Existence of Positive Solutions for Quasi-Linear Differential Equations

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Abstract—Sufficient conditions for the existence of at least one positive solution for the following quasilinear differential equation

\[ (\varphi_p(x'))' + c(t)f(x) = 0 \]

where \( \varphi_p(u) = |u|^{p-2}u, p \geq 2 \) is a constant, \( c \in C(R^+, R^+) \), \( f(x) > 0 \) for \( x > 0 \), \( f(x) = 0 \) for \( x \leq 0 \). The method used in this paper is the shooting method. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The existence of positive solutions for the following second-order quasi-linear differential equation

\[ (\varphi_p(x'))' + f(t, x) = 0, \quad 0 < t < T, \]  

where \( \varphi_p(u) = |u|^{p-2}u, p \geq 2 \) is a constant, \( f(t, 0) = 0, t \in R^+(0, +\infty) \), \( f(t, x) \geq 0 \) for \( x > 0 \) and \( f \) is continuous, has been discussed by many authors (see, e.g., [1–5]). If \( p = 2, f(t, x) = c(t)f(x) \), then (1) reduces to

\[ x'' + c(t)f(x) = 0. \]  

This equation often arises in the study of positive radial solutions for the following nonlinear elliptic equation

\[ \Delta u + h(|x|)f(u) = 0. \]  

In this paper, the existence of positive solutions for the following Dirichlet boundary problem

\[ (\varphi_p(x'))' + c(t)f(x) = 0, \]  
\[ x(0) = x(T) = 0, \]  

is studied, where \( T > 0 \) and the following assumptions are assumed

(A1) \( \varphi_p(u) = |u|^{p-2}u, p \geq 2 \) is a constant,

(A2) \( c \in C(R^+, R^+) \), and \( c(t) \) is not identically zero in any subinterval of \( (0, +\infty) \) and \( \int_0^T c(t) \, dt < +\infty, \)

(A3) \( f(x) > 0 \) for \( x > 0 \), \( f(x) = 0 \) for \( x \leq 0 \) and \( f(x^{1/(p-1)}) \) is locally Lipschitz,

(A4) \( \lim_{x \to 0^+} f(x) / x^{p-1} = 0, \)

(A5) \( \lim_{x \to +\infty} f(x) / x^{p-1} = +\infty. \)
The method used here is the forward shooting method combined with the generalized form of the Sturm comparison theorem. Our method is different to the methods used in [1-5].

The main result of this paper is as follows.

**Theorem 1.** Let Assumptions (A1)-(A5) hold, then the boundary value problem (4),(5) has at least one solution.

**2. PROOF OF THEOREM 1**

For the proof of Theorem 1, we need the following lemmas.

**Lemma 1.** The solution of the following initial value problem

\[(\varphi_p (x'))' + c(t)f(x) = 0,\]
\[x(0) = 0, \quad x'(0) = a,\]

with \(a > 0\), exists on \([0,+\infty)\) and is unique.

**Proof.** Since \(p \geq 2\), \(f(x)/x^{p-1} \to 0\) as \(x \to 0^+\) and \(f(x^{1/(p-1)})\) is locally Lipschitz, by applying Proposition A1 and Proposition A2 in [6], one obtains the uniqueness of solution (6),(7). For the global existence of solution of (6),(7), one applies the uniqueness and the fact \(f(x) \geq 0\) to obtain

\[0 \leq x(t) \leq \int_0^t a \, ds = at, \quad t \geq 0,
\]

which implies that \(x(t)\) exists on \(R^+\).

**Lemma 2.** Generalized Sturm Comparison Theorem. Consider the following nonlinear second-order differential equations

\[(\varphi_p (x'))' + Q(t)\varphi_p(x) = 0,\]
\[(\varphi_p (y'))' + q(t)\varphi_p(y) = 0,\]

with \(t \in I = [t_1,t_2]\), \(t_1 < t_2\), \(p > 1\), \(Q(t) < q(t), t \in (t_1,t_2)\).

If \(x(t)\) is a solution of (8) satisfying

\[x(t_1) = x(t_2) = 0, \quad x(t) \neq 0, \quad t \in (t_1,t_2),\]

then any solution \(y(t)\) of (9) has at least one zero in \(I\).

**Proof.** Since if \(x(t)\) is a solution of (8), \(-x(t)\) is also a solution of (8), therefore, we can assume without loss of generality that \(x(t) > 0, t \in (t_1,t_2)\). It is easy to see \(x'(t_1) > 0, x'(t_2) < 0\) and \(u(t) = \varphi_p(x'(t)/x(t))\) is a solution of

\[u'(t) + (p-1)|u(t)|^{p/(p-1)} + Q(t) = 0, \quad t \in (t_1,t_2),\]

and satisfies boundary conditions

\[\lim_{t \to t_1^+} u(t) = +\infty, \quad \lim_{t \to t_2^-} u(t) = -\infty.\]

If \(y(t)\) has no zero in \(I\), say \(y(t) > 0, t \in I\), then the function

\[v(t) = r(t) \varphi_p \left( \frac{y'(t)}{y(t)} \right)\]
is a solution of the following equation
\[ v'(t) + (p-1)|v(t)|^{p/(p-1)} + q(t) = 0, \quad t \in I. \] (12)

From (11) and the continuity of \( v(t) \), there exists a \( t^+ \in (t_1, t_2) \), such that
\[ u(t^+) = v(t^+), \quad u(t) < v(t), \quad t \in (t^+, t_2). \] (13)

On the other hand, from \( Q(t) \leq q(t) \), for \( t \in (t_1, t_2) \), one has
\[ u'(t) = -(p-1)|v(t)|^{p/(p-1)} - Q(t) \geq -(p-1)|v(t)|^{p/(p-1)} - q(t). \] (15)

It follows from the comparison theorem of differential equations of first order \([7]\), one obtains
\[ u(t) \geq v(t), \quad t \in (t^+, t_2), \]
but this contradicts (14).

Let \( Q(t) \equiv q(t), t \in I \), then one has the following results.

**Corollary 1.** If a nonzero solution of (8) has two zeros in \( I \), then any other solution of (8) has at least one zero in \( I \).

**Corollary 2.** If the inequality \( Q(t) < q(t) \) holds at least one point in \( I \) and if \( x(t), y(t) \) are solutions of (8) and (9), respectively, with \( x(t_1) = y(t_1) = 0, x(t_2) = 0, x(t) \neq 0, t \in (t_1, t_2) \), then there exists a \( t^* \in (t_1, t_2) \), such that \( y(t^*) = 0 \).

**Lemma 3.** (See \([8]\).) Consider an eigenvalue problem
\[ (\varphi_p(x'))' + \lambda \varphi_p(x) = 0, \]
\[ x(0) = x(T) = 0. \] (16) (17)

Then, the eigenvalues of (16), (17) are \( \lambda = \lambda_n(p) = (n\pi_p/T)^p, n = 1, 2, 3, \ldots, \) where
\[ \pi_p = \int_0^{(p+1)/p} \frac{ds}{(1 - s^p/p - 1)^{1/p}}, \quad p > 1, \]
and the corresponding eigenfunction \( x_n(t) \) has \( n \) zeros in \([0, T]\).

**Corollary 3.** Let \( J = [T_1, T_2] \) with \( T_1 < T_2 \) be a closed interval, then there exists a sufficiently large positive number \( M > 0 \), such that for any solution \( x(t) \) of the following equation
\[ (\varphi_p(x'))' + M \varphi_p(x) = 0, \]
\( x(t) \) has at least two zeros in \( J \).

Corollary 3 is a direct consequence of Corollary 2 and Lemma 2.

**Lemma 4.** Let \( x(t) = x(t,a) \) be the solution of (6), (7). Then, for \( a > 0 \) sufficiently large, there exists a unique \( \tau(a) > 0 \) such that \( x' (\tau(a)) = 0, x'(t) > 0, t \in [0, \tau(a)], x'(t) < 0, t > \tau(a) \) and
\[ \lim_{a \to +\infty} \tau(a) = 0, \]
\[ \lim_{a \to +\infty} x(\tau(a)) = +\infty. \] (19) (20)
PROOF. Since $x(t)$ satisfies (6), (7), if $x'(t) \geq 0$ for $t \in [0, t_1]$, then $x(t)$ satisfies

$$x(t) = \int_0^t \left( a^{p-1} - \int_0^u c(s)f(x(s)) \, ds \right)^{1/(p-1)} \, du, \quad t \in [0, t_1].$$

(21)

If the lemma were false, then there exists a $\tau_0 > 0$, such that $a_k \to +\infty$,

$$x_k(t) > 0, \quad x_k'(t) > 0, \quad t \in (0, \tau_0),$$

where $x_k(t) = x_k(t, a)$. For fixed $t_1 > 0$, we claim that

$$\lim_{k \to +\infty} \sup_{t \in [0, t_1]} x_k(t) = +\infty.$$  (22)

Suppose that this is not the case. Then there exists a constant $M > 0$ such that

$$x_k \left( \frac{\tau_0}{2} \right) \leq M, \quad k \in \mathbb{N}.$$  (23)

It follows from (21) and (23) that

$$x_k \left( \frac{\tau_0}{2} \right) \geq \int_0^{\tau_0/2} \left( a_k^{p-1} - \int_0^u c(s) \max_{0 \leq s \leq \tau_0/2} f(x_k(s)) \, ds \right)^{1/(p-1)} \, du$$

$$\geq \int_0^{\tau_0/2} \left( a_k^{p-1} - C \right)^{1/(p-1)} \, du,$$

for some constant $C > 0$. But by (23), this is impossible. Therefore, (22) holds. By choosing a subsequence of $a_k$ if necessary, one may assume

$$\lim_{k \to +\infty} x_k \left( \frac{\tau_0}{2} \right) = +\infty.$$  (24)

By (A2), there exists a subinterval $J_0 = (t', t'')$ of $(\tau_0/2, \tau_0)$ such that $c(t) \geq c_0 > 0$, $t \in J_0$. Denote

$$M_k = \inf \left\{ \frac{f(x_k(t))}{\varphi_p(x_k(t))} : t \in J_0 \right\}.$$  

Then,

$$M_k \geq \inf \left\{ \frac{f(x)}{\varphi_p(x)} : x \geq x_k \left( \frac{\tau_0}{2} \right) \right\}.$$  

It follows from $x_k'(t) > 0$, $t \in [0, t_1]$, (24), and (A5) that

$$\lim_{k \to +\infty} M_k = +\infty.$$  (25)

By (6), $x_k$ satisfies

$$(\varphi_p(x'))' + c(t)h_k(t)\varphi_p(x) = 0, \quad t \in J_0,$$

where

$$h_k(t) = \frac{f(x_k(t))}{\varphi_p(x_k(t))} \quad \text{and} \quad c(t)h_k(t) \geq c_0M_k, \quad t \in J_0.$$  (26)

Let $y_k(t)$ be a nonzero solution of

$$(\varphi_p(y'))' + c_0M_k\varphi_p(y) = 0,$$  (27)

then Corollary 3 implies that $y_k(t)$ has at least two zeros in $J_0$ if $k$ is sufficiently large. Corollary 2 now implies that $x_k(t)$ has at least one zero in $J_0$. This contradicts the assumption $x_k(t) > 0,$
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t \in (0, +\infty). Hence, there exists a \tau(a) > 0 such that \(x'(\tau(a)) = 0\). The uniqueness of \(\tau(a)\) is a direct consequence of the fact \(c(t)f(x) \geq 0\) and \(f(x) > 0\) for \(x > 0\), \(c(t) \neq 0\) on any subinterval of \([0, +\infty)\).

Next, we show (19). If (19) were false, there exists a \(\tau_0 > 0\) such that for some \(a_k \to +\infty, k \to +\infty,\)

\[
x_k(t) > 0, \quad x'_k(t) > 0, \quad t \in (0, \tau_0).
\]

(28)

Letting \(\tau_0 = t_1\) in the above discussion, one can show that for \(k\) sufficiently large, \(x_k(t)\) has at least one zero in \((\tau_0, \tau_0)\). This contradicts assumption (28). Hence, (19) holds.

Lastly, we show (20). Suppose (20) does not hold. Then there exist \(M_1 > 0, M_2 > 0, a_k \to +\infty, k \to +\infty,\) such that for \(t \in [0, \tau(a_k)]\),

\[
0 \leq x_k(t) \leq x_k(\tau(a_k)) \leq M_1, \quad 0 \leq f(x_k(t)) \leq f(M_2).
\]

(29)

(30)

Since \(x_k(t)\) satisfies

\[
a^p_k - 1 = \int_0^{\tau(a_k)} c(t)f(x_k(t)) \, dt.
\]

(31)

Let \(a_k \to +\infty\) in (31), then by (19),

\[
a^p_k - 1 \leq \int_0^{\tau(a_k)} c(t)f(x_k(t)) \, dt \leq \int_0^{\tau(a_k)} c(t)f(M_2) \, dt < +\infty,
\]

(32)

but the left side of (32) is unbounded, this is impossible. Hence, (20) holds.

If \(x(t)\) has a zero \(t_b\) in \((0, \infty)\), then \(t_b\) is the unique zero of \(x(t)\) in \((0, \infty)\) and \(x'(t_b) < 0\). From the implicit function theorem, \(t_b = t_b(a)\) is a \(C^1\) function on \((0, +\infty)\).

**Lemma 5.** Assume all the conditions in Lemma 4 are satisfied, then for a sufficiently large, the function \(t_b = t_b(a)\) is defined and satisfies

\[
\lim_{a \to +\infty} t_b(a) = 0.
\]

(33)

**Proof.** If the lemma were false, there would be a point \(t^0 \in (0, T_k)\), with \(a_k \to \infty\) such that

\[
x_k(t) > 0, \quad x'_k(t) < 0, \quad t \in \left(t_k, t^0\right).
\]

(34)

By Lemma 3, for sufficiently large \(k\), we may assume \(t_k < t^0/2\). We claim that

\[
\lim_{k \to \infty} \sup x_k \left(\frac{t^0}{2}\right) < \infty.
\]

(35)

If (35) were not true, then by using a similar method used in the proof of Lemma 3 again, \(x_k(t)\) has at least one zero in \((t_k, t^0/2)\) when \(k\) is large enough, which contradicts (34).

By (6), (7), it is not difficult to prove that

\[
x_k(t_k) = x_k \left(\frac{t^0}{2}\right) + \int_{t_k}^{t^0/2} \left\{ - \int_{u/2}^{s} c(u)f(x_k(u)) \, du + \left|x'_k \left(\frac{t^0}{2}\right)\right|^{p-1} \right\}^{1/(p-1)} \, ds
\]

\[
< x_k \left(\frac{t^0}{2}\right) + \int_{t_k}^{t^0/2} \left|x'_k \left(\frac{t^0}{2}\right)\right| \, ds
\]

\[
= x_k \left(\frac{t^0}{2}\right) + \left(\frac{t^0}{2} - t_k\right) \left|x'_k \left(\frac{t^0}{2}\right)\right|.
\]

(36)
By Lemma 3, \( x_k(t_{a_k}) \to \infty \), and (36) implies
\[
\lim_{k \to \infty} \left| x_k \left( \frac{t^0}{2} \right) \right| = \infty. \tag{37}
\]
Since \( x_k''(t) \leq 0 \),
\[
\lim_{k \to \infty} \left| x_k \left( \frac{t^0}{2} \right) \right| = \infty. \tag{38}
\]
By (6), (7), and (35), there exists a constant \( C_1 > 0 \) such that
\[
x_k \left( \frac{t^0}{2} \right) = x_k \left( t^0 \right) + \int_{t^0/2}^{t^0} \left\{ \left| x_k'(t^0) \right|^{p-1} \right. \left. ds - \int_{t^0/2}^t c(u)f(x_k(u)) du \right\}^{1/(p-1)} du
\]
\[
> \int_{t^0/2}^{t^0} \left\{ \left| x_k'(t^0) \right|^{p-1} \right. \left. du - C_1 \right\}^{1/(p-1)} ds.
\]
Hence, \( x_k(t^0/2) \to \infty \), this contradicts (35).

**Lemma 6.** If Conditions (A1)–(A3) hold, then for small \( a > 0 \), \( t_b = t_b(a) \) is defined and
\[
\lim_{a \to 0} t_b(a) = \infty. \tag{39}
\]
Conditions (A1)–(A3) and Lemma 1 guarantee the existence of solution of (6), (7) on \([0, \infty)\).

**Proof.** If (39) were false, there would be a point \( t^0 > 0 \) and a sequence \( a_k \to 0 \) such that
\[
x_k \left( t^0 \right) \leq 0. \tag{40}
\]
Since if \( x > 0 \), \(-c(t)f(x) \leq 0\), hence, \( x_k'' \leq 0 \), \( x_k(t) \leq x_k(t_{a_k}) < a_k t_{a_k} \leq a_k t^0 \), we have \( x_k(t) \to 0 \), \( t \in [0, t_{a_k}] \).

Defining the function
\[
f^*(x) = \begin{cases} f(x), & x > 0, \\ 0, & x \leq 0, \end{cases}
\]
then \( x_k(t) \) satisfies the following equation:
\[
(\phi_p (x'))' + c(t)h_k(t)\phi_p(x) = 0, \quad t \in [0, t^0], 
\]
where \( h_k(t) = f^*(x_k(t))/\phi_p(x_k(t)) \). Let \( C^0 = \max_{0 \leq t \leq t^0} c(t) \). Since \( a_k \to 0 \), \( x_k(t) \to 0 \) for \( t \in [0, t^0] \), by (A3), there exist \( \varepsilon_k \to 0 \), such that \( f(x_k(t)) \leq \varepsilon_k \phi_p(x_k(t)) \), for \( t \in [0, t^0] \).

Now, let \( y_k(t) \) be a solution of
\[
(\phi_p (y'))' + \varepsilon_k C^0 \phi_p(y) = 0, \quad t \in [0, t^0], 
\]
then Lemma 2 implies \( y_k(t) \) has no zero in \((0, t^0)\) for \( k \) sufficiently large, but Corollary 2 implies \( y_k(t) \) has at least one zero in \((0, t^0)\). This is a contradiction.

**Lemma 7.** Define \( I = \{ a > 0 : t_b = t_b(a) > 0, x(t_b) = 0 \} \). Then,

(i) \( I \) is an open set and \( t_b(a) \) is a continuous function of \( a \);
(ii) if \( \{a_1, a_2\} \subset (0, +\infty) \) is a connected component of \( I \), then
\[(a)
\lim_{a \to a_2^-} t_b(a) = 0,
\]
\[(b)
\lim_{a \to a_1^+} t_b(a) = +\infty.
\]
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PROOF.

(i) Since \( x'(t_b) < 0 \), by the implicit function theorem, \( I \) is open and \( t_b \in C^1 \subset C^0 \).

(ii) (a) If \( a_2 = +\infty \), the result follows from (33) of Lemma 5. Let \( 0 < a_2 < +\infty \), then there exist a point \( t_0 > 0 \) and a sequence \( \{a_k\}_{k=1}^\infty \), \( a_k \to a_2 \) with \( t_b(a_k) \to t_0 \). Then, \( x(t_0) = x(t_0,a_2) = 0 \), i.e., \( a_2 \in I \), a contradiction to \( (a_1,a_2) \) is a connected component of \( I \). (a) is proved.

The proof of (b) is similar to the proof of (a), so we omit it.

PROOF OF THEOREM 1. Let \( I_T = \{x_\alpha(T) : x_\alpha(t) \text{ is the solution of (4) satisfying } x_\alpha(0) = 0, x'_\alpha(0) = \alpha \} \), then from the uniqueness and continuity of solutions of ordinary differential equations to the initial values, we show the equation \( t_b(a) = T \) has at least one solution. From Lemma 5, the set \( I \) defined in Lemma 7 is not empty: \( I \neq \emptyset \) and there exists \( a_0 \geq 0 \) such that \( (a_0, \infty) \subset I \) and \( \{t_b(a) : a \in (a_0, +\infty) \} \supset (0, +\infty) \). From the continuity of \( t_b(a) \), there exist \( a_T \in (a_0, +\infty) \) such that \( t_b(a_T) = T \). Let \( a = a_T \), then (6),(7) has a solution \( x(t) \) satisfying \( x(0) = x(T) = 0, x(t) > 0, t \in (0,T) \), which is the solution of (4),(5).

REMARK. With a slight modification of the previous arguments, we can relax (A1) to (A1)': \( f \in C(R), f(x) \geq 0 \) for \( x > 0 \); or suppose \( c(t)f(x) \) be replaced by \( f(t,x) \) if we impose some conditions on \( f(t,x) \), and the boundary condition

(a) \( x(0) = x(T) = 0 \)

can be replaced by

(b) \( x'(0) = x'(T) = 0 \), or

(c) \( x(0) = x'(T) = 0 \).

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