conditions in G give rise to an ω-sequence unbounded in ω_1^{CK}. This sequence is called C_G (or just C when G is clear from context). The t's and u's are thought of as putting together a subtree of $[\omega_1^{CK}]^{<\omega}$, namely, $T_G \equiv \{\sigma \mid \exists\, (c, t, u, d) \in G\, [\sigma \in t \wedge u(\sigma) = 1]\}$. $d(\sigma)$ is thought of as the *delay* attached to the node σ of T_G. D_G is the map put together by the d's in G. (Note that $\operatorname{dom}(D_G)$ includes certain nodes $\notin T_G$, a point we return to in the proof of Sublemma 2.9.)

Here is the main idea behind the delays. Any infinite path through T_G must infinitely often jump to a higher ordinal mentioned in the range of C_G. Any given node can specify an ordinal delay before which the next jump occurs, and in going from node to node the path may avoid the next jump, but only at the expense of decreasing the delay; thus it eventually must jump again, at which time it is free to specify a new delay.

The above discussion easily gives a proof of

PROPOSITION 1.1. *If G is M-generic on $\mathscr{D}$ then for P any (infinite) path through T_G, the map $n \mapsto$ (the unique $\sigma \in P$ of length $n + 1$)$^{\cdot}$ is an unbounded map: $\omega \to \omega_1^{CK}$.*

For the remainder of Section 1, *fix G M-generic on $\mathscr{D}$. We take $G \in L_{\omega_1^{CK}+29}$.* The presence of C_G clearly makes $M[G]$ nonadmissible (if G is available as a predicate). $\mathfrak{A}$ is a carefully built inner model of $M[G]$ over which $T \equiv T_G$ is $\mathbf{\Sigma}_1$, but over which no map resembling $C \equiv C_G$ is $\mathbf{\Sigma}_1$.

Notation. Let r be a fixed Σ_1-over $L_{\omega_1^{CK}}$ map: $[\omega_1^{CK}]^{<\omega} \to^{1-1} \omega$. $T(\alpha) \equiv T_G(\alpha) \equiv \{\sigma \in T \mid \sigma = \varnothing$ or $(\forall\, \tau \subseteq \sigma,\ \tau \neq \varnothing)\ (\tau^{\cdot} < \alpha$ and $D_G(\tau) < \alpha)\}$. So $T = \bigcup_{\alpha < \omega_1^{CK}} T(\alpha)$. $P_{n,\alpha}$ denotes P_n if $r^{-1}(n) \in T(\alpha)$ and denotes $\varnothing$ otherwise.[1] For any structure $\mathscr{B}$, $\operatorname{FODO}(\mathscr{B}) \equiv$ all subsets of the domain of $\mathscr{B}$ first-order definable over $\mathscr{B}$ with parameters from $\mathscr{B}$.

DEFINITION OF $\mathfrak{A}$. $\mathfrak{A} \equiv \mathfrak{A}(\omega_1^{CK})$, where $\mathfrak{A}(0) = \varnothing$, $\mathfrak{A}(\lambda) = \bigcup_{\alpha<\lambda} \mathfrak{A}(\alpha)$ for $\lambda \in \mathrm{Lim}$, and $\mathfrak{A}(\alpha + 1) = \operatorname{FODO}(\mathfrak{A}(\alpha), \in, \hat{W}, P_{n,\alpha})_{n<\omega}$. $\mathfrak{A}$ is clearly a transitive

[1] We write "$r^{-1}(n) \in T(\alpha)$" to mean "$r^{-1}(n)$ is defined and is in $T(\alpha)$."

set, $\mathbf{\Sigma}_1$ over $M[G]$ (allowing W and G to appear in the defining formula, of course).

Notation. For H an arbitrary generic subset of $\mathscr{S} \times \mathscr{D}$ we use $\mathfrak{A}_H$ to denote the $\mathfrak{A}$ built above from H.

We now use arguments like those in Steel [9] to show that if S is Sacks-generic over $\mathfrak{A}$ then $L_{\omega_1^{CK}}(S) \cap \{P \mid \exists\, n\ (P \text{ is a path through } T_n)\} = \{P_n \mid r^{-1}(n) \in T\}$.

DEFINITION. For $q = (u, v)$ and $q^* = (u^*, v^*) \in Q$, and $\alpha < \omega_1^{CK}$,

q^* is an *$\omega\alpha$-absolute retagging* of q iff $u = u^*$ and $\forall\, \sigma \in u[(v(\sigma) <_R i(\omega\alpha) \to v^*(\sigma) = v(\sigma)) \wedge (v(\sigma) \geqslant_R i(\omega\alpha) \to v^*(\sigma) \geqslant_R i(\omega\alpha))]$.

q^* is a *distinguished-path-preserving $\omega\alpha$-absolute retagging of q* if, in addition,

$$\forall\, \sigma \in u[v(\sigma) = \infty \leftrightarrow v^*(\sigma) = \infty].$$

Note that these properties are symmetric in q and q^*.

Given N a finite subset of ω, and $f, g \in \mathscr{S}$, then f is an N-good $\omega\alpha$-absolute retagging of g iff

$$\forall\, n[f(n) \text{ is an } \omega\alpha\text{-absolute retagging of } g(n)]$$

and

$$\forall\, n[n \notin N \to f(n) \text{ is a distinguished-path-preserving } \omega\alpha\text{-absolute retagging of } g(n)].$$

The following lemma is a routine generalization of a lemma of Steel.

LEMMA 1.2. *Let N be a finite subset of ω. Suppose f is an N-good $\omega\alpha$-absolute retagging of g and $\beta < \alpha$. Then*

(i) $\forall f' \leqslant f\ \exists\, g' \leqslant g$ *[g' is an N-good $\omega\beta$-absolute retagging of f'],*

(ii) $\forall g' \leqslant g\ \exists f' \leqslant f$ *[f' is an N-good $\omega\beta$-absolute retagging of g'].*

Terminology. For $p = (f, e) \in \mathscr{S} \times \mathscr{D}$ and $p' = (g, e) \in \mathscr{S} \times \mathscr{D}$, we say that p' is an N-good $\omega\alpha$-absolute retagging of p iff that property holds for f and g.

Until further notice, $\Vdash$ denotes $\Vdash_{\mathscr{S}\times\mathscr{D}, L_{\omega_1^{CK}}}$. $\mathscr{L}$ denotes the ramified language appropriate to this forcing. We now need a specialized analysis for the forcing of Σ_1 facts over $\mathfrak{A}$. More specifically, let $\hat{\mathscr{L}}$ be the ramified language appropriate to the construction of $\mathfrak{A}$. In particular, $\hat{\mathscr{L}}$ has terms for $\hat{W}$ and for each $P_{n,\alpha}$, and the appropriate abstraction terms of rank $< \omega_1^{CK}$ based on these. Now, of course, for $\mathscr{F} \in \hat{\mathscr{L}}$ there is a natural way of embedding statements of the form "$\mathfrak{A} \models \mathscr{F}$" into $\mathscr{L}$, so we already know what "$p \Vdash \mathfrak{A} \models \mathscr{F}$" means.

However, the symmetry arguments we need are facilitated by giving a direct syntactic description of when $p \mid\vdash \mathfrak{A} \models \mathscr{F}$, where $\mathscr{F}$ is a ranked sentence of $\hat{\mathscr{L}}$. First note that, letting $\mathbb{1}_{\mathscr{S}}$ denote $\lambda n[\mathbb{1}_Q]$, i.e., the weakest condition of $\mathscr{S}$, it is easy to see that for $(f, e) \in \mathscr{S} \times \mathscr{D}$ and $\sigma \in [\omega_1^{CK}]^{<\omega}$, and any $\alpha < \omega_1^{CK}$,

$$(f, e) \mid\vdash \sigma \in T(\alpha) \leftrightarrow (\mathbb{1}_{\mathscr{S}}, e) \mid\vdash \sigma \in T(\alpha)$$

and

$$(f, e) \mid\vdash \sigma \notin T(\alpha) \leftrightarrow (\mathbb{1}_{\mathscr{S}}, e) \mid\vdash \sigma \notin T(\alpha).$$

Definition of $\mid\vdash'$. (i) $(f, e) \mid\vdash' \mathfrak{A} \models \rho \in P_{n,\alpha}$ iff $(\mathbb{1}, e) \mid\vdash r^{-1}(n) \in T(\alpha)$ and $f(n) \mid\vdash_{Q, L_{\omega_1^{CK}}} \rho \in \underline{P}$ (where $\underline{P}$ is the term for the distinguished path built by forcing over Q).

(ii) $(f, e) \mid\vdash' \mathfrak{A} \models (n, \rho) \in \hat{W}$ iff $f(n) \mid\vdash_{Q, L_{\omega_1^{CK}}} \rho \in \underline{U}$ (where $\underline{U}$ is the term for the tree built by forcing over Q).

(iii) $p \mid\vdash' \mathfrak{A} \models \sim\mathscr{F}$ iff no $q \leqslant p$ has $q \mid\vdash' \mathfrak{A} \models \mathscr{F}$.

(iv) $p \mid\vdash' \mathfrak{A} \models \mathscr{F} \wedge \mathscr{G}$ iff $p \mid\vdash' \mathfrak{A} \models \mathscr{F}$ and $p \mid\vdash' \mathfrak{A} \models \mathscr{G}$.

(v) $p \mid\vdash' \mathfrak{A} \models \exists^{\alpha} x \mathscr{F}(x)$ iff there is a term τ of rank $<\alpha$ $[p \mid\vdash' \mathfrak{A} \models \mathscr{F}(\tau)]$.

(vi) $p \mid\vdash' \mathfrak{A} \models \tau \in \nu$ iff when ν is the abstraction term for $\mathscr{F}$ we have [rank $(\tau) <$ rank $(\nu) \wedge p \mid\vdash' \mathfrak{A} \models \mathscr{F}(\tau)$] or [rank $(\tau) \geqslant$ rank $(\nu) \wedge$ ($\exists$ term τ' of rank $<$ rank (ν)) $(p \mid\vdash' \mathfrak{A} \models \tau = \tau' \wedge \mathscr{F}(\tau'))$].

(vii) $p \mid\vdash' \mathfrak{A} \models \tau = \nu$ iff for $\delta =$ rank $(\tau) \cup$ rank (ν), $p \mid\vdash' \sim \exists^{\delta} x$ $[x \in \tau \not\leftrightarrow x \in \nu]$.

It is routine to check, by induction on the rank of $\mathscr{F}$ (suitably defined), that whenever H is $L_{\omega_1^{CK}}$-generic on $\mathscr{S} \times \mathscr{D}$, then $\mathfrak{A}_H \models \mathscr{F}$ iff $\exists p \in H[p \mid\vdash' \mathfrak{A} \models \mathscr{F}]$. (The rank of a formula $\mathscr{F}$ is an ordinal depending both on the ordinals mentioned in $\mathscr{F}$ and on the syntactic complexity of $\mathscr{F}$. For a term, τ, we say rank of $\tau = \alpha$ to mean that τ is a definition of a set over $\mathfrak{A}(\alpha)$.)

Lemma 1.3. *Assume*

(0) $\alpha > 0$.

(i) *$\mathscr{F}$ is a sentence of $\hat{\mathscr{L}}$ of rank $\leqslant \alpha$.*

(ii) *N is a finite subset of ω, and $(\mathbb{1}, e) \mid\vdash \sigma \notin T$ whenever $r(\sigma) \in N$.*

(iii) *$p \equiv (f, e)$, and $p' \equiv (f', e)$ is an N-good $\omega\alpha$-absolute retagging of p.*

Then $p \mid\vdash' \mathfrak{A} \models \mathscr{F}$ iff $p' \mid\vdash' \mathfrak{A} \models \mathscr{F}$.

Proof. Fix N, and restrict attention to $\mathscr{R} \equiv \{q \in \mathscr{S} \times \mathscr{D} \mid q \leqslant (\mathbb{1}, e)\}$. One uses clauses (i)–(vii) to check the assertion by simultaneous transfinite induction on α and the rank of $\mathscr{F}$. The only interesting cases are clauses (i), (ii), and (iii).

Clause (ii) goes through as any $\omega\alpha$-absolute retagging of a condition in Q does not change forcing of statements of form $\rho \in \underset{\sim}{U}$.

If there is a $\sigma \in [\omega_1^{CK}]^{<\omega}$ and $r(\sigma) = n \notin N$, then (i) goes through because $f'(n)$ is a distinguished-path-preserving retagging of $f(n)$. If $r(\sigma) = n \in N$, or $r^{-1}(n)$ is undefined, then (i) goes through as every condition in $\mathscr{R} \Vdash \mathfrak{A} \models \rho \notin P_{n,\alpha}$.

Clause (iii) is immediate from Lemma 1.2. ▌

Corollary 1.4. *Let S be Sacks-generic over $\mathfrak{A}$. Then* $\{P \in L_{\omega_1^{CK}}(S) \mid \exists\, n\ [P$ *a path through* $T_n]\} = \{P_n \mid r^{-1}(n) \in T\}$.

Proof. We need only show that whenever $P \in L_{\omega_1^{CK}}(S)$ and P is a path through T_n then $P = P_n$ and $r^{-1}(n) \in T$. Suppose P counterexamples this. So there is a Sacks condition, X, in $\mathfrak{A}$ such that S satisfies X, and a $\delta < \omega_1^{CK}$ with $X \Vdash_{\text{Sacks},\mathfrak{A}}$ $\{\delta\}^S$ is a path through T_n. Of course, by $\Vdash_{\text{Sacks},\mathfrak{A}}$ we mean the canonical weak forcing over $\mathfrak{A}$ given rise to by the set of Sacks conditions in $\mathfrak{A}$. However, Sacks' argument shows that there is a Σ_1 formula $\mathscr{G}$ such that $\forall\, Y(\mathfrak{A} \models \mathscr{G}(Y) \rightarrow Y \Vdash_{\text{Sacks},\mathfrak{A}} \{\delta\}^S$ is a path through $T_n)$, and $\{Y \mid \mathfrak{A} \models \mathscr{G}(Y)\}$ is dense above X. ($\mathscr{G}$ is the formula saying that *every* path S' through Y makes $\{\delta\}^{S'}$ a path through T_n.) So S satisfies such a $Y \in \mathfrak{A}$. As $\mathscr{G}$ is Σ_1, there is an ordinal $\alpha < \omega_1^{CK}$ and a sentence $\mathscr{F}$ of $\hat{\mathscr{L}}$ of rank α which says that $\mathfrak{A} \models \mathscr{G}(\underset{\sim}{Y})$. ($\underset{\sim}{Y}$ is a name for Y.) So if $p \in \mathscr{S} \times \mathscr{D}$ and $p \Vdash \mathscr{F}$ then for any generic H satisfying p and any Sacks generic S' over $\mathfrak{A}_H$, with S' satisfying the denotation of $\underset{\sim}{Y}$ in $\mathfrak{A}_H$, we have that $\{\delta\}^{S'}$ is a path through $(T_n)_H$. So, in particular, such a path exists in $(T_n)_H$.

Case I. $r^{-1}(n)$ is undefined or $r^{-1}(n) \notin T$. Let $p \in G$ be such that $p \Vdash' \mathfrak{A} \models \mathscr{F}$, and if $r(\sigma) = n$, also $p \Vdash' \sigma \notin T$. Let $N = \{n\}$. We assume that we have arranged the $X, Y, \mathscr{F}, \mathscr{G}$ above so that for some particular $\rho \in \omega^{<\omega}, \rho \neq \varnothing$, $Y \Vdash_{\text{Sacks},\mathfrak{A}} \{\delta\}^S$ is a path through ρ in T_n. Now, just let $p' = (f', e)$ be an N-good $\omega\alpha$-absolute retagging of p such that $(f'(n))(\rho) = i(\delta)$ for some $\delta < \omega^{CK}$. By Lemma 1.3, $p' \Vdash' \mathfrak{A} \models \mathscr{F}$, so, in particular, there is a path through ρ in $(T_n)_H$ whenever H is generic and $p' \in H$, which is absurd as $(T_n)_H$ is wellfounded below ρ for any such H.

Case II. $r^{-1}(n) \in T$. Then there is some $\rho \notin P_n$ with $\{\delta\}^S$ a path through ρ in T_n. Let $p = (f, e) \in G$ such that $p \Vdash \mathfrak{A} \models \mathscr{F}$ where, once again, we have made the arrangements as in Case I for this particular ρ. As $\rho \notin P_n$, $(f(n))(\rho) \neq \infty$. So apply Lemma 1.3 with $N = \varnothing$ to $p' = (f', e)$ chosen so that $(f'(n))(\rho) = i(\delta)$ for some $\delta < \omega_1^{CK}$. The same contradiction ensues as in Case I. ▌

Note that it is immediate from Corollary 1.4 that $\{P \in \mathfrak{A} \mid \exists\, n\ [P$ a path through $T_n]\} = \{P_n \mid r^{-1}(n) \in T\}$.

Corollary 1.5. *If S is Sacks generic over $\mathfrak{A}$ then $\omega_1^S > \omega_1^{CK}$.*

Proof. By Corollary 1.4, T is $\mathbf{\Sigma}_1$ over $L_{\omega_1^{CK}}(S)$, so as $L_{\omega_1^S}(S)$ is admissible and satisfies $\mathbf{\Sigma}_1$-DC, there is a path through T in $L_{\omega_1^S}(S)$. So by Proposition 1.1, $\omega_1^S > \omega_1^{CK}$. ▌

For the sake of expositing the main result, the proofs of the next two lemmas are deferred to Section 2.

LEMMA 1.6 (Little Lemma). *$\mathfrak{A}$ has no largest hyperdegree.*

LEMMA 1.7 (Main Lemma). *$\mathfrak{A}$ is admissible.*

Given the above, we can now prove

COROLLARY 1.8. *If S is any real such that $\omega_1^S > \omega_1^{CK}$ then S is a nonminimal upper bound for the hyperdegrees in $\mathfrak{A}$.*

Proof. As $W, G \in L_{\omega_1^{CK}+29}$, $W, G \in L_{\omega_1^S}(S)$, so as $\mathfrak{A} \subseteq L_{\omega_1^{CK}}[W \times G]$, $\mathfrak{A} \in L_{\omega_1^S}(S)$. So S is certainly an upper bound. Now, the Barwise-compactness with type-omitting construction of [8, Theorem 3.19] can be carried out in $L_{\omega_1^S}(S)$ to produce reals $T, U \leqslant_h S$, with (T, U) an exact pair[2] for the hyperdegrees in $\mathfrak{A}$. By Lemma 1.6 we must have $T <_h S$ and $U <_h S$, and each of T and U is an upper bound on the hyperdegrees in $\mathfrak{A}$. ∎

Putting Corollaries 1.5 and 1.8 together immediately yields the main result of this paper.

THEOREM 1.9. *$\mathfrak{A}$ is a countable admissible set with ordinal ω_1^{CK} such that if S is Sacks generic over $\mathfrak{A}$ then $\omega_1^S > \omega_1^{CK}$ and S is a nonminimal upper bound for the hyperdegrees in $\mathfrak{A}$.*

Directly from Lemma 1.7 we get an example of how strange an admissible set $\mathfrak{A}$ is.

COROLLARY 1.10. (i) $\mathfrak{A} \not\models \boldsymbol{\Sigma}_1$-DC.

(ii) *There is no function $f: \omega_1^{CK} \to \mathfrak{A}$, $\boldsymbol{\Delta}_1$ over $\mathfrak{A}$, such that $\mathfrak{A} = \bigcup \{f(\alpha) \mid \alpha < \omega_1^{CK}\}$.*[3]

Proof. (i) If $\mathfrak{A} \models \boldsymbol{\Sigma}_1$-DC then it would have a path through T, so $\mathfrak{A}$ could not be admissible.

(ii) If such an f existed one would also have a path through T in $\mathfrak{A}$ defined by

$P(n+1) = P(n) \frown \alpha$, where α is the least ordinal α which has a path through $T_{r(P(n)\frown\alpha)}$ in $f(\beta)$, where β is the least ordinal β such that there is such an α in $f(\beta)$. ∎

By some further analysis, one sees that the same results hold over $\mathfrak{A}$ for a

[2] That is, $\forall$ reals $X[(X \leqslant_h T \wedge X \leqslant_h U) \leftrightarrow X \in \mathfrak{A}]$.

[3] In the terminology of Barwise [2], this says that $\mathfrak{A}$ is nonresolvable.

certain general class of forcings. Now, because of Corollary 1.10(ii) and because $\mathfrak{A} \models\!\!\!\!/\ $ Power Set, there are some difficulties in giving a general theory of forcing over $\mathfrak{A}$, mainly in providing parameters for all of $\mathfrak{A}$'s elements. The difficulties can be surmounted by virtue of $\mathfrak{A} = \bigcup_{X \in \mathfrak{A}} L_{\omega_1^{CK}}[X]$, so that a satisfactory general formulation can be made even for nondefinable classes over $\mathfrak{A}$, and this formulation will reduce to the usual ones in more civilized situations. (Of course, many of these forcings do drastic damage to $\mathfrak{A}$.) For the present purposes it suffices to say that a forcing over $\mathfrak{A}$ is based on some poset $\subseteq \mathfrak{A}$, a notion of genericity, and an appropriate language for building $\mathfrak{A}[H]$, such that for generic H we always have $\mathfrak{A}[H] \models \mathscr{F}$ iff $\exists\, p \in H\ (p \Vdash \mathscr{F})$.

DEFINITION. Suppose $\underset{\sim}{x}$ is a name for a real in the language appropriate for forcing over $\mathfrak{A}$ with a partial ordering $\mathscr{P}$. We say that *$\mathscr{P}$ is $\boldsymbol{\Sigma}_1$-honest for $\underset{\sim}{x}$* iff for each $\boldsymbol{\Delta}_0$-formula $\mathscr{G}(n, \underset{\sim}{x})$ (with terms from the ramified language as parameters in $\mathscr{G}$) and each $Y \in \mathscr{P}$ there is a $\boldsymbol{\Sigma}_1$-formula $\mathscr{F}(n, X)$ with parameters in $\mathfrak{A}$ such that for each $n < \omega$

$$\{X \in \mathfrak{A} \mid \mathfrak{A} \models \mathscr{F}(n, X)\}$$

is a dense subset of

$$\{X \leqslant Y \mid X \Vdash_{\mathscr{P},\mathfrak{A}} \mathscr{G}(n, \underset{\sim}{x})\}.$$

PROPOSITION 1.11. *Suppose $\mathscr{P}$ is any forcing over $\mathfrak{A}$ which is $\boldsymbol{\Sigma}_1$-honest for $\underset{\sim}{x}$. Then if H is $\mathfrak{A}$-generic on $\mathscr{P}$, and $x \subseteq \omega$ is the denotation of $\underset{\sim}{x}$ in $\mathfrak{A}[H]$ then $\{P \in L_{\omega_1^{CK}}(x) \mid \exists\, n\ (P$ a path through $T_n)\} \subseteq \{P_n \mid r^{-1}(n) \in T\}$.*

Proof. Suppose $P = \{\alpha\}^x$ is a path in T_n. Take any $\rho \in P$. Let $Y \in H$, $Y \Vdash_{\mathscr{P},\mathfrak{A}} \{\alpha\}^x$ is a path through ρ in T_n. By $\boldsymbol{\Sigma}_1$-honesty and the admissibility of $\mathfrak{A}$, there is in $\mathfrak{A}$ a subtree V of T_n such that each level of V is nonempty and for all $\rho' \in V$, $\rho' \supseteq \rho$ and there is some $X \leqslant Y$ with $X \Vdash_{\mathscr{P},\mathfrak{A}} \rho' \in \{\alpha\}^x$. Using Lemma 1.3 one can show that any subtree of T_n in $\mathfrak{A}$ with more than one node on infinitely many levels has some node tagged with some $\imath(\beta)$. So all but finitely many levels of V have exactly one point. So there is a path in $\mathfrak{A}$ through ρ in T_n. Thus $r^{-1}(n) \in T$, and $\rho \in P_n$. As ρ was arbitrary in P, $P = P_n$. ∎

This immediately yields

THEOREM 1.12. *Let x be as in Proposition 1.11. Then x is not a minimal upper bound for $\mathfrak{A}$'s hyperdegrees.*

Proof. If $\omega_1^x = \omega_1^{CK}$ then by Proposition 1.11, $\{p \in L_{\omega_1^{CK}}(x) \mid \exists\, n\ (P$ a path through $T_n)\} \subsetneq \{P_n \mid r^{-1}(n) \in T\}$, so x is not an upper bound. If $\omega_1^x > \omega_1^{CK}$ then by Corollary 1.8, x is a nonminimal upper bound. ∎

Finally, note that one property of a forcing which would usually be needed to show that it preserves ω_1^{CK} is $\mathbf{\Sigma}_1$-honesty. But to produce a minimal upper bound for $\mathcal{A}$'s hyperdegrees, a forcing would have to preserve ω_1^{CK} without having this property.

2. The Admissibility of $\mathcal{A}$

This section investigates the interaction of the delayed collapse with the forcing of $\mathbf{\Sigma}_1$ formulas. Main Lemma 1.7 is proved. Little Lemma 1.6 falls out along the way. To prove 1.7 three main sublemmas are required: Factoring (2.2), Positivization (2.8), and Delay (2.9).

The factoring sublemma deals with breaking an M-generic object G on $\mathscr{D}$ into pieces. For $\alpha \leqslant \omega_1^{CK}$, let $\mathscr{D} \restriction \alpha = \{(c, t, u, d) \in \mathscr{D} \mid \text{range } (c) \subseteq \alpha,\ t \text{ is a subtree of } [\alpha]^{<\omega}, \text{ and range } (d) \subseteq \alpha\}$. Let $G \restriction \alpha = G \cap (\mathscr{D} \restriction \alpha)$. For most α, $G \restriction \alpha$ is *not* generic on $\mathscr{D} \restriction \alpha$. (Since, for example, there is a $k < \omega$ such that length $(c) < k$ for any c from $G \restriction \alpha$.) Nonetheless, we would like to view $G \restriction \alpha$ as an M-generic object. To do this, we introduce partial orderings $\mathscr{D}_\alpha$ with conditions which are able to say, for a given $\sigma \in [\omega_1^{CK}]^{<\omega}$, that the delay attached to σ is $> \alpha$ but that further decisions about σ (and all extensions of σ) are to be deferred; these decisions will then be made by a condition in some $\mathscr{D}_\beta$ for $\beta > \alpha$. $G \restriction \alpha$ will be (intertranslatable with) a generic object on $\mathscr{D}_\alpha$. In essence, what we are doing here is giving a concrete description of the complete subalgebra generated by $\mathscr{D} \restriction \alpha$ in the Boolean algebra corresponding to $\mathscr{D}$.

The Partial Orderings $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta/H$

$\mathscr{D}_{\omega_1^{CK}} \equiv \mathscr{D}$. Now, let $*$ be a new symbol. Fix $\alpha < \omega_1^{CK}$. $(c, t, u, d) \in \mathscr{D}_\alpha$ iff

(i)′ $c \in [\alpha \cap \text{Lim}]^{<\omega}$ or $c = c'^\frown *$ and $c' \in [\alpha \cap \text{Lim}]^{<\omega}$.

(ii)′ t is a finite nonempty subtree of $[\alpha]^{<\omega}$.

(iii)′ $\text{dom}(u) = \{\sigma \in t \mid d(\sigma) \neq *\}$, $u\colon \text{dom}(u) \to 2$, $u(\varnothing) = 1$, and whenever $\sigma \in t$ and $u(\sigma) = 0$ then for no $\tau \supsetneq \sigma$ is $\tau \in t$.

(iv)′ $d\colon t - \{\varnothing\} \to \alpha \cup \{*\}$ and whenever $d(\sigma) = *$ then for no $\tau \supsetneq \sigma$ is $\tau \in t$, and if $\sigma \neq \varnothing$, $d(\sigma) \neq *$, $d(\sigma^\frown\delta) \neq *$, and $u(\sigma) = u(\sigma^\frown\delta) = 1$ then

$$\textit{either } d(\sigma^\frown\delta) < d(\sigma),$$
$$\textit{or } \exists j \in \text{dom}(c)\,[c(j) \neq * \wedge \sigma\cdot < c(j) \leqslant \delta].$$

$(c', t', u', d') \leqslant (c, t, u, d)$ iff $c' \supseteq c$, $t' \supseteq t$, $u' \supseteq u$, and $d' \supseteq d$.

Note that $\mathscr{D} \restriction \alpha \subseteq \mathscr{D}_\alpha$.

Notation. If $p \equiv (c, t, u, d)$ and $q \equiv (c', t', u', d')$ are compatible in some $\mathscr{D}_\alpha$, $\alpha \leqslant \omega_1^{CK}$, then $(c \cup c', t \cup t', u \cup u', d \cup d')$ is obviously the weakest condition extending them both. It is denoted p *inf* q.

DEFINITION OF f_α. $f_\alpha : \bigcup_{\beta \geqslant \alpha} \mathscr{D}_\beta \to \mathscr{D}_\alpha$ by $f_\alpha(c, t, u, d) = (c', t', u', d')$, where

(a) $c' = c$ if range $(c) \subseteq \alpha$,

$= (c \restriction j)^\frown *$, where j is least j $[c(j) \notin \alpha]$ if range $(c) \nsubseteq \alpha$.

(So range $(c') \subseteq \alpha \cup \{*\}$.)

(b) t' is the largest subtree of t such that $\forall \sigma \in t'$ ($\sigma = \varnothing$ or $[\sigma^\cdot < \alpha$ and $\forall \tau \subsetneq \sigma(\tau = \varnothing$ or $d(\tau) \in \alpha)]$).

(c) For $\sigma \in t'$,

$$u'(\sigma) = u(\sigma) \text{ if } \sigma = \varnothing \text{ or } d(\sigma) \in \alpha,$$
$$= \text{undefined if } d(\sigma) \notin \alpha.$$

(d) For $\sigma \in t' - \{\varnothing\}$,

$$d'(\sigma) = d(\sigma) \quad \text{if} \quad d(\sigma) \in \alpha,$$
$$= * \quad \text{otherwise.}$$

Note that by (c) and (d), u' does not comment on whether or not σ (or any extension of σ) $\in T$ if $d'(\sigma) = *$.

It is not hard to see that $f_{\omega_1^{CK}} =$ identity, and for $\alpha \leqslant \beta \leqslant \omega_1^{CK}$, $f_\alpha = f_\alpha \circ f_\beta$. The map $f_\alpha : \mathscr{D}_\beta \to \mathscr{D}_\alpha$ is surjective. Also, if $p \leqslant q$ in $\mathscr{D}_\beta$ then $f_\alpha(p) \leqslant f_\alpha(q)$ in $\mathscr{D}_\alpha$.

DEFINITION OF H_α. If $\alpha \leqslant \beta \leqslant \omega_1^{CK}$ and $H \subseteq \mathscr{D}_\beta$, then $H_\alpha \equiv f_\alpha'' H$ (i.e., $\{f_\alpha(p) \mid p \in H\}$).

Note that if $\alpha \leqslant \beta \leqslant \gamma \leqslant \omega_1^{CK}$ and $H \subseteq \mathscr{D}_\gamma$ then $H_\alpha = (H_\beta)_\alpha$.

For G M-generic on $\mathscr{D}$ and $\alpha < \omega_1^{CK}$, G_α is almost the same object as $G \restriction \alpha$; the latter is obtained from the former by discarding the final $*$ at the end of C_{G_α} and dropping all nodes from the t's of G_α with $d(\sigma) = *$; the former is obtained from the latter as follows: First, add a $*$ to the longest c in $G \restriction \alpha$. Then whenever $\sigma \in [\alpha]^{<\omega}$ and $\delta < \alpha$ and σ appears in some $(c, t, u, d) \in G \restriction \alpha$ with $u(\sigma) = 1$, but $\sigma^\frown\delta \notin t'$ for any $(c', t', u', d') \in G \restriction \alpha$, then add a condition to G_α with $\sigma^\frown\delta$ in its tree, and delay $*$ attached to $\sigma^\frown\delta$.

DEFINITION OF $\mathscr{D}_\beta/H$. For $\alpha \leqslant \beta \leqslant \omega_1^{CK}$ and H M-generic on $\mathscr{D}_\alpha$, $\mathscr{D}_\beta/H \equiv \{p \in \mathscr{D}_\beta \mid f_\alpha(p) \in H\}$.

Remark. There will be $p, q \in \mathscr{D}_\beta/H$ such that $p \neq q$ but $p \Vdash_{\mathscr{D}_\beta/H, M[H]} q \in \underset{\sim}{G}$ and vice versa. One could, of course, mod out by this equivalence relation, though we will not bother with this, here.

Now, we are almost in position to prove the Factoring Sublemma. First we need

PROPOSITION 2.1. *Let* $\alpha \leqslant \beta \leqslant \omega_1^{CK}$, $q \in \mathscr{D}_\beta$. *Let* $p = f_\alpha(q)$. *Then*

(i) *If* $p' \leqslant p$ *in* $\mathscr{D}_\alpha$ *then there is a* $q' \leqslant q$ *in* $\mathscr{D}_\beta$ *with* $f_\alpha(q') = p'$.

(ii) *If* $p' \geqslant p$ *in* $\mathscr{D}_\alpha$ *then there is a* $q' \geqslant q$ *in* $\mathscr{D}_\beta$ *with* $f_\alpha(q') = p'$.

(iii) f_α *preserves (finite) infs.*

Proof. We give an intuitive description, omitting details, for (i). q may attach delays between α and β to nodes in $[\alpha]^{<\omega}$. By inserting $*$'s, f_α prevents p' from making any comment on these nodes, so anything about σ's $\in [\alpha]^{<\omega}$ added to p by p' is still compatible with q. (There is no danger from extending the Cohen part of p, as doing so only makes it easier to extend q.)

The proofs of (ii) and (iii) are routine. ∎

SUBLEMMA 2.2 (Factoring). *Let* $\alpha \leqslant \beta \leqslant \omega_1^{CK}$. *Then* G *is* M*-generic on* $\mathscr{D}_\beta$ M G_α *is* M*-generic on* $\mathscr{D}_\alpha$ *and* G *is* $M[G_\alpha]$*-generic on* $\mathscr{D}_\beta/G_\alpha$.

Proof. The statement is trivial if $\alpha = \beta$. So assume $\alpha < \beta$.

($\Rightarrow$): First note that the trivial requirements for genericity are direct from Proposition 2.1(ii) and (iii). So let E be any dense open subset of $\mathscr{D}_\alpha$ definable over M. Suppose $q \Vdash_{\mathscr{D}_\beta, M} \underset{\sim}{G}_\alpha \cap E = \varnothing$. Let $p = f_\alpha(q)$. Let $p' \leqslant p$ in $\mathscr{D}_\alpha$, $p' \in E$. By 2.1(i), take $q' \leqslant q$ such that $f_\alpha(q') = p'$. Then $q' \Vdash_{\mathscr{D}_\beta, M} p' \in \underset{\sim}{G}_\alpha \cap E$, a contradiction.

To show G $M[G_\alpha]$-generic on $\mathscr{D}_\beta/G_\alpha$, suppose E is dense open in $\mathscr{D}_\beta/G_\alpha$, E definable over $M[G_\alpha]$ and $G \cap E = \varnothing$. Take $w \in G_\alpha$ such that $w \Vdash_{\mathscr{D}_\alpha, M} \underset{\sim}{E}$ is a dense open subset of $\mathscr{D}_\beta/\underset{\sim}{H}$, (where H is a symbol for the generic object over $\mathscr{D}_\alpha$). Take $v \in G$ such that $w = f_\alpha(v)$. Take $z \leqslant v$, $z \in G$, $z \Vdash_{\mathscr{D}_\beta, M} \underset{\sim}{G} \cap \underset{\sim}{E} = \varnothing$. So $y = f_\alpha(z) \Vdash_{\mathscr{D}_\alpha, M} \underset{\sim}{E}$ is a dense open subset of $\mathscr{D}_\beta/\underset{\sim}{H}$. So there is a $y' \leqslant y$ in $\mathscr{D}_\alpha$ and a $q \leqslant z$ such that $y' \Vdash_{\mathscr{D}_\alpha, M} q \in \underset{\sim}{E}$. Let $p = f_\alpha(q)$. As $y' \Vdash_{\mathscr{D}_\alpha, M} \underset{\sim}{E} \subseteq \mathscr{D}_\beta/\underset{\sim}{H}$, $y' \Vdash_{\mathscr{D}_\alpha, M} p \in \mathscr{D}_\beta/\underset{\sim}{H}$, so y' is certainly compatible with p. Let $p' = y'$ inf p. By Proposition 2.1(i), take $q' \leqslant q$ such that $f_\alpha(q') = p'$. So $p' \Vdash_{\mathscr{D}_\alpha, M} q' \in \underset{\sim}{E}$, but $q' \Vdash_{\mathscr{D}_\beta, M} \underset{\sim}{E} \cap \underset{\sim}{G} = \varnothing$. So $q' \Vdash_{\mathscr{D}_\beta, M} p' \notin \underset{\sim}{G}_\alpha$, an absurdity. ∎

($\Leftarrow$): Let E be a dense open subset of $\mathscr{D}_\beta$ definable over M. It suffices to show that for every $x \in \mathscr{D}_\alpha$, $x \Vdash_{\mathscr{D}_\alpha, M} E \cap \mathscr{D}_\beta/\underset{\sim}{H}$ is dense in $\mathscr{D}_\beta/\underset{\sim}{H}$. Suppose not. Then there is a $y \in \mathscr{D}_\alpha$ and a $q \in \mathscr{D}_\beta$ such that $y \Vdash_{\mathscr{D}_\alpha, M}$ ($q \in \mathscr{D}_\beta/\underset{\sim}{H}$ and q has no extension in $E \cap \mathscr{D}_\beta/\underset{\sim}{H}$). $p = f_\alpha(q)$ is compatible with y. So let $p' = p$ inf y. By Proposition 2.1(i), take $q' \leqslant q$ such that $f_\alpha(q') = p'$. Now find $w \leqslant q'$, $w \in E$. So $f_\alpha(w) \leqslant p$. Therefore, $f_\alpha(w) \Vdash_{\mathscr{D}_\alpha, M}$ (q has no extension in $E \cap \mathscr{D}_\beta/\underset{\sim}{H}$). So $f_\alpha(w) \Vdash_{\mathscr{D}_\alpha, M} w \notin \mathscr{D}_\beta/\underset{\sim}{H}$, an absurdity. ∎

COROLLARY 2.3. *If* G *is* M*-generic on* $\mathscr{D}$ *and* $\alpha < \omega_1^{CK}$ *then* $M[G_\alpha]$ *is admissible and* $\mathscr{O}\!\ell_G(\alpha) \in M[G_\alpha]$.

Proof. $\mathscr{D}_\alpha \in M$ (in fact, $\in L_{\omega_1^{CK}}$), so as G_α is M-generic on $\mathscr{D}_\alpha$, $M[G_\alpha]$ is admissible. It is obvious from the construction of $\mathcal{O}_G(\alpha)$ that $\mathcal{O}_G(\alpha) \in M[G \upharpoonright \alpha]$, and $G \upharpoonright \alpha$ is easy to get from G_α. ▌

COROLLARY 2.4 (Little Lemma 1.6). *If G is M-generic on $\mathcal{D}$ then $\mathcal{O}_G$ has no largest hyperdegree.*

Proof. Suppose x were the largest hyperdegree of $\mathcal{O}_G$. Then $x \in M[G_\alpha]$ for some $\alpha < \omega_1^{CK}$. But it is easy to see that as G is $M[G_\alpha]$-generic on $\mathcal{D}/G_\alpha$, $y \equiv \{n \mid \langle \alpha + n \rangle \in T_G(\alpha + \omega)\}$ is Cohen generic over $M[G_\alpha]$, and $y \in \mathcal{O}_G$. So $y \notin M[G_\alpha]$. $L_{\omega_1^x}(x) = L_{\omega_1^{CK}}(x) \subseteq M[G_\alpha]$. So $y \nleqslant_h x$, a contradiction. ▌

Terminology. For $\alpha < \omega_1^{CK}$ and H M-generic on $\mathcal{D}$, one can talk about $\mathcal{O}_H(\alpha)$ within $M[H_\alpha]$ simply by constructing $\mathcal{O}_H(\alpha)$ within $M[H_\alpha]$. In fact, even if H is only M-generic on $\mathcal{D}_\beta$, for $\beta \geqslant \alpha$, this same construction makes sense for H_α. For $\mathcal{F} \in \hat{\mathcal{L}}$ (the ramified language from Section 1 for building $\mathcal{O}$), with the terms in $\mathcal{F}$ of rank $< \alpha$, we say $M[H] \models \mathcal{O}(\alpha) \models \mathcal{F}$ to mean that when this construction is carried out from H_α one gets a model of $\mathcal{F}$. (Note that $\mathcal{O}(\alpha) \models \mathcal{F}$ is $\mathbf{\Delta}_1$ over $M[G_\alpha]$.) We also use $T_H(\alpha)$, in this context, to mean the part of T constructed in this way from H_α.

The following is immediate from Sublemma 2.2.

COROLLARY 2.5. *Suppose $p \in \mathcal{D}_\beta$, $\alpha \leqslant \beta \leqslant \omega_1^{CK}$, $\mathcal{F}$ a $\mathbf{\Sigma}_1$ formula with terms of rank $< \alpha$ in $\hat{\mathcal{L}}$. Then $p \Vdash_{\mathcal{D}_\beta, M} \mathcal{O}(\alpha) \models \mathcal{F}$ iff $f_\alpha(p) \Vdash_{\mathcal{D}_\alpha, M} \mathcal{O}(\alpha) \models \mathcal{F}$.*

COROLLARY 2.6. *For $p \in \mathcal{D}$, $\mathcal{F}$ a $\mathbf{\Sigma}_1$ formula of $\hat{\mathcal{L}}$ with terms of rank $< \alpha < \omega_1^{CK}$, the predicate*

$$p \Vdash \mathcal{O}(\alpha) \models \mathcal{F}$$

is $\mathbf{\Sigma}_1$ over M.

Proof. Immediate from Corollary 2.5 as "$\mathcal{O}(\alpha) \models \mathcal{F}$" is $\mathbf{\Delta}_1$ over $M[H_\alpha]$ and $\mathcal{D}_\alpha \in M$, so the forcing relation for $\mathcal{D}_\alpha$ over M for $\mathbf{\Sigma}_1$ formulas is $\mathbf{\Sigma}_1$ over M.

Notation. For α any ordinal, $\check{\alpha}$ is the least primitive recursively closed ordinal $> \alpha \cup \omega$. (Actually, primitive recursive closure is far more than what we will need.)

PROPOSITION 2.7. *Let $\alpha < \omega_1^{CK}$ and H be M-generic on $\mathcal{D}$. Then $\{(\delta, T_H(\delta)) \mid \delta < \alpha\} \in \mathcal{O}_H(\check{\alpha})$.*

Proof. By construction of $\mathcal{O}_H$, $T_H(\alpha + \omega) \in \mathcal{O}_H(\alpha + \omega + 1)$. It suffices to compute $D_H(\sigma)$ (for $\sigma \in T_H(\alpha + \omega)$) within $L_{\check{\alpha}}(T_H(\alpha + \omega))$. Now, by genericity of H, for $\sigma \in T_H(\alpha + \omega)$, $\sigma \neq \varnothing$, we have $D_H(\sigma) = 0$ iff $\{\beta \mid \sigma^{\cdot} < \beta < \sigma^{\cdot} + \omega$ and $\sigma^\frown\beta \in T_H(\alpha + \omega)\} = \varnothing$. (This is the only place we use the fact that range $(C_H) \subseteq \mathrm{Lim}$.) $D_H(\sigma) = 1$ iff $\{\beta \mid \sigma^{\cdot} < \beta < \sigma^{\cdot} + \omega,\ \sigma^\frown\beta \in T_H(\alpha + \omega)$, and

$D_H(\sigma\hat{}\beta) \neq 0\} = \varnothing$. By a similar analysis one computes $D_H(\sigma)$ for all $\sigma \in T_H(\alpha + \omega)$ in an iteration of length $\alpha + \omega$. ∎

The Positivization Sublemma (2.8) says, in effect, that the negative parts of conditions (i.e., those that directly say $\sigma \notin T$) do not help to force Σ_1 statements. This follows the intuition that a Σ_1 statement can say "look for some $\sigma \in T$" but cannot say "check that $\sigma \notin T$." However, the negative parts are not totally ineffectual—they can prevent a Σ_1 thing from happening, which will be crucial later on.

DEFINITION OF p^+. For $p \equiv (c, t, u, d) \in \bigcup \{\mathscr{D}_\alpha \mid \alpha \leqslant \omega_1^{CK}\}$, $p^+ \equiv (c', t', u', d')$ where

(a) $c' = c$,

(b) $t' = t - \{\sigma \in t \mid u(\sigma) = 0\}$,

(c) $u' = u \upharpoonright t'$,

(d) $d' = d \upharpoonright t' - \{\varnothing\}$.

Notice that t' is a subtree of t since σ is a terminal node of t whenever $u(\sigma) = 0$. If $p \in \mathscr{D}_\alpha$, so is p^+.

Next we define a map that, in effect, puts back what is lost in going from p to p^+.

DEFINITION OF N_p. Let $p, q \in \bigcup \{\mathscr{D}_\alpha \mid \alpha \leqslant \omega_1^{CK}\}$, $p \equiv (c, t, u, d)$, $q \equiv (c_1, t_1, u_1, d_1)$. Then $N_p(q) = (c_1', t_1', u_1', d_1')$, where

(a) $c_1' = c_1$.

(b) t_1' is the largest subtree of t_1 such that $\forall \sigma \in t_1'$, $\forall \tau \subsetneq \sigma$ [if $u(\tau)$ is defined then $u(\tau) \neq 0$].

(c) For $\sigma \in t_1'$,

$$
\begin{aligned}
u_1'(\sigma) &= 0 && \text{if } u(\sigma) \text{ is defined, and } u(\sigma) = 0,\\
&= u_1(\sigma) && \text{otherwise.}
\end{aligned}
$$

(d) For $\sigma \in t_1' - \{\varnothing\}$,

$$
\begin{aligned}
d_1(\sigma) &= d(\sigma) && \text{if } u(\sigma) \text{ is defined, and } u(\sigma) = 0,\\
&= d_1(\sigma) && \text{otherwise.}
\end{aligned}
$$

Note that if $p, q \in \mathscr{D}_\alpha$ then $N_p(q) \in \mathscr{D}_\alpha$. Also, when all the conditions mentioned are in $\bigcup \{\mathscr{D}_\alpha \mid \alpha \leqslant \omega_1^{CK}\}$ we have

(i) $q \leqslant p^+ \Rightarrow N_p(q)$ is compatible with p.

(ii) $q' \leqslant q \Rightarrow N_p(q') \leqslant N_p(q)$.

(iii) N_p preserves (finite) infs.

(iv) $p' \geqslant N_p(q) \Rightarrow \exists\, q' \geqslant q\, [p' = N_p(q')]$.

The crucial point in the following proof is that one can translate a generic object containing p^+ to one containing p in a way which only results in thinning T.

SUBLEMMA 2.8 (Positivization). *Let $\alpha \leqslant \beta < \omega_1^{CK}$ and let $\mathscr{F}$ be a $\mathbf{\Delta}_0$ formula of $\mathscr{L}$ with terms of rank $< \alpha$. Let H be M-generic on $\mathscr{D}_\alpha$, $p \in \mathscr{D}_\beta/H$, and suppose $p \Vdash_{\mathscr{D}_\beta/H, M[H]} \mathscr{A}(\beta) \models \exists\, x\mathscr{F}(x)$. Then for any $M[H]$-generic J on $\mathscr{D}_{\check\beta}/H$ with $p^+ \in J_\beta$ we have $M[J] \models \mathscr{A}_{\check\beta} \models \exists\, x\mathscr{F}(x)$.*

Proof. It is routine to check the following

CLAIM. (i) $(\forall\, q \in \mathscr{D}_\beta/H)\, [N_p(q) \in \mathscr{D}_\beta/H]$.

(ii) $(\forall\, q \leqslant p^+ \text{ in } \mathscr{D}_\beta/H)\, (\forall\, p' \leqslant p \text{ inf } N_p(q) \text{ in } \mathscr{D}_\beta/H)\, (\exists\, q' \leqslant q \text{ in } \mathscr{D}_\beta/H)$ $[N_p(q') = p']$.

From the claim and the properties of N_p listed above, it is routine to prove that if I is $M[H]$-generic on $\mathscr{D}_\beta/H$ and $p^+ \in I$, then $N_p''I$ is $M[H]$-generic on $\mathscr{D}_\beta/H$. By property (i), $p \in N_p''I$. Now, let J be as stated. By 2.7, $\{(\delta, T_J(\delta)) \mid \delta < \beta\} \in \mathscr{A}_J(\check\beta)$. Now it is easy to obtain $\{(\delta, T_{N_p''J_\beta}(\delta)) \mid \delta < \beta\}$ from the above sequence. As $T_{N_p''J_\beta}(\delta) \subsetneq T_{J_\beta}(\delta)$, all the $P_{n,\delta}$'s necessary to construct $\mathscr{A}_{N_p''J_\beta}(\beta)$ are in $\mathscr{A}_J(\beta)$. So $\mathscr{A}_J(\check\beta)$ has in it $\mathscr{A}_{N_p''J_\beta}(\beta)$. But as $p \in N_p''J_\beta$ and $N_p''J_\beta$ is $M[H]$-generic on $\mathscr{D}_\beta/H$, $\mathscr{A}_{N_p''J_\beta}(\beta) \models \exists\, x\mathscr{F}(x)$. As each term in $\mathscr{F}$ has rank $<\alpha$, these terms have the same meaning in $\mathscr{A}_J$ as in $\mathscr{A}_{N_p''J_\beta}$. So, by the upward persistence of $\mathbf{\Sigma}_1$ statements, $M[J] \models \mathscr{A}(\check\beta) \models \exists\, x\mathscr{F}(x)$. ∎

Remark. We will only use Sublemma 2.8 for $p \in (\mathscr{D}/H) \cap (\mathscr{D} \restriction \beta)$. In this case, the conclusion can be simply restated as $p^+ \Vdash_{\mathscr{D}_{\check\beta}/H, M[H]} \mathscr{A}(\check\beta) \models \exists\, x\mathscr{F}(x)$.

Now we come to the main sublemma of Section 2, the Delay Sublemma (2.9). Roughly speaking, $\mathscr{A}$ is admissible because, even though C_G destroys ω_1^{CK}, there is nothing that goes on in $\mathscr{A}$ which enables one, in a $\mathbf{\Sigma}_1$ fashion, to get his hands on any of C_G's values. The Delay Sublemma is the precise formulation of this idea. It basically says that if $p \equiv (c, t, u, d)$ makes a $\mathbf{\Sigma}_1$ thing happen, then by sufficiently increasing the delays in d and throwing out c, one still makes the $\mathbf{\Sigma}_1$ thing happen. (Thus, the fact that the $\mathbf{\Sigma}_1$ thing happens gives no information about c.)

As in the proof of Sublemma 2.8, the technique is to show that from a generic object containing the altered p, a generic object can be constructed which contains the original p, and this can be done in a way which only thins T. We define maps that play roles analogous to those of $p \mapsto p^+$ and N_p in Sublemma 2.8. Here is the first real interaction of the delays with the values of c. Also, in the proof of Delay Sublemma 2.9 there is a claim analogous to that in the proof of Sublemma 2.8. It is in the proof of this claim that the first use is made of a condition's ability to comment negatively on the question, "Is $\sigma \in T$?"

Now, some auxiliary definitions.

Let $p \equiv (c, x, y, d)$, $\alpha < \omega_1^{CK}$, $\sigma \in [\omega_1^{CK}]^{<\omega} - \{\varnothing\}$, where

$$c \in [\omega_1^{CK} \cap \mathrm{Lim}]^{<\omega},\ d\colon \{\tau \mid \varnothing \neq \tau \subseteq \sigma\} \to \omega_1^{CK},$$

x and y are arbitrary, and assume $\alpha \in$ range (c).

Let $n \equiv$ length (c).

$k_\alpha(\sigma, p) \equiv$ least $j <$ length σ $[\sigma(j) \geqslant \alpha$ or $d(\sigma \upharpoonright j + 1) \geqslant \alpha]$, if any such j, undefined otherwise.

$j_\alpha(\sigma, p) \equiv$ "number of jumps of σ after leaving $\mathscr{D} \upharpoonright \alpha$"

$$\equiv \text{cardinality } \{j \mid k_\alpha(\sigma, p) \leqslant j < \text{length}(\sigma) - 1 \wedge \exists\, m < n\ [\sigma(j) < c(m) \leqslant \sigma(j + 1)]\} \quad \text{if } k_\alpha(\sigma, p) \text{ is defined,}$$
$$\equiv n \text{ otherwise.}$$

So $j_\alpha(\sigma, p)$ is always $\leqslant n$.

Definition of $p^{\alpha,\beta}$. Let $p \equiv (c, t, u, d) \in \mathscr{D} \upharpoonright \beta$, $\alpha < \beta < \omega_1^{CK}$, $n \equiv$ length c, and assume $\alpha \in$ range (c). $p^{\alpha,\beta} \equiv (c', t', u', d')$, where

(a) $c' = c \upharpoonright (1 + \text{least } j\ [c(j) = \alpha])$. (So $(c')^{\cdot} = \alpha$).

(b) $t' = t$.

(c) $u' = u$.

(d) For $\sigma \in t' - \{\varnothing\}$, $d'(\sigma) = \beta \cdot [n - j_\alpha(\sigma, p)] + d(\sigma)$.

The idea of the definition is to push up the delays attached to those $\sigma \in t$ which are beyond $T(\alpha)$ in such a way that no evidence of the values of c above α remains. This is accomplished by simulating possible increases in the delays corresponding to jumps by going to an increased ordinal in a lower copy of β. It is not hard to check that $p^{\alpha,\beta} \in \mathscr{D} \upharpoonright \beta \cdot \omega$. (By pushing up certain delays, nodes above α at which the delay previously increased now have a decreasing delay, so truncating c does not hurt. One needs here that $\alpha \in$ range (c).)

Definition of $R_{\alpha,\beta,p}(q)$. Let p, α, β, n be as in the above definition, $q \equiv (c_1, t_1, u_1, d_1) \in \mathscr{D}$. For $\tau \in t_1$, let $\hat{d}_1(\tau) = d_1(\tau) - \beta \cdot [n - j_\alpha(\tau, (c, t_1, u_1, d_1))]$ (if the first ordinal $\geqslant$ the second, undefined otherwise). We say τ *respects the coding* if $\forall j <$ length (τ) $\{\hat{d}_1(\tau)$ is defined $\wedge$ $[(j = k_\alpha(\tau, (c, t_1, u_1, d_1)) \wedge \tau(j) < \alpha) \to \hat{d}_1(\tau \upharpoonright j + 1) \geqslant \alpha]$ $\wedge$ $[j > 0 \to [\hat{d}_1(\tau \upharpoonright j + 1) < \hat{d}_1(\tau \upharpoonright j)$ or $\exists m < n\ ((\tau \upharpoonright j)^{\cdot} < c(m) \leqslant (\tau \upharpoonright j + 1)^{\cdot})]]\}$. $R_{\alpha,\beta,p}(q) \equiv (c_1', t_1', u_1', d_1')$, where

(a) $c_1' = c^\frown *$.

(b) $t_1' =$ largest subtree of t_1 such that $\forall \sigma \in t_1'$ $\{\sigma = \varnothing$ or $[\sigma^{\cdot} < \beta \wedge \forall \tau \subsetneq \sigma$ $(\tau = \varnothing$ or $(\tau$ respects the coding $\wedge$ $\hat{d}_1(\tau) < \beta))]\}$.

(c) For $\sigma \in t_1'$,

$$\begin{aligned} u_1'(\sigma) &= u(\sigma) && \text{if } \sigma \text{ respects the coding and } \hat{d}_1(\sigma) < \beta, \\ &= \text{undefined} && \text{if } \sigma \text{ respects the coding and } \hat{d}_1(\sigma) \geqslant \beta, \\ &= 0 && \text{if } \sigma \text{ does not respect the coding.} \end{aligned}$$

(d) For $\sigma \in t_1' - \{\varnothing\}$,

$$\begin{aligned} d_1'(\sigma) &= \hat{d}_1(\sigma) && \text{if } \sigma \text{ respects the coding and } \hat{d}_1(\sigma) < \beta, \\ &= * && \text{if } \sigma \text{ respects the coding and } \hat{d}_1(\sigma) \geqslant \beta, \\ &= \alpha && \text{if } \sigma \text{ does not respect the coding.} \end{aligned}$$

Now, one can check that if $p = (c, t, u, d)$ is as in the definition of $p^{\alpha,\beta}$, then $R_{\alpha,\beta,p}(p^{\alpha,\beta}) = (c^\frown *, t, u, d)$. Also, if $q' \leqslant q$ then $R_{\alpha,\beta,p}(q') \leqslant R_{\alpha,\beta,p}(q)$ in $\mathscr{D}_\beta$. The effect of (c) and (d) just above is to eliminate those σ from T for which q fails to follow the coding scheme used by $p \mapsto p^{\alpha,\beta}$.

SUBLEMMA 2.9 (Delay). *Let* $\alpha < \beta < \omega_1^{CK}$, $p = (c, t, u, d) \in \mathscr{D} \restriction \beta$, *and* $\alpha \in$ *range* (c). *Let H be M-generic on* $\mathscr{D}_\alpha$, *and assume* $p \in \mathscr{D}_\beta/H$. *Suppose* $\mathscr{F}$ *is* Δ_0 *in* $\hat{\mathscr{L}}$ *with terms of rank* $< \alpha$ *and that* $p \Vdash_{\mathscr{D}_\beta/H, M[H]} \mathscr{A}(\beta) \models \exists x \mathscr{F}(x)$. *Then* $p^{\alpha,\beta} \Vdash_{\mathscr{D}_{\check{\beta}}/H, M[H]} \mathscr{A}(\check{\beta}) \models \exists x \mathscr{F}(x)$.

Proof. First of all, it is straightforward to check that $f_\alpha(p^{\alpha,\beta}) = f_\alpha(p)$, so $p^{\alpha,\beta} \in \mathscr{D}_{\check{\beta}}/H$. Suppose the sublemma fails. Then, by Sublemma 2.2, there is an $M[H]$-generic J on $\mathscr{D}/H$ such that $p^{\alpha,\beta} \in J$ and $\mathscr{A}_J(\check{\beta}) \not\models \exists x \mathscr{F}(x)$.

Claim. (i) $\forall q \leqslant p^{\alpha,\beta}$ in $\mathscr{D}/H$ $[R_{\alpha,\beta,p}(q) \in \mathscr{D}_\beta/H]$,

(ii) $(\forall q \leqslant p^{\alpha,\beta}$ in $\mathscr{D}/H)$ $(\forall p' \leqslant R_{\alpha,\beta,p}(q)$ in $\mathscr{D}_\beta/H)$ $(\exists q' \leqslant q$ in $\mathscr{D}/H)$ $[R_{\alpha,\beta,p}(q') = p']$.

(*Proof of Claim.* For (i) the only danger is from (c) and (d) of the definition of $R_{\alpha,\beta,p}$ when σ does not respect the coding. But H cannot prohibit making $u(\sigma) = 0$ and $d(\sigma) = \alpha$. (Note that this is the only place where we use the fact that D_G can take arbitrary values on σ when $u(\sigma) = 0$.)

For (ii), the main point is that for any node on which the delays of q violate the coding scheme, $R_{\alpha,\beta,p}(q)$ ensures that no extension of these nodes is in p'.)

Let $J' = \{q \in J \mid q \leqslant p^{\alpha,\beta}\}$. It is routine to show from the claim that $R''_{\alpha,\beta,p}J'$ is $M[H]$-generic on $\{z \in \mathscr{D}_\beta/H \mid z \leqslant (c^\frown *, t, u, d)\}$, so closing $R''_{\alpha,\beta,p}J'$ under weakening of conditions gives I, an $M[H]$-generic object on $\mathscr{D}_\beta/H$, and $p \in I$. By the nature of $R_{\alpha,\beta,p}$, it is clear that $T_I(\beta) \subseteq T_J(\beta \cdot \omega)$, so all the $P_{n,\delta}$ needed to build $\mathscr{A}_I(\beta)$ are in $\mathscr{A}_J(\beta \cdot \omega)$, and by Proposition 2.7, $\{(\delta, T_I(\delta)) \mid \delta < \beta\} \in \mathscr{A}_J(\check{\beta})$. So, finally, $\mathscr{A}_I(\beta) \in \mathscr{A}_J(\check{\beta})$. As $p \in I$, $\mathscr{A}_I(\beta) \models \exists x \mathscr{F}(x)$, so $\mathscr{A}_J(\check{\beta}) \models \exists x \mathscr{F}(x)$, a contradiction. ∎

Now, finally, we can prove that $\mathcal{O}\!\ell$ is admissible. Given all the above, the proof is constructed from the following simple intuition: Suppose one knows that $\sigma \in T$. Then some $\delta < \sigma^{\cdot} + \omega$ has $\sigma^\frown\delta \in T$ with $D_G(\sigma^\frown\delta) = 0$. So any $\sigma^\frown\delta^\frown\eta \in T$ has $\eta \geqslant$ the next value of C_G above $\sigma^{\cdot}$. Now, there are two ways a $\mathbf{\Sigma}_1$ formula might attempt to define a value $\geqslant$ next value of C_G above $\sigma^{\cdot}$. One would be to fix δ and say "Take any η such that $\sigma^\frown\delta^\frown\eta \in T$." This could fail, since, as a condition can put $u(\sigma^\frown\delta) = 0$, $\sigma^\frown\delta$ might not be in T. So if the formula is to be sure of success, it cannot say the above. The other approach is to say "Find some δ such that $\sigma^\frown\delta \in T$ and $D_G(\sigma^\frown\delta) = 0$. Then find any η such that $\sigma^\frown\delta^\frown\eta \in T$, and take value η." But by the Delay Sublemma, a $\mathbf{\Sigma}_1$ formula cannot say this, for if a condition making $D_G(\sigma^\frown\delta) = 0$ forces the formula to take value η, so would some condition making $D_G(\sigma^\frown\delta) > 0$. So the best the formula could do would be to say "Find some δ and η such that $\sigma^\frown\delta^\frown\eta \in T$, and take value η." But since it might be that $D_G(\sigma^\frown\delta) > 0$, η might not be $\geqslant$ next value of C_G.

Proof of Main Lemma 1.7 ($\mathcal{O}\!\ell$ is admissible). We have a particular G M-generic on $\mathscr{D}$. Assume $\mathcal{O}\!\ell = \mathcal{O}\!\ell_G \models \forall n < \omega\ \exists x \mathscr{F}(n, x)$, where $\mathscr{F}$ is $\mathbf{\Delta}_0$ in $\hat{\mathscr{L}}$. We need to show that $\mathcal{O}\!\ell(\theta) \models \forall n < \omega\ \exists x \mathscr{F}(n, x)$, for some $\theta < \omega_1^{CK}$. (This suffices as $\mathcal{O}\!\ell$ is locally countable by 2.7.)

Let $p_0 \equiv (c_0, t, u, d) \in G$, $p_0 \Vdash_{\mathscr{D},M} \mathcal{O}\!\ell \models \forall n < \omega\ \exists x \mathscr{F}(n, x)$.

Let $\alpha_0 >$ rank of all parameters in $\mathscr{F}$ and large enough so that $p_0 \in \mathscr{D} \restriction \alpha_0$.

Take $p = (c, t, u, d) \in G$ such that $c^{\cdot} = \alpha > \alpha_0$.

By 2.6, the following can be done in $M[G_\alpha]$:

Fix $n < \omega$. Let $q_n'(0) = p$. Now, for $i < \omega$, assume $q_n(j)$ has been defined for all $j < i$, and $q_n'(j) \leqslant p$ has been defined for all $j \leqslant i$.

Let $q_n(i) \leqslant q_n'(i) \in \mathscr{D}/G_\alpha$, such that for some $\beta_n(i) < \omega_1^{CK}$, $q_n(i) \Vdash_{\mathscr{D},M} \mathcal{O}\!\ell(\beta_n(i)) \models \exists x \mathscr{F}(n, x)$.

Let $q_n'(i + 1) \leqslant p$ in $\mathscr{D}/G_\alpha$ such that for each $j \leqslant i$ and each σ in the t from $q_n(j)$, if it is compatible with G_α to have $\sigma \notin T$, then $q_n'(i + 1) \Vdash_{\mathscr{D},M} \sigma \notin T$.

Let $\beta < \omega_1^{CK}$ be large enough so that each $\beta_n(i) < \beta$ and each $q_n(i) \in \mathscr{D} \restriction \beta$. Let $\theta = \beta^{\cdot}$

CLAIM. $\mathcal{O}\!\ell(\theta) \models \forall n < \omega\ \exists x \mathscr{F}(n, x)$.

Proof of Claim. Fix $n < \omega$. It is easy to see that for any $q \in G$, there is some i such that $(q_n(i)^+)^{\alpha,\check{\beta}}$ is compatible with q. (Note that in order to arrange the $q_n(i)$ so that this be true, we have, once again, made use of the ability of conditions to comment negatively on "$\sigma \in T$?") So, for some i, $(q_n(i)^+)^{\alpha,\check{\beta}} \in G$. As $q_n(i) \Vdash_{\mathscr{D},M} \mathcal{O}\!\ell(\beta) \models \exists x\, \mathscr{F}(n, x)$, $q_n(i) \Vdash_{\mathscr{D}_\beta/G_\alpha, M[G_\alpha]} \mathcal{O}\!\ell(\beta) \models \exists x\, \mathscr{F}(n, x)$, so by 2.8, (and the remark following 2.8), $q_n(i)^+ \Vdash_{\mathscr{D}^{\vee}_{\beta}/G_\alpha, M[G_\alpha]} \mathcal{O}\!\ell(\check{\beta}) \models \exists x\, \mathscr{F}(n, x)$. By 2.9, $(q_n(i)^+)^{\alpha,\check{\beta}} \Vdash_{\mathscr{D}_{\check{\beta}}^{\vee}/G_\alpha, M[G_\alpha]} \mathcal{O}\!\ell(\check{\beta}) \models \exists x \mathscr{F}(n, x)$. As $(q_n(i)^+)^{\alpha,\check{\beta}} \in G$, $\mathcal{O}\!\ell_G(\check{\beta}) \models \exists x \mathscr{F}(n, x)$. $\blacksquare$ for Claim; $\blacksquare$ for Main Lemma 1.7.

Technical Comments. The intuitive remarks used to motivate the proof just above yield an argument showing that if $\mathscr{D}$ did not allow $u(\sigma) = 0$, the resulting $\mathfrak{A}$ would not be admissible. Also, it is possible to show that if the range of the d's were constrained to be $\subseteq \delta$, for some fixed $\delta < \omega_1^{CK}$, $\mathfrak{A}$ would not be admissible. However, allowing $d(\sigma) > 0$ when $u(\sigma) = 0$ is purely a technical device to control the way G is broken up into the G_α's. By changing all these to 0 one still builds the same inner model $\mathfrak{A}$. Using this idea one can show, as a corollary to the fact that the construction we have used gives an admissible $\mathfrak{A}$, that a construction based upon a forcing which demands $d(\sigma) = 0$ when $u(\sigma) = 0$ still produces an admissible $\mathfrak{A}$.

3. Remarks and Open Questions

Sacks' question "Does every countable set of hyperdegrees have a minimal upper bound?" remains open. The results here seem to give evidence for a negative answer. Perhaps the hyperdegrees in $\mathfrak{A}$ provide a counterexample, though this is not known. A possible approach for getting some other countable admissible $\mathscr{B}$, similar to $\mathfrak{A}$, for which it would be possible to prove that $\mathscr{B}$'s hyperdegrees have no minimal upper bound, would be to modify the present construction to obtain

(*) For any upper bound S on $\mathscr{B}$'s hyperdegrees, if $\{P \leqslant_h S \mid P$ a path through some T_n with $n \notin r''T\}$ is infinite, then S is nonminimal.

Perhaps one could get (*) by insuring that for every such S there is an upper bound $S' <_h S$ with some new $P \nleqslant_h S'$. It is not possible to get (*) by arranging matters so that any such new P is an upper bound, (and so that any two different such P are mutually generic), for then the real coding T_n would itself be an upper bound on $\mathscr{B}$'s hyperdegrees.

References

1. K. J. Barwise, Infinitary logic and admissible sets, *J. Symbolic Logic* **34** (1969), 226–252.
2. K. J. Barwise, "Admissible Sets and Structures," Springer-Verlag, New York/Berlin, 1975.
3. P. J. Cohen, "Set Theory and the Continuum Hypothesis," Benjamin, New York, 1966.
4. H. Friedman, "Subsystems of Set Theory and Analysis," Ph.D. Dissertation, M.I.T., 1967.
5. H. J. Keisler, "Model Theory for Infinitary Logic," North-Holland, Amsterdam, 1971.
6. G. E. Sacks, "Degrees of Unsolvability," Princeton Univ. Press, Princeton, N.J., 1963; 2nd ed., 1966.

7. G. E. SACKS, Forcing with perfect closed sets, *in* "Proceedings, Symposia Pure Math. XIII," pp. 331–355, Amer. Math. Soc., Providence, R.I., 1971.
8. G. E. SACKS, Countable admissible ordinals and hyperdegrees, *Advances in Math.* 20 (1976), 213–262.
9. J. STEEL, Forcing with tagged trees (Abstract No. 718-E1), *Notices Amer. Math. Soc.* 21 No. 7 (1974), A-627–A-628.
10. J. STEEL, "Subsystems of Analysis and the Axiom of Determinacy," Ph.D. Dissertation, Department of Mathematics, University of California, Berkeley, 1977.

Printed in Belgium