ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE @ DIRECT•

J. Differential Equations 189 (2003) 292–317

Journal of Differential Equations

http://www.elsevier.com/locate/jde

A singular multi-dimensional piston problem in compressible flow

Shuxing Chen

Institute of Mathematics, Fudan University, Shanghai 200433, China

Received September 22, 2000; revised June 28, 2001

Abstract

This paper concerns the multi-dimensional piston problem, which is a special initial boundary value problem of multi-dimensional unsteady potential flow equation. The problem is defined in a domain bounded by two conical surfaces, one of them is shock, whose location is also to be determined. By introducing self-similar coordinates, the problem can be reduced to a free boundary value problem of an elliptic equation. The existence of the problem is proved by using partial hodograph transformation and nonlinear alternating iteration. The result also shows the stability of the structure of shock front in symmetric case under small perturbation.

Keywords: Piston problem; Quasilinear hyperbolic system; Conservation laws; Potential flow; Compressible flow; Fluid dynamics; Partial hodograph transformation; Domain decomposition

1. Introduction

In the study of mathematical theory of compressible fluid dynamics, the piston problem is a basic prototype problem. As described in [8–10,20,21], giving a long tube closed by a piston at one end and open at the other end, and assuming that the gas in the tube is static with uniform pressure p_0 and density ρ_0, then any motion of the piston will cause a corresponding motion of the air. Generally, if the piston is pulled back, then a rarefaction wave will be formed, and otherwise, if the piston is pushed forward, then the push will cause a compressed wave moving into the air. Particularly, if the initial velocity of the piston is positive, then ahead of the piston

E-mail address: sxchen@fudan.ac.cn.

PII: S0022-0396(02)00060-8

there will be a shock front, moving into the air faster than the piston. Such phenomena are verified by physical experiments, and also elaborately studied in its mathematical aspects. It is natural to study the multi-dimensional version of such piston problems.

Suppose there is uniform static gas filling up the whole space outside a given body with moving boundary. The body is called piston in the sequel. Starting from the initial time, the piston gradually expands and its boundary moves into the air as in the one-dimensional case. Then, away from the piston there is a shock front moving into the air. Ahead of the shock front the state of the air is kept unchanged, while the location of the moving shock and the flow field in between the shock and the path of the piston are to be determined. Such a problem is called multi-dimensional piston problem, which is an initial boundary value problem for compressible flow equation. For such problems, the most interesting case is that when the location of the piston at the initial time degenerates into a single point, because such a case offers a good model problem of multi-dimensional hyperbolic conservation laws and is related to the study of explosive wave in physics. In the meantime, this is also the most difficult case due to the appearance of singularity. In the sequel we are going to study such a singular case. The main result in this paper is the existence of the solution and the stability of the moving shock when the motion of the piston is not symmetric. We noticed that some related problems were studied in [1–7,11–13,16,19].

Our main assumption for the motion of the piston is that the velocity of the piston in each direction is independent of time t. Due to this fact we can use self-similar coordinates to study such a problem. Moreover, since the normal component of the relative velocity behind the shock front is subsonic, the reduced equation in the self-similar coordinates (ξ, η) is elliptic, rather than hyperbolic in the original time-space coordinates. Besides, since the location of the shock front is unknown, the whole mathematical problem is a free boundary value problem for the elliptic equation.

To get rid of the difficulty caused by the free boundary, we use the partial hodograph transformation, which played the crucial role in proving the existence of solution to the initial value problem of multi-dimensional quasilinear potential flow equation with discontinuous data (cf. [15,17]). Its main idea is to change the position of the unknown function of the problem and one coordinate variable, which is the radius r in our problem. Such a transformation will let the free boundary become a fixed one. However, the above transformation will also change the given path of the piston to a new free boundary. To avoid the appearance of the new free boundary, we use the method of domain decomposition. That is, we will decompose the annular domain into a set of overlapped domains, and correspondingly introduce a set of auxiliary boundary value problems on these domains. Like the Schwatz alternating iteration, we establish a set of sequences of the solutions to these auxiliary problems by solving the problems alternatively. In each step, the value of the solution of the previous problem is taken as the data to determine the solution of the next problem. Finally, by establishing the convergence of these sequences, we obtain the existence and stability of the original problem consequently.

The whole paper will be arranged as follows. In Section 2 we give the mathematical formulation of the singular multi-dimensional piston problem and describe the results of the paper precisely. In Section 3 we solve the problem in symmetric case. Starting from Section 4 we treat the nonsymmetric case. In Section 4 we introduce the partial hodograph transformation and the method of domain decomposition. Then based on them we establish auxiliary boundary value problems. In Sections 5 and 6 we discuss the property of the solution to these problems, and then establish a set of sequences of the solutions by using nonlinear alternating iteration. Finally, we prove the convergence of these sequences and complete the proof of the main theorem in Section 7.

2. Formulation and result

In the whole paper, we use the unsteady potential flow equation to describe the motion of the compressible flow. As it is well known, when the strength of all possible shock is weak, the equation can offer a good description of the motion of the flow (see [14,18]). The unsteady potential flow equation in two space dimension can be written as

$$\frac{\partial}{\partial t}H + \frac{\partial}{\partial x}(\Phi_x H) + \frac{\partial}{\partial y}(\Phi_y H) = 0, \tag{2.1}$$

where Φ is the potential satisfying $\nabla\Phi = (u, v)$, H is the inverse function of $i(\rho)$ with i, ρ being the enthalpy and the density, respectively. For polytropic gas $i(\rho) = \frac{\gamma}{\gamma-1}\rho^{\gamma-1}$, and $H(s)$ can be taken as $(\frac{\gamma-1}{\gamma}s)^{\frac{1}{\gamma-1}}$, where γ is the adiabatic exponent satisfying $1<\gamma<3$. From Bernoulli's relation, we have

$$\Phi_t + \tfrac{1}{2}|\nabla\Phi|^2 + i(\rho) = const., \tag{2.2}$$

where the constant can be simply taken as zero. Therefore, in (2.1) the function H reads

$$H = H\left(-\Phi_t - \frac{1}{2}|\nabla\Phi|^2\right) = \left(\frac{\gamma-1}{\gamma}\left(-\Phi_t - \frac{1}{2}|\nabla\Phi|^2\right)\right)^{\frac{1}{\gamma-1}}.$$

As in the one space-dimensional case the motion of the piston will produce a shock, which moves faster than the piston. On the shock, the R-H condition and the entropy condition should be satisfied. If Σ is the shock front, on which the parameters of the flow has jump, then the R-H condition on Σ is

$$\Phi \text{ is continuous},$$

$$n_t[H] + n_x[\Phi_x H] + n_y[\Phi_y H] = 0, \tag{2.3}$$

where (n_t, n_x, n_y) is the vector normal to Σ, $[\cdot]$ stands for the jump of the function in the bracket. Besides, the entropy condition means the density behind shock is greater than the density ahead of shock, or equivalently, the comparative normal velocity is supersonic ahead of shock, and is subsonic behind shock.

Assume that the state of the gas at the initial time is characterized by $\rho = \rho_0$, $(u, v) = (0, 0)$, and the piston is located at the origin. Starting from $t = 0$, the piston expanded with velocity depending on $\theta = \arctan y/x$. Assume that the velocity is independent of time t, then the path of the piston can be described by

$$B\colon\ \mu\left(\frac{x}{t}\cdot\frac{\theta}{t}\right) = 0 \tag{2.4}$$

which is a conical surface in (t, x, y) space. Then the boundary value condition on B is

$$-\frac{x}{t}\mu_\xi - \frac{y}{t}\mu_\eta + \mu_\xi \Phi_x + \mu_\eta \Phi_y = 0,$$

which means the gas could not go into the piston, nor produce vacuum near the piston.

Since the whole problem is invariant under the dilation $t \to \alpha t$, $x \to \alpha x$, $y \to \alpha y$, we can only consider the self-similar solution of (2.1). Take $\xi = x/t$, $\eta = y/t$, $\Phi(t, x, y) = t\psi(x/t, y/t)$, we have

$$\Phi_x = \psi_\xi, \quad \Phi_y = \psi_\eta,$$

$$-\Phi_t - \tfrac{1}{2}|\nabla\Phi|^2 = -\psi + \xi\psi_\xi + \eta\psi_\eta - \tfrac{1}{2}(\psi_\xi^2 + \psi_\eta^2).$$

Then (2.1) becomes

$$\begin{aligned}(a^2 - (\psi_\xi - \xi)^2)\psi_{\xi\xi} &- 2(\psi_\xi - \xi)(\psi_\eta - \eta)\psi_{\xi\eta} \\ &+ (a^2 - (\psi_\eta - \eta)^2)\psi_{\eta\eta} = 0,\end{aligned} \tag{2.5}$$

where $a = (H/H')^{\frac{1}{2}}$ is the sonic speed.

The boundary condition on Σ is

$$\begin{aligned}&\psi \text{ is continuous,} \\ &\sigma_\xi[(\psi_\xi - \xi)H] + \sigma_\eta[(\psi_\eta - \eta)H] = 0,\end{aligned} \tag{2.6}$$

where $\sigma(\xi, \eta) = 0$ is the equation of the surface of Σ. Since the gas is static ahead of the shock front, then ψ is constant there. By using Bernoulli's relation we have

$$\psi = \psi_0\left(= -\frac{\gamma}{\gamma - 1}\rho_0^{\gamma-1} = -\frac{a_0^2}{\gamma - 1}\right). \tag{2.7}$$

Therefore, the condition on continuity of ψ on Σ can be replaced by (2.7).

Noticing $\theta = \arctan\, y/x = \arctan\, \eta/\xi$, the boundary condition on B is

$$\psi_\xi(\psi_\xi - \xi) + \mu_\eta(\psi_\eta - \eta) = 0. \tag{2.8}$$

It is convenient to work in polar coordinate system (r, θ) instead of (ξ, η). By using the transformation $r = (\xi^2 + \eta^2)^{\frac{1}{2}},\ \theta = \arctan\, \eta/\xi$, the equation becomes

$$\begin{aligned}(a^2 - (\psi_r - r)^2)\psi_{rr} - 2(\psi_r - r)\frac{\psi_\theta}{r^2}\psi_{r\theta} + \left(a^2 - \frac{\psi_\theta^2}{r^2}\right)\frac{1}{r^2}\psi_{\theta\theta} \\ + \frac{\psi_r}{r}\left(a^2 + \frac{\psi_\theta^2}{r^2}\right) - \frac{2}{r^2}\psi_\theta^2 = 0.\end{aligned} \tag{2.9}$$

Correspondingly, if Σ and B are denoted by $r = s(\theta)$ and $r = b(\theta)$, the boundary conditions are

$$(\psi_r - r) - \frac{1}{r^2}\psi_\theta\, b_\theta = 0 \quad \text{on } r = b(\theta), \tag{2.10}$$

$$\psi = \psi_0, \quad \psi_r[(\psi_r - r)H] + \frac{\psi_\theta}{r^2}[\psi_\theta H] = 0 \quad \text{on } r = s(\theta), \tag{2.11}$$

where the argument of H is

$$-\Phi_t - \tfrac{1}{2}|\nabla\Phi|^2 = -\psi_0 + r\psi_r - \tfrac{1}{2}\left(\psi_r^2 + \frac{1}{r^2}\psi_\theta^2\right). \tag{2.12}$$

The main result in this paper is the existence of solution with a shock front outside the piston, provided $b(\theta)$ is a small perturbation of a constant b_0. More precisely, we have

Theorem 2.1. *Assume that the path $b(\theta)$ of the piston satisfies*

$$||b(\theta) - b_0||_{C^{2+\alpha}} \leqslant \varepsilon_0, \tag{2.13}$$

where $b_0 = \min b(\theta)$, and ε_0 is sufficiently small. Then we can find a function $s(\theta)$ defined in $0 \leqslant \theta \leqslant 2\pi$, a function $\psi(r, \theta)$ defined in $b(\theta) \leqslant r \leqslant s(\theta),\ 0 \leqslant \theta \leqslant 2\pi$, such that (2.9)–(2.11) *is satisfied. Moreover, if we denote by $s(\theta),\ \psi_B(r)$ the location of the shock and the solution of the problem with $b(\theta) = b_0$, then*

$$||s(\theta) - s_0||_{C^{2+\alpha}} \leqslant C\varepsilon_0, \tag{2.14}$$

$$||\psi(r, \theta) - \psi_B(r)||_{C^{2+\alpha}} \leqslant C\varepsilon_0. \tag{2.15}$$

3. Symmetric case

In symmetric case, $b(\theta) = b_0$, then the potential $\psi(r,\theta)$ and the function $s(\theta)$ describing the location of shock front will also be independent of θ. Therefore, (2.9) becomes an ordinary differential equation

$$(a^2 - (\psi_r - r)^2)\psi_{rr} + \frac{a^2}{r}\psi_r = 0, \quad r \in (b_0, s_0), \tag{3.1}$$

while the boundary conditions are

$$\psi_r = b_0 \quad \text{on } r = b_0, \tag{3.2}$$

$$\psi = \psi_0, \quad [(\psi_r - r)H] = 0 \quad \text{on } r = s_0, \tag{3.3}$$

where s_0 is unknown, and will be determined together with ψ. Besides, according to the entropy condition, we have

$$\psi_r > 0, \quad a > r - \psi_r > 0. \tag{3.4}$$

Eqs. (3.1)–(3.4) is a free boundary value problem of ordinary differential equation. To solve it we first analyze condition (3.3) under restriction (3.4). Using the expression of H we obtain

$$\frac{\gamma - 1}{\gamma}\left(-\psi_0 + r\psi_r - \frac{1}{2}\psi_r^2\right) = \left(\frac{\rho_0 r}{r - \psi_r}\right)^{\gamma-1}.$$

Denoting $r - \psi_r$ by w and using (2.7), we have

$$\psi_r\left(r - \frac{1}{2}\psi_r\right)(r - \psi_r)^{\gamma-1} + \psi_0(r^{\gamma-1} - (r - \psi_r)^{\gamma-1}) = 0. \tag{3.5}$$

According to (3.4) we only need to consider (3.5) in $0 < \psi_r < r$. Eq. (3.5) can also be written as

$$(r - \tfrac{1}{2}\psi_r)(r - \psi_r)^{\gamma-1} + \psi_0\psi_r^{-1}(r^{\gamma-1} - (r - \psi_r)^{\gamma-1}) = 0.$$

Let $\psi_r \to 0$, it leads to

$$r^{\gamma} + \psi_0(\gamma - 1)r^{\gamma-2} = 0.$$

Namely, $r = (-(\gamma - 1)\psi_0)^{\frac{1}{2}} = a_0$. Therefore, we only need to consider (3.5) in $r \geqslant r_0$. Notice that (3.5) indicates the condition satisfied by ψ_r (the velocity in the radial direction) downward to the shock, so the graph of (3.5) on (r, ψ_r) plane is called shock polar. In the sequel we always denote it by Γ (see Fig. 1).

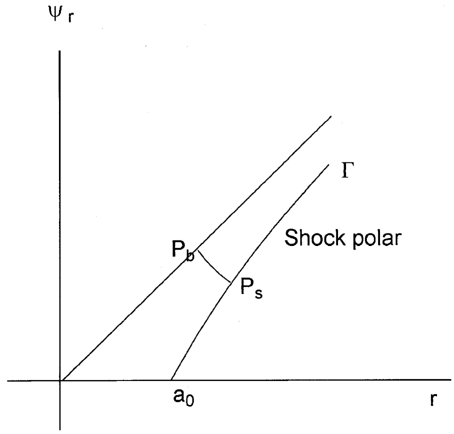

Fig. 1. Shock polar.

Lemma 3.1. *For all possible state ψ, which can be connected with the state $\psi = \psi_0$ by a shock front moving outward, the corresponding (r, ψ_r) must fall onto the shock polar Γ. The shock polar locates below the diagonal $r = \psi_r$. It is increasing, and takes diagonal as its asymptote.*

Proof. We only need to verify the property of the shock polar. Denoting $w = r - \psi_r$ and differentiating (3.5) with respect to r yields

$$(2r - 2ww_r)w^{\gamma-1} - (r^2 - w^2)(\gamma - 1)w^{\gamma-2}w_r - 2a_0^2(r^{\gamma-2} - w^{\gamma-2}w_r) = 0,$$

$$w_r((\gamma - 3)w^{\gamma} - 2a_0^2 w^{\gamma-2} - (\gamma - 1)r^2 w^{\gamma-2}) = 2a_0^2 r^{\gamma-2} - 2rw^{\gamma-1}.$$

It is easy to see that the coefficient of w_r in the above equality is negative due to $1 < \gamma \leqslant 3$, and the right-hand side is positive, then we have $w_r < 0$, namely, $r - \psi_r$ is decreasing, and then ψ_r is increasing on Γ.

Write (3.5) as

$$w^{\gamma-1} = 2\psi_0 \frac{r^{\gamma-1} - w^{\gamma-1}}{r^2 - w^2},$$

we see that the right-hand side tends to zero when $r \to +\infty$, due to $\gamma - 1 < 2$. Hence the shock polar Γ takes the diagonal $r = \psi_r$ as its asymptote.

For any point P_s on Γ, we denote $s_0 = r(P_s)$, $\chi_0 = \psi_r(P_s) > 0$. Taking the initial data as

$$\psi|_{r=s_0} = \psi_0, \quad \psi_r|_{r=s_0} = \chi_0,$$

and integrating (3.1) in $r < s_0$, we obtain a solution of (3.1). □

Lemma 3.2. *There exists an integral curve of* (3.1) *with initial data* $\psi = \psi_0$ *and* ψ_r *determined by the location of the starting point on the shock polar* Γ. *The curve intersect with the diagonal* $r = \psi_r$.

Proof. By continuity

$$r > \psi_r > 0, \quad r - \psi_r < a \tag{3.6}$$

is valid near $r = s_0$. Now we prove that if (3.6) is valid in $s_0 > r > r_0$ and is violated at $r = r_0$, then we must have $r = \psi_r$ at $r = r_0$. In fact, the validity of (3.6) in $r_0 < r < s_0$ implies

$$\psi_{rr} = -\frac{a^2}{r(a^2 - w^2)}\psi_r < 0. \tag{3.7}$$

Hence along the backward integral curve starting from any point on Γ, r is decreasing, and ψ_r is increasing. Therefore, $\psi_r > 0$, $r - \psi_r < a$ are also valid at $r = r_0$. It means that the only possibility, which let (3.6) be violated at $r = r_0$ is $r = \psi_r$. Namely, the integral curve of (3.1) on (r, ψ_r) plane can be extended up to its intersection with the diagonal $r = \psi_r$. The lemma is thus proved. □

Based on these two lemmas, we establish the existence of problem (3.1)–(3.3).

Theorem 3.1. *For any point* $b_0 > 0$ *there is a unique solution* $\psi(r)$ *of* (3.1)–(3.3). *Corresponding to each solution, the curve* $(r, \psi_r(r))$ *is decreasing in* (b_0, s_0), *it intersects with the diagonal* $r = \psi_r$ *at* $r = b_0$ *and with* Γ at $r = s_0$.

Proof. The left end point of Γ is $(a_0, 0)$. Through the end point the solution of (3.1) satisfying $\psi(a_0) = \psi_0$ is $\psi \equiv \psi_0$, which is the interval $(0, a_0)$ on the r-axis. As shown in Fig. 1 starting from any point P_s on Γ, we have a solution of (3.1) satisfying $\psi(r_{P_s}) = \psi_0$. The integral curve ℓ intersects with the diagonal $r = \psi_r$ at P_b. By the property of ordinary differential equation the coordinates of P_b is a continuous function of P_s. When P_s runs to the point $(a_0, 0)$, the corresponding integral curve sweeps the domain bounded by Γ, r-axis, the diagonal and the integral curve ℓ. Besides, we confirm that for each point in between O and P_b on the diagonal there is one and only one integral curve passing through the point. In fact, two integrals ℓ_1 and ℓ_2 of (3.1) intersect at P_b. Since on the diagonal, Eq. (3.1) becomes

$$a^2\psi_{rr} + \frac{1}{r}a^2\psi_r = 0, \tag{3.8}$$

ψ_{rr} on ℓ_1 and ℓ_2 takes the same value.

On the other hand, by differentiating (3.1) we can obtain a second-order differential equation for $\psi_r(r)$ (or denoted by $\chi(r)$):

$$(a^2 - (\chi - r)^2)\chi_{rr} - 2(\chi - r)\chi_r^2 + \frac{a^2}{r}\chi_r + \left(\chi_r + \frac{\chi}{r}\right)a_\chi^2\chi_r = 0. \tag{3.9}$$

By uniqueness of solution to (3.9) with the initial data

$$\chi(r_{P_b}) = \chi_{P_b}, \quad \chi_r(r_{P_b}) = -\frac{1}{r_{P_b}}\chi_{P_b},$$

ℓ_1 and ℓ_2 must coincide. This also means the one-to-one correspondence of P_b and P_s. Since P_s can be any point on Γ and the slope of all integrals of (3.1) on (r, ψ_r) plane is bounded, for any positive number b_0 we can find s_0 and a solution of (3.1), satisfying the boundary conditions (3.2) and (3.3). Hence the theorem is proved. □

4. Partial hodograph transformation and domain decomposition

From now on we consider problem (2.9)–(2.11) in nonsymmetric case. Assume that $b_0 = \min b(\theta)$, and $||b(\theta) - b_0||_{C^{2+\alpha}} < \varepsilon_0$ with ε_0 being sufficiently small, we expect that the solution of (2.9)–(2.11) is also a perturbation of problem (3.1)–(3.3). In the sequel we call the solution of (3.1)–(3.3) as background solution, and denote it by ψ_B.

Problem (2.9)–(2.11) is a free boundary value problem. When $(r - \psi_r)^2 + \frac{1}{r^2}\psi_\theta^2 < a^2$, Eq. (2.9) is elliptic. Since ψ_B satisfies (3.4), then (2.9) is elliptic for ψ_B, and is also elliptic for the small perturbation of ψ_B. To avoid the difficulty caused by the moving boundary $r = s(\theta)$, we introduce a partial hodograph transformation to fix it. The transformation is

$$T : \begin{cases} \sigma = \theta, \\ p = -\psi(r, \theta), \end{cases} \tag{4.1}$$

which changes the position of the unknown function ψ and the variable r. Since ψ equals a constant ψ_0 on the shock front, T transforms the shock front to a fixed boundary $p = -\psi_0$. The inverse of T is

$$T^{-1} : \begin{cases} \theta = \sigma, \\ r = u(p, \sigma). \end{cases} \tag{4.2}$$

Here we always have $p > 0$, because of $\psi_r > 0$ and $p|_{r=s(\theta)} = -\psi_0 > 0$. By chain rule

$$\psi_r = -\frac{1}{u_p}, \quad \psi_\theta = \frac{u_\sigma}{u_p},$$

$$\psi_{rr} = \frac{1}{u_p^3}u_{pp}, \quad \psi_{r\theta} = -\frac{u_\sigma}{u_p^3}u_{pp} + \frac{1}{u_p^2}u_{p\sigma},$$

$$\psi_{\theta\theta} = \frac{1}{u_p}u_{\sigma\sigma} - \frac{2u_\sigma}{u_p^2}u_{p\sigma} + \frac{u_\sigma^2}{u_p^3}u_{pp}.$$

Then we have

$$\left(a^2-\left(\frac{1}{u_p}+u\right)^2\right)\left(\frac{1}{u_p^3}u_{pp}\right)-2\left(\frac{1}{u_p}+u\right)\left(\frac{u_\sigma}{u_p}\right)\left(\frac{u_\sigma}{u_p^3}u_{pp}-\frac{1}{u_p^2}u_{p\sigma}\right)\frac{1}{u^2}$$
$$+\frac{1}{u^2}\left(a^2-\frac{1}{u^2}\frac{u_\sigma^2}{u_p^2}\right)\left(\frac{u_{\sigma\sigma}}{u_p}-\frac{2u_\sigma}{u_p^2}u_{p\sigma}+\frac{u_\sigma^2}{u_p^3}u_{pp}\right)$$
$$-\frac{1}{uu_p}\left(a^2+\frac{u_\sigma^2}{u^2u_p^2}\right)-\frac{2}{u^2}\frac{u_\sigma^2}{u_p^2}=0$$

or

$$E_{11}u_{pp}+2E_{12}u_{p\sigma}+E_{22}u_{\sigma\sigma}+Q(u,u_p,u_\sigma)=0, \tag{4.3}$$

where

$$E_{11}=\left(a^2-\left(\frac{1}{u_p}+u\right)^2\right)-2\left(\frac{1}{u_p}+u\right)\left(\frac{u_\sigma^2}{u^2\,u_p}\right)+\left(a^2-\frac{1\,u_\sigma^2}{u^2u_p^2}\right)\frac{u_\sigma^2}{u^2},$$

$$E_{12}=-\left(\frac{1}{u_p}+u\right)\frac{u_\sigma}{u^2}-\left(a^2-\frac{1}{u^2}\frac{u_\sigma^2}{u_p^2}\right)\frac{u_\sigma u_p}{u^2},$$

$$E_{22}=\left(a^2-\frac{1}{u^2}\frac{u_\sigma^2}{u_p^2}\right)\frac{u_p^2}{u^2},$$

$$Q(u,u_p,u_\sigma)=-\frac{1}{u}\left(a^2u_p^2+\frac{u_\sigma^2}{u^2}\right)-\frac{2}{u^2}u_\sigma^2u.$$

The boundary conditions will also have new forms in the new coordinates. However, in the new coordinates system the path of the piston becomes unknown. Denote it by $p=g(\sigma)$, the boundary condition on it is

$$u=b(\sigma),\quad u_p=-\frac{1}{u}-\frac{u_\sigma b_\sigma}{u^3}. \tag{4.4}$$

On the shock front $p=\psi_0$, the condition is

$$\left(\frac{1}{u_p}+u\right)H-u\rho_0+\frac{u_\sigma^2}{u^2u_\rho}H=0. \tag{4.5}$$

In what follows, problem (2.9)–(2.11) is called (NL), and problem (4.3)–(4.5) is called $(NL)^*$. In fact, (NL) is equivalent to $(NL)^*$. If one of them is solved, then the solution of another one is also obtained.

Since the boundary $p = g(\sigma)$ is unknown, the problem $(NL)^*$ is also a free boundary value problem. To avoid the appearance of any new free boundary, we try to only consider (2.9) near $r = b(\theta)$, and only consider (4.3) near $p = -\psi_0$. To this end we decompose the annular domain $b(\theta) < r < s(\theta)$ to a set of overlapped annuluses. In the inner annulus, which is adjacent to the path of the piston and is denoted by Ω_a, we keep the original coordinates (r, θ) and consider (2.9). While in other annuluses, which are denoted by $\tilde{\Omega}_{b_1}, \ldots, \tilde{\Omega}_{b_k}$, the partial hodograph transformation (4.1) is applied. More precisely, we introduce constants r_1, r_2 and monotonically decreasing sequences $\{\alpha_\ell\}, \{\beta_\ell\}$ with $1 \leqslant \ell \leqslant k$ satisfying (see Fig. 2)

$$\begin{aligned} & b_0 \leqslant b(\theta) < r_2 < r_1 < b_0 + \delta, \\ & \alpha_1 = -\psi_B(r_2), \\ & \alpha_\ell > \beta_{\ell-1} > \alpha_{\ell+1} > \beta_\ell, \quad 1 < \ell < k, \\ & \beta_k = -\psi_0. \end{aligned} \tag{4.6}$$

The annulus $b(\theta) < r < r_1,\ 0 \leqslant \theta \leqslant 2\pi$ on (r, θ) plane is denoted by Ω_a, and the annulus $\beta_\ell < p < \alpha_\ell,\ 0 \leqslant \sigma \leqslant 2\pi$ is denoted by Ω_{b_ℓ}. The image of Ω_{b_ℓ} under the transformation T^{-1} is $\tilde{\Omega}_{b_\ell}$. All Ω_a and $\tilde{\Omega}_{b_\ell} (1 \leqslant \ell \leqslant k)$ form an overlapped covering to the domain $b(\theta) < r < s(\theta)$. Besides, we assume that $\beta_\ell - \alpha_\ell \leqslant \delta$, $|\alpha_\ell - \beta_{\ell-1}| \geqslant \frac{\delta}{4}$ for all ℓ, and δ is small, so that the corresponding boundary value problems of Eq. (2.9) in Ω_a or Eq. (4.3) in $\tilde{\Omega}_{b_\ell}$ as well as their linearization are well posed. In the next section we will explain it once more.

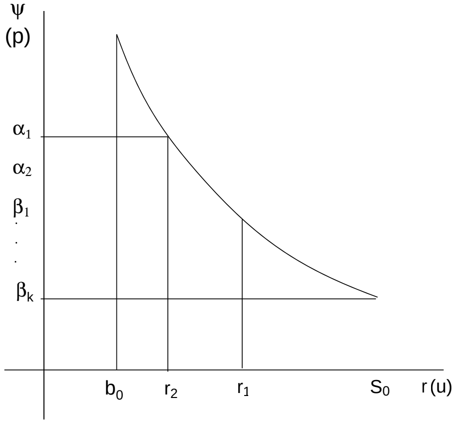

Fig. 2. Domain decomposition.

Corresponding to the $k+1$ annuluses, we introduce a set of auxiliary boundary value problem as follows:

$$(NL)^{(a)}:\begin{cases}\text{Eq. (2.9)} \quad \text{in } \Omega_a,\\ \text{boundary condition} \quad (2.10) \quad \text{on } r=b(\theta),\\ \psi=d(\theta) \quad \text{on } r=r_1,\end{cases} \tag{4.7}$$

$$(NL)^{(b_\ell)}:\begin{cases}\text{Eq. (4.3)} \quad \text{in } \Omega_{b_\ell},\\ u=q_{\ell_1}(\sigma) \quad \text{on } p=\alpha_\ell,\\ u=q_{\ell_2}(\sigma) \quad \text{on } p=\beta_\ell,\end{cases} \tag{4.8}$$

where $1\leqslant\ell\leqslant k-1$,

$$(NL)^{(b_k)}:\begin{cases}\text{Eq. (4.3)} \quad \text{in } \Omega_{b_k},\\ u=q_{k1}(\sigma) \quad \text{on } p=\alpha_k,\\ \text{boundary condition (4.5)} \quad \text{on } p=-\psi_0.\end{cases} \tag{4.9}$$

All these problems are defined in a domain with fixed boundary. The solvability of the problems $(NL)^{(a)}$, $(NL)^{(b_\ell)}$ and the corresponding estimates of their solutions will be given in the next section. To emphasize the dependence on the corresponding data, the above three problems will also be denoted by $(NL)^{(a)}\{b(\theta),d(\theta)\}$, $(NL)^{(b_\ell)}\{q_{\ell 1}(\sigma),q_{\ell 2}(\sigma)\}$, and $(NL)^{(b_k)}\{q_{k1}(\sigma)\}$, respectively. As we will see in the following sections, the solvability and the corresponding estimates will lead us to obtain the solution of the problem (NL) (and $(NL)^*$).

5. Problem $(NL)^{(a)}$

First let us prove the existence of solution to $(NL)^{(a)}$ and indicate some of its properties. Under the transformation

$$\tau:\begin{cases}\tilde\theta=\theta,\\ \dfrac{\tilde r-b_0}{r_1-b_0}=\dfrac{r-b(\theta)}{r_1-b(\theta)},\end{cases} \tag{5.1}$$

the boundary $r=b(\theta)$ is mapped onto $\tilde r=b_0$, and the boundary $r=r_1$ is unchanged, where r_1 is given in the last section. Consider the linearization of the problem $(NL)^{(a)}$ at $\psi=\psi_B(r),\ b(\theta)=b_0,\ d(\theta)=\psi_{10}$. Since at the background solution, $\tilde r_r=1,\ \tilde r_\theta=0$, and $(\psi_B)_\theta=(\psi_B)_{r\theta}=(\psi_B)_{\theta\theta}=0$, we obtain an equation for the perturbation $\dot\psi$:

$$\mathscr{L}^{(a)}\dot\psi\equiv A_{11}\dot\psi_{rr}+A_{22}\dot\psi_{\theta\theta}+B_1\dot\psi_r+C\dot\psi=f, \tag{5.2}$$

where

$$A_{11} = a^2 - (\psi_r - r)^2, \quad A_{22} = \frac{a^2}{r^2},$$

$$B_1 = \frac{a^2}{r} + 2(r - \psi_r)\psi_{rr} + \left(\psi_{rr} + \frac{\psi_r}{r}\right)(\gamma - 1)(r - \psi_r),$$

$$C = -(\gamma - 1)\left(\psi_{rr} + \frac{\psi_r}{r}\right).$$

Correspondingly, the linearized boundary conditions are

$$\dot{\psi}_r - \frac{b_\theta}{r^2}\dot{\psi}_\theta = g \quad \text{on } r = b, \tag{5.3}$$

$$\dot{\psi} = h \quad \text{on } r = r_1. \tag{5.4}$$

The linearized problem (5.2)–(5.4) is denoted by $L^{(a)}$, which is a linear elliptic boundary problem because $A_{11} > 0$ and $A_{22} > 0$.

Lemma 5.1. *There is $\delta > 0$, such that the solution of $L^{(a)}$ uniquely exists, and*

$$||\dot{\psi}||_{C^{2+\alpha}[b_0, r_1; 0, 2\pi]} \leqslant C_1(||f||_{C^\alpha[b_0, r_1; 0, 2\pi]} + ||g||_{C^{1+\alpha}(0, 2\pi)} + ||h||_{C^{2+\alpha}(0, 2\pi)}), \tag{5.5}$$

$$||\dot{\psi}||_{C^{2+\alpha}[b_0, r_1 - \frac{\delta}{10}; 0, 2\pi]} \leqslant C_2(||f||_{C^\alpha[b_0, r_1; 0, 2\pi]} + ||g||_{C^{1+\alpha}(0, 2\pi)} + ||h||_{C^0(0, 2\pi)}) \tag{5.6}$$

provided $|r_1 - b_0| < \delta$.

Proof. First, let us show that the solution $\dot{\psi}$ of the linearized problem (5.2)–(5.4) monotonically depends on its boundary value on $r = r_1$, provided f and g vanish. In fact, making a transformation of unknown function $v = e^{K(r - b_0)^2}\dot{\psi}$ for the problem $L^{(a)}$,

$$\mathscr{L}^{(a)}\dot{\psi} = \mathscr{L}^{(a)}(e^{-K(r - b_0)^2}v) = e^{-K(r - b_0)^2}\mathscr{L}_K^{(a)}v,$$

where

$$\mathscr{L}_K^{(a)}v = \mathscr{L}^{(a)}v - 4K(r - b_0)A_{11}v_r + ((4K^2(r - b_0)^2 - 2K)A_{11} - 2K(r - b_0)B_1)v. \tag{5.7}$$

Obviously, v satisfies the elliptic equation

$$\mathscr{L}_K^{(a)}v = 0,$$

provided $\dot{\psi}$ satisfies $\mathscr{L}^{(a)}\dot{\psi} = 0$. When δ is sufficiently small and $K = \delta^{-1}$, the coefficient of v in $\mathscr{L}_K^{(a)} v$ satisfies

$$(4K^2(r-b_0)^2 - 2K)A_{11} - 2K(r-b_0)B_1 < (4 - 2\delta^{-1})A_{11} + 2|B_1| < 0.$$

Hence, v cannot take its positive maximum or negative minimum in the domain Ω_a, nor on the boundary of Ω_a. This fact implies that v depends on its boundary value on $r = r_1$ monotonically. Hence it is also true for the solution $\dot{\psi}$ of the problem $L^{(a)}$.

The above argument indicates that 0 is not an eigenvalue of the elliptic operator $\mathscr{L}^{(a)}$ under homogeneous boundary conditions corresponding to (5.3) and (5.4), provided δ is sufficiently small. Namely, problem (5.2)–(5.4) is uniquely solvable. Finally, (5.5) and (5.6) are just the generalized global and interior Schauder estimates. □

Remark 5.1. In this paper we will often use two kinds of small constants with different scale. The small constant δ (or δ_i) is used to restrict the domain, where a boundary value problem under consideration is given. Usually, we choose δ small enough to ensure the well posedness of the corresponding boundary value problem, while the constant ε (or ε_i) is used to describe the perturbation of the data. The constant ε is small, and is also small enough with respect to δ. Usually, ε is chosen when δ has been fixed.

Lemma 5.2. *Assume that* $\delta,\ \varepsilon$ *are sufficiently small in the sense of Remark* 5.1. $|r_1 - b_0| < \delta,\ \|b(\theta) - b_0\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon,\ \|d(\theta) - \psi_{10}\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$ *with* $\psi_{10} = \psi_B(r_1)$, *then the problem* $(NL)^{(a)}\{b(\theta), d(\theta)\}$ *has unique solution* $\psi(r,\theta)$. *Moreover,*

$$\|\psi(r,\theta) - \psi_B(r)\|_{C^{2+\alpha}(\Omega_a)} \to 0 \quad \text{when } \varepsilon \to 0. \tag{5.8}$$

Proof. When $b(\theta) = b_0$, the function $\psi_B(r)$ is the solution of the nonlinear problem $(NL)^{(a)}\{b_0,\ \psi_{10}\}$. For $|r_1 - b_0| < \delta$, the linearization $L^{(a)}$ of nonlinear problem $(NL)^{(a)}$ at $b(\theta) = b_0,\ \psi = \psi_{10}$ has estimate (5.5), where the constant C_1 is uniform with respect to $b(\theta)$. Then the implicit function theorem implies that the problem $(NL)^{(i)}\{b(\theta), d(\theta)\}$ has an unique solution, which is a small perturbation of $\psi = \psi_B(r)$, provided $\|b(\theta) - b_0\|_{C^{2+\alpha}} < \varepsilon, \|d(\theta) - \psi_{10}\|_{C^{2+\alpha}} < \varepsilon$ for sufficiently small ε. Finally, (5.8) follows from the conclusion of the implicit function theorem. □

The following lemma is called comparison principle.

Lemma 5.3. *Assume that* $\delta,\ \varepsilon$ *are sufficiently small in the sense of Remark* 5.1, $|r_1 - b_0| < \delta, \|b(\theta) - b_0\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon, \|d_j(\theta) - \psi_{10}\|_{C^{2+\alpha}(0,2\pi)} < \varepsilon$ $(j = 1,2)$, *and* $\psi_j(r,\theta)$ *is the solution of the problem* $(NL)^{(a)}\{b(\theta), d_j(\theta)\}$, *then the comparison principle is valid, i.e.* $d_2 \geqslant d_1$ *implies* $\psi_2 \geqslant \psi_1$.

Proof. Assume $f = g = 0$ in problem (5.2)–(5.4), the boundary $r = b_0$ is replaced by $r = b(\theta)$, and all the coefficients have small perturbation, then by using the same argument as in Lemma 5.1, we confirm that a nonnegative datum on the boundary $r = r_1$ corresponds to a nonnegative solution of the perturbed problem.

For the nonlinear problem $(NL)^{(a)}$, we denote the solution with boundary condition $\psi|_{r=r_1} = d_j(\theta)$ by $\psi_j(r, \theta)$. Also denote $\dot{\psi} = \psi_2 - \psi_1$, then $\dot{\psi}$ satisfies

$$A_{11}(\psi_1)\dot{\psi}_{rr} + D_0\dot{\psi}_{r\theta} + A_{22}(\psi_1)\dot{\psi}_{\theta\theta} + (B_1(\psi_1) + D_1)\dot{\psi}_r + D_2\dot{\psi}_\theta + (C(\psi_1) + D_3)\dot{\psi} = 0, \tag{5.9}$$

where $A_{11}(\psi_1),\ A_{22}(\psi_1),\ B_1(\psi_1),\ C(\psi_1)$ are the coefficients in (5.2) with ψ replaced by ψ_1, $D_j (j = 0, 1, 2, 3)$ are small quantities with factors $\dot{\psi}_{rr}, \dot{\psi}_r, \dot{\psi}$ or $\psi_{i\theta\theta}, \psi_{ir\theta}, \psi_{i\theta}$. Correspondingly, the boundary conditions for $\dot{\psi}$ are

$$\dot{\psi}_r - \frac{b_\theta}{r^2}\dot{\psi}_\theta = 0 \quad \text{on } r = b(\theta), \tag{5.10}$$

$$\dot{\psi} = d_2 - d_1 \quad \text{on } r = r_1. \tag{5.11}$$

In view of estimate (5.5), the coefficients of Eq. (5.9) are small perturbations of the corresponding coefficients of (5.2), provided that $||d_j(\theta) - \psi_{10}||_{C^{2+\alpha}}$ and $||b(\theta) - b_0||_{C^{2+\alpha}}$ are sufficiently small. Therefore, 0 is not an eigenvalue of the operator in the left-hand side of (5.9), and the comparison principle is available. Namely, $d_2 - d_1 \geqslant 0$ on $r = r_1$ implies $\dot{\psi} \geqslant 0$ in Ω_a.

Let $\tilde{b_0} = \max b(\theta)$, then from Theorem 3.1 we know that the solution $\tilde{\psi}_B(r)$ of (3.1)–(3.3) with b_0 replaced by $\tilde{b_0}$ is defined on $(\tilde{b_0}, \tilde{s}_0)$, where $\tilde{s}_0 > s_0$. Moreover,

$$\tilde{w}(r) < w(r), \quad \tilde{\psi}_{Br}(r) > \psi_{Br}(r) \quad \text{on } \tilde{b_0} \leqslant r \leqslant s_0$$

and

$$\begin{aligned}\tilde{\psi}_B(r) &= \psi_0 - \int_r^{\tilde{s}_0} \tilde{\psi}_{Br}(r)\, dr \\ &= \psi_0 - \int_{s_0}^{\tilde{s}_0} \tilde{\psi}_{Br}(r)\, dr - \int_r^{s_0} \tilde{\psi}_{Br}(r)\, dr \\ &< \psi_0 - \int_r^{s_0} \psi_{Br}(r)\, dr = \psi_B(r).\end{aligned}$$

On the other hand, extending $\tilde{\psi}_B(r)$ to $(b_0, \tilde{b_0})$ by using (3.1) and the data at $r = \tilde{b_0}$, we know $\tilde{\psi}_B(r) < \psi_B(r)$ also holds in $(b_0, \tilde{b_0})$. In the sequel the functions $\psi_B(r)$ and $\tilde{\psi}_B(r)$ will be applied to bound the solution of (NL) from above and below. □

Lemma 5.4. *Assume that* $b_0 = \min b(\theta)$, $\tilde{b}_0 = \max b(\theta)$, $||b(\theta) - b_0||_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, $\psi(r,\theta) = (NL)^{(a)}\{b(\theta), d(\theta)\}$, $\tilde{\psi}_B(r)$ *is the solution of* (3.1)–(3.3) *with* b_0 *replaced by* $\tilde{b}_0$, $\tilde{\psi}_B(r_1) \leqslant d(\theta) \leqslant \psi_B(r_1)$, *then*

$$\tilde{\psi}_B(r) \leqslant \psi(r,\theta) \leqslant \psi_B(r) \quad \text{in } \Omega_a.$$

Besides

$$||\psi(r,\theta) - \psi_B(r)||_{C^{2+\alpha}(\Omega_a^-)} \leqslant C(||d(\theta) - \psi_{10}||_{C(0,2\pi)} + \varepsilon), \tag{5.12}$$

where $\Omega_a^- = \{(r,\theta)\colon\ b(\theta) \leqslant r \leqslant r_1 - \frac{1}{10}\delta,\ 0 \leqslant \theta \leqslant 2\pi\}$.

Proof. Set $\Delta_1\psi = \psi - \psi_B$. It satisfies

$$\begin{aligned} &A_{11}(\psi_B)(\Delta_1\psi)_{rr} + A_{22}(\psi_B)(\Delta_1\psi)_{\theta\theta} + \tilde{D}_0(\Delta_1\psi)_{r\theta} + (B_1(\psi_B) + \tilde{D}_1)(\Delta_1\psi)_r \\ &\quad + \tilde{D}_2(\Delta_1\psi)_\theta + (C(\psi_B) + \tilde{D}_3)\Delta_1\psi = 0, \end{aligned} \tag{5.13}$$

where $\tilde{D}_j$ are small quantities as D_j in (5.9). In view of $w_B = r - \psi_{Br}$ being positive for $r > r_0$, $\psi_{Br} \leqslant r$ on $r = b(\theta)$. Hence

$$(\Delta_1\psi)_r \geqslant 0 \quad \text{on } r = b(\theta). \tag{5.14}$$

Besides,

$$\Delta_1\psi = d(\theta) - \psi_{10} \quad \text{on } r = r_1. \tag{5.15}$$

Again noticing that the coefficients of (5.13) are small perturbation of corresponding coefficients of (5.2), and using the argument similar to the proof of Lemma 5.3, we know that $\Delta_1\psi$ cannot attain its positive maximum inside the domain and on the boundary. Namely, we have $\psi(r,\theta) \leqslant \psi_B(r)$.

Similarly, we can prove $\psi(r,\theta) \geqslant \tilde{\psi}_B(r)$.

To prove (5.12), we use the transformation (5.1). Let $\psi_B^*(r,\theta) = \tau^{-1}(\psi_B(\tilde{r}))$, then ψ_B^* is defined in whole Ω_a. Since ψ_B satisfies Eq. (3.1), we have

$$e^2(a^2 - (e(\psi_B^*)_r - r)^2)(\psi_B^*)_{rr} + \frac{a^2}{\tilde{r}} e(\psi_B^*)_r = 0, \tag{5.16}$$

where $e = \frac{\partial r}{\partial \tilde{r}} = \frac{r_1 - b(\theta)}{r_1 - b_0} = 1 + \frac{b_0 - b(\theta)}{r_1 - b_0} = 1 + O(\varepsilon)$. Denote $\Delta\psi^* = \psi(r,\theta) - \psi_B^*$, it satisfies

$$\begin{aligned} &A_{11}(\psi_B)(\Delta\psi^*)_{rr} + A_{22}(\psi_B)(\Delta\psi^*)_{\theta\theta} + D_0^*(\Delta\psi^*)_{r\theta} + (B_1(\psi_B) + D_1^*)(\Delta\psi^*)_r \\ &\quad + D_2^*(\Delta\psi^*)_\theta + (C(\psi_B) + D_3^*)\Delta\psi^* = f, \end{aligned} \tag{5.17}$$

and corresponding boundary conditions

$$(\Delta\psi^*)_r = g \quad \text{on } r = b(\theta), \tag{5.18}$$

$$\Delta\psi^* = d(\theta) - \psi_{10} \quad \text{on } r = r_1, \tag{5.19}$$

where $||f||_{C^{2+\alpha}}$, $||g||_{C^{2+\alpha}}$ are quantities $O(\varepsilon)$. Therefore, we have

$$||\Delta\psi^*||_{C^{2+\alpha}(\Omega_a^-)} \leqslant C(||d(\theta) - \psi_{10}||_{C^0(0,2\pi)} + \varepsilon),$$

which leads to (5.12) directly. □

6. Problem $(NL)^{(b_\ell)}$

Now let us consider the nonlinear problem $(NL)^{(b_\ell)}$ $(1 \leqslant \ell \leqslant k)$ in the annulus Ω_{b_ℓ} with fixed boundary $p = \alpha_\ell$ and β_ℓ as defined in (4.8) and (4.9). The inverse function of $\psi_B(r)$ is also called background solution of $(NL)^*$, and is denoted by $u_B(p)$. Linearizing (4.3) at the background solution $u = u_B(p)$ and using $u_{B\sigma} = 0$, $a^2 = (\gamma - 1)(-p - \frac{u}{u_p} - \frac{1}{2}\frac{1}{u_p^2})$, we obtain

$$\left(a^2 - \left(\frac{1}{u_p} + u\right)^2\right)\dot{u}_{pp} + \frac{a^2 u_p^2}{u^2}\dot{u}_{\sigma\sigma} + F_1\dot{u}_p + F_0\dot{u} = f, \tag{6.1}$$

where

$$F_1 = 2\left(\frac{1}{u_p} + u\right)\frac{u_{pp}}{u_p^2} - \frac{2a^2}{u}u_p + (\gamma - 1)\left(u_{pp} - \frac{u_p^2}{u}\right)\left(\frac{u}{u_p^2} + \frac{1}{u_p^3}\right) - 2a^2\frac{u_p}{u},$$

$$F_0 = -2\left(\frac{1}{u_p} + u\right)u_{pp} + \frac{a^2u_p^2}{u^2} - (\gamma - 1)\left(u_{pp} - \frac{u_p^2}{u}\right)\frac{1}{u_p} + \frac{u_p^2}{u^2}a^2.$$

The linearization of the boundary conditions on $p = \alpha_\ell (1 \leq \ell \leqslant k)$ and $p = \beta_\ell (1 \leq \ell \leqslant k - 1)$ is simple, while the linearization of the boundary condition on $p = -\psi_0 (= \beta_k)$ is

$$\left(-\frac{\dot{u}_p}{u_p^2} + \dot{u}\right)H - \dot{u}\rho_0 + \left(\frac{1}{u_p} + u\right)H'\left(\frac{u\dot{u}_p}{u_p^2} + \frac{\dot{u}_p}{u_p^3} - \frac{\dot{u}}{u_p}\right) = 0.$$

In view of $H = \rho$, $H' = \rho/a^2$, the condition can be written as

$$\gamma_1\dot{u}_p + \gamma_0\dot{u} = g, \tag{6.2}$$

where

$$\gamma_1 = -\frac{1}{u_p^2} + a^{-2}\left(\frac{1}{u_p} + u\right)\left(\frac{u}{u_p^2} + \frac{1}{u_p^3}\right),$$

$$\gamma_0 = \left(1 - \frac{\rho_0}{\rho}\right) + \left(\frac{1}{u_p} + u\right)a^{-2}\left(-\frac{1}{u_p}\right).$$

Since $\frac{1}{u_p} + u = r - \psi_r$ is the relative velocity, which is less than the sound speed behind the shock front, we have

$$\gamma_1 = \frac{\psi_r^2}{a^2}((r - \psi_r)^2 - a^2) < 0,$$

$$\gamma_0 = 1 - \frac{\rho_0}{\rho} + (r - \psi_r)a^{-2}(\psi_r) > 0.$$

Noticing the direction $\frac{\partial}{\partial p}$ points inward of the domain $\beta_k \leqslant p \leqslant \alpha_k,\ 0 \leqslant \theta \leqslant 2\pi$, the sign of the coefficients in condition (6.2) satisfies the requirement of maximum principle for elliptic boundary value problems. Therefore, assume $\frac{\delta}{2} \leqslant |\beta_\ell - \alpha_\ell| \leqslant \delta$ for $1 \leqslant \ell \leqslant k$, and δ is small enough, all linearized problems

$$L^{(b_\ell)} : \begin{cases} \text{Eq. (6.1)} & \text{in } \Omega_{b_\ell}, \\ \dot{u}_\ell = q_{\ell 1} & \text{on } p = \alpha_\ell, \\ \dot{u}_\ell = q_{\ell 2} & \text{on } p = \beta_\ell \end{cases} \tag{6.3}$$

for $1 \leqslant \ell \leqslant k - 1$, and

$$L^{(b_k)} : \begin{cases} \text{Eq. (6.1)} & \text{in } \Omega_{b_k}, \\ \dot{u}_k = q_{k1} & \text{on } p = \alpha_k, \\ \text{condition (6.2)} & \text{on } p = \beta_k (= -\psi_0) \end{cases} \tag{6.4}$$

are well posed. Meanwhile, the following properties for the solution u_ℓ of the nonlinear problem $NL^{(b_\ell)}$ and the solution $\dot{u}_\ell$ of the linear problem $L^{(b_\ell)}$ the following properties can be verified by the similar method as we did for ψ and $\dot{\psi}$ in Section 5.

Lemma 6.1. *There is a $\delta > 0$, such that the solution $\dot{u}_\ell$ of $L^{(b_\ell)}$ uniquely exists, and satisfies*

$$||\dot{u}_\ell||_{C^{2+\alpha}(\Omega_{b_\ell})} \leqslant C_1(||f||_{C^\alpha(\Omega_{b_\ell})} + ||q_{\ell 1}||_{C^{2+\alpha}(0,2\pi)} + ||q_{\ell 2}||_{C^{2+\alpha}(0,2\pi)}), \tag{6.5}$$

$$||\dot{u}_\ell||_{C^{2+\alpha}(\Omega^-_{b_\ell})} \leqslant C_2(||f||_{C^\alpha(\Omega_{b_\ell})} + ||q_{\ell 1}||_{C^0(0,2\pi)} + ||q_{\ell 2}||_{C^0(0,2\pi)}) \tag{6.6}$$

for $1 \leqslant \ell \leqslant k-1$, *and*

$$||\dot{u}_k||_{C^{2+\alpha}(\Omega_{b_k})} \leqslant C_1(||f||_{C^\alpha(\Omega_{b_k})} + ||g||_{C^{1+\alpha}(0,2\pi)} + ||q_{k1}||_{C^{2+\alpha}(0,2\pi)}), \tag{6.7}$$

$$||\dot{u}_k||_{C^{2+\alpha}(\Omega^-_{b_k})} \leqslant C_2(||f||_{C^\alpha(\Omega_{b_k})} + ||g||_{C^{1+\alpha}(0,2\pi)} + ||q_{k1}||_{C^0(0,2\pi)}), \tag{6.8}$$

where $\Omega^-_{b_\ell} = [\beta_\ell + \frac{1}{10}\delta, \alpha_\ell - \frac{1}{10}\delta; 0, 2\pi]$ *for* $1 \leqslant \ell \leqslant k-1$ *and* $\Omega^-_{b_k} = [\beta_k, \alpha_k - \frac{1}{10}\delta; 0, 2\pi]$.

Lemma 6.2. (a) *Assume that* $||q_{\ell 1}(\sigma) - u_B(\alpha_1)||_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, $||q_{\ell 2}(\sigma) - u_B(\beta_1)||_{C^{2+\alpha}(0,2\pi)} < \varepsilon$ *hold for* $1 \leqslant \ell \leqslant k-1$, *then* $(NL)^{(b_\ell)}\{q_{\ell 1}(\sigma), q_{\ell 2}(\sigma)\}$ *has a unique solution. Moreover*,

$$||u(p,\sigma) - u_B(p)||_{C^{2+\alpha}(\Omega_{b_\ell})} \to 0 \quad \text{when } \varepsilon \to 0. \tag{6.9}$$

(b) *Assume that* $||q_{k1}(\sigma) - u_B(\alpha_2)||_{C^{2+\alpha}(0,2\pi)} < \varepsilon$, *then* $(NL)^{(b_k)}\{q_{k1}(\sigma)\}$ *has a unique solution. Moreover*,

$$||u(p,\sigma) - u_B(p)||_{C^{2+\alpha}(\Omega_{b_k})} \to 0 \quad \text{when } \varepsilon \to 0. \tag{6.10}$$

Proof. The proof for cases (a) and (b) are similar, so we only prove the case (b). The three equations in the problem $(NL)^{(b_k)}$ can be regarded as a map from $C^{2+\alpha}(\Omega_{b_k})$ to $C^\alpha(\Omega_{b_k}) \times C^{1+\alpha}(0,2\pi) \times C^{2+\alpha}(0,2\pi)$. For $(0,0,u_B(\alpha_2)) \in C^\alpha \times C^{1+\alpha} \times C^{2+\alpha}$, it has been known that the problem has a solution $u_B(p)$, which is just the inverse of $\psi_B(r)$. Besides, the linearized problem has estimate (6.7). According to the implicit function theorem, there is an $\varepsilon > 0$, such that $(NL)^{(e_k)}$ has a unique solution, provided $||q_{k1}(\sigma) - u_B(\alpha_2)||_{C^{2+\alpha}} < \varepsilon$. Finally, (6.10) obviously follows from the implicit function theorem. □

Lemma 6.3. *Assume that* δ, ε *are sufficiently small in the sense of Remark* 5.1, *then*

(a) *If* $u^{(j)}_\ell(p,\sigma) = (NL)^{(b_\ell)}\{q^{(j)}_{\ell 1}(\sigma), q^{(j)}_{\ell 2}(\sigma)\}$, $||q^{(j)}_{\ell 1}(\sigma) - u_B(\alpha_\ell)||_{C^{2+\alpha}} < \varepsilon$, $||q^{(j)}_{\ell 2}(\sigma) - u_B(\beta_\ell)||_{C^{2+\alpha}} < \varepsilon$, *for* $j = 1,2$, $1 \leqslant \ell \leqslant k-1$, *then*

$$q^{(2)}_{\ell 1}(\sigma) \geqslant q^{(1)}_{\ell 1}(\sigma), q^{(2)}_{\ell 2}(\sigma) \geqslant q^{(1)}_{\ell 2}(\sigma) \Rightarrow u^{(2)}_\ell(p,\sigma) \geqslant u^{(1)}_\ell(p,\sigma). \tag{6.11}$$

Besides,

$$\begin{aligned} &||u^{(j)}_\ell(p,\sigma) - u_B(p)||_{C^{2+\alpha}(\Omega^-_{b_\ell})} \\ &\quad \leqslant C(||q^{(j)}_{\ell 1}(\sigma) - u_B(\alpha_\ell)||_{C^0(0,2\pi)} + ||q^{(j)}_{\ell 2}(\sigma) - u_B(\beta_\ell)||_{C^0(0,2\pi)} + \varepsilon). \end{aligned} \tag{6.12}$$

(b) *If* $||q_{k1}^{(j)}(\sigma) - u_B(\alpha_k)||_{C^{2+\alpha}} < \varepsilon$, $u_k^{(j)}(p,\sigma) = NL^{(e_k)}\{q_{k1}^{(j)}\}$, *then*

$$q_{k1}^{(2)}(\sigma) \geqslant q_{k1}^{(1)}(\sigma) \Rightarrow u_k^{(2)}(p,\sigma) \geqslant u_k^{(1)}(p,\sigma). \tag{6.13}$$

Besides,

$$||u_k^{(j)}(p,\sigma) - u_B(p)||_{C^{2+\alpha}(\Omega_{b_k}^-)} \leqslant C(||q_{k1}^{(j)}(\sigma) - u_B(\alpha_k)||_{C^0(0,2\pi)} + \varepsilon). \tag{6.14}$$

Proof. Take $\dot{u}_\ell = u_\ell^{(1)} - u_\ell^{(2)} (1 \leqslant \ell \leqslant k)$, *then* $\dot{u}_\ell$ *satisfies*

$$\left(a^2 - \left(\frac{1}{u_p} + u\right)^2\right)\dot{u}_{pp} + \frac{a^2u_p^2}{u^2}\dot{u}_{\sigma\sigma} + D_0\dot{u}_{p\sigma} + (F_1 + D_1)\dot{u}_p + D_2\dot{u}_\sigma + (F_0 + D_3)\dot{u} = 0, \tag{6.15}$$

where $\dot{u}$ stands for $\dot{u}_\ell$ for notational simplification. On the boundaries, $\dot{u}_\ell$ is nonnegative on $p = \alpha_\ell, \beta_\ell$, $\dot{u}_k$ is nonnegative on $p = \alpha_k$. Moreover, $\dot{u}_k$ satisfies

$$(\gamma_1 + \mu_1)\dot{u}_p + \mu_2\dot{u}_\sigma + (\gamma_0 + \mu_0)\dot{u} = 0 \tag{6.16}$$

on $p = \beta_k$. Here all D_j, μ_j in the above equalities are small quantities $O(\varepsilon)$. Therefore, the coefficients in (6.15), (6.16) are small perturbation of the corresponding coefficients in (6.1), (6.2). It turns out that the comparison principle holds for small δ, then the estimate (6.12), (6.14) can be derived from Lemma 6.1. □

7. Solution to nonlinear problem (NL) and $(NL)^*$

Based on the discussion of $(NL)^{(a)}$ and $(NL)^{(b_\ell)}$, we are able to construct the solution of (NL) and $(NL)^*$. The main idea in this step is by alternatively solving $(NL)^{(a)}$ and $(NL)^{(b_\ell)}$ $(\ell = 1, \ldots, k)$ to establish sequences of approximate solutions $\{\psi^{(n)}\}$ for $(NL)^{(a)}$ and approximate solutions $\{u_\ell^{(n)}\}$ to $(NL)^{(b_\ell)}$, so that these sequences are convergent and the limit of them solves the nonlinear problem (NL) and $(NL)^*$.

First, let us describe the method of determining $r_1, \alpha_\ell, \beta_\ell (0 \leqslant \ell \leqslant k)$ again. By using Lemmas 5.2–5.4 we take $r_1 > \tilde{b}_0 = \max b(\theta)$, so that $|r_1 - b(\theta)| < \delta$ and the solution ψ of the problem $(NL)^{(a)}$ uniquely exists in the domain $\Omega_a : b(\theta) \leqslant r \leqslant r_1$. According to these lemmas we have the solution ψ of $(NL)^{(a)}(b(\theta), \psi_{10})$ satisfies $\psi \leqslant \psi_B$ and the value of ψ in Ω_a depends monotonically on its boundary value on $r = r_1$. Then by using Lemma 6.2, we choose k and $\{\alpha_\ell\}, \{\beta_\ell\}$ for $1 \leqslant \ell \leqslant k$, so that (4.6) is satisfied, and the solution to problem $(NL)^{(b_\ell)}$ uniquely exists in the domain Ω_{b_ℓ} and satisfies the comparison principle with respect to the data on the boundary.

Let us assume $k = 2$ for notations simplification in the remaining part of this paper. In fact, the case $k > 2$ can be treated in the same way. Besides, we also use the following notations: r as a function of p, θ will be denoted by $\psi^{-1}(-p, \theta)$, provided $p = -\psi(r, \theta)$; while p as a function of r, σ will be denoted by $u^{-1}(r, \sigma)$, provided $r = u(p, \sigma)$.

The sequences of approximate solutions will be established as follows. Denote $\psi^{(0)} = \psi_B$, $u_1^{(0)} = u_{1'}^{(0)} = u_2^{(0)} = u_B$, then we define $\psi^{(1)}(r, \theta)$ as the solution of the problem $(NL)^{(a)}\{b(\theta), \psi_B(r_1)\}$. For $n \geqslant 1$, we define

$$u_1^{(n)}(p, \sigma) = (NL)^{(b_1)}\{(\psi^{(n)})^{-1}(-\alpha_1, \sigma), (u_2^{(n-1)}(\beta_1, \sigma)\},$$

$$u_2^{(n)}(p, \sigma) = NL^{(b_2)}\{u_1^{(n)}(\alpha_2, \sigma)\},$$

$$u_{1'}^{(n)}(p, \sigma) = NL^{(b_1)}\{u_1^{(n)}(\alpha_1, \sigma), u_2^{(n)}(\beta_1, \sigma)\},$$

$$\psi^{(n+1)}(r, \theta) = NL^{(a)}\{b(\theta), -(u_{1'}^{(n)})^{-1}(r_1, \theta)\}$$

inductively.

Lemma 7.1. *If* $||b(\theta) - b_0||_{C^{2+\alpha}} < \varepsilon$ *with* ε *being sufficiently small,* $\tilde{b_0} = \max b(\theta)$, $\psi_B(r), \tilde{\psi}_B(r)$, $u_B(p), \tilde{u}_B(p)$ *are the corresponding background solutions for* (NL) *and* $(NL)^*$ *in symmetric case. Then the sequences* $\{\psi^{(n)}\}, \{u_1^{(n)}\}, \{u_2^{(n)}\}, \{u_{1'}^{(n)}\}$ *established above are well defined, which satisfy*

$$\tilde{\psi}_B(r) \leqslant \psi^{(n)}(r, \theta) \leqslant \psi_B(r) \quad \text{in } \Omega_a^-,$$
$$||\psi^{(n)}(r, \theta) - \psi_B(r)||_{C^{2+\alpha}(\Omega_a^-)} \leqslant C\varepsilon \tag{7.1}$$

and

$$\tilde{u}_B(p) \geqslant u_1^{(n)}(p, \sigma), u_{1'}^{(n)}(p, \sigma) \geqslant u_B(p) \quad \text{in } \Omega_{b_1}^-,$$
$$\tilde{u}_B(p) \geqslant u_2^{(n)}(p, \sigma) \geqslant u_B(p) \quad \text{in } \Omega_{b_2}^-,$$
$$||u_1^{(n)}(p, \sigma) - u_B(p)||_{C^{2+\alpha}(\Omega_{b_1}^-)} \leqslant C\varepsilon,$$
$$||u_{1'}^{(n)}(p, \sigma) - u_B(p)||_{C^{2+\alpha}(\Omega_{b_1}^-)} \leqslant C\varepsilon,$$
$$||u_2^{(n)}(p, \sigma) - u_B(p)||_{C^{2+\alpha}(\Omega_{b_2}^-)} \leqslant C\varepsilon. \tag{7.2}$$

Moreover, $\{\psi^{(n)}\}$ *is monotone decreasing with respect to n, and other three sequences are monotone increasing with respect to n.*

Proof. Without loss of generality we assume

$$||\tilde{\psi}_B - \psi_B||_{C^0} < \varepsilon, \quad ||\tilde{u}_B - u_B||_{C^0} < \varepsilon.$$

Lemma 5.4 indicates that the solution $\psi^{(1)}(r, \theta)$ of the nonlinear problem $(NL)^{(a)}\{b(\theta), \psi_B(r_1)\}$ exists and satisfies $\tilde{\psi}_B \leqslant \psi^{(1)} \leqslant \psi_B$. Under the assumptions of the lemma we have

$$||\psi^{(1)} - \psi_B||_{C^0(\Omega_a)} \leqslant \varepsilon, \tag{7.3}$$

$$||\psi^{(1)} - \psi_B||_{C^{2+\alpha}(\Omega_a^-)} \leqslant C_2 \varepsilon. \tag{7.4}$$

Because $\psi^{(1)}$ is a small perturbation of ψ_B, $(\psi^{(1)})^{-1}$ is well defined and $(\psi^{(1)})^{-1}(-p, \sigma) \geqslant (\psi_B)^{-1}(-p) = u_B(p)$. Moreover, from (7.3) and $\psi_r^{(1)} > \kappa > 0$, we have

$$||(\psi^{(1)})^{-1}(-\alpha_1, \sigma) - u_B(\alpha_1)||_{C^0(0,2\pi)} < \frac{\varepsilon}{\kappa}. \tag{7.5}$$

Taking $(\psi^{(1)})^{-1}(-\alpha_1, \sigma)$ and $u_B(\beta_1)$ as the data on the boundary $p = \alpha_1$ and β_1, respectively, for the nonlinear problem $(NL)^{(b_1)}$, we obtain the solution $u_1^{(1)}(p, \sigma)$ of $(NL)^{(b_1)}$ by Lemma 6.2, while Lemmas 6.1 and 6.3 imply the following estimates:

$$||u_1^{(1)} - u_B||_{C^{2+\alpha}(\Omega_{b_1}^-)} \leqslant C_2 ||(\psi^{(1)})^{-1}(-\alpha_1, \sigma) - u_B(\alpha_1)||_{C^0(0,2\pi)} \leqslant C_2 \frac{\varepsilon}{\kappa}, \tag{7.6}$$

$$\tilde{u}_B(\alpha_1) \geqslant u_1^{(1)}(\alpha_1, \sigma) \geqslant u_B(\alpha_1). \tag{7.7}$$

Hence by using comparison principle established in the above sections we have

$$\tilde{u}_B(p) \geqslant u_1^{(1)}(p, \sigma) \geqslant u_B(p) \tag{7.8}$$

in Ω_{b_1}, as well as on $p = \alpha_2$.

Next, we solve the problem $(NL)^{(b_2)}\{u_1^{(1)}(\alpha_2, \sigma)\}$. Lemmas 6.2 and 6.3 imply the existence of the problem, and

$$\tilde{u}_B(p) \geqslant u_1^{(1)}(p, \sigma) \geqslant u_B(p) \quad \text{in } \Omega_{b_2}^-,$$

$$||u_2^{(1)} - u_B||_{C^{2+\alpha}(\Omega_{b_2}^-)} \leqslant C_2 \varepsilon. \tag{7.9}$$

Returning to domain Ω_{b_1}, we solve another Dirichlet problem $(NL)^{(b_1)}$ with data $u_1^{(1)}(\alpha_1,\sigma), u_2^{(1)}(\beta_1,\sigma)$ on $p=\alpha_1$ and p_1, respectively. That is

$$u_{1'}^{(1)}(p,\sigma) = NL^{(b_1)}\{u_1^{(1)}(\alpha_1,\sigma), u_2^{(1)}(\beta_1,\sigma)\}$$

which also satisfies (7.6) and (7.7) with $u_1^{(1)}$ being replaced by $u_{1'}^{(1)}$.

In view of $(u_{1'}^{(1)})_p > 0$, we know $(u_{1'}^{(1)})^{-1} \geqslant -\psi_B$ on $r=r_1$, meanwhile,

$$|| - (u_{1'}^{(1)})^{-1}(r_1,\theta) - \psi_B(r_1)||_{C^0(0,2\pi)} < \varepsilon. \tag{7.10}$$

In addition, we can solve the problem $NL^{(a)}\{b(\theta), -(u_{1'}^{(1)})^{-1}(r_1,\theta)\}$ in Ω_a by Lemma 5.2. Its solution $\psi^{(2)}(r,\theta)$ satisfies $\psi^{(2)} \leqslant \psi^{(1)}$ according to Lemma 5.3 and satisfies $\psi^{(2)} \geqslant \tilde{\psi}_B$ according to Lemma 5.4. Therefore, we have

$$||\psi^{(2)}(r_2,\theta) - \psi_B(r_2)||_{C^0(0,2\pi)} < \varepsilon, \tag{7.11}$$

$$\begin{aligned} ||\psi^{(2)}(r,\theta) - \psi_B(r)||_{C^{2+\alpha}(\Omega_a^-)} &\leqslant C_2|| - (u_{1'}^{(1)})^{-1}(r_1,\theta) - \psi_B(r_1)||_{C^0(0,2\pi)} \\ &\leqslant C_2\varepsilon. \end{aligned} \tag{7.12}$$

Notice that (7.11) and (7.12) are the same as (7.3) and (7.4) with index replaced by 2. Therefore, the sequence $\{\psi^{(n)}\}, \{u_1^{(n)}\}, \{u_2^{(n)}\}, \{u_{1'}^{(n)}\}$ can be established.

By the same procedure, we obtain $u_1^{(2)}(p,\sigma)$, $u_2^{(2)}(p,\sigma)$, $u_{1'}^{(2)}(p,\sigma)$, $\psi^{(3)}(r,\theta)$ and so on. Then (7.1) and (7.2) can be proved by induction.

To prove the monotonicity of the sequences $\{\psi^{(n)}\}$ and $\{u^{(n)}\}$, we also verify inequalities

$$\begin{aligned} &\psi^{(n+1)}(r,\theta) \leqslant \psi^{(n)}(r,\theta), \\ &u_1^{(n+1)}(p,\sigma) \geqslant u_1^{(n)}(p,\sigma), \\ &u_2^{(n+1)}(p,\sigma) \geqslant u_2^{(n)}(p,\sigma), \\ &u_{1'}^{(n+1)}(p,\sigma) \geqslant u_{1'}^{(n)}(p,\sigma) \end{aligned} \tag{7.13}$$

by induction. According to the process of establishing these sequences, we have $\psi^{(1)}(r,\theta) \leqslant \psi_B(r)$ from Lemma 5.4. Furthermore, the inequalities $u_1^{(1)}(p,\sigma) \geqslant u_B(p)$, $u_{1'}^{(1)}(p,\sigma) \geqslant u_B(p)$, $u_1^{(2)}(p,\sigma) \geqslant u_B(p)$ follow from Lemma 6.3. It means that (7.13) is true for $n=0$.

Now assuming (7.13) be valid with n replaced by $n-1$, we prove that it is also valid when the index is n. First, because of $u_p < 0$, the last inequality of (7.13) with index n replaced by $n-1$ means $(u_{1'}^{(n)})^{-1}(r_1,\theta) \geqslant (u_{1'}^{(n-1)})^{-1}(r_1,\theta)$, which implies

$$\psi^{(n+1)}(r,\theta) \leqslant \psi^{(n)}(r,\theta) \quad \text{in } \Omega_a \tag{7.14}$$

according to Lemma 5.3. Hence $(\psi^{(n+1)})^{-1}(-\alpha_1,\sigma) \geqslant (\psi^{(n)})^{-1}(-\alpha_1,\sigma)$. Combining it with the hypothesis of induction $u_2^{(n)}(\beta_1,\sigma) \geqslant u_2^{(n-1)}(\beta_1,\sigma)$ we have

$$u_1^{(n+1)}(p,\sigma) \geqslant u_1^{(n)}(p,\sigma) \quad \text{in } \Omega_{b_1} \tag{7.15}$$

according to part (a) of Lemma 6.3. Next, by using (7.15) on the line $p = \alpha_2$ and the boundary condition on $p = \beta_2$, the inequality

$$u_2^{(n+1)}(p,\sigma) \geqslant u_2^{(n)}(p,\sigma) \quad \text{in } \Omega_{b_2} \tag{7.16}$$

is also valid according to the part (b) of Lemma 6.3. Furthermore, the facts (7.16) on the line $p = \beta_1$ and (7.15) on the line $p = \alpha_1$ indicate

$$u_{1'}^{(n+1)}(p,\sigma) \geqslant u_{1'}^{(n)}(p,\sigma) \quad \text{in } \Omega_{b_1}. \tag{7.17}$$

Therefore, since all equalities in (7.13) with index n hold, the monotonicity of the sequences in the lemma is proved by induction.

Proof of Theorem 2.1. As we proved in Lemma 7.1, the sequences $\{u_1^{(n)}(p,\sigma)\}$, $\{u_2^{(n)}(p,\sigma)\}$, $\{u_{1'}^{(n)}(p,\sigma)\}$ and $\{\psi^{(n)}(r,\theta)\}$ are bounded and monotone with respect to n, then these sequences are convergent. Denote their limits by $u_1(p,\sigma)$, $u_2(p,\sigma)$, $u_{1'}(p,\sigma)$, $\psi(r,\theta)$, respectively. Since the $C^{2+\alpha}$ norm of $\psi^{(n)}$ on $r = u_B(\alpha_1)$ is dominated by its C^0 norm on $r = r_1$, the $C^{2+\alpha}$ norm of $u_1^{(n)}, u_{1'}^{(n)}$ in $\Omega_{b_1}^-$ is dominated by their C^0 norm on $p = \alpha_1$, $p = \beta_1$, the $C^{2+\alpha}$ norm of $u_2^{(n)}$ in $\Omega_{b_2}^-$ is dominated by its C^0 norm on $p = \alpha_2$, so the $C^{2+\alpha}$ norm of $\psi^{(n)}, u_1^{(n)}, u_{1'}^{(n)}, u_2^{(n)}$ are uniformly bounded in corresponding domains. The fact also implies ψ, u_1,$u_{1'}$, u_2 are $C^{2+\alpha}$ function there.

On the other hand, ψ satisfies (2.9), (2.10) and $\psi(r_1,\theta) = -(u_1)^{-1}(r_1,\theta)$; $u_1, u_{1'}$ satisfies (4.3) and $u_1 = u_{1'}$ on the boundary $p = \alpha_1$, $p = \beta_1$; u_2 satisfies (4.3), (4.5) and $u_1(\alpha_2,\sigma) = u_2(\alpha_2,\sigma)$. Notice that (4.3) is the another form of Eq. (2.9) in the coordinates (p,σ), then both $\psi(r,\theta)$ and $-(u_1)^{-1}(r,\theta)$ satisfy Eq. (2.9) in the overlapped domain $\Omega_a \cap T^{(-1)}(\Omega_{b_1})$. Besides, these two functions coincide on the boundaries $r = r_1$ and $r = u(\alpha_1,\theta)$. Since the domain Ω_a is chosen so small that the linearized problem $L^{(a)}$ does not have nonnegative eigenvalue in Ω_a, there is no nonnegative eigenvalue in its subdomain $\Omega_a \cap T^{(-1)}(\Omega_{b_1})$ either. By the uniqueness of

Dirichlet problem for the nonlinear elliptic equation (2.9), the functions $\psi(r,\theta)$ and $u_1^{-1}(r,\theta)$ coincide on the whole domain $\Omega_a \cap T^{(-1)}(\Omega_{b_1})$. Moreover, $u_1(p,\sigma), u_{1'}(p,\sigma), u_2(p,\sigma)$ coincide on the corresponding overlapped domain. Therefore, viewing functions $u_1^{-1}(r,\theta), u_2^{-1}(r,\theta)$ as extensions of $\psi(r,\theta)$, we obtain the solution of (NL) in the whole domain $b(\theta) < r < \psi^{-1}(-\psi_0,\theta)$.

As for the stability, (2.15) can be obtained by taking limit in (7.1), (7.2). Besides, in view of $s(\theta) = \psi^{-1}(-\psi_0,\theta)$, (2.14) can also be obtained in the mean time.

Acknowledgments

The author is grateful to the Institute of Mathematical Science of CUHK, where he worked on this paper in his visiting period. This work is also partially supported by NNSF of China, the Key Grant of NMST of China and the Doctoral Foundation of NEC.

References

[1] S. Alinhac, Existence d'ondes de rarefaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels, Comm. Partial Differential Equations 14 (1989) 173–230.

[2] S. Canic, B. Keyfitz, Riemann problems for the two-dimensional unsteady transonic small disturbance equation, SIAM J. Appl. Math. 58 (1998) 636–665.

[3] S. Canic, B. Keyfitz, G. Lieberman, A proof of existence of perturbed steady transonic shocks via a free boundary problem, Comm. Pure Appl. Math. 53 (2000) 484–511.

[4] S. Canic, B. Keyfitz, E.H. Kim, Free boundary problems for the unsteady transonic small disturbance equation: transonic regular reflection, Method Appl. Anal. 7 (2000) 313–335.

[5] G.Q. Chen, Remarks on spherically symmetric solutions of the compressible Euler equations, Proc. Roy. Soc. Edinburgh 127A (1997) 243–259.

[6] S. Chen, Non-symmetric conical supersonic flow, International Series of Numerical Mathematics, Vol. 129, Birkhauser Verlag, Basel, Switzerland, 1999, pp. 140–158.

[7] S. Chen, Existence of stationary supersonic flow past a pointed body, Arch. Rational Mech. Anal. 156 (2001) 141–181.

[8] R. Courant, K.O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers Inc., New York, 1948.

[9] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer, Berlin, Heidelberg, New York, 2000.

[10] T.T. Li, Global Classical Solutions for Quasilinear Hyperbolic Systems, Wiley, Masson, New York, Paris, 1994.

[11] W.C. Lien, T.P. Liu, Nonlinear stability of a self-similar 3-d gas flow, Comm. Math. Phys. 304 (1999) 524–549.

[12] A. Majda, The stability of multi-dimensional shock fronts, Mem. Amer. Math. Soc. 273 (1983).

[13] A. Majda, The existence of multi-dimensional shock fronts, Mem. Amer. Math. Soc. 281 (1983).

[14] A. Majda, One perspective on open problems in multi-dimensional conservation laws, IMA Vol. Math. Appl. 29 (1991) 217–237.

[15] A. Majda, E. Thomann, Multi-dimensional shock fronts for second order wave equations, Comm. Partial Differential Equations 12 (1987) 777–828.
[16] G. Metivier, Ondes Soniques, J. Math. Pure Appl. 70 (1991) 197–268.
[17] M. Mnif, Probléme de Riemann pour une loi de conservation scalaire hyperbolique d'order deux, Comm. Partial Differential Equations 22 (1997) 1589–1627.
[18] C.S. Morawetz, Potential theory for regular and Mach reflection of a shock at a wedge, Comm. Pure Appl. Math. 47 (1994) 593–624.
[19] D.G. Schaeffer, Supersonic flow past a nearly straight wedge, Duke Math. J. 43 (1976) 637–670.
[20] J. Smoller, Shock Waves and Reaction-diffusion Equations, Springer, Berlin, New York, 1983.
[21] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, London, Sydney, Toronto, 1974.