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Balanced Tableaux

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We study a new class of tableaux defined by a certain condition on hook-ranks. Many connections with the classical theory of standard Young tableaux are developed, as well as applications to the problem of enumerating reduced decompositions of permutations in S_n.

1. INTRODUCTION

In this paper we define and study a new class of combinatorial objects called *balanced tableaux.* They are obtained by labeling the Ferrers diagram of an integer partition, so that certain rank conditions (to be described in Section 2) are satisfied. Balanced tableaux have much in common with the more familiar *standard Young tableaux* (which are defined by a simpler rank condition), although the connections between these two classes of objects do not seem to lie on the surface. Our main theorem proves that for a fixed shape, standard tableaux and balanced tableaux are equinumerous, and obtain explicit bijections between these two families. Along the way, we study numerous other relationships between the two kinds of tableaux.

We were motivated in this work by a conjecture [17] of R. P. Stanley, subsequently proved in [18]. Stanley studied the number of maximal chains in the so-called *weak order* (sometimes called the *weak Bruhat order*) of the symmetric group S_n. In particular, he proved that for each n, this number is equal to the number f_λ of standard Young tableaux of shape $\lambda = \lambda^{[n]} = \{n-1, n-2, \ldots, 2, 1\}$. In subsequent sections, this shape will be called a *staircase.* Stanley's argument was based on properties of Schur

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functions, and did not yield an explicit correspondence between maximal chains and standard tableaux. In this paper we give a bijection between balanced tableaux of staircase shape and maximal chains in the weak order of S_n. Using this correspondence, we can view balanced tableaux (of staircase shape) as a natural encoding of inversions in a maximal chain. We also give two different bijections between standard staircase tableaux and maximal chains in S_n, thus providing a purely combinatorial basis for Stanley's result. This approach yields additional information about the number of maximal chains in subintervals of S_n, and settles a conjecture about these numbers made in [18].

One of our mappings (from standard staircase tableaux to balanced staircase tableaux) relies heavily on techniques introduced by M. P. Schützenberger, in particular his theory of *promotion* and *evacuation* of tableaux. (See [13, 14, 16].) We make several new contributions to this theory. Our correspondence constructs a maximal chain associated with a standard staircase tableau by iterative application of Schützenberger's evacuation operator.

To describe the inverse correspondence (from balanced staircase tableaux to standard staircase tableaux) we introduce a variant of the *Robinson–Schensted–Knuth correspondence* ([11, 12, 6]; see [16]), which associates to each word ω in an ordered alphabet a pair $(P(\omega), Q(\omega))$ of tableaux with certain properties. There is an obvious way to encode balanced staircase tableaux (and hence, by our correspondence, maximal chains in the weak order of S_n) as words of length $\binom{n}{2}$ in an alphabet $\mathcal{N} = \{1, 2,..., n-1\}$. These words are known as *reduced decompostions* in S_n. When applied to such words, our modified Robinson–Schensted–Knuth correspondence is injective in its second component, with image equal to the set of all standard staircase tableaux. Hence we obtain a second correspondence between balanced and standard tableaux (of staircase shape), and a second explicit bijection which verifies Stanley's theorem. A correspondence essentially equivalent to this has been obtained independently by Lascoux and Schützenberger in their study of the so-called "nil-plactic monoid" [8].

Most of the results in this paper were obtained in 1982 and announced (without proofs) in the proceedings of the 1983 Boulder conference on algebraic combinatorics [5]. The present paper contains complete proofs, as well as additional results including

- some explicit correspondences for special classes of shapes,
- some results on symmetries of the generalized Robinson–Schensted–Knuth correspondence,
- more detailed information about the number of maximal chains in subintervals of the weak order.

The remaining sections are organized as follows: Section 2 introduces the basic definitions, and proves some elementary facts about balanced tableaux. Section 3 illustrates the fundamental bijection in some relatively easy special cases. Section 4 makes the connection between balanced staircase tableaux and maximal chains in the weak order of S_n. Section 5 summarizes some background material from Schützenberger's theory of promotion and evacuation, and uses these ideas to define the mapping Γ from standard staircase tableaux to maximal chains in the weak order (and hence balanced staircase tableaux). Section 6 constructs the inverse mapping Ψ, using ideas based on the Robinson–Schensted–Knuth correspondence. Section 6 concludes by showing that Ψ is a bijection, thus completing the combinatorial proof of Stanley's theorem. Section 7 concerns technical properties of the mapping Γ, and proves that Γ and Ψ are inverses. Section 8 applies the methods of Sections 5 and 6 to count maximal chains in arbitrary subintervals of S_n, and applies these results to settle the conjecture of Stanley mentioned above. Section 9 proves the main theorem on balanced tableaux for arbitrary shapes.

2. Preliminaries

We begin by reviewing some of the basic terminology of partitions and tableaux.

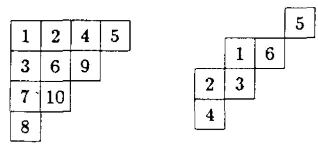
Let $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ be a partition of n, that is, $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$. The *Ferrers diagram* of λ is a left-justified array of cells (usually represented by dots or squares) with λ_1 in the first row, λ_2 in the second, etc. We will usually not distinguish between a partition and its diagram. A *tableau* T of shape λ is an assignment of integers to the cells in the diagram of λ. The entry in the ith row and jth column of T will be denoted by t_{ij}. If T is a tableau of shape λ, we also write $\lambda = \lambda(T)$.

Let λ and μ be partitions such that $\mu_i \leqslant \lambda_i$ for each i. Then the diagram of μ fits entirely inside the diagram of λ. The *skew diagram* λ/μ is the array of cells obtained by removing the cells of μ from the cells of λ. A *skew tableau* of shape λ/μ is an assignment of integers to the cells of the skew diagram λ/μ.

A tableau (ordinary or skew) with n cells is said to be *standard* if the labels t_{ij} form a permutation of $\{1, 2, ..., n\}$, and are increasing along rows and columns. Figure 2.1 illustrates standard tableaux of ordinary and skew shape.

To each cell (i, j) in a Ferrers diagram (ordinary or skew), we associate the *hook* H_{ij}, which is the subdiagram consisting of all cells

$$\{(i, j') \mid j' \geqslant j\} \cup \{(i', j) \mid i' \geqslant i\}.$$



$\lambda = \{4, 3, 2, 1\}$ $\quad\quad$ $\lambda = \{4, 3, 2, 1\}$, $\mu = \{3, 1\}$

FIG. 2.1. Standard tableaux (ordinary and skew).

The *hook height* $h_H(i, j)$ of cell (i, j) is the number of cells below (and including) (i, j) in H_{ij}. Similarly, the *hook width* $h_W(i, j)$ of (i, j) is the number of cells to the right of (and including) (i, j). The *hook length* $h(i, j)$ is defined to be the cardinality of H_{ij}, that is,

$$h(i, j) = h_H(i, j) + h_W(i, j) - 1.$$

If λ is an ordinary partition, the *conjugate partition* λ^* is defined by

$$\lambda_j^* = \mathrm{card}\{i \mid \lambda_i \geqslant j\}.$$

Thus for ordinary partitions,

$$\begin{aligned} h_H(i, j) &= \lambda_j^* - i + 1 \\ h_W(i, j) &= \lambda_i - j + 1 \\ h(i, j) &= (\lambda_i - j) + (\lambda_j^* - i) + 1. \end{aligned}$$

Let T be a tableau, and let (i, j) be a cell in the corresponding Ferrers diagram. The *hook rank* $r(i, j)$ of label t_{ij} is defined to be the number of labels $t_{i'j'}$ in H_{ij} which are less than or equal to t_{ij}. Thus, for example, $r(i, j) = 1$ if and only if the "corner" label t_{ij} is the smallest label in H_{ij}.

DEFINITION 2.1. An ordinary tableau is said to be *balanced* if $r(i, j) = h_H(i, j)$ for all cells (i, j).

Figure 2.2 illustrates a balanced tableaux. Notice that standard tableaux can also be defined by a rank condition: T is standard if $r(i, j) = 1$ for all cells (i, j). Despite the similarity in definition, the two classes of tableaux do not seem to be related in an obvious way.

For any partition λ, let $\mathscr{S}(\lambda)$ denote the set of standard tableaux of shape λ, and let $\mathscr{B}(\lambda)$ denote the set of balanced tableaux of shape λ. Let f_λ and b_λ denote the cardinalities of $\mathscr{S}(\lambda)$ and $\mathscr{B}(\lambda)$, respectively. We can now state our main result:

THEOREM 2.2. *$b_\lambda = f_\lambda$, for all partitions λ.*

$$\begin{array}{|c|c|c|c|}\hline 6&7&3&10\\ \hline 4&5&1\\ \cline{1-3} 8&9\\ \cline{1-2} 2\\ \cline{1-1}\end{array}$$

FIG. 2.2. Balanced tableau of shape $\lambda = \{4, 3, 2, 1\}$.

Theorem 2.2 will proved in Section 8, by exhibiting an explicit bijection between $\mathscr{S}(\lambda)$ and $\mathscr{B}(\lambda)$. Figure 2.3 illustrates this correspondence when $\lambda = \{3, 2, 1\}$. In each pair the standard tableau is on the left and the balanced tableau is on the right.

One might hope to prove Theorem 2.2 by methods similar to those used to derive standard facts about the f_λ's. For example, one might seek a recurrence for the b_λ's analogous to the well-known formula

$$f_\lambda = \sum_{\lambda^-} f_{\lambda^-} \tag{2.1}$$

where the sum is over all shapes obtained by deleting a border cell from λ.

$$\begin{array}{lll}1&2&3\\4&5\\6\end{array} \longleftrightarrow \begin{array}{lll}3&2&1\\5&4\\6\end{array} \qquad \begin{array}{lll}1&2&3\\4&6\\5\end{array} \longleftrightarrow \begin{array}{lll}3&2&1\\5&6\\4\end{array}$$

$$\begin{array}{lll}1&2&4\\3&5\\6\end{array} \longleftrightarrow \begin{array}{lll}4&2&1\\5&3\\6\end{array} \qquad \begin{array}{lll}1&2&4\\3&6\\5\end{array} \longleftrightarrow \begin{array}{lll}3&4&1\\5&6\\2\end{array}$$

$$\begin{array}{lll}1&2&5\\3&4\\6\end{array} \longleftrightarrow \begin{array}{lll}4&3&5\\2&1\\6\end{array} \qquad \begin{array}{lll}1&2&5\\3&6\\4\end{array} \longleftrightarrow \begin{array}{lll}3&5&1\\4&6\\2\end{array}$$

$$\begin{array}{lll}1&2&6\\3&4\\5\end{array} \longleftrightarrow \begin{array}{lll}4&3&6\\2&1\\5\end{array} \qquad \begin{array}{lll}1&2&6\\3&5\\4\end{array} \longleftrightarrow \begin{array}{lll}4&5&6\\2&1\\3\end{array}$$

$$\begin{array}{lll}1&3&4\\2&5\\6\end{array} \longleftrightarrow \begin{array}{lll}4&2&3\\5&1\\6\end{array} \qquad \begin{array}{lll}1&3&4\\2&6\\5\end{array} \longleftrightarrow \begin{array}{lll}3&4&2\\5&6\\1\end{array}$$

$$\begin{array}{lll}1&3&5\\2&4\\6\end{array} \longleftrightarrow \begin{array}{lll}4&2&5\\3&1\\6\end{array} \qquad \begin{array}{lll}1&3&5\\2&6\\4\end{array} \longleftrightarrow \begin{array}{lll}3&5&2\\4&6\\1\end{array}$$

$$\begin{array}{lll}1&3&6\\2&4\\5\end{array} \longleftrightarrow \begin{array}{lll}4&2&6\\3&1\\5\end{array} \qquad \begin{array}{lll}1&3&6\\2&5\\4\end{array} \longleftrightarrow \begin{array}{lll}4&5&6\\2&3\\1\end{array}$$

$$\begin{array}{lll}1&4&5\\2&6\\3\end{array} \longleftrightarrow \begin{array}{lll}3&5&4\\2&6\\1\end{array} \qquad \begin{array}{lll}1&4&6\\2&5\\3\end{array} \longleftrightarrow \begin{array}{lll}3&5&6\\2&4\\1\end{array}$$

FIG. 2.3. Bijection between $\mathscr{S}(\lambda)$ and $\mathscr{B}(\lambda)$.

Formula (2.1) is obvious for standard tableaux, since the largest label must occur on a border cell, and removing it gives a standard tableau of shape λ^-. For balanced tableaux, however, this is not the case: the largest label need not occur on the border, and even when it does its removal leaves the tableau unbalanced. Surprisingly, formula (2.1) also holds for balanced tableaux (as it must, by virtue of Theorem 2.2), yet we know of no simple direct proof.

We conclude this section with several elementary lemmas concerning balanced tableaux. Throughout this discussion n will always denote the number of cells in λ. The first lemma shows that certain inequalities in a balanced tableau are always forced.

LEMMA 2.3. *Let T be a balanced tableau shape λ, and let (i,j) be a cell in the diagram of T. Then*

$$h_H(i,j) = h_H(i,j+1) \Rightarrow t_{ij} < t_{i\,j+1} \tag{2.2}$$

and

$$h_W(i,j) = h_W(i+1,j) \Rightarrow t_{ij} > t_{i+1\,j}. \tag{2.3}$$

Proof. Trivially, if T is balanced, the operation of transposing T and replacing each label i by $n+1-i$ preserves the property of being balanced. Thus by symmetry it suffices to prove (2.2). If $h_H(i,j)=1$ then t_{ij} must be the smallest element of H_{ij}, and the conclusion follows immediately. Next assume that $t_{kj} < t_{k\,j+1}$ has been proved for all $k<i$. We will shown that $t_{ij} > t_{i\,j+1}$ is impossible. If $h_H(i,j+1)=h$, then $t_{i\,j+1}$ is larger than exactly h entries in $H_{i\,j+1}$. If $t_{ij} > t_{i\,j+1}$ then t_{ij} is larger than all of these entries, as well as $t_{i\,j+1}$ itself. This means that t_{ij} dominates at least $h+1$ entries in its own hook, which is a contradiction. Hence $t_{ij} < t_{i\,j+1}$, and the lemma follows by induction. ∎

Lemma 2.3 shows that a balanced tableau of shape λ may be decomposed into rectangular "zones" (in which $h_H(i,j)$ and $h_W(i,j)$ are constant) such that in each zone the labels increase along rows and decrease along columns (see Fig. 2.4). As a special case we have:

COROLLARY 2.4. *If T is a balanced tableau of rectangular shape, and T^* is obtained by reflecting T about a horizontal axis, then T^* is standard, and conversely.* ∎

Next we prove a lemma which shows that the largest label in a balanced tableau can be added or deleted (with care) under special circumstances.

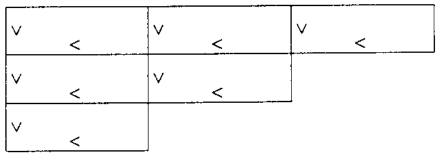


FIG. 2.4. The "zone" effect.

LEMMA 2.5. *Let T be a balanced tableau of shape λ, with n cells. Suppose that $\lambda_i = j-1 < \lambda_{i-1} - 1$. Let λ^+ denote the diagram obtained by adding cell (i,j) to the diagram of λ, i.e., $\lambda^+ = \{\lambda_1, \lambda_2, \ldots, \lambda_i + 1, \ldots\}$. Let T^+ be the tableau obtained from T by*

(i) *transposing columns j and $j+1$,*

(ii) *defining $t_{ij}^+ = n+1$.*

Then T^+ is a balanced tableau. Furthermore, the correspondence $T \to T^+$ is a bijection from $\mathscr{B}(\lambda)$ to the set $\{T^+ \in \mathscr{B}(\lambda^+) \mid t_{ij}^+ = n+1\}$.

Proof. For each $k < i$, we have $h_H(k,j) = h_H(k,j+1)$, by hypothesis. By Lemma 2.3, $t_{k\,j} < t_{k\,j+1}$ for all $k < i$. Since T is balanced, t_{kj} dominates exactly $h(k,j)$ entries in its hook. In T^+, the entry t_{kj} dominates precisely the same entries as it does in T, and has the same hook height. Thus each of the hooks $H_{k\,j+1}$, $k < i$, remains balanced in T^+. A similar argument shows that the hooks H_{kj}, $k < i$, remain balanced, and the proof is complete. ∎

Figure 2.5 illustrates the process of adding n to a balanced tableau, under the assumptions of Lemma 2.5. We will call this the *column–exchange insertion-deletion process.* The process can also be reversed, i.e., we have a canonical procedure for deleting the largest label from a balanced tableau, if this label occurs in a row whose length has multiplicity 1 in λ. When λ has distinct parts, the procedure can be applied regardless of the position of n, and we obtain an analog of Formula (2.1) for balanced tableaux:

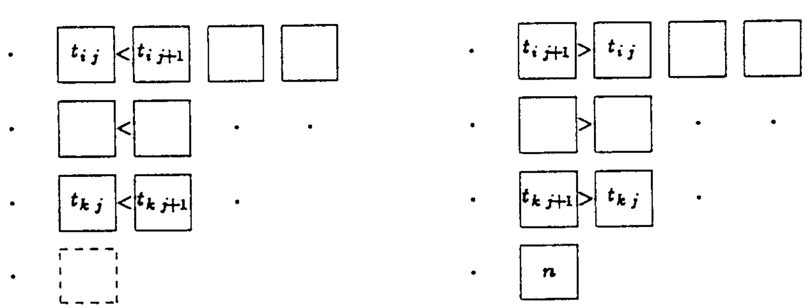


FIG. 2.5. "Column-exchange" insertion-deletion.

COROLLARY 2.6. *If λ has distinct parts, then*

$$b_\lambda = \sum_{\lambda^-} b_{\lambda^-}. \tag{2.4}$$

We know of no straightforward way to prove formula (2.4) if λ has parts of multiplicity greater than one.

3. Some Special Cases of the Correspondence

In this section we exhibit a bijection between $\mathscr{B}(\lambda)$ and $\mathscr{S}(\lambda)$ in several easy special cases. One such case has already been noted in the previous section:

LEMMA 3.1. *If λ is a rectangular shape, i.e., $\lambda = \{k, k, \ldots, k\}$ for some integer k, then the map $T \to T^*$ described in Corollary 2.4 is a bijection from $\mathscr{B}(\lambda)$ to $\mathscr{S}(\lambda)$.*

Another easy case occurs when λ is *hook shape*, i.e., $\lambda = \{k, 1, 1, 1, \ldots, 1\}$ for some integer k. A mapping from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$ can be defined as follows: suppose that λ has m rows and n cells. If $T \in \mathscr{S}(\lambda)$, define T^* to be the tableau obtained from T by (i) exchanging labels 1 and m, (ii) sorting the labels in rows 2 through m into decreasing order, and, finally, (iii) sorting the labels in columns 2 through $n-m$ into decreasing order. It is then easy to prove:

LEMMA 3.2. *If λ is a hook shape, the map $T \to T^*$ just defined is a bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$.*

The next case is somewhat less trivial. Let λ be a *two-rowed shape*, that is, $\lambda = \{m_1, m_2\}$ with $m_1 \geqslant m_2$. Given $T \in \mathscr{S}(\lambda)$ define

$$k_0 = \max\{k \mid t_{2k} = 2k\}.$$

By convention, we assume $t_{20} = 0$, so that k_0 is always defined. In the language of random walks (where a particle takes a positive step at time k if k lies in the first row of T, and a negative step otherwise), k_0 represents the "last equalization," i.e., the largest k such that the values $\{1, 2, \ldots, 2k\}$ are split evenly between the two rows. Note that $k_0 = m_1 = m_2$ if the row lengths m_1 and m_2 are equal, i.e., λ is rectangular.

1	2	4	5	9	10	12	14
3	6	7	8	11	13		

Standard

3	6	7	8	10	12	9	14
1	2	4	5	11	13		

Balanced

FIG. 3.1. Two-rowed tableaux.

LEMMA 3.3. *If $\lambda = \{m_1, m_2\}$, and $T \in \mathscr{S}(\lambda)$, construct a tableau T^* as follows:*

(i) *for each $i = 1, 2, \dots, k_0$, exchange labels t_{1i} and t_{2i},*

(ii) *if $m_1 > m_2$, perform the cyclic shift*

$$t_{1\,k_0+1} \leftarrow t_{1\,k_0+2} \leftarrow \cdots \leftarrow t_{1\,m_1+1} \leftarrow t_{1\,k_0+1}.$$

Then T^* is a balanced tableau, and the map $T \to T^*$ is a bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$.

Proof. We leave the details of this argument to the reader. Figure 3.1 gives an example of the construction. ∎

The next case is considerably more complex, and makes use of ideas which will ultimately be used to prove the general case.

Let λ be a partition whose diagram is a rectangle with the bottom-right corner cell removed, that is, $\lambda = \{q, q, q, \dots, q, q-1\}$ for some integer $q > 2$. We will call such a shape a *notched rectangle*.

DEFINITION 3.4. Let T be a standard tableau of shape λ where λ is a

1	2	4	5
3	7	8	12
6	9	13	15
10	11	14	

T

$\Longrightarrow$

10	11	14	
6	9	13	15
3	7	8	12
1	2	4	5

T'

$\Longrightarrow$

10	11	14
6	8	12
2	4	5

T''

$\Longrightarrow$

10	11	14	15
6	8	12	7
2	4	5	1
3	9	13	

T^*

FIG. 3.2. Construction of T^* from T, by "evacuation."

notched rectangle having p rows and q columns. The *evacuation path* in T is a sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_{p+q})$ of cells (i, j) constructed as follows:

(i) $\pi_1 = (p, q)$,

(ii) if $\pi_i = (a, b)$, then π_{i+1} is the cell (a', b') adjacent to (a, b) such that $t_{a'b'} = \max\{t_{a-1\,b}, t_{a\,b-1}\}$, $i = 1, 2, \ldots, p+q-1$.

For purposes of the definition, assume that $t_{i0} = t_{0i} = 0$ for all i and j. The first tableau in Fig. 3.2 illustrates the construction of an evacuation path. In the path π we *circle* the labels on those cells which are lowest in columns $1, 2, \ldots, q-1$. The left-hand tableau in Fig. 3.2 illustrates the circling procedure.

Now we construct another tableau T^*, as follows.

(i) Reflect T about a horizontal line, obtaining a labeled array T'.

(ii) Remove the cells of the evacuation path. The remaining cells form two connected pieces, which fit together to form a $(p-1) \times (q-1)$ rectangle T''.

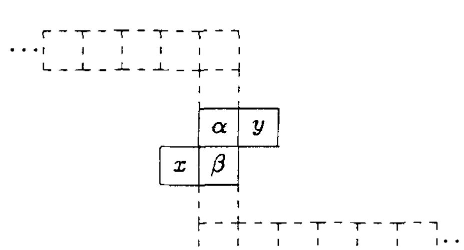
(iii) At the bottom of T'' add an mth row, consisting of the $q-1$ circled elements in the evacuation path π.

(iv) To the right of T'' add a kth column consisting of the $p-1$ remaining uncircled elements in π.

The last tableau in Fig. 3.2 illustrates the construction of T^* from T''.

LEMMA 3.5. *If λ is a notched rectangle, and T is a tableau of shape λ, the tableau T^* constructed by the method just described is balanced, and the map $T \to T^*$ is a bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$.*

Proof. First we claim that the entries of T'' increase along rows and decrease along columns. This is clearly true within the "upper" and "lower" pieces of T'' (hereafter denoted by T''_+ and T''_-). Let x and y be adjacent element in a row of T'' which lie in different pieces, i.e., they are separated by the evacuation path in T. Then there exist elements $\alpha < \beta$ in π such that x is to the left of β, y is to the right of α, and α lies immediately above β (see illustration):

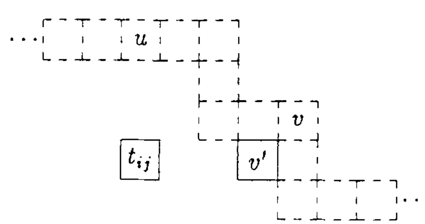


Thus $x<\beta$ and $\alpha<y$. From the construction of π it follows that $\alpha>x$. Hence $x<\alpha<y$. A similar argument (which we omit) proves that T'' is decreasing along columns.

Hence by Corollary 2.4, T'' is a balanced tableau. Let t_{ij} be an element of T'', and let u and v denote the elements added below and to the right of t_{ij} in steps (iii) and (iv). We must show that the hook H_{ij} remains balanced when u and v are added. Since $h_H(i,j)$ increases by one, this will follow if we can show that exactly one of $\{u, v\}$ is less than t_{ij}.

At this point it is convenient to distinguish two cases, according to whether t_{ij} lies in T''_+ or T''_-. If t_{ij} is in T''_+, it was *below* the evacuation path in T.

We claim that $u<t_{ij}<v$. Clearly, since u lies above t_{ij} in column j of T, we have $u<t_{ij}$. By construction, v is the rightmost element of row $i-1$ which lies in the evacuation path, since all of the other elements in row $i-1$ of π are circled. Let $v=t_{i-1\,k}$, and let $v'=t_{i\,k-1}$. Then $v'=t_{i\,k-1}<t_{i-1\,k}=v$, for otherwise π would include cell $(i, k-1)$ rather than $(i-1, k)$ (see illustration):



Thus $t_{ij}\leqslant v'<v$, and the claim is proved. A similar argument shows that if t_{ij} lies above π then $v<t_{ij}<u$, and this completes the proof that T^* is balanced.

We must also show that the map is reversible. Given a balanced tableau B of shape λ, define a tableau B^* by the following operations:

(i) For each $j=1, 2,..., q-1$, sort column j of B^*. (By Lemma 2.3 the elements in the first $p-1$ rows are already ordered, so we are just inserting b_{jp} into its proper place in column j.)

(ii) Slide column q down one cell.

(iii) For each $i=2, 3,..., p$, sort row i of T^*. (Again, this just inserts the last element into its proper position in row i.)

(iv) Reflect the resulting tableau about a horizontal axis.

We claim (but leave to the reader the tasks of verifying) that $B=T^*$, steps (i)–(iii) recover T', and step (iv) recovers T. Thus $T^{**}=T$. It is not difficult to show (again we omit the details) that $B^{**}=B$ for any balanced tableau of shape λ, and this completes the proof. ∎

We have included Lemma 3.5 and its rather lengthy proof to illustrate some of the complexity which seems to underlie this problem. It would be extremely interesting to extend these methods to tableaux of arbitrary shape. We believe such extensions exist, though we have not been able to find them. The notion of evacuation path turns out to play a central role in the proof of the general case, but in a quite different way (see Sections 5, 6, and 7).

4. Staircase Shapes and Maximal Chains in the Weak Order

In this section we establish the connection between balanced staircase tableaux and maximal chains in the weak order of S_n. We begin by reviewing the basic properties of the weak order. The reader is referred to [1] for a more complete exposition of these ideas, most of which can be extended to any Coxeter group.

If σ is a permutation in S_n, the *length* $l(\sigma)$ of σ is defined to be the smallest integer k such that σ can be expressed as the product of k adjacent transpositions. We define $\theta \leqslant \sigma$ if $\sigma = \theta\psi$, with $l(\theta) = l(\sigma) + l(\psi)$. This defines a partial order on S_n, known as the *weak order*, sometimes referred to as the weak *Bruhat* order. There is a simple combinatorial way to represent the weak order on S_n. Think of permutations σ as acting on rearrangements of $\{1, 2,..., n\}$, with composition defined from left to right, and identify σ with the sequence $[\sigma_1, \sigma_2,..., \sigma_n]$. Then $\sigma\tau$ covers σ in the weak order if and only if τ transposes two adjacent increasing elements of σ. Figure 4.1 illustrates the weak order of S_4.

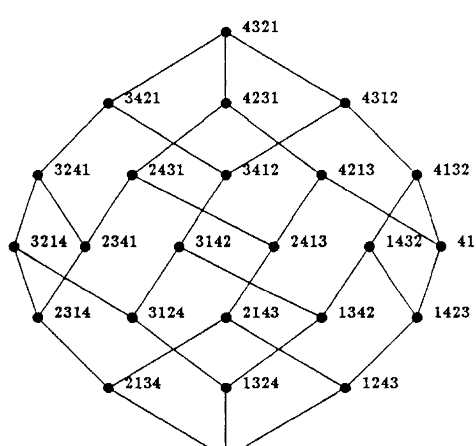


FIG. 4.1. The weak order of S_4.

An *inversion* in σ is a pair (σ_i, σ_j) such that $i<j$ and $\sigma_i > \sigma_j$. It is not difficult to show that the poset $(S_n, \leqslant)$ is a *ranked lattice*, with rank function

$$r(\sigma) = l(\sigma) = \text{number of inversions in } \sigma.$$

In fact the lattice structure can be characterized completely by the *inversion sets*

$$\mathscr{I}(\sigma) = \{(\sigma_i, \sigma_j) | i<j \text{ and } \sigma_i > \sigma_j\}.$$

In particular,

$$\sigma \leqslant \theta \Leftrightarrow \mathscr{I}(\sigma) \subseteq \mathscr{I}(\theta).$$

Moreover, θ covers σ if and only if $\mathscr{I}(\theta)$ contains exactly one more inversion than $\mathscr{I}(\sigma)$, which occurs precisely when θ is obtained from σ by exchanging a pair of adjacent increasing elements. The bottom element $\hat{\mathbf{0}}$ in the ordering is the identity permutation $[1, 2,..., n]$ and the top element $\hat{\mathbf{1}}$ is the permutation $[n, n-1,..., 1]$. A maximal chain from $\hat{\mathbf{0}}$ to $\hat{\mathbf{1}}$ has $L+1$ elements, where $L = \binom{n}{2}$.

Maximal chains from $\hat{\mathbf{0}}$ to σ in S_n can also be interpreted as *reduced decompositions* of σ, that is, sequences $(\tau_1, \tau_2,..., \tau_l)$ of minimum length l such that $\sigma = \tau_1 \tau_2 \cdots \tau_l$ and each τ_i is a transposition of adjacent positions. Let $\mathscr{N} = \{1, 2, 3,...\}$, and let $\mathscr{N}^*$ denote the set of all words in the alphabet $\mathscr{N}$. If we identify letter x with the transposition which exchanges positions x and $x+1$, then each reduced decomposition is represented by a word in $\mathscr{N}^*$. For each $n>1$, let $\mathscr{R}(n)$ denote the set of all words which represent reduced decompositions of $\hat{\mathbf{1}}$ in S_n. Let $\mathscr{C}(n)$ denote the set of all maximal chains from $\hat{\mathbf{0}}$ to $\hat{\mathbf{1}}$ in the weak order of S_n. Thus there is a natural bijection

$$\mathscr{R}(n) \leftrightarrow \mathscr{C}(n).$$

Stanley [17] raised (and later answered) the question of counting the number $C_n = |\mathscr{C}(n)| = |\mathscr{R}(n)|$. On the basis of (limited) numerical evidence, he made the remarkable conjecture

$$C_n = f_\lambda \tag{4.1}$$

where $\lambda = \lambda^{[n]}$ denotes the staircase shape $\{n-1, n-2,..., 2, 1\}$. In [18] he proved this conjecture, using Schur function expansions and other arguments involving symmetric functions. Thus, writing $\mathscr{S}(n) = \mathscr{S}(\lambda^{[n]})$ we have

THEOREM 4.1. *For any positive integer n,*

$$C_n = |\mathscr{C}(n)| = |\mathscr{R}(n)| = |\mathscr{S}(n)|.$$

Stanley's argument did not yield a purely combinatorial (i.e., bijective) proof of Theorem 4.1. Our goal is to explain all of the implied relationships combinatorially.

We begin by describing a natural way to encode maximal chains as balanced staircase tableaux. As a consequence, it will follow that

$$|\mathscr{C}(n)| = |\mathscr{B}(n)|$$

where, by definition, $\mathscr{B}(n) = \mathscr{B}(\lambda^{[n]})$. Let $L = \binom{n}{2}$, and let

$$\Gamma = \{\hat{\mathbf{0}} = \sigma^{(0)} < \sigma^{(1)} < \cdots < \sigma^{(L)} = \hat{\mathbf{1}}\}$$

be a maximal chain in S_n. For each $k = 1, 2,..., L$, let $\tau^{(k)}$ denote the transposition such that $\sigma^{(k)} = \sigma^{(k-1)}\tau^{(k)}$. Note that $\tau^{(k)}$ transposes a pair of elements which are adjacent and increasing in $\sigma^{(k-1)}$. Denote these two elements by $a(k)$ and $b(k)$, assuming that $a(k) < b(k)$. Clearly the pairs

$$(a(k), b(k)), \qquad k = 1, 2,..., L$$

include each of the $\binom{n}{2}$ pairs $\{(i, j) | i < j\}$ exactly once.

Now construct a tableau $T[\Gamma] = (t_{ij})$ of shape $\lambda = \{n-1, n-2,..., 2, 1\}$ as follows. Write $\bar{b}(k) = n + 1 - b(k)$ for each k, and set

$$t_{\bar{b}(k)a(k)} = k, \qquad \text{for } k = 1, 2,..., L.$$

In other words, $t_{ij} = k$ if the kth transposition exchanges $(n+1-i)$ and j. An easy way to visualize this contruction is to insert the numbers $1, 2,..., n$ on the "steps" of the staircase, and use these numbers to index the rows and columns. For example, the chain

$$\hat{\mathbf{0}} = [1234] < [1324] < [3124] < [3142] < [3412] < [4312] < [4321] < \hat{\mathbf{1}}$$

may be represented by the diagram

$$\begin{array}{ccccc}
 & 1 & 2 & 3 & \\
1 = \bar{4} & 4 & 3 & 5 & 4 \\
2 = \bar{3} & 2 & 1 & 3 & \\
3 = \bar{2} & 6 & 2 & & \\
 & 1 & & &
\end{array}$$

For example, the pair (2, 4) is exchanged in step 3, hence $t_{\bar{4}2} = t_{12} = 3$. The reader should note that $T[\Gamma]$ is just a particular way of labeling inversion in the permutation $\hat{\mathbf{1}}$.

THEOREM 4.2. *For any maximal chain $\Gamma \in \mathscr{C}(n)$, the tableau $T[\Gamma]$ constructed in this manner is balanced, and the mapping $\Gamma \to T[\Gamma]$ is a bijection from $\mathscr{C}(n)$ to $\mathscr{B}(n)$.*

The proof of Theorem 4.2 depends on the following lemma:

LEMMA 4.3. *Let T be a tableau of shape $\lambda^{[n]} = \{n-1, n-2, ..., 2, 1\}$. Then T is balanced if and only if for all positive integers i, j such that $i+j \leqslant n$, and for all k such that $k > i$ and $\bar{k} = n+1-k > j$, exactly one of $\{t_{kj}, t_{i\bar{k}}\}$ is less than t_{ij}.*

Proof. The set $H_{ij} - \{(i,j)\}$ can be partitioned into disjoint pairs

$$\{(k,j), (i,\bar{k}) \mid k > i, \bar{k} > j\}.$$

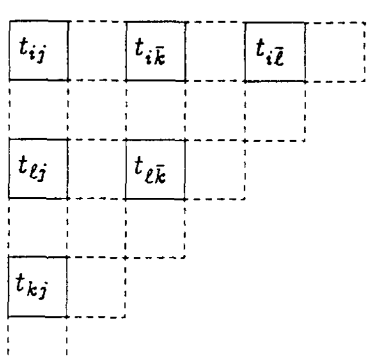
Hence if exactly one of $\{t_{kj}, t_{i\bar{k}}\}$ is less than t_{ij} for each k, the hook H_{ij} is balanced. If this is true for all (i,j), then T is balanced. Conversely, suppose T is balanced, and suppose (for example) that t_{kj} and $t_{i\bar{k}}$ are both less than t_{ij}. Since H_{ij} is balanced, there must exist an l such that t_{lj} and $t_{i\bar{l}}$ are both greater than t_{ij}, with $l > i$, $\bar{l} > j$. Thus

$$t_{kj}, t_{i\bar{k}} < t_{ij} < t_{lj}, t_{i\bar{l}}.$$

Suppose that $l < k$ (if $l > k$ the argument is similar). The condition stated in the lemma clearly holds for all (i,j) with $h(i,j) \leqslant 3$. We assume, inductively, that

(i) t_{lj} lies between t_{kj} and $t_{l\bar{k}}$,

(ii) $t_{i\bar{k}}$ lies between $t_{l\bar{k}}$ and $t_{i\bar{l}}$.



Consider the entry $t_{l\bar{k}}$. It follows from (i) that

$$t_{l\bar{k}} > t_{lj} > t_{ij}.$$

On the other hand, (ii) implies

$$t_{lk} < t_{ik} < t_{ij}$$

which is a contradiction. This completes the proof of the converse. ∎

Remark. If we write $c = \bar{i}$, $a = j$, and $b = \bar{k}$, then the inequalities $i + j \leqslant n$, $k > i$, $\bar{k} > j$ imply $1 \leqslant a < b < c \leqslant n$, and conversely. Thus Lemma 4.3 can be restated as follows: T is balanced if and only if $t_{\bar{c}a}$ lies between $t_{\bar{b}a}$ and $t_{\bar{c}b}$, for all integers a, b, c such that $1 \leqslant a < b < c \leqslant n$.

Proof of Theorem 4.2. Let Γ be a maximal chain, and let $T = T[\Gamma]$ be constructed as described. If a, b, c are such that $1 \leqslant a < b < c \leqslant n$, it is clear that the exchange of (a, c) must occur between the exchanges of (a, b) and (b, c). In other words, $t_{\bar{c}a}$ is between $t_{\bar{b}a}$ and $t_{\bar{c}b}$, for all $a < b < c$, and Lemma 4.3 shows that T is balanced. Clearly Γ is uniquely determined by $T[\Gamma]$, so it remains to show that $T \to T[\Gamma]$ is surjective.

Let T be an arbitrary balanced staircase tableau. Define a sequence of transpositions $\tau^{(1)}, \tau^{(2)}, ..., \tau^{(L)}$ by setting $\tau^{(k)} = (a, b)$ if $t_{\bar{b}a} = k$, and define $\sigma^{(k)} = \tau^{(1)} \cdots \tau^{(k)}$. We claim that the sequence $\sigma^{(1)}, \sigma^{(2)}, ..., \sigma^{(L)}$ is a maximal chain in $(S_n, \leqslant)$, in other words, each $\tau^{(k)}$ transposes a pair of increasing elements, with each such pair occurring exactly once. It suffices to show that if $\tau^{(k)} = (a, b)$, then a and b are adjacent in $\sigma^{(k-1)}$. This follows from the fact that previous adjacent transpositions could not have inverted a and b and hence $a < b$. Suppose that a and b are not adjacent in $\sigma^{(k-1)}$, and that k is minimal with this property. Then all previous transpositions $\tau^{(i)}$, $i < k$, exchange adjacent increasing elements. If $a < c < b$, Lemma 4.3 shows that exactly one of the pairs (a, c), (c, b) has been transposed prior to time k. Hence no such c remains between a and b at time k. If $a < b < d$, suppose that $t_{\bar{d}b} < k$, i.e., d has been exchanged with b prior to time k. Again by Lemma 4.3, $t_{\bar{d}a}$ must be between $t_{\bar{d}b}$ and $t_{\bar{b}a} = k$. Hence $t_{\bar{d}a} < k$, which implies that d has also been exchanged with a, and hence cannot lie between a and b at time k. A similar argument shows that if $d < a < b$, then d cannot remain between a and b at time k. Hence a and b must be adjacent, and the proof is complete. ∎

Remark. One can show that balanced tableaux of arbitrary shape satisfy a condition analogous to the one described in Lemma 4.3, using external border cells to define triples of entries which must be balanced. However, the obvious converse does not hold: balanced tableaux are not characterized by this condition. It would be interesting to find the "correct" generalization of Lemma 4.3 to arbitrary shapes.

The next three sections will be devoted to proving that $C_n = f_\lambda = b_\lambda$, where $\lambda = \lambda^{[n]} = \{n-1, n-2, ..., 2, 1\}$.

5. Promotion and Evacuation of Tableaux

In this section we will construct a bijection

$$\mathscr{S}(n) \leftrightarrow \mathscr{C}(n)$$

where, as before, $\mathscr{S}(n)$ denotes the set of standard tableaux of shape $\lambda^{[n]} = \{n-1, n-2, ..., 2, 1\}$, and $\mathscr{C}(n)$ denotes the collection of maximal chains from $\hat{0}$ to $\hat{1}$ in $(S_n, \leqslant)$. Theorem 4.1 follows as a consequence of this bijection, and Theorem 2.2 follows in the special case when λ is a staircase.

We will need to introduce an operation on tableaux introduced by Schützenberger in [13], and studied much more extensively in [14–16]. Let T be a standard tableau of any shape (ordinary or skew), with n cells. It will be convenient to assume temporarily that T has labels $k+1$, $k+1, ..., k+n$ for some integer k. Let (p, q) be the cell in T such that $t_{pq} = k+n$. Define the *evacuation path* $\pi = [\pi_1, \pi_2, ..., \pi_s]$ of T exactly as in Definition 3.4, with the convention that $t_{ij} = 0$ for all cells (i, j) above and to the left of T. If T is an ordinary standard tableau, π_s will be the upper-leftmost cell, and will have label k. Otherwise (if T is a skew tableau), π_s will be some cell on the inner border, but may not be the cell with label k.

Now define a new tableau T^{∂} by

(1) removing label $n+k$ from cell π_1,

(2) shifting labels downward along the evacuation path

$$t_{\pi_1} \leftarrow t_{\pi_2} \leftarrow \cdots \leftarrow t_{\pi_s},$$

(3) setting $t_{\pi_s} = k$.

Thus T^{∂} has labels $\{k, k+1, ..., k+n-1\}$.

We will refer to a single application of operator ∂ as an *elementary promotion* of T. The inverse operation ∂^{-1} may be defined by similar rules: first remove the *smallest* label, then shift labels *upward* along the evacuation path (defined analogously), and finally add a new label $n+k+1$. Next we introduce two more operators, defined using ∂:

Definition 5.1. If T is a standard tableau with n cells, define T^P to be the result of applying ∂^n to T, and then adding n to each entry.

Definition 5.2. If T is a standard tableau with n cells, define T^S by setting

$$t_{ij}^S = q \Leftrightarrow t_{ij}^{\partial^q} \leqslant k \qquad \text{but } t_{ij}^{\partial^{q-1}} > k.$$

We will call P the *promotion operator*, and S the *evacuation operator*. It is customary (and convenient) to think of T^S as follows: each time ∂ is applied to T, a *new* label is introduced, and these labels are all less than k. If the new labels are suppressed, the remaining labels determine a nested sequence of skew shapes, each one obtained from its predecessor by deleting a single cell on the inner border. Such a sequence of shapes always determines (and is determined by) a standard tableau, obtained by numbering the deleted cells in order. The definition of standard tableaux by nested sequences of shapes is a familiar construction, and we will use it several times in subsequent sections.

Note that if T has labels $\{k+1, k+2, \ldots, k+n\}$, then T^P has labels $\{k+1, k+2, \ldots, k+n\}$, while T^S has labels $\{1, 2, \ldots, n\}$. Every tableau is "order-isomorphic" (by translation of labels) to a unique tableau having labels $\{1, 2, \ldots, n\}$. Figure 5.1 illustrates the action of ∂, S, and P.

Schutzenberger obtained many remarkable results concerning the operators ∂, S and P, including generalizations to other labeled structures besides tableaux (see [14, 15]). Perhaps the most striking result is the following:

THEOREM 5.3 [13, 14]. *The operator S is an involution.*

We now are equipped to define the basic correspondence between standard tableaux of staircase shape and maximal chains in $(S_n, \leqslant)$.

Let T be a standard tableau of shape $\lambda = \{n-1, n-2, \ldots, 2, 1\}$. Let ∂ act on T until all of the original labels have been evacuated, that is, $L = \binom{n}{2}$ times. For $k = 1, 2, \ldots, L$, let $\pi^{(k)}$ denote the evacuation path for the kth

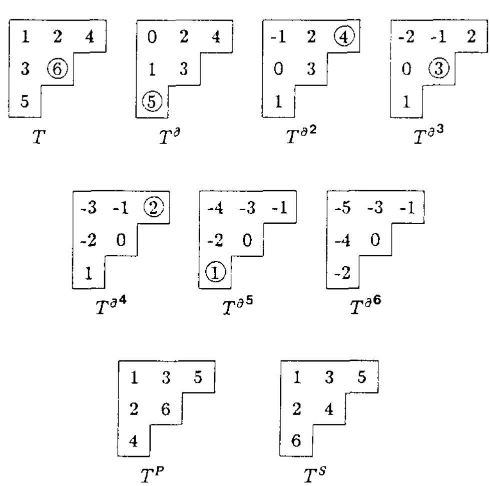


FIG. 5.1. Schützenberger's ∂, P, and S operators.

iteration, and let (p_k, q_k) denote the initial cell $\pi_1^{(k)}$ in $\pi^{(k)}$. Define a sequence $\Gamma(T)$ of permutations $\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(L)}$ by setting

(i) $\sigma^{(0)} = [1, 2, \ldots, n]$,

(ii) $\sigma^{(k+1)}$ is the permutation obtained from $\sigma^{(k)}$ by transposing positions q_k and $q_k + 1$ in $\sigma^{(k)}$.

THEOREM 5.4. *The sequence* $\Gamma(T) = \sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(L)}$ *is a maximal chain from* $\hat{\mathbf{0}}$ *to* $\hat{\mathbf{1}}$ *in the weak order. Furthermore, the mapping*

$$T \to \Gamma(T)$$

is a bijection from the set $\mathcal{S}(n) = \mathcal{S}(\lambda^{[n]})$ *of standard staircase tableau to the set* $\mathcal{C}(n)$ *of maximal chains in* $(S_n, \leqslant)$.

As indicated in Section 4, we will often need to think of $\Gamma(T)$ as a word

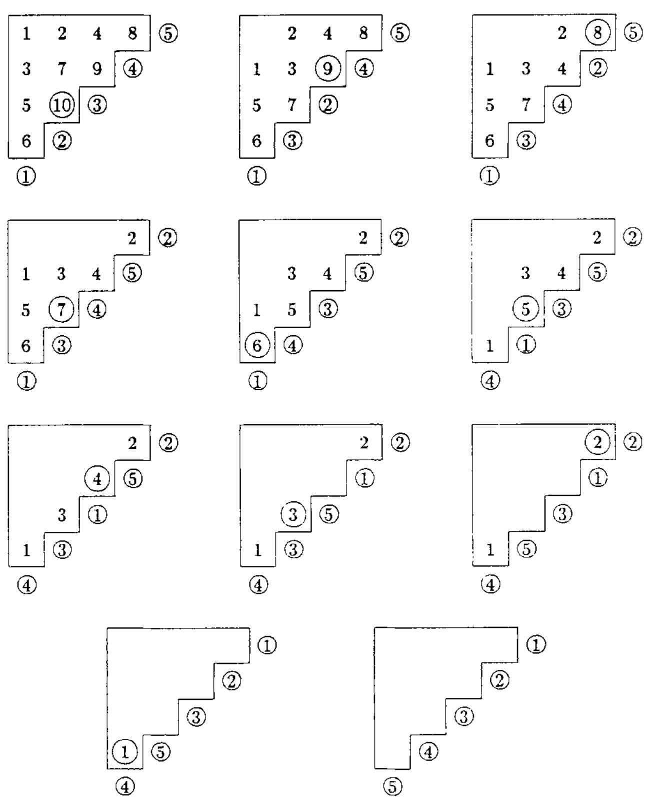


FIG. 5.2. Algorithm for constructing a maximal chain in S_n from a standard staircase tableau. Evacuation word is $\Gamma(T) = 1\ 4\ 2\ 3\ 2\ 1\ 2\ 4\ 3\ 2$.

in the alphabet $\{1, 2, ..., n-1\}$, in which case it is convenient to write $\Gamma(T) = \gamma_1 \gamma_2 \cdots \gamma_L$, where γ_i denotes the transposition obtained when i is evacuated. In this notation (the virtues of which will become apparent later) one obtains the chain $\sigma^{(0)}, \sigma^{(1)}, ..., \sigma^{(L)}$ by applying the letters of $\Gamma(T)$ in reverse order. We call $\Gamma(T)$ the *evacuation word* of T.

Figure 5.2 illustrates the algorithm. The largest element of the evacuation path $\pi^{(k)}$ is circled in each case, and the permutation $\sigma^{(k)}$ is represented by labels on the "steps" of the staircase.

The proof of Theorem 5.4 involves many steps, and most of Sections 6 and 7 will be devoted to proving it. It is not obvious, for example, that $\Gamma(T)$ even represents a maximal chain in S_n, since the adjacent pairs transposed by $\sigma^{(k+1)}$ might not be increasing. We will prove this in Section 7.

Before proceeding with the details of the proof, we will define another map Ψ from maximal chains (or reduced decompositions) to staircase tableaux, which turns out to be the inverse of Γ. We will describe Ψ and its properties in the next section, first digressing to review some important related material.

6. The Robinson–Schensted–Knuth Correspondence

Basic Properties of the Correspondence

In [12] Schensted defined a bijective correspondence (anticipated by Robinson [11]) between permutations $\sigma \in S_n$ and pairs $(P(\sigma), Q(\sigma))$ of standard tableaux of the same shape, with n cells. Thus permutations may be coded by pairs of standard tableaux, and there is now a vast literature concerning the symmetries and other remarkable properties of this correspondence (cf. [16, 7]).

The correspondence was extended by Schensted and later Knuth [6] to the case where the range (and even the domain) of σ is allowed to have repeated elements.[1] In these extended versions of the correspondence, $P(\sigma)$ (and in the most general case $Q(\sigma)$) may have repeated entries, but the rows increase weakly and the columns increase strictly. Such tableaux are called *column strict*.

For the sake of completeness and further motivation, we will briefly describe the correspondence when the range (but not the domain) is allowed to have repeated elements. Recall that $\mathcal{N}^*$ denotes the monoid

[1] These "multipermutations" are represented by nonnegative integral matrices, in the same sense that ordinary permutations are represented by permutation matrices.

consisting of all words in the alphabet $\mathcal{N} = \{1, 2, 3,...\}$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ be a word in $\mathcal{N}^*$. Define a sequence

$$P^{(1)}, P^{(2)},..., P^{(n)}$$

of column strict tableaux as follows:

(i) $P^{(1)}$ is the singleton tableau with entry σ_1, and

(ii) for $k > 1$, $P^{(k)}$ is the tableau obtained from $P^{(k-1)}$ by inserting σ_k according to the following *insertion algorithm*.

DEFINITION 6.1. SCHENSTED–KNUTH INSERTION. Let P be a column strict tableau, and let $x = x_0$ be a positive integer. Let $P_1, P_2,..., P_l$ denote the rows of P. Insert x_0 into P as follows: if $x_0 \geqslant z$ for all $z \in P_1$, place x_0 at the end of P_1 and stop. Otherwise, let x_1 denote the smallest element of P_1 such that $x_1 > x_0$. Replace x_1 by x_0 in P_1. At this point we say that x_1 has been "bumped" from P_1. Now iterate the procedure: in general, if x_i has been bumped from row P_i, insert it into row P_{i+1} by the same rule. The algorithm terminates when for some i, $x_i \geqslant z$ for all $z \in P_{i+1}$, in which case x_i is added to the end of the row.

It is easy to verify (by induction) that each tableau contructed in the sequence $P^{(1)}, P^{(2)},..., P^{(n)}$ is column strict. The tableaux $P^{(k)}$ determine a nested sequence $\lambda^{(0)} = \Phi, \lambda^{(1)}, \lambda^{(2)},..., \lambda^{(n)}$ of shapes, each one obtained from its predecessor by adding a single cell. Define another tableau $Q(\sigma)$ which records the order in which these cells are added, i.e., for each $k \geqslant 1$ assign label k to the unique cell of $\lambda^{(k)}/\lambda^{(k-1)}$. Clearly $Q(\sigma)$ is a standard tableau.

DEFINITION 6.2. ROBINSON–SCHENSTED–KNUTH (RSK) CORRESPONDENCE. To each $\sigma \in \mathcal{N}^*$, associate the pair $(P(\sigma), Q(\sigma))$, where $P(\sigma) = P^{(n)}$, as constructed in Definition 6.1, and $Q(\sigma)$ is the tableau (defined above) which records the order in which cells are added during the insertion process.

For example, if $\sigma = 2\,3\,2\,1\,2\,3$, the construction of $P(\sigma)$ and $Q(\sigma)$ is illustrated in Fig. 6.1. It is relatively easy to reverse the steps in the RSK

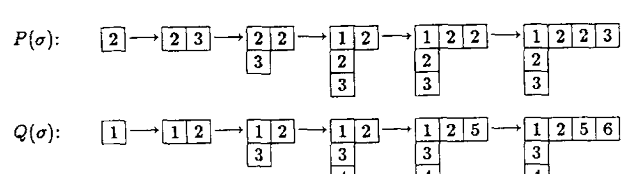


FIG. 6.1. RSK insertion ($\sigma = 2\,3\,2\,1\,2\,3$).

insertion process, and using this fact one can prove the following result, which is central to the theory:

THEOREM 6.3. *The map $\sigma \leftrightarrow (P(\sigma), Q(\sigma))$ is a bijection from $\mathcal{N}^*$ to the set of all pairs (P, Q) such that P and Q have the same shape, P is column strict, and Q is standard.*

If σ is a permutation of $\{1, 2,..., n\}$, it is clear that both $P(\sigma)$ and $Q(\sigma)$ are standard tableaux. An important symmetry of the correspondence was discovered by Schützenberger in [13]:

THEOREM 6.4. *If σ is a permutation of $\{1, 2,..., n\}$, and*

$$\sigma \leftrightarrow (P(\sigma), Q(\sigma))$$

then

$$\sigma^{-1} \leftrightarrow (Q(\sigma), P(\sigma)).$$

In [6] Knuth gave conditions under which two arbitrary words σ and σ' have the same P-tableau, i.e., $P(\sigma) = P(\sigma')$.

DEFINITION 6.5. Let σ and σ' be words in $\mathcal{N}^*$. We say that $\sigma \sim \sigma'$ if σ' can be obtained from σ by a sequence of transformations of subwords, each of which is of the following type:

$$\begin{aligned} yxz &\leftrightarrow yzx, \qquad x < y \leqslant z \\ xzy &\leftrightarrow zxy, \qquad x \leqslant y < z. \end{aligned} \tag{6.1}$$

Transformations of type (6.1) are known as *elementary Knuth transformations*, and the relation $\sim$ is known as *Knuth equivalence*. (See [16, 9] for an important elaboration of these ideas.) Knuth's result is the following:

THEOREM 6.6. *Let σ and σ' be words in $\mathcal{N}^*$. Then $P(\sigma) = P(\sigma')$ if and only if $\sigma \sim \sigma'$.*

One of the goals of this section is to generalize the Robinson–Schensted–Knuth correspondence, preserving the validity of Theorem 6.6. Thus it seems worthwhile to sketch the main ideas in the proof of Theorem 6.6, although we will not attempt to give all the details. In effect, one imitates the insertion algorithm (Definition 6.1) by applying elementary Knuth transformations to σ in a canonical way. To explain this precisely, another definition is needed.

1	2	4	5
3	7	8	
6	10		
9			

FIG. 6.2. $\rho(T) = 9\ 6\ 10\ 3\ 7\ 8\ 1\ 2\ 4\ 5$).

DEFINITION 6.7. If T is a tableau, the *bottom-up reading word* of T is the word $\rho(T)$ obtained by reading the rows of T from bottom to top, reading the elements in each row from left to right.

For example, if T is the tableau in Fig. 6.2, then

$$\rho(T) = 9\ 6\ 10\ 3\ 7\ 8\ 1\ 2\ 4\ 5.$$

A word in $\mathcal{N}^*$ of the form $\rho(T)$ for some column strict tableau T is called a *tableau word.* If T has l rows, and ρ_i is the word whose letters from the ith row of T, then the expression $\rho_l \rho_{l-1} \cdots \rho_2 \rho_1$ is called the *row factorization* of $\rho(T)$. There is usually no harm if we blur the distinction between rows of T and "row words" ρ_i, and we will do so frequently. The following lemma is easy to check:

LEMMA 6.8. *If T is any column strict tableau, then $P(\rho(T)) = T$.*

COROLLARY 6.9. *Let θ and ω be words in $\mathcal{N}^*$. Then*

$$P(\theta\omega) = P(\rho(P(\theta))\,\omega).$$

Proof. By definition, $P(\theta\omega)$ is obtained by inserting the letters of ω into $P(\theta)$, while $P(\rho(P(\theta))\,\omega)$ is obtained by inserting the same letters into $P(\rho(P(\theta)))$. By Lemma 6.8, the latter two tableaux are the same. ∎

The next lemma explains precisely how the insertion process may be imitated by Knuth transformations. Its proof is straightforward, and omitted.

LEMMA 6.10. *Let $\phi \in \mathcal{N}^*$ be a word whose letters are weakly increasing, i.e., $\phi = \rho(R)$ for a single-row tableau R. Let $x \in \mathcal{N}$, and let x' denote the element which is bumped from R by x, according to the Schensted–Knuth insertion rules. (We assume that such an x' exists.) Let R' denote the resulting row after x' has been replaced by x, and let $\phi' = \rho(R')$. Then*

$$\phi x \sim x' \phi'. \tag{6.2}$$

If $\rho = \rho(P)$ is an arbitrary tableau word with row factorization

$\phi_l\phi_{l-1}\cdots\phi_2\phi_1$, and $x_0 \in \mathcal{N}$, we can apply Lemma 6.10 repeatedly and obtain

$$\begin{aligned}\rho(P)\,x_0 &= \phi_l\phi_{l-1}\cdots\phi_2\phi_1 x_0\\ &\sim \phi_l\phi_{l-1}\cdots\phi_2 x_1\phi_1'\\ &\sim \phi_l\phi_{l-1}\cdots x_2\phi_2'\phi_1'\\ &\ \ \vdots\end{aligned}$$

and so forth, until some $x_i \geqslant z$ for all $z \in \phi_{i+1}$. Here x_i denotes the element bumped from row ϕ_i by x_{i-1}, for each i, and ϕ_i' denotes the resulting row. At this point we have transformed $\rho(P)\,x_0$ into the tableau word $\rho(P')$, where P' represents the result of inserting x_0 in P. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ then each σ_i can be inserted in $P(\sigma_1\sigma_2\cdots\sigma_{i-1})$ by the same process, and we conclude:

COROLLARY 6.11. *If $\sigma \in \mathcal{N}^*$, then $\sigma \sim \rho(P(\sigma))$.*

In other words, $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ can be transformed into $\rho(P(\sigma))$ using elementary Knuth transformations. In the example illustrated by Fig. 6.1, the successive transformations are

$$\begin{array}{llccccccccl} & \sigma = & & 2 & 3 & 2 & 1 & 2 & 3 & & \\ \text{Insert } 2: & \sim & 2 & * & 3 & 2 & 1 & 2 & 3 & & \\ \text{Insert } 3: & \sim & 2 & 3 & * & 2 & 1 & 2 & 3 & & \\ \text{Insert } 2: & \sim & 3 & 2 & 2 & * & 1 & 2 & 3 & & \\ \text{Insert } 1: & \sim & 3 & 2 & 1 & 2 & * & 2 & 3 & & \\ \text{Insert } 2: & \sim & 3 & 2 & 1 & 2 & 2 & * & 3 & & \\ \text{Insert } 3: & \sim & 3 & 2 & 1 & 2 & 2 & 3 & & & = \rho(T) \end{array}$$

An immediate application of Corollary 6.11 is the following corollary, which is the easy half of Theorem 6.6.

COROLLARY 6.12. *If $P(\sigma) = P(\sigma')$, then $\sigma \sim \sigma'$.*

The converse of Corollary 6.12 is somewhat more complicated. The major steps are sketched in the proof of Lemma 6.13, and this argument completes the proof of Theorem 6.6:

LEMMA 6.13. *If σ and σ' differ by an elementary Knuth transformation, then $P(\sigma) = P(\sigma')$.*

Proof. Let $\sigma = \theta abc$ and $\sigma' = \theta pqr$, and suppose that $abc \leftrightarrow pqr$ is a

Knuth transformation. It suffices to show that $P(\sigma) = P(\sigma')$ in this case. By Corollary 6.9 we may also assume that θ is a tableau word, with row decomposition $\theta_l \theta_{l-1} \cdots \theta_2 \theta_1$. Consider θ_1 first. Let

$$\rho(P(\theta_1 abc)) = a'b'c'\theta_1'$$

where θ_1' is the result of inserting a, b, c in the first row, and the word $a'b'c'$ represents the sequence of letters bumped during the process. Depending on the situation, a', b', and c' may be "empty" letters (for example, if a, b, or c are added at the end of a row). Similarly, define

$$\rho(P(\theta_1 pqr)) = p'q'r'\theta_1''.$$

Then

(i) $\theta_1' = \theta_1''$, and

(ii) either $a'b'c' = p'q'r'$, or $a'b'c' \leftrightarrow p'q'r'$ is a Knuth transformation.

This is proved by a straightforward argument based on (6.1), which we omit here, and Lemma 6.13 follows by induction. ∎

It follows from Lemmas 6.8 and 6.11 that $\rho(P(\sigma))$ is the unique tableau word in the Knuth equivalence class containing σ.

Dual Knuth Equivalence

If σ and σ' are permutations, Theorems 6.4 and 6.6 also characterize when σ and σ' have the same Q-tableau.

DEFINITION 6.14. If σ is a permutation in S_n, a dual Knuth transformation of σ is a transposition $(i, i+1)$ of letters such that either $i-1$ or $i+2$ occurs between the occurrences of i and $i+1$ in σ. Permutations σ and σ' are said to be *dual Knuth equivalent* if σ' may be obtained from σ by a sequence of dual Knuth transformations.

An immediate consequence of Theorems 6.4 and 6.6 is the following:

COROLLARY 6.15. *If σ and σ' are permutations in S_n, then $Q(\sigma) = Q(\sigma')$ if and only if σ and σ' are dual Knuth equivalent.*

It is important to note that if $\sigma = \rho(T)$ for some T, then $Q(\sigma)$ has a special form, described in the following easy lemma.

LEMMA 6.16. *If T is a column strict tableau of shape λ, and $\sigma = \rho(T)$, then $Q(\sigma)$ is the standard tableau constructed by these steps:*

$$
\underset{T}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 8 \\ \hline 3 & 7 & 9 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}}
\longrightarrow
\underset{\textit{bottom-up reading order}}{\begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 \\ 1 \end{array}}
\longrightarrow
\underset{Q(\rho(T))}{\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 8 & 9 \\ \hline 2 & 6 & 7 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}}
$$

FIG. 6.3. Dual reading tableau.

(i) *Label the cells of* λ *with the integers* $1, 2, \ldots, n$ *in bottom-up reading order, i.e., from left to right in each row, beginning with the bottom row.*

(ii) *Sort the columns into increasing order.*

The construction of $Q(\rho(T))$ is illustrated in Fig. 6.3. $Q(\rho(T))$ will be called the *dual reading tableau* of T. Notice that $Q(\rho(T))$ depends only on the shape of T. Hence by Corollary 6.15 we have:

COROLLARY 6.17. *If* T *and* T' *are standard tableaux of the same shape, then* $\rho(T)$ *and* $\rho(T')$ *are dual Knuth equivalent.*

This fact will have important applications in Section 7.

An Analog of the RSK Correspondence for Reduced Decompositions

Let $\omega \in \mathcal{N}^*$, and let $\Pi(\omega)$ denote the permutation in S_n obtained by interpreting ω as a product of transpositions (in the sense of Section 4) and evaluating the product in S_n. The following is a classical result (cf. [2]):

LEMMA 6.18. *If* ω *and* ω' *are words in* $\mathcal{N}^*$, *then* $\Pi(\omega) = \Pi(\omega')$ *if and only if* ω' *can be obtained from* ω *by a sequence of transformations of the form*

$$
\begin{array}{ll}
x\ x+1\ x \leftrightarrow x+1\ x\ x+1 & \\
xy \leftrightarrow yx & |x-y| \geqslant 2. \\
xx \leftrightarrow \Phi &
\end{array}
\tag{6.3}
$$

Here Φ denotes the empty word. Lemma 6.18 of course just says that the set X of adjacent transpositions together with the *Coxeter relations* (6.3) is a presentation of S_n. It is natural to ask whether there exists an analog of the Robinson–Schensted–Knuth correspondence in this context, with Coxeter relations (6.3) playing the role of elementary Knuth transformations (6.1). We will show that under certain circumstances (in particular, for words ω which represent reduced decompositions) this is the case. The appropriate vehicle turns out to be a hybrid set of relations which we call *Coxeter–Knuth relations.*

DEFINITION 6.19. Two words ω and ω' in $\mathcal{N}^*$ will be said to be *Coxeter–Knuth equivalent* (written $\omega \approx \omega'$) if ω can be transformed into ω' by a sequence of operations of the form

$$\begin{array}{lll} x\ x+1\ x \leftrightarrow x+1\ x\ x+1 & & \\ \quad yxz \leftrightarrow yzx & x<y\leqslant z, \quad |x-z|\geqslant 2 & \quad (6.4) \\ \quad xzy \leftrightarrow zxy & x\leqslant y<z, \quad |x-z|\geqslant 2. & \end{array}$$

Notice that the Coxeter–Knuth relations are valid in S_n, but are weaker than the Coxeter relations. For reduced decompositions ω, we will construct a mapping

$$\omega \rightarrow (\tilde{P}(\omega), \tilde{Q}(\omega))$$

where $\tilde{P}(\omega)$ is row and column strict and $\tilde{Q}(\omega)$ is standard, and such that $\tilde{P}(\omega)=\tilde{P}(\omega')$ if and only if ω and ω' are Coxeter–Knuth equivalent.

Let $\omega=\omega_1\omega_2\cdots\omega_n$ be a word in $\mathcal{N}^*$. Define an analog $\tilde{P}(\omega)$ of the P-tableau by forming the sequence

$$\tilde{P}^{(1)}, \tilde{P}^{(2)}, ..., \tilde{P}^{(n)}$$

of tableaux as before, with $\tilde{P}^{(k)}$ constructed from $\tilde{P}^{(k-1)}$ by inserting ω_k according to the following rule:

DEFINITION 6.20. COXETER–KNUTH INSERTION. Suppose that $\tilde{P}$ is a tableau with rows $\tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_l$, and x_0 is to be inserted in $\tilde{P}$. For each $i\geqslant 0$ add x_i to row $\tilde{P}_{i+1}$ as before, bumping x_{i+1} to the next row, following the same rules as in Definition 6.1 except in this special case: if $x_i=x$ bumps $x_{i+1}=x+1$ from row $\tilde{P}_{i+1}$, and x is already present in $\tilde{P}_{i+1}$, the value $x_i=x$ in $\tilde{P}_{i+1}$ is changed from x to $x+1$.

In other words, if x is inserted into a row containing $x\ x+1$, a copy of $x+1$ is bumped to the next row, but the original $x+1$ remains unchanged.

DEFINITION 6.21. GENERALIZED RSK CORRESPONDENCE. If $\omega\in\mathcal{N}^*$, let $\tilde{P}(\omega)$ be the tableau $\tilde{P}^{(n)}$ constructed by successive insertion of the letters in ω, using Definition 6.20. Let $\tilde{Q}(\omega)$ be the standard tableau which records the growth of cells in $\tilde{P}(\omega)$, as in Definition 6.2.

For example, if $\omega=2\ 3\ 2\ 1\ 2\ 3$, Fig. 6.4 illustrates the construction of $\tilde{P}(\omega)$ and $\tilde{Q}(\omega)$. The reader should compare the result of this process with the tableaux constructed in Fig. 6.1, using the ordinary RSK correspondence.

An easy calculation shows that an analog of Lemma 6.8 holds for $\tilde{P}$:

$$\tilde{P}(\omega):\quad \begin{array}{c}2\end{array} \longrightarrow \begin{array}{cc}2&3\end{array} \longrightarrow \begin{array}{cc}2&3\\3&\end{array} \longrightarrow \begin{array}{cc}1&3\\2&\\3&\end{array} \longrightarrow \begin{array}{cc}1&2\\2&3\\3&\end{array} \longrightarrow \begin{array}{ccc}1&2&3\\2&3&\\3&&\end{array}$$

$$\tilde{Q}(\omega):\quad \begin{array}{c}1\end{array} \longrightarrow \begin{array}{cc}1&2\end{array} \longrightarrow \begin{array}{cc}1&2\\3&\end{array} \longrightarrow \begin{array}{cc}1&2\\3&\\4&\end{array} \longrightarrow \begin{array}{cc}1&2\\3&5\\4&\end{array} \longrightarrow \begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array}$$

FIG. 6.4. Coxeter–Knuth insertion ($\omega = 2\,3\,2\,1\,2\,3$).

LEMMA 6.22. *If T is any row and column strict tableau, then $\tilde{P}(\rho(T)) = T$.*

The next lemma states two basic properties of the correspondence.

LEMMA 6.23. *Let $\omega \in \mathcal{N}^*$ be a word which represents a reduced decomposition in S_n. Then*:

(1) *ω and $\rho(\tilde{P}(\omega))$ are Coxeter–Knuth equivalent*; *in particular, both represent the same element of S_n.*

(2) *$\tilde{P}(\omega)$ is row and column strict, and $\tilde{Q}(\omega)$ is standard.*

Proof. Assume inductively that both statements are true for all words shorter than ω, and write $\omega = \theta x_0$, with $x_0 \in \mathcal{N}$. By definition, $\tilde{P}(\omega)$ is the result of inserting x_0 into $\tilde{P}(\theta)$. By the inductive hypothesis, $\tilde{P}(\theta)$ is row and column strict, and $\theta \approx \rho(\tilde{P}(\theta))$. Hence $\omega \approx \rho(\tilde{P}(\theta))\, x_0$. Write $\rho(\tilde{P}(\theta)) = \phi_l \phi_{l-1} \cdots \phi_2 \phi_1$, where the letters in ϕ_k form the rows of $\tilde{P}(\theta)$. By an argument analogous to Lemma 6.10, one can show that

$$\phi_1 x_0 \approx x_1 \phi_1'$$

where x_1 denotes the letter bumped from the first row (by Coxeter–Knuth insertion) and ϕ_1' represents the first row after the bump has taken place. More generally, $\phi_{i+1} x_i \approx x_{i+1} \phi_{i+1}'$, for $i \geq 0$, and we can iterate the process until $\rho(\tilde{P}(\theta))\, x_0$ has been transformed into $\rho(\tilde{P}(\omega))$. Hence $\omega \approx \rho(\tilde{P}(\theta))\, x_0 \approx \rho(\tilde{P}(\omega))$, and this proves (1). To see that (2) holds, first note that the bumping process trivially preserves weak inequalities in the rows. Since ω is reduced, and $\rho(\tilde{P}(\omega))$ and ω have the same number of letters, $\rho(\tilde{P}(\omega))$ must also be reduced. Hence the rows of $\tilde{P}(\omega)$ cannot contain repeated letters, i.e., $\tilde{P}(\omega)$ is row strict.

To prove that $\tilde{P}(\omega)$ is column strict, it suffices to consider the column inequalities in two successive rows, i.e., the result of a single bumping step. Suppose $x = x_i$ bumps element $z = x_{i+1}$ from row $\tilde{P}_{i+1}$. Let u be the element to the left of z, and let v be the element below z in $\tilde{P}$, as illustrated:

$$\begin{array}{l|c|c|c|c|c|c|c|c|l} \tilde{P}_{i+1} & & u' & & u & z & & & & \longleftarrow \quad x = x_i \\ \tilde{P}_{i+2} & & z' & & w & v & & & & \end{array}$$

Thus $u \leqslant x < z < v$, by hypothesis. If $z \neq u+1$, the bumping proceeds exactly as in the ordinary RSK case, i.e., x replaces z and z moves to the next row, where (since $z < v$) it bumps an element z' which lies at least as far to the left as v. If this element is v itself, only one column is changed, and the resulting column inequality is $x < z$, which is strict. On the other hand, if z' is strictly to the left of v, let u' denote the element above z' in row $\tilde{P}_{i+1}$ (as shown above). Then $u' \leqslant u < z$. Hence in this case the two original column inequalities $u' < z'$ and $z < v$ are replaced by $u' < z$ and $x < v$, both of which are strict, as claimed.

It remains to consider the case when $z = u+1$. Again z is bumped to row $\tilde{P}_{i+2}$ but also remains in row $\tilde{P}_{i+1}$. The only possible difficulty arises if z in turn bumps the element v directly below it, leading to the nonstrict column inequality $z \leqslant z$. However, this can never happen: if w denotes the element below u, we have $u < w$, hence $w \geqslant u+1 = z$. If z bumps v, then $u+1 \leqslant w \leqslant z = u+1$, hence $w = z$. This is a contradiction, since we are assuming ω is reduced. Hence the column inequalities remain strict in every case, and the proof is complete. ∎

THEOREM 6.24. *If ω is a reduced decomposition in S_n, then $\tilde{P}(\omega) = \tilde{P}(\omega')$ if and only if ω and ω' are Coxeter–Knuth equivalent.*

Proof. If $\tilde{P}(\omega) = \tilde{P}(\omega')$ then by Lemma 6.23, $\omega \approx \rho(\tilde{P}(\omega)) = \rho(\tilde{P}(\omega')) \approx \omega'$ and the theorem is proved in one direction. Conversely, suppose that ω and ω' differ by an elementary Coxeter–Knuth transformation. We will prove that $\tilde{P}(\omega) = \tilde{P}(\omega')$, by reproducing the steps used in the proof of Lemma 6.13.

As before, it suffices to assume that $\omega = \theta abc$ and $\omega' = \theta pqr$, where $abc \leftrightarrow pqr$ is an elementary Coxeter–Knuth transformation. Furthermore, since the analogs of Lemmas 6.8 and 6.9 hold, we may assume that θ is a tableau word, i.e., $\theta = \rho(\tilde{P}(\theta))$. If $\theta = \theta_l \theta_{l-1} \cdots \theta_2 \theta_1$ is the row decomposition of θ, then each θ_k consists of strictly increasing letters, since ω is reduced. Let $a'b'c'$, $p'q'r'$, θ_1', and θ_1'' be defined as in the proof of Lemma 6.13, i.e., $a'b'c'$ and $p'q'r'$ are the "output sequences" obtained when abc and pqr are inserted into θ_1, and θ_1'' are the resulting rows. We again claim that

(i) $\theta_1' = \theta_1''$, and

(ii) either $a'b'c' = p'q'r'$ or $a'b'c' \leftrightarrow p'q'r'$ is a Coxeter–Knuth transformation.

This time we will present the argument in detail.

To simplify terminology, we adopt two conventions. First, if letters x and y appear in a row, with x in column i and y in column j, we say that x *lies to the left* of y if $i \leqslant j$, and x *lies strictly to the left of* y if $i < j$. The phrase "to the right of" should be understood similarly. Second, if x is inserted into row R by Coxeter–Knuth insertion, we say that a *special bump* takes place if $x(x+1)$ is present in R and the exceptional rule is followed in Definition 6.20. We assume, further, that $a'b'c'$ and $p'q'r'$ each contain three letters, that is, both abc and pqr actually produce three bumps. We leave the other cases (which are easier) to the reader.

Case 1. Suppose that

$$abc = yxz$$

$$pqr = yzx$$

with $x < y < z$. Note that while the standard Coxeter–Knuth relation allows $x < y \leqslant z$, we are assuming that ω is reduced, hence $y = z$ is impossible.

Consider the output of the bumping process in each case. Let $\alpha = a' = p'$ be the letter bumped first by y (in both sequences). Let $\beta = b'$ (the letter bumped by x in the first sequence), and $\gamma = q'$ (the letter bumped by z in the second sequence). Clearly β occurs to the left of γ, since $x < z$. Note that β and γ cannot coincide, since y is present after the first insertion, and $x < y < z$. Hence β lies to the left of y, and γ lies strictly to the right of y. We conclude that β lies strictly to the left of γ. Since the insertions of x and z occur in different positions, it is easy to see that the input yxz produces the output sequence

$$a'b'c' = \alpha\beta\gamma$$

while yzx produces

$$p'q'r' = \alpha\gamma\beta.$$

Furthermore, we have already argued that $\beta < \alpha < \gamma$, so the transformation $a'b'c' \leftrightarrow p'q'r'$ is a Coxeter–Knuth transformation (of the same type as the original).

Finally, we must show that $\theta'_1 = \theta''_1$, i.e., the resulting row is the same independent of the order in which x and z are inserted. Since β and γ are in different positions, this is clear unless switching xz and zx changes a "special" bump into a "nonspecial" bump, or vice versa. This is only possible if β and γ are adjacent, with $\beta = y$, $\gamma = y + 1$. However, $y < z$ and $z < \gamma$ implies $y < z < y + 1$, a contradiction. This completes the proof of Case 1.

Case 2. Suppose that

$$abc = xzy$$

$$pqr = zxy$$

with $x<y<z$. Let $\alpha=a'$ be the element first bumped by x, and let $\beta=p'$ be the element first bumped by z.

Subcase 2.1. $\alpha\neq\beta$. Clearly this implies $\alpha<\beta$. We claim that (apart from one exceptional situation) the output sequences are

$$a'b'c'=\alpha\beta\gamma$$

$$p'q'r'=\beta\alpha\gamma$$

where γ is such that $\alpha<\gamma<\beta$ and the resulting rows θ_1' and θ_1'' are identical. It is easy to convince oneself of this unless (1) α and β are adjacent, (2) $\beta=\alpha+1$, and (3) $z=\alpha$, i.e., the initial bump by z is special. In this case, one can check directly that the output sequences are

$$a'b'c'=\alpha\ \alpha+1\ \alpha$$

$$p'q'r'=\alpha+1\ \alpha\ \alpha+1$$

and θ_1' and θ_1'' are both equal to the row obtained by replacing $\alpha\beta$ by xy.

Subcase 2.2: $\alpha=\beta$. Let ε_1 and ε_2 denote the letters immediately to the left and right of α (either one of these may be the "empty" letter). First note that neither of the two initial bumps can be special, since $\varepsilon_1\leqslant x<z<\alpha$. With this in mind, it is easy to show that

$$a'b'c'=\alpha\varepsilon_2 z$$

$$p'q'r'=\alpha z\varepsilon_2$$

with $z<\alpha<\varepsilon_2$, while θ_1' and θ_1'' are both equal to the row obtained by replacing $\alpha\varepsilon_2$ by xy.

This completes the proof of case 2.

Case 3. Suppose that

$$abc=x\ x+1\ x$$

$$pqr=x+1\ x\ x+1.$$

Let $\alpha=a'$ be the element first bumped by x. Let ε_1 and ε_2 be the elements immediately to the left and right of α.

Subcase 3.1. $\varepsilon_1<x$, $x+1<\alpha$. In this case the initial bump by x is not special, and one can check that the output is

$$a'b'c'=\alpha\ \varepsilon_2\ x+1$$

$$p'q'r'=\alpha\ x+1\ \varepsilon_2$$

and θ'_1 and θ''_1 are both equal to the row obtained by replacing $\alpha\varepsilon_2$ by $x\ x+1$. Note that one must use the inequality $x+1<\varepsilon_2$, which follows since $x<\varepsilon_2$ and $x+1=\varepsilon_2$ would imply that ω' is not reduced.

Subcase 3.2: $\varepsilon_1<x$, $x+1=\alpha$. This case can only happen if $\varepsilon_2=x+2$, since otherwise ω' is not reduced. It is easy to check that the output is

$$a'b'c'=x+1\ \ x+2\ \ x+1$$

$$p'q'r'=x+2\ \ x+1\ \ x+2$$

while θ'_1 and θ''_1 are both obtained by replacing $\alpha\varepsilon_2$ by $x\ x+1$.

Subcase 3.3: $\varepsilon_1=x$. This can only occur if $\varepsilon_1=x$, $\alpha=x+1$, and $\varepsilon_2=x+2$, since otherwise ω' is not reduced. The output sequences are again

$$a'b'c'=x+1\ \ x+2\ \ x+1$$

$$p'q'r'=x+2\ \ x+1\ \ x+2$$

while θ'_1 and θ''_1 both remain equal to θ.

This completes the proof of Theorem 6.24. ∎

Next we consider the analog of Theorem 6.3. Let $\mathscr{R}\subseteq\mathscr{N}^*$ denote the collection of words which are reduced decompositions in S_n, for some n. Let $\mathscr{S}$ denote the collection of all standard tableaux, and let $\mathscr{T}_R$ denote the set of all row and column strict tableaux T such that $\rho(T)\in\mathscr{R}$, i.e., the tableau word of T is reduced. By Lemma 6.23, the correspondence

$$\omega\to(\tilde{P}(\omega),\tilde{Q}(\omega))$$

maps $\mathscr{R}\to\mathscr{T}_R\times\mathscr{S}$.

THEOREM 6.25. *The correspondence $\omega\to(\tilde{P}(\omega),\tilde{Q}(\omega))$ is a bijection between $\mathscr{R}$ and the set of all pairs of tableaux (P,Q) such that P and Q have the same shape, $P\in\mathscr{T}_R$, and $Q\in\mathscr{S}$.*

Proof. It suffices to show that if $P\in\mathscr{T}_R$ and (i,j) is any border cell of P, there is a unique tableau $\hat{P}\in\mathscr{T}_R$, and a unique $\alpha\in\mathscr{N}$ such that

(1) the shape of $\hat{P}$ is obtained by deleting cell (i,j) from the shape of P, and

(2) P is obtained by inserting α into $\hat{P}$ (by Coxeter–Knuth insertion).

In other words, we can reverse the insertion process (uniquely) in such a way that border cells disappear in any desired order. If this order is specified by an arbitrary standard tableau, we obtain the inverse mapping which proves Theorem 6.25.

Let $P \in \mathscr{T}_R$, with rows $P_1, P_2, \ldots, P_l$, and let $x = P_{k\lambda(k)}$ be the last entry in row P_k. We will show how to delete x from P so that (1) and (2) hold. Define two sequences $x = x_k, x_{k-1}, \ldots, x_2, x_1$ and $y = y_k, y_{k-1}, \ldots, y_2, y_1$ as follows:

(i) $x_k = y_k = x$,

(ii) If y_{i+1} and x_{i+1} have been defined, let x_i be the right rightmost element of row P_i such that $x_i < x_{i+1}$, and let x_i^+ be the element immediately to the right of x_i. If no such element exists, let $x_i^+ = \infty$. Also:

Case 1. If $x_i^+ = x_{i+1} = x_i + 1$, let $y_i = x_i^+$.

Case 2. Otherwise let $y_i = x_i$.

Notice that since P is column strict, there is always at least one $x_i \in P_i$ which satisfies $x_i < x_{i+1}$. Notice also that $x_i \leqslant y_i$ for all i, and y_i always lies to the right of y_{i+1}. Now define $\alpha = x_1$, and construct $\hat{P}$ as follows: first delete x_k from row P_k; then for each $i < k$ replace the occurrence of y_i in P_i by x_{i+1}. We claim that this construction works, namely:

(a) $\hat{P} \in \mathscr{T}_R$, and the shape of $\hat{P}$ is obtained by deleting cell (k, λ_k) from the shape of P,

(b) inserting α in $\hat{P}$ yields P,

(c) no other choice of $\hat{P}$ and α satisfies (a) and (b).

To prove (a) it suffices to prove that $\hat{P}$ is column strict and weakly increasing along rows, and also that $\rho(P)$ may be transformed into $\rho(\hat{P})\,\alpha$ by Coxeter–Knuth transformations. The latter fact implies $\rho(\hat{P})\,\alpha$ is reduced. Hence $\hat{P}$ is row strict, $\rho(\hat{P})$ is reduced, and $\hat{P} \in \mathscr{T}_R$.

By assumption, P is a row and column strict tableau. In the construction of $\hat{P}$, it is easy to verify the inequalities $y_i \leqslant x_{i+1} \leqslant x_i^+$ for each i. This implies that the rows of $\hat{P}$ are (at least weakly) increasing. To verify the column inequalities, consider two adjacent rows P_i and P_{i+1}. If y_i and y_{i+1} lie in the same column of P, these elements are replaced by x_{i+1} and x_{i+2}, respectively, and by definition $x_{i+1} < x_{i+2}$. On the other hand, if y_i and y_{i+1} lie in different columns, let u be the element in P_i which lies above y_{i+1}, and let v be the element in P_{i+1} which lies below y_i. (In the latter case, if there are no such elements, let $v = \infty$.) Then by construction, we have $u < y_{i+1} \leqslant x_{i+2}$. Furthermore, $x_{i+1} < v$, since $x_{i+1} \leqslant y_{i+1}$ and y_{i+1} lies strictly to the left of v in P_{i+1}. This proves that $\hat{P}$ is column strict.

To show that $\rho(P) \approx \rho(\hat{P})\,\alpha$, one must check that for $i = k, k-1, \ldots, 2$ the process of moving x_i from its original position in P_i to its new position (previously occupied by y_{i-1}) in row P_{i-1} can be carried out in $\rho(P)$ by Coxeter–Knuth transformations. The details of this argument are straightforward, and are left to the reader. This completes the proof that $\hat{P} \in \mathcal{T}_R$.

Assertion (b) above is an immediate consequence of the definition of $\hat{P}$, i.e., one can show that if $\alpha = x_1$ is inserted into $\hat{P}$, then for each $i \geqslant 1$, x_i bumps element x_{i+1} from row P_i, leaving y_i in its place. Hence inserting α into $\hat{P}$ yields P.

Finally, if α' and $\hat{P}'$ also satisfy (a) and (b), let $\{x_i'\}$ and $\{y_i'\}$ denote the corresponding sequences which occur when α' is inserted into $\hat{P}'$. By hypothesis, $x_k' = x_k = y_k = y_k'$. Given x_{i+1}' and y_{i+1}', it is straightforward to check that x_i' and y_i' are uniquely determined by the rules (i) and (ii) above, and hence $\alpha' = \alpha$, $\hat{P}' = \hat{P}$. This completes the proof of Theorem 6.25. ∎

Theorem 6.25 immediately yields one of our main results: an explicit bijection between the set $\mathcal{R}(n)$ of all reduced decompositions of the element $\hat{1}$ in S_n, and the set $\mathcal{S}(n)$ of standard tableaux of staircase shape $\{n-1, n-2, \ldots, 2, 1\}$.

THEOREM 6.26. *The map*

$$\Psi\colon \omega \to \tilde{Q}(\omega)$$

is a bijection from $\mathcal{R}(n)$ *to* $\mathcal{S}(n)$.

Proof. If $\omega \in \mathcal{R}(n)$, then ω has $\binom{n}{2}$ letters ω_i, and $1 \leqslant \omega_i \leqslant n-1$ for each i. Clearly the entries in $\tilde{P}(\omega)$ obey the same inequalities, and by Lemma 6.23, $\tilde{P}(\omega)$ is both row and column strict. But there is only one such tableau with $\binom{n}{2}$ cells, namely, the tableau $\tilde{P}_{\hat{1}}$ illustrated in Fig. 6.5. Since $\omega \approx \rho(\tilde{P}(\omega))$, it follows that $\omega \in \mathcal{R}(n)$ if and only if $\tilde{P}(\omega) = \tilde{P}_{\hat{1}}$. Hence $\omega \to (\tilde{P}(\omega), \tilde{Q}(\omega))$ maps $\mathcal{R}(n)$ bijectively onto the set $\{\tilde{P}_{\hat{1}}\} \times \mathcal{S}(n)$, and the theorem follows. ∎

In Section 7, we will show that $\Psi = \Gamma^{-1}$, where $\Gamma\colon \mathcal{S}(n) \to \mathcal{R}(n)$ is the operator defined in Section 5.

We conclude this section by applying Theorems 6.25 to the study of *descent sets* of words and tableaux. If $\omega \in \mathcal{N}^*$, a descent of ω is an index i

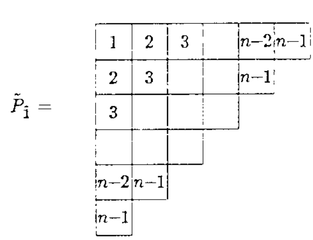


FIGURE 6.5

such that $\omega_i > \omega_{i+1}$. If T is a standard tableau, a descent of T is an entry i such that $i+1$ appears in a lower row than i. The *descent sets* of ω and T are the sets $D(\omega)$ and $D(T)$ consisting of the integers i which are descents of ω and T, respectively. If $D \subseteq \{1, 2,..., n-1\}$, define

$$\mathscr{R}(D) = \{\omega \in \mathscr{R}(n) | D(\omega) = D\}$$

$$\mathscr{S}(D) = \{T \in \mathscr{S}(n) | D(T) = D\}.$$

A central result proved in [18] states that

$$|\mathscr{R}(D)| = |\mathscr{S}(D)|$$

for all $D \subseteq \{1, 2,..., n-1\}$. Our methods yield a direct combinatorial proof of this fact:

THEOREM 6.27. *If $D \subseteq \{1, 2,..., n-1\}$, then $\omega \in \mathscr{R}(D)$ if and only if $\Psi(\omega) \in \mathscr{S}(D)$.*

The steps in the proof are of some interest in their own right, and are summarized in the statement of the following lemma:

LEMMA 6.28. *If $\omega \in \mathscr{R}(n)$, and x is a letter occurring in ω, let ξ_x^ω denote the bumping path (i.e., the sequence of cells along which labels are displaced) obtained when x is inserted during the construction of $\tilde{P}(\omega)$. Let r_x^ω denote the row index of the last (lowest) cell in ξ_x^ω. For each $k \leqslant r_x^\omega$, let $\xi_x^\omega(k)$ denote the column index of the cell of ξ_x^ω in row k. Let $\omega_i = u$ and $\omega_{i+1} = v$.*

(1) *If $u < v$, then $\xi_v^\omega(k) > \xi_u^\omega(k)$ for $k \leqslant r_v^\omega$. Hence $r_u^\omega \geqslant r_v^\omega$, and $i \notin D(\Psi(\omega))$.*

(2) *If $u > v$, then $\xi_v^\omega(k) \leqslant \xi_u^\omega(k)$ for $k \leqslant r_u^\omega$. Hence $r_u^\omega < r_v^\omega$, and $i \in D(\Psi(\omega))$.*

In other words, ξ_u^ω lies (weakly) to the right of ξ_v^ω if uv is a descent, and ξ_v^ω lies (strictly) to the right of ξ_u^ω if uv is a nondescent.

The straightforward proof of Lemma 6.28 is omitted. We note that results analogous to Theorem 6.27 and Lemma 6.28 hold for the RSK correspondence, and can be proved in essentially the same way (see [16]).

7. PROPERTIES OF THE MAP $T \to \Gamma(T)$

This section contains a proof that the mapping $T \to \Gamma(T)$ is a bijection from $\mathscr{S}(n)$ to $\mathscr{C}(n)$. Toward this end, we will first derive a number of elementary facts about the action of ∂ on T, some of which are of interest in their own right. Throughout this section, T denotes a standard tableau of staircase shape $\{n-1, n-2, \ldots, 2, 1\}$, with labels $\{k+1, k+2, \ldots, k+L\}$, where $L = \binom{n}{2}$. Let M denote the maximum label in T, namely, $k+L$.

Orientations of Pairs

For each integer i appearing in T, it turns out to be important to note the relative position of i, $i+1$, and $i+2$, and to characterize the effect ∂ has on these three entries.

DEFINITION 7.1. If T is a standard tableau, and x and y are distinct entries in T occupying cells (i, j) and (i', j'), respectively, we say that x is *below* y, and write $x \prec y$, if $i \geqslant i'$ and $j \leqslant j'$. Similarly, we say that x is *above* y, and write $y \prec x$, if $i \leqslant i'$ and $j \geqslant j'$.

The relation $\prec$ defines a partial order on the entries of T which depends only on their location, not their value. The following is trivial to check:

Lemma 7.2. *For any standard tableau T, if $x = i$ and $y = i+1$ or $y = i+2$, then y is above x in T if and only if y is not below x. In other words, if $|x-y| \leqslant 2$ then $\{x, y\}$ is a comparable pair in the partial order just defined.*

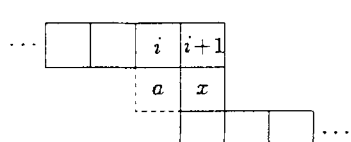
For example, in the tableau in Fig. 6.2, $3 \prec 2$, $7 \prec 8$, $9 \prec 8$, and $3 \prec 5$.

DEFINITION 7.3. If $x \notin \{i, i+1\}$, we say that x is *between* i and $i+1$ if either $i \prec x \prec i+1$ or $i+1 \prec x \prec i$. We also say that i and $i+1$ are *separated* by x.

For example, in the tableau in Fig. 6.2, 7 is between 5 and 6, and 2 is between 3 and 4.

LEMMA 7.4. *Let i and $i+1$ be entries in T such that $i \prec i+1$, and $i+1 < M$. Then $i \prec i+1$ in T^{∂}. Similarly, if $i+1 \prec i$ in T, then $i+1 \prec i$ in T^{∂}.*

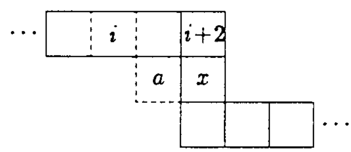
Proof. If $i \prec i+1$ in T, but ∂ moves $i+1$ to a row below the row containing i, then $i+1$ must occupy a "corner" of the evacuation path, with i immediately to the left, as illustrated:



However, this configuration is impossible, since $i < a$ implies $i+1 < a$. Thus if an evacuation path contains x it must contain a rather than $i+1$. A similar argument applies when $i+1 \prec i$. ▌

LEMMA 7.5. *If $i \prec i+1$ and $i \prec i+2$ in T, and $i+2 < M$, then $i \prec i+1$ and $i \prec i+2$ in T^∂. Similarly, if $i+1 \prec i$ and $i+2 \prec i$ in T, then both of these relations hold in T^∂.*

Proof. By symmetry we need only consider the case when $i \prec i+2$ in T. If $i \prec i+2$, but $i+2 \prec i$ in T^∂, then $i+2$ must occupy a corner of the evacuation path, with i in the same row, as illustrated:

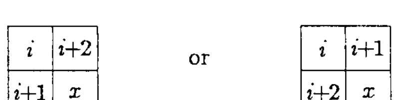


Here i may be adjacent to $i+2$, or separated from $i+2$ by $i+1$. But $i < a < i+2$ implies $a = i+1$, contradicting the assumption that $i \prec i+1$. Hence $i \prec i+2$ in T^∂. ∎

If i, $i+1$, and $i+2$ are entries in T, there are six possible orientations, which can be classified as follows:

Type 1:	$i \prec i+1 \prec i+2$
Type 2:	$i+2 \prec i+1 \prec i$
Type 3:	$i+1 \prec i \prec i+2$
Type 4:	$i+2 \prec i \prec i+1$
Type 5:	$i+1 \prec i+2 \prec i$
Type 6:	$i \prec i+2 \prec i+1$

We will describe the effect of ∂ on each of these six cases. Lemma 7.4 shows that types 1 and 2 are preserved by the action of ∂. Lemma 7.5 shows that in types 5 and 6 the orientation of i and $i+2$ cannot change, and hence these types are also preserved. In cases 3 and 4, the proof of Lemma 7.5 shows that the orientation of i and $i+2$ changes if and only if i, $i+1$, and $i+2$ occur in a configuration of the type illustrated in Fig. 7.1, where it is assumed that the cell labeled x lies in the evacuation path. When this happens, we will say that $i+1$ and $i+2$ lie in a *critical configuration.* One can



TYPE 3 TYPE 4

FIG. 7.1. Critical configurations (x lies in the evacuation path).

check that configurations of types 3 and 4 are transformed by ∂ into configurations of types 5 and 6, respectively. To summarize:

LEMMA 7.6. *If i, $i+1$, $i+2$ appear in T, and $i+2<M$, then the possible effects of ∂ on the orientation of i, $i+1$, $i+2$ are as follows*:

Orientation in T		*Possible orientation in T^∂*
Type 1	$\longrightarrow$	Type 1 *only*
Type 2	$\longrightarrow$	Type 2 *only*
Type 3	$\longrightarrow$	Type 3 *or* Type 5
Type 4	$\longrightarrow$	Type 4 *or* Type 6
Type 5	$\longrightarrow$	Type 5 *only*
Type 6	$\longrightarrow$	Type 6 *only*

In particular, notice that if i and $i+1$ are separated by $i+2$ in T, the same relation holds in T^∂.

Dual Knuth Transformations

DEFINITION 7.7. Let $\Delta_{i\,i+1}$ denote the operator which acts on a tableau T by exchanging entries i and $i+1$. Note that $\Delta_{i\,i+1}$ can only be applied to T if i and $i+1$ are both present in T, and do not lie in the same row or column.

DEFINITION 7.8. An application of $\Delta_{i\,i+1}$ to T is called a *dual Knuth transformation* if the entries i and $i+1$ are separated, in the sense of Definition 7.3, by either $i-1$ or $i+2$.

This definition coincides exactly with the definition of dual Knuth transformations given in the previous section:

LEMMA 7.9. *The operator $\Delta_{i\,i+1}$ is a dual Knuth transformation of T in the sense of Definition* 7.7 *if and only if it is a dual Knuth transformation of $\rho(T)$ in the sense of Definition* 6.14.

In the next several lemmas we will describe the effect of dual Knuth transformations on the evacuation sequence T, T^∂, T^{∂^2},.... Recall that T is assumed throughout to be a standard tableau of staircase shape.

DEFINITION 7.10. If π is an evacuation path in T, the *horizontal portion* of π is the set

$$\pi_H=\{(i,j)\in\pi\,|\,(i,j-1)\in\pi\}.$$

Similarly, the *vertical portion* of π is the set

$$\pi_V = \{(i, j) \in \pi \mid (i-1, j) \in \pi\}.$$

Note that $\pi_H \cap \pi_V = \emptyset$, and $\pi = \pi_H \cup \pi_V \cup \{(1, 1)\}$. For example, if π is as illustrated in Fig. 7.2, then π_H consists of the cells marked H, and π_V consists of the cells marked V.

LEMMA 7.11. *Let M be the largest label in T, and let $\pi^{(M)}$ be the path obtained by evacuating M from T. Let $\pi^{(M-1)}$ be the path obtained by evacuating $M-1$. If $M-1 \prec M$ in T, then*

$$\pi^{(M-1)} \cap \pi_H^{(M)} = \emptyset.$$

Similarly, if $M \prec M-1$ in T, then

$$\pi^{(M-1)} \cap \pi_V^{(M)} = \emptyset.$$

Proof. Suppose, for example, that $M-1 \prec M$ and $\pi^{(M-1)}$ contains a cell (i, j) in the horizontal portion of $\pi^{(M)}$. Suppose further that (i, j) is the rightmost such cell in row i, in other words, $\pi^{(M-1)}$ contains cell $(i+1, j)$ rather than cell $(i, j+1)$. Let $a = t_{i\,j-1}$, $b = t_{i+1\,j-1}$, and $x = t_{i+1\,j}$, as shown in the left-hand diagram of Fig. 7.3. Then $a < b$ since a and b lie in a column of T. Thus the evacuation path $\pi^{(M-1)}$ will choose b rather than a, once it reaches x. Thus $\pi^{(M-1)}$ can never reach cell (i, j), and we have a contradiction. ∎

Thus if $\pi^{(M-1)}$ starts below $\pi^{(M)}$, it must remain below $\pi^{(M)}$, in the sense that it can never touch or cross the horizontal barriers created by the cells in $\pi_H^{(M)}$. We will frequently use the words "above" and "below" in this sense when speaking about evacuation paths. An argument similar to the proof of Lemma 7.11 also yields the following:

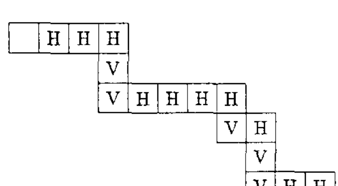
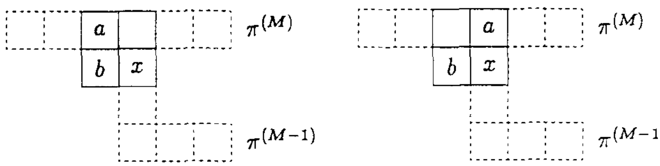


FIGURE 7.2



Original T *After evacuating*

FIGURE 7.3

COROLLARY 7.12. *Let M be the largest label in T, and suppose that $M-1 \prec M \prec M-2$. If $\pi^{(M)}$, $\pi^{(M-1)}$, and $\pi^{(M-2)}$ are defined as in* Lemma 7.11, *then*

$$\pi^{(M-1)} \cap \pi_H^{(M)} = \varnothing$$

and

$$\pi^{(M-2)} \cap \pi_V^{(M)} = \varnothing$$

and, as a consequence,

$$\pi^{(M-1)} \cap \pi^{(M-2)} = \{(1, 1)\}.$$

A similar statement holds if $M-2 \prec M \prec M-1$.

In the following lemmas, let $\bar{j}$ denote the label which is added when j is evacuated from T. In other words, $\bar{j} = k - M + j = -L + j$, for $j = k+1$, $k+2, \ldots, k+L$.

LEMMA 7.13. *Suppose $i+2 \leqslant M$, and $i+2$ lies between i and $i+1$ in T. Then*

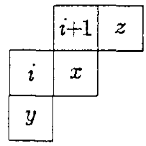
$$\Delta_{i\,i+1} \partial = \partial \Delta_{i\,i+1}$$

where both sides should be regarded as operators on T. If $i+2 = M$, then

$$\Delta_{i\,i+1} \partial^3 = \partial^3 \Delta_{\overline{i+1}\,\overline{i+2}}.$$

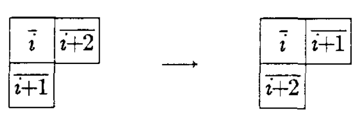
In other words, exchanging i and $i+1$ commutes with ∂ as long as both elements remain in T. After i and $i+1$ disappear, the residual effect of $\Delta_{i\,i+1}$ is limited to the elements $\overline{i+1}$ and $\overline{i+2}$.

Proof. First we show that $\Delta_{i\,i+1} \partial = \partial \Delta_{i\,i+1}$. Since labels $x \neq i$, $i+1$ bear the same order relation to both i and $i+1$, exchanging i and $i+1$ has no effect on the choice of cells in any evacuation path, unless a situation of the form



occurs, with x in π. Since $i+2$ is between i and $i+1$, we must have $x=i+2$. However, it is easy to see that if such a configuration occurs, it is impossible for π to contain x, since the elements labeled y and z in the above diagram must both be greater than x. This proves that $\Delta_{i\,i+1}\partial = \partial\Delta_{i\,i+1}$.

By Lemma 7.6, the condition that $i+2$ is between i and $i+1$ is preserved by the action of ∂, and we can iterate the process until $i+2$ has been evacuated, namely, $M-i-1$ times. When $i+2$ is evacuated, i, $i+1$, and $i+2$ must all lie on the border of T, with $i+2$ between i and $i+1$. Assume (without loss of generality) that $i \prec i+1$. Let $\pi^{(i+2)}$, $\pi^{(i+1)}$, and $\pi^{(i)}$ denote the paths obtained by evacuating $i+2$, $i+1$, and i, respectively. By Corollary 7.12, $\pi^{(i+1)}$ and $\pi^{(i)}$ intersect only in cell $(1, 1)$. Hence if i and $i+1$ are transposed and ∂ is applied two more times, the roles of $\pi^{(i+1)}$ and $\pi^{(i)}$ are reversed. Furthermore, it is easy to see that the resulting tableau differs from the original only in the locations of labels, $\bar{i}$, $\overline{i+1}$, and $\overline{i+2}$, as shown below:



T^{∂^3} $\qquad$ $T^{\Delta_{i\,i+1}\partial^3}$

Hence $\Delta_{i\,i+1}\partial^3 = \partial^3\Delta_{\overline{i+1}\;\overline{i+2}}$, and the proof is complete. ∎

Next we consider the effect of dual Knuth transformations in which i and $i+1$ are separated by $i-1$. Here the situation is somewhat more complicated. The effect of $\Delta_{i\,i+1}$ depends on whether or not the action of ∂ involves configurations which are critical in the sense of Fig. 7.1.

LEMMA 7.14. *Suppose that i and $i+1$ are separated by $i-1$ in T. Then*

$$\Delta_{i\,i+1}\partial = \begin{cases} \partial\Delta_{i-1\,i} & \text{if } i \text{ and } i+1 \text{ lie in a critical configuration} \\ \partial\Delta_{i\,i+1} & \text{otherwise.} \end{cases}$$

If $i+1=M$, then $\Delta_{i\,i+1}\partial^3 = \partial^3\Delta_{\bar{i}\;\overline{i+1}}$.

Proof. *Case* 1. First assume $i+1<M$. As in the proof of Lemma 7.13, we argue that exchanging i and $i+1$ does not affect the construction of π, unless i and $i+1$ occur in T in a configuration of the form

$$\begin{array}{|c|c|}\hline \alpha & i \\ \hline i+1 & x \\ \hline\end{array} \qquad \text{or} \qquad \begin{array}{|c|c|}\hline \alpha & i+1 \\ \hline i & x \\ \hline\end{array}$$

with x in the evacuation path. If no such configuration occurs, we have $\Delta_{i\,i+1}\partial=\partial\Delta_{i\,i+1}$ as in the proof of Lemma 7.13. If such a configuration does occur, the entry labeled α must equal $i-1$, since i and $i+1$ are separated by $i-1$. In other words, the configuration is critical in the sense of Fig. 7.1. In T^{∂} we then have

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{i-1} & i+1 \\ \hline\end{array} \qquad \text{or} \qquad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i-1} \\ \hline \multicolumn{1}{|c|}{i} & i+1 \\ \hline\end{array}$$

On the other hand, in $T^{\Delta_{i\,i+1}\partial}$ the corresponding configurations are

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i-1} \\ \hline \multicolumn{1}{|c|}{i} & i+1 \\ \hline\end{array} \qquad \text{or} \qquad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{i-1} & i+1 \\ \hline\end{array}$$

In other words, $T^{\Delta_{i\,i+1}\partial}=T^{\partial\Delta_{i-1\,i}}$ as claimed.

Case 2. Next suppose that $i+1=M$. There are two cases, depending on how $i-1$, i, and $i+1$ are situated on the border of T. Without loss of generality, assume that $i \prec i+1$.

Subcase 2.1. Entries i and $i+1$ lie in a border cell of T, while $i-1$ does not. In this case, i and $i+1$ must lie in adjacent cells, and $i-1$ is between them, as illustrated:

$$\begin{array}{cc}\hline \multicolumn{1}{|c|}{i-1} & \multicolumn{1}{c|}{i+1} \\ \hline \multicolumn{1}{|c|}{i} & \\ \cline{1-1}\end{array} \qquad \text{or} \qquad \begin{array}{cc}\hline \multicolumn{1}{|c|}{i-1} & \multicolumn{1}{c|}{i} \\ \hline \multicolumn{1}{|c|}{i+1} & \\ \cline{1-1}\end{array}$$

We will refer to such configurations on the border as *tight*. Let $\pi^{(i+1)}$, $\pi^{(i)}$, and $\pi^{(i-1)}$ denote the evacuation paths in T obtained by evacuating $i+1$, i, and $i-1$, respectively. Let $\tilde{\pi}^{(i+1)}$, $\tilde{\pi}^{(i)}$, and $\tilde{\pi}^{(i-1)}$ denote the corresponding paths in $T^{\Delta_{i\,i+1}}$. Clearly, $\pi^{(i+1)}$ and $\tilde{\pi}^{(i+1)}$ agree after the initial cell, and in each case, $i-1$ occupies the cell previously occupied by $i+1$. Furthermore, it is not difficult to check (using Lemma 7.11) that $\tilde{\pi}^{(i-1)}=\pi^{(i)}$ and $\tilde{\pi}^{(i)}=\pi^{(i-1)}$. The latter two paths are separated (in the sense of Corollary 7.12) by $\tilde{\pi}^{(i+1)}=\pi^{(i+1)}$, and hence intersect only in cell $(1, 1)$.

Hence $T^{\Delta_{i\,i+1}\partial^3}$ and T^{∂^3} agree except on cells labeled by $\overline{i-1}$, $\bar{i}$, and $\overline{i+1}$. As in the proof of Lemma 7.13, one can check that these labels differ by a transposition of $\bar{i}$ and $\overline{i+1}$.

Subcase 2.2. Entries $i-1$, i, and $i+1$ all lie in cells along the border of T. Assume without loss of generality that $i \prec i-1 \prec i+1$. Since $i \prec i+1$, the path $\pi^{(i)}$ lies below $\pi^{(i+1)}$, in the sense of Lemma 7.11. Similarly, $\pi^{(i-1)}$ lies above $\pi^{(i)}$. If $\pi^{(i+1)}$ and $\pi^{(i)}$ have only cell $(1, 1)$ in common, then trivially $\tilde{\pi}^{(i)} = \pi^{(i+1)}$ and $\tilde{\pi}^{(i+1)} = \pi^{(i)}$. It follows that $T^{\Delta_{i\,i+1}\partial^3}$ and T^{∂^3} differ only by a transposition of $\bar{i}$ and $\overline{i+1}$, and the proof is complete. On the other hand, if $\pi^{(i+1)}$ and $\pi^{(i)}$ intersect nontrivially, the situation may be represented as in Fig. 7.4.

Here the cell labeled a represents the first intersection of $\pi^{(i+1)}$ and $\pi^{(i)}$. By Lemma 7.11, π^{i-1} is trapped by the initial segments of $\pi^{(i+1)}$ and $\pi^{(i)}$, and must therefore contain the cell labeled d. It subsequently passes through $\pi^{(i+1)}$ at the cell labeled c, and remains above $\pi^{(i+1)}$ thereafter, as shown in Fig. 7.4. If $\Delta_{i\,i+1}$ is applied to T, the initial segments of $\pi^{(i+1)}$ and $\pi^{(i)}$ are interchanged, as before. A careful analysis (omitted here) shows that the terminal portions of $\pi^{(i)}$, $\pi^{(i+1)}$, and $\pi^{(i+2)}$ are rearranged as shown in Fig. 7.4, namely, $\pi^{(i+1)} = \tilde{\pi}^{(i+1)}$, $\pi^{(i-1)} = \tilde{\pi}^{(i)}$, $\pi^{(i)} = \tilde{\pi}^{(i-1)}$. Furthermore, $T^{\Delta_{i\,i+1}\partial^3}$ and T^{∂^3} agree in all cells except those labeled by $\overline{i-1}$, $\bar{i}$, and $\overline{i+1}$, and these cells differ by an application of $\Delta_{\bar{i}\,\overline{i+1}}$. This completes the proof of Lemma 7.14. ∎

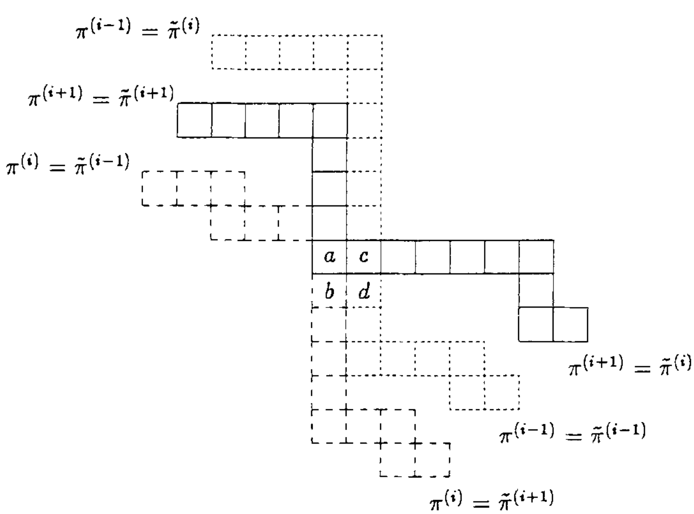


FIG. 7.4. Interaction of evacuation paths.

Proof That the Mapping $T \to \Gamma(T)$ Is Well-Defined

We will show that for any $T \in \mathscr{S}(n)$, the sequence $\Gamma(T)$ represents a maximal chain from $\hat{\mathbf{0}}$ to $\hat{\mathbf{1}}$ in the weak order $(S_n, \leqslant)$. In other words, $\Gamma(T) \in \mathscr{R}(n)$. As before, we think of $\Gamma(T)$ as a word in the alphabet $N = \{1, 2, \ldots, n-1\}$, where letter x represents the transposition of adjacent positions x and $x+1$.

Let T_0 be the staircase tableau with $n-1$ rows defined by $t_{ij} = \binom{i+j}{2} - i + 1$. In this case, T_0 is equal to its own dual reading tableau. For example, if $n = 5$, then T_0 is illustrated in Fig. 7.5. It is easy to check that $\Gamma(T_0) \in \mathscr{R}(n)$, i.e., $\Gamma(T_0)$ represents the maximal chain which reverses $[1\ 2 \cdots n]$ by moving n to the left, then $n-1$, then $n-2$, etc. For example, if $n = 5$, then

$$\Gamma(T_0) = 4\ 3\ 4\ 2\ 3\ 4\ 1\ 2\ 3\ 4.$$

Recall that the transpositions in $\Gamma(T_0)$ should be read in reverse order. If T is an arbitrary tableau, Lemmas 6.17 and 7.9 show that T may be obtained from T_0 by a sequence of dual Knuth transformations. We claim that each such transformation preserves the property that $\Gamma(T)$ is a reduced decomposition. This is the content of the next lemma.

LEMMA 7.15. *Let T be a staircase tableau such that $\Gamma(T) \in \mathscr{R}(n)$. Let $\Delta = \Delta_{i\,i+1}$ be a dual Knuth transformation of T. Then $\Gamma(T)$ and $\Gamma(T^{\Delta})$ are Coxeter–Knuth equivalent, and $\Gamma(T^{\Delta}) \in \mathscr{R}(n)$.*

Proof. *Case* 1. Suppose first that i and $i+1$ are separated in T by $i+2$. Let γ_i and γ_{i+1} denote the letters in positions i and $i+1$ of $\Gamma(T)$. As T is evacuated, $i+2$ continues to separate i and $i+1$, by Lemma 7.6. In particular, i and $i+1$ occupy nonadjacent cells (separated by $i+2$) when they reach the outer boundary of T, hence γ_i and γ_{i+1} represent transpositions of disjoint pairs of elements. Lemma 7.13 shows that $\Gamma(T)$ and $\Gamma(T^{\Delta})$ differ only by a transposition of γ_i and γ_{i+1}. Furthermore, γ_{i+2} lies between γ_i and γ_{i+1}. Hence $\Gamma(T)$ and $\Gamma(T^{\Delta})$ differ by the Coxeter–Knuth relation

$$\gamma_i \gamma_{i+1} \gamma_{i+2} = \gamma_{i+1} \gamma_i \gamma_{i+2}.$$

1	3	6	10
2	5	9	
4	8		
7			

FIG. 7.5. Dual reading tableau of a staircase.

Hence $\Gamma(T)$ and $\Gamma(T^{\Delta})$ are Coxeter–Knuth equivalent, and hence $\Gamma(T^{\Delta}) \in \mathscr{R}(n)$.

Case 2. Next suppose that i and $i+1$ are separated by $i-1$, and i and $i+1$ lie in a critical configuration (Fig. 7.1). By Lemma 7.14, we have $T^{\Delta_{i\,i+1}\partial} = T^{\partial\Delta_{i-1\,i}}$. Furthermore, in T^{∂} the elements $i-1$ and i are separated by $i+1$. Hence the effect of $\Delta_{i\,i+1}$ on $\Gamma(T)$ is the same as that of $\Delta_{i-1\,i}$ on $\Gamma(T^{\partial})$. The latter operation is a dual Knuth transformation of the type considered in Case 1, and we conclude that $\Gamma(T)$ and $\Gamma(T^{\Delta})$ differ by the Coxeter–Knuth relation

$$\gamma_{i-1}\gamma_i\gamma_{i+1} = \gamma_i\gamma_{i-1}\gamma_{i+1}$$

where γ_{i+1} lies between γ_i and γ_{i-1}. A similar argument applies if i and $i+1$ are separated by $i-1$, in a noncritical configuration which becomes critical after some iteration of ∂.

Case 3. Suppose that i and $i+1$ are separated by $i-1$, noncritically, and no critical configurations are introduced by ∂. There are two cases, according to whether or not i and $i+1$ are adjacent when they reach the outer boundary of T.

Subcase 3.1. If i and $i+1$ are adjacent on the boundary, then $i-1$ must be adjacent to both, i.e., they form a tight border configuration. If $i \prec i+1$, and $\gamma_i = x$, then $\gamma_{i+1} = x+1$ and $\gamma_{i-1} = x$. By Lemma 7.14, the corresponding three letters in $\Gamma(T^{\Delta})$ are $x+1$, x, and $x+1$, and the remaining letters in $\Gamma(T)$ and $\Gamma(T^{\Delta})$ coincide. Thus $\Gamma(T)$ and $\Gamma(T^{\Delta})$ differ by the Coxeter–Knuth relation.

$$x(x+1)\,x = (x+1)\,x(x+1)$$

occurring in positions $i-1$, i, $i+1$.

Subcase 3.2. If i and $i+1$ are nonadjacent on the boundary, then $i-1$ must lie between them, and hence γ_i and γ_{i+1} are separated by γ_{i-1}. Hence $\Gamma(T)$ and $\Gamma(T^{\Delta})$ differ by the Coxeter–Knuth relation

$$\gamma_{i-1}\gamma_i\gamma_{i+1} = \gamma_{i-1}\gamma_{i+1}\gamma_i.$$

This completes the proof. ▮

COROLLARY 7.16. *If $T \in \mathscr{S}(n)$, then $\Gamma(T) \in \mathscr{R}(n)$.*

Proof That Γ and Ψ Are Inverses

In fact the proof of Lemma 7.15 yields much more. If we carefully examine the relationship between dual Knuth transformations of T and

Coxeter–Knuth transformations of $\Gamma(T)$, we can show that Γ and Ψ are inverses, a result already mentioned in Section 6. The next corollary summarizes the arguments used in the proof of Lemma 7.15.

COROLLARY 7.17. *Let $T \in \mathscr{S}(n)$, and let $\Delta = \Delta_{i\,i+1}$ be a dual Knuth transformation of T. Then $\Gamma(T^{\Delta}) = \Gamma(T)^{\tilde{\Delta}}$ where $\tilde{\Delta}$ operates on $\gamma = \Gamma(T)$ as follows:*

Case 1. *If i and $i+1$ are separated by $i+2$ in T, then*

$$\tilde{\Delta}\colon \gamma_i \gamma_{i+1} \gamma_{i+2} \leftrightarrow \gamma_{i+1} \gamma_i \gamma_{i+2}.$$

Case 2. *If i and $i+1$ are separated by $i-1$, and a critical configuration occurs, then*

$$\tilde{\Delta}\colon \gamma_{i-1} \gamma_i \gamma_{i+1} \leftrightarrow \gamma_i \gamma_{i-1} \gamma_{i+1}.$$

Case 3.1. *If i and $i+1$ are separated by $i-1$, never critically, and reach the border in a tight configuration, then*

$$\tilde{\Delta}\colon x(x+1)\,x \leftrightarrow (x+1)\,x(x+1)$$

in positions $i-1$, i, $i+1$.

Case 3.2. *If i and $i+1$ are separated by $i-1$, never critically, and reach the border in a nontight configuration, then*

$$\tilde{\Delta}\colon \gamma_{i-1} \gamma_i \gamma_{i+1} \leftrightarrow \gamma_{i-1} \gamma_{i+1} \gamma_i.$$

THEOREM 7.18. *The maps Γ and Ψ are inverses.*

Proof. Again, we can easily verify that $\Psi(\Gamma(T)) = T$ in special cases, for example, if $T = T_0$ is a dual reading tableau (see Fig. 7.5). The next step is to show that for arbitrary $T \in \mathscr{S}(n)$, if $\Psi(\Gamma(T)) = T$, and Δ is a dual Knuth transformation of T, then

$$\Psi(\Gamma(T)^{\tilde{\Delta}}) = \Psi(\Gamma(T))^{\Delta} \tag{7.1}$$

where $\tilde{\Delta}$ is defined as in Corollary 7.17. Then

$$\Psi(\Gamma(T^{\Delta})) = \Psi(\Gamma(T)^{\tilde{\Delta}}) = \Psi(\Gamma(T))^{\Delta} = T^{\Delta}.$$

By Lemmas 6.17 and 7.9, every tableau $T \in \mathscr{S}(n)$ can be obtained from T_0 by a sequence of Δ transformations. Hence $\Psi(\Gamma(T)) = T$ holds for all T, and the proof is finished. To prove (7.1) we let $\Gamma(T) = \gamma = \gamma_1 \gamma_2 \cdots \gamma_L$, and consider each of the cases in Corollary 7.17 in turn. For notational convenience, define $\tilde{\gamma} = \gamma^{\tilde{\Delta}}$ and $\tilde{T} = \Psi(\gamma^{\tilde{\Delta}})$.

Case 1. Assume first that $i \prec i+2 \prec i+1$ in T, and write $\gamma_i\gamma_{i+1}\gamma_{i+2} = acb$. It follows from Lemma 7.4 (or Lemma 6.28) that $a < b < c$, and thus $\tilde{\Delta}$ transforms acb into cab. Since $\tilde{\Delta}$ is a Coxeter–Knuth transformation, it follows from Theorem 6.24 that T and $\tilde{T}$ differe at most by a permutation of i, $i+1$, and $i+2$. Since i is a descent of $\gamma^{\tilde{\Delta}}$ but $i+1$ is not, we have $i+1 \prec i$ and $i+1 \prec i+2$ in $\tilde{T}$, by Lemma 6.28. Thus either $i+1 \prec i+2 \prec i$ or $i+1 \prec i \prec i+2$. We must distinguish between these two possibilities. In the notation of Lemma 6.28, we know that ξ^{γ}_c lies strictly to the right of ξ^{γ}_a. We clim that $\xi^{\gamma}_c = \xi^{\tilde{\gamma}}_c$, in other words the bumping path of c is the same for both γ and $\tilde{\gamma}$. This is clear unless $\xi^{\tilde{\gamma}}_c$ and ξ^{γ}_a have a common cell. But if the paths have a common cell, they must agree in all subsequent cells. In particular, the cell labeled by i in T is again labeled by i in $\tilde{T}$, contradicting the fact that $i+1 \prec i$ in $\tilde{T}$. This proves that $i+1 \prec i+2 \prec i$ in $\tilde{T}$, and hence T and $\tilde{T}$ differ by a transposition of i and $i+1$, as claimed. If $i+1 \prec i+2 \prec i$ in T, the argument is similar.

Case 2. Suppose that $i \prec i-1 \prec i+1$ in T, in a critical configuration. We may assume that $\gamma_{i-1}\gamma_i\gamma_{i+1} = cab$, with $a < b < c$, so that $\tilde{\Delta}$ transforms cab into acb. Arguing as in Case 1, we conclude that T and $\tilde{T}$ differ by a permutation of $i-1$, i, and $i+1$. Clearly, the position of $i-1$ must be the same in both T and $\tilde{T}$, since the configuration of $i-1$, i, and $i+1$ is critical. Furthermore, $i \in D(\tilde{T})$ implies $i+1 \prec i$ in $\tilde{T}$. Hence T and $\tilde{T}$ differ by a transposition of i and $i+1$, and we are done. If $i \prec i-1 \prec i+1$, non-critically, but the configurations becomes critical after some iteration of ∂, then again $\tilde{\Delta}$ transforms $\gamma_{i-1}\gamma_i\gamma_{i+1} = cab$ into acb. This implies $i+1 \prec i$ and $i-1 \prec i$ in $\tilde{T}$, but it is no longer so obvious that the position of $i-1$ remains the same in both T and $\tilde{T}$. Arguing as in Case 1, we claim that $\xi^{\gamma}_c \neq \xi^{\tilde{\gamma}}_c$. This follows since in $\tilde{T}$, label i occupies the cell labeled by $i+1$ in T. Hence by the remarks made in Case 1, the bumping paths $\xi^{\tilde{\gamma}}_a$ and ξ^{γ}_c must have a common cell, and thus coincide from that cell onward. The terminal cell is labeled $i-1$ in both T and $\tilde{T}$, hence $i-1$ is fixed and $\tilde{T} = T^{\Delta_{ii+1}}$, as claimed. The argument is similar if we assume $i+1 \prec i-1 \prec i$ in T.

Case 3.1. If $i \prec i-1 \prec i+1$ in T, noncritically, but these labels reach the border in a tight configuration, then $\gamma_{i-1}\gamma_i\gamma_{i+1} = (x+1)\,x(x+1)$, and $\tilde{\Delta}$ transforms $(x+1)\,x(x+1)$ into $x(x+1)\,x$. As a consequence, $i-1 \notin D(\tilde{\gamma})$ and $i \in D(\tilde{\gamma})$, which implies $i-1 \prec i$ and $i+1 \prec i$ in $\tilde{T}$. Hence either $i+1 \prec i-1 \prec i$ or $i-1 \prec i+1 \prec i$. We claim that $\xi^{\gamma}_{x+1} = \xi^{\tilde{\gamma}}_x$ (these symbols stand for the bumping paths of γ_{i-1} and $\tilde{\gamma}_{i-1}$, respectively). This will show that the same cell is labeled $i-1$ in both T and $\tilde{T}$, which implies $i+1 \prec i-1 \prec i$ in $\tilde{T}$ and hence $\tilde{T} = T^{\Delta_{ii+1}}$. The claim is easy to verify by showing that γ_{i-1} and $\tilde{\gamma}_{i-1}$ each bump the same element from the first row, and hence ξ^{γ}_{x+1} and $\xi^{\bar{\gamma}}_{x}$ agree in all subsequent rows. The argument is similar if $i+1 \prec i-1 \prec i$ in T.

Case 3.2. Assume $i \prec i-1 \prec i+1$ in T, noncritically, and the border configuration is nontight. Then $\gamma_{i-1}\gamma_i\gamma_{i+1} = bac$, with $a<b<c$, and $\tilde{\Delta}$ transforms bac into bca. Trivially, $i-1$ appears in the same cell in both T and $\tilde{T}$, and $i+1 \prec i$ in $\tilde{T}$. Hence $\tilde{T} = T^{\Delta_{ii+1}}$, and the proof is complete. ∎

It is possible to give a somewhat more direct proof of Theorem 7.18, based on the following two lemmas. The first is an immediate consequence of the reverse-insertion process defined in the proof the Theorem 6.25.

Lemma 7.19. *If* $\omega = \omega_1\omega_2 \cdots \omega_L \in \mathscr{R}(n)$, *and* $\tilde{Q}(\omega)$ *has entry* L *in cell* $(n-k, k)$, *then* $\omega_L = k$.

The second lemma has a well-known counterpart (with essentially the same statement) for the ordinary RSK correspondence, and holds for reduced words of arbitrary length. We will not give an independent proof here, but will rather show how it can be derived from Theorem 7.18. If T is a standard tableau with n cells, let $\hat{T}$ denote the tableau obtained from T by (i) applying ∂^{-1} to T, (ii) suppressing the cell labeled $n+1$, and (iii) reducing each of the labels by 1.

Lemma 7.20. *If* $\omega = \omega_1\omega_2 \cdots \omega_k \in \mathscr{R}$, *let* $\hat{\omega} = \omega_2 \cdots \omega_k$. *If* $\tilde{Q}(\omega) = T$, *then* $\tilde{Q}(\hat{\omega}) = \hat{T}$.

Proof. First augment ω to obtain a maximal length reduced decomposition $\omega' = \omega_1 \cdots \omega_k\omega_{k+1} \cdots \omega_L$. Write $\delta(\omega') = \omega_2 \cdots \omega_k\omega_{k+1} \cdots \omega_L\overline{\omega_1}$, where $\overline{\omega_1} = n - \omega_1$. Then $\delta(\omega')$ is also a reduced decomposition, and it is immediate from Theorem 7.18 and the definition of Γ that $\Psi(\delta(\omega')) = \tilde{Q}(\delta(\omega'))$ is obtained by first computing $T^{\partial^{-1}}$, and then translating the labels from $\{2, 3, \ldots, L+1\}$ to $\{1, 2, \ldots, L\}$. Restricting $\delta(\omega')$ to the segment $\omega_2 \cdots \omega_k$ yields $\hat{T}$ as defined above. ∎

We mention several other corollaries concerning symmetries of the Coxeter–Knuth correspondence. These all can be derived from Theorem 7.18 by methods similar to those used to prove Lemma 7.20.

Corollary 7.21. *If* $\omega = \omega_1\omega_2 \cdots \omega_L \in \mathscr{R}(n)$, *let* $\bar{\omega} = \bar{\omega}_1\bar{\omega}_2 \cdots \bar{\omega}_L$, where $\bar{\omega}_i = n - \omega_i$ *for each* i. *Then* $\omega \in \mathscr{R}(n)$, *and* $\tilde{Q}(\bar{\omega}) = \tilde{Q}(\omega)^{\mathrm{t}}$, *where* t *denotes transpose.*

COROLLARY 7.22. *If* $\omega=\omega_1\omega_2\cdots\omega_k\in\mathscr{R}$, *let* $\omega^{\text{rev}}=\omega_k\omega_{k-1}\cdots\omega_2\omega_1$. *Then*

$$\tilde{P}(\omega^{\text{rev}})=\tilde{P}(\omega)^{\text{t}}$$
$$\tilde{Q}(\omega^{\text{rev}})=\tilde{Q}(\omega)^{S}$$

where S *is the evacuation operator defined in Definition* 5.2.

Another corollary is the following surprising fact about Schützenberger's promotion operator P (see Definition 5.1).

COROLLARY 7.23. *If* $T\in\mathscr{S}(n)$, i.e., T *is a standard staircase tableau, then* $T^P=T^{\text{t}}$.

For example, see Fig. 5.1. It is possible to prove Corollary 7.23 without using the full machinery of the Coxeter–Knuth correspondence (see [4]), but there does not seem to be any easy direct argument.

8. Reduced Decompositions of Arbitrary Permutations

Let $\sigma\in S_n$, and let $\omega=\omega_1\omega_2\cdots\omega_l$ be a reduced decomposition of σ. By Theorem 6.24, if $\omega'\in\mathscr{R}$ is such that $\tilde{P}(\omega)=\tilde{P}(\omega')$, then $\omega\approx\omega'$ and hence $\Pi(\omega)=\Pi(\omega')=\sigma$. As before, let $\mathscr{R}(\sigma)$ denote the set of all reduced decompositions of σ. Thus $\mathscr{R}(\sigma)$ can be partitioned according to the values of the first component of the bijection

$$\omega\leftrightarrow(\tilde{P}(\omega),\tilde{Q}(\omega)).$$

As an immediate consequence we have

$$|\mathscr{R}(\sigma)|=\sum_{\lambda\in\mathscr{M}(\sigma)}f_\lambda \tag{8.1}$$

where $\mathscr{M}(\sigma)$ is the multiset of shapes occurring among the tableaux $\tilde{P}(\omega)$ as ω ranges over $\mathscr{R}(\sigma)$. The purpose of this section is to obtain more detailed information about the multiset $\mathscr{M}(\sigma)$.

Let $\mathbf{N}(\sigma)$ denote the matrix of zeros and ones whose ijth element is equal to 1 if $i>j$ and j appears to the right of i in σ. Thus the ones in $\mathbf{N}(\sigma)$ represent pairs $i>j$ which are inversions in σ. Let $r(\sigma)$ and $c(\sigma)$ denote the row and column sum vectors of $\mathbf{N}(\sigma)$, and let $\bar{r}(\sigma)$ and $\bar{c}(\sigma)$ denote the partitions obtained by arranging $r(\sigma)$ and $c(\sigma)$ in nonincreasing order. Clearly $\bar{r}(\sigma^{-1})=\bar{c}(\sigma)$, and $\bar{c}(\sigma^{-1})=\bar{r}(\sigma)$. A familiar combinatorial inequality (see, e.g., [10]) states that $\bar{c}(\sigma)\preccurlyeq\bar{r}(\sigma)^*$, where the star denotes conjugation and

$\preccurlyeq$ denotes the *majorization* or *dominance* order on partitions. In other words,

$$\begin{aligned} \bar{c}_1 &\leqslant \bar{r}_1^* \\ \bar{c}_1+\bar{c}_2 &\leqslant \bar{r}_1^*+\bar{r}_2^* \\ \bar{c}_1+\bar{c}_2+\bar{c}_3 &\leqslant \bar{r}_1^*+\bar{r}_2^*+\bar{r}_3^* \\ &\cdots \end{aligned} \tag{8.2}$$

and so forth. In [18] Stanley showed that

$$|\mathscr{R}(\sigma)| = \sum_{\bar{c}(\sigma) \preccurlyeq \lambda \preccurlyeq \bar{r}(\sigma)^*} m_\lambda(\sigma) f_\lambda \tag{8.3}$$

where the $m_\lambda(\sigma)$'s are integers (possibly negative). He showed further that $m_\lambda(\sigma)=1$ when $\lambda=\bar{c}(\sigma)$ or $\lambda=\bar{r}(\sigma)^*$, and used this fact in his proof of Theorem 4.1. We will show that these results follow readily from our approach. In addition, we will prove that $m_\lambda(\sigma)\geqslant 0$ for all λ, a result conjectured but not proved in the original version of [18].

THEOREM 8.1. (1) *Let $T\in\mathscr{T}_R$, and let $\sigma=\Pi(\rho(T))$, in other words $\rho(T)\in\mathscr{R}(\sigma)$. Then*

$$\bar{c}(\sigma)\preccurlyeq\lambda(T)\preccurlyeq\bar{r}(\sigma)^*.$$

Here $\lambda(T)$ denotes the shape of T.

(2) *Let $\sigma\in S_n$, and let $\lambda=\bar{r}(\sigma)^*$. Then there exists a unique tableau $T\in\mathscr{T}_R$ such that $\rho(T)\in\mathscr{R}(\sigma)$ and $\lambda(T)=\lambda$. A similar statement holds if $\lambda=\bar{c}(\sigma)$.*

For the proof of Theorem 8.1 we need several lemmas. First let $\sigma\in S_n$ and let $\omega=\omega_1\omega_2\cdots\omega_l\in\mathscr{R}(\sigma)$. When the letters of ω are multiplied from left to right, each ω_i creates a new inversion in σ. Construct a new matrix $\tilde{\mathbf{N}}(\sigma)$ by replacing the ones in $\mathbf{N}(\sigma)$ by the integers $1, 2, \ldots, l$, according to the order in which new inversions are introduced. We call $\tilde{\mathbf{N}}(\sigma)$ the *ω-labeling* of $\mathbf{N}(\sigma)$. When $\sigma=\hat{1}$, this is essentially the encoding of maximal chains in S_n as balanced staircase tableaux, described in Theorem 4.2.

LEMMA 8.2. *Let $\sigma\in S_n$, and let $\omega=\omega_1\omega_2\cdots\omega_l$ be a reduced decomposition of σ. Suppose that ω contains a subword $\omega_{m+1}\omega_{m+2}\cdots\omega_{m+j}$ whose letters are strictly increasing. Then the labels $m+1, m+2, \ldots, m+j$ appear in different rows of $\tilde{\mathbf{N}}(\sigma)$.*

Proof. For $i=m+1, m+2, \ldots, m+j$ let (x_i, y_i) denote the pair of elements transposed by ω_i, with $x_i<y_i$. Since $\omega_{m+1}<\omega_{m+2}<\cdots<\omega_{m+j}$

the y_i's must be distinct. This means that the corresponding labels in $\tilde{\mathbf{N}}(\sigma)$ appear in different rows. ∎

An analogous result holds for decreasing subwords: in this case the corresponding labels in $\tilde{\mathbf{N}}(\sigma)$ appear in different columns. We note that the conclusion of Lemma 8.2 remains valid even if the letters of ρ are not adjacent, provided that for each i, the letters ω_i and ω_{i+1} are not separated by another occurence of ω_i.

LEMMA 8.3. *Let* $\mathbf{M}$ *be a matrix of zeros and ones, and let* $\bar{r}$ *denote the sorted row sum vector of* $\mathbf{M}$. *Suppose that the ones in* $\mathbf{M}$ *have been colored with k colors, so that no color appears twice in the same row. For* $i = 1, 2, \ldots, k$ *let* λ_i *denote the set number of entries colored with the ith color, and let* $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. *Then* $\lambda \preccurlyeq \bar{r}^*$. *A corresponding statement holds if the roles of rows and columns are interchanged.*

Proof. Permute the entries in each row of $\mathbf{M}$ so that color 1 lies in the first column, color 2 lies in the second column, etc. The resulting matrix has column sums $\lambda_1, \lambda_2, \ldots, \lambda_k$, and hence it follows from (8.2) that $\lambda \preccurlyeq \bar{r}^*$. ∎

Proof of Theorem 8.1. The proof is based largely on ideas used in [18] to prove Eq. (8.3).

(1) Suppose $T \in \mathcal{T}_R$, and $\rho = \rho(T)$ has row factorization $\rho_l \rho_{l-1} \cdots \rho_2 \rho_1$, where each ρ_i is an increasing sequence of length λ_i. In $\tilde{\mathbf{N}}(\sigma)$, the integers corresponding to letters in ρ_i lie in different rows, for each i, by Lemma 8.2. Color these entries with color i, for each i. By Lemma 8.3, we have $\lambda \preccurlyeq \bar{r}^*$, as claimed. Next let $\gamma_j = c_{1j} c_{2j} \cdots c_{qj}$ denote the jth column of T. It is easy to see that the letters of γ_j form a decreasing subsequence of ρ, in which no pair c_{ij} and $c_{i+1\,j}$ is separated by an occurrence of c_{ij}. By the remarks following Lemma 8.2, the labels corresponding to letters in column γ_j appear in different columns of $\tilde{\mathbf{N}}(\sigma)$, for each j. Hence Lemma 8.3 shows that $\lambda^* \preccurlyeq \bar{c}(\sigma)^*$. By well-known results on the majorization order (see, for example, [10]), this is equivalent to showing that $\bar{c}(\sigma) \preccurlyeq \lambda$, and the proof of part (1) is complete. The inequality $\bar{c}(\sigma) \preccurlyeq \lambda$ can also be obtained directly from the inequality $\lambda \preccurlyeq \bar{r}(\sigma)^*$, by the

$$T_0 = \begin{array}{ccccc} ② & ⑧ & ④ & ⑨ & ⑦ \\ \hline 1 & 2 & 4 & 6 & 7 \\ & 3 & & 7 & 8 \\ & 4 & & 8 & \\ & 5 & & & \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 7 \\ \hline 3 & 7 & 8 \\ \cline{1-3} 4 & 8 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} = T$$

FIGURE 8.1

following argument: if $\rho(T)\in\mathscr{R}(\sigma)$, then Corollary 7.22 shows that $\rho(T^{\mathrm{t}})\approx\rho(T)^{\mathrm{rev}}\in\mathscr{R}(\sigma^{-1})$. Hence $\lambda^*\preccurlyeq\bar{r}(\sigma^{-1})^*=\bar{c}(\sigma)^*$, and we again conclude $\bar{c}(\sigma)\preccurlyeq\lambda$.

To prove (2), suppose that σ has been given. We will show how to contruct a tableau $T\in\mathscr{T}_R$ such that $\lambda(T)=\bar{r}(\sigma)^*$ and $\rho(T)\in\mathscr{R}(\sigma)$. Let $K=\{k_1,k_2,...\}$ denote the set of integers k such that $r_k(\sigma)>0$. In other words, K consists of those σ_i which have a smaller σ_j to their right. Note that $|K|=\bar{r}_1(\sigma)^*$. Assume that the k_i's have been indexed in the order they appear as a subsequence of σ. First construct an array T_0 consisting of $|K|$ columns, as follows: if $k_i\in K$ appears in position p_i of σ, define the entries in the ith column of T_0 to be $p_i, p_i+1, p_i+2,..., p_i+r_i(\sigma)-1$. Let T be the tableau obtained from T_0 by left-justifying each of the rows. For example, if $\sigma=2\,8\,1\,4\,3\,9\,7\,5\,6$, then $K=\{2,8,4,9,7\}$, and Fig. 8.1 illustrates the construction of T_0 and T. It is clear from the definition that $\lambda(T)=\bar{r}(\sigma)^*$, and the rows and columns of T obviously increase strictly. If $\rho_l\rho_{l-1}\cdots\rho_2\rho_1$ denotes the row factorization of $\rho(T)$, one can easily show that the permutation $\sigma\rho_1^{-1}=\hat{\sigma}$ is obtained from σ by shifting each of the elements in K one position to the right. Hence $\bar{r}(\hat{\sigma})$ is obtained from $\bar{r}(\sigma)$ by reducing each $\bar{r}_i(\sigma)$ by 1, which in turn implies that $\bar{r}(\hat{\sigma})^*=\{\bar{r}_2(\sigma)^*,\bar{r}_3(\sigma)^*,...\}$. If $\hat{T}$ is the tableau obtained by removing the first row from T, then $\hat{T}$ is constructed from $\hat{\sigma}$ in the same way that T is constructed from σ. Inductively, we may assume that $\hat{T}$ is the unique tableau in $\mathscr{T}_R$ such that $\rho(\hat{T})\in\mathscr{R}(\hat{\sigma})$ and $\lambda(\hat{T})=\bar{r}(\hat{\sigma})^*$. It follows that $\rho(\hat{T})\,\rho_1=\rho(T)$ is a reduced decomposition of σ, and hence T has all the desired properties. It is easy to see that if T' is any other tableau in $\mathscr{T}_R$ with $\lambda_1(T')=\bar{r}_1(\sigma)^*$ and $\rho(T')\in\mathscr{R}(\sigma)$, then the first row of T' must consist of the entries $p_1, p_2,..., p_{|K|}$, as defined above. In other words ρ_1 is uniquely determined by σ, and the result follows by induction.

To prove the corresponding statement for $\bar{c}(\sigma)$, recall that $\bar{c}(\sigma)=\bar{r}(\sigma^{-1})$. To show that $T\in\mathscr{T}_R$ exists, with $\rho(T)\in\mathscr{R}(\sigma)$ and $\lambda(T)=\bar{c}(\sigma)$, it suffices to construct $U\in\mathscr{T}_R$ with $\rho(U)\in\mathscr{R}(\sigma^{-1})$ and $\lambda(U)=\bar{r}(\sigma^{-1})^*$, using the results of the previous paragraph. Then we can take $T=U^{\mathrm{t}}$, since $\rho(U^{\mathrm{t}})\approx\rho(U)^{\mathrm{rev}}\in\mathscr{R}(\sigma)$ (by Corollary 7.22) and $\lambda(U^{\mathrm{t}})=\lambda(U)^*=\bar{r}(\sigma^{-1})=\bar{c}(\sigma)$. The uniqueness of T follows from the uniqueness of U, and we are done. ∎

COROLLARY 8.4. *If $\sigma\in S_n$, then*

$$|\mathscr{R}(\sigma)|=\sum_{\bar{c}(\sigma)\preccurlyeq\lambda\preccurlyeq\bar{r}(\sigma)^*} m_\lambda(\sigma)f_\lambda$$

where $m_\lambda(\sigma)\geqslant 0$ for all λ, and $m_\lambda(\sigma)=1$ if $\lambda=\bar{c}(\sigma)$ or $\lambda=\bar{r}(\sigma)^*$.

Proof. By the remarks at the beginning of this section, $|\mathscr{R}(\sigma)|$ can be expressed as a sum of terms $f_{\lambda(T)}$, one for each distinct tableau T of the form $\tilde{P}(\omega)$, for $\omega \in \mathscr{R}(\sigma)$. For each such T we have $\rho(T) \in \mathscr{R}(\sigma)$ by Lemma 6.23. Hence by the first part of Theorem 8.1 we have $\bar{c}(\sigma) \preccurlyeq \lambda(T) \preccurlyeq \bar{r}(\sigma)^*$. By the second part of Theorem 8.1 there is exactly one such tableau of shape $\bar{r}(\sigma)^*$, and exactly one of shape $\bar{c}(\sigma)$. This completes the proof. ∎

We can view Corollary 8.4 in a somewhat different way by looking at the mappings Γ and Ψ defined in Sections 5 and 6 for staircase tableaux. If $\sigma \in S_n$ let $\tau = \tau_{l+1} \cdots \tau_L$ be a reduced decomposition of $\sigma^{-1}\hat{\mathbf{1}}$. Then for any reduced decomposition $\omega = \omega_1 \omega_2 \cdots \omega_l \in \mathscr{R}(\sigma)$ we have $\omega\tau = \omega_1 \cdots \omega_l \tau_{l+1} \cdots \tau_L \in \mathscr{R}(\hat{\mathbf{1}})$, and conversely. If we fix the word $\tau = \tau_{l+1} \cdots \tau_L$, and count reduced decompositions $\omega\tau \in \mathscr{R}(\hat{\mathbf{1}})$ which agree with τ in their last $L-l$ positions, the result is clearly $|\mathscr{R}(\sigma)|$. Each such reduced decomposition corresponds under Ψ to a standard staircase tableau $\Psi(\omega\tau)$ in which the labels $\{1, 2, ..., l\}$ occupy a subtableau T of shape $\lambda = \lambda(T)$, and the labels $\{l+1, ..., L\}$ occupy a skew tableau S of shape $\lambda^{[n]}/\lambda$. Clearly, every possible standard tableau T of shape λ arises as a subtableau of some $\Psi(\omega\tau)$ under this correspondence. If S denotes the skew tableau defined above, we will write $\tau = \Gamma(S)$, extending slightly the notation introduced in Section 5.

As a consequence of these remarks, we can interpret the frequency $m_\lambda(\sigma)$ with which f_λ appears in (8.3) as counting the number of skew tableaux S of shape $\lambda^{[n]}/\lambda$ whose evacuation $\Gamma(S)$ agrees exactly with τ. We state this result as a corollary:

COROLLARY 8.5. *Let $\sigma \in S_n$, and let τ be any reduced decomposition of $\sigma^{-1}\hat{\mathbf{1}}$. Let $n_\lambda(\tau)$ denote the number of skew tableaux S of shape $\lambda^{[N]}/\lambda$ such that $\Gamma(S) = \tau$. Then $n_\lambda(\tau) = m_\lambda(\sigma)$, where $m_\lambda(\sigma)$ is the coefficient appearing in* (8.3). *As a consequence, $n_\lambda(\tau)$ depends only on σ, and not on the choice of τ.*

9. BALANCED TABLEAUX OF ARBITRARY SHAPE

In this section we will prove Theorem 2.2: for any shape λ, the number of balanced tableaux of shape λ equals the number of standard tableaux of shape λ. We will do this by constructing and explicit bijection between the sets $\mathscr{B}(\lambda)$ and $\mathscr{S}(\lambda)$. The construction makes use of the bijection Γ already defined for staircase shapes. The idea is this: given a standard tableau T of arbitrary shape λ, "pack" T in a standard staircase tableau T^+ in a canonical way. Then apply Γ to T^+, obtaining a reduced decomposition. Then apply the map described in Theorem 4.2 to get a balanced staircase tableau B^+. Finally, "unpack" B^+ canonically, and get a balanced tableau B of shape λ. We begin by defining a canonical packing of a standard tableau.

DEFINITION 9.1. Let T be a standard tableau of shape λ. Let $\lambda^+ = \lambda^{[N]}$ denote the smallest staircase shape which contains λ. Let n denote the number of cells in λ, and let L denote the number of cells in λ^+. Let T^+ be the tableau obtained from T by assigning labels $\{n+1,..., L\}$ to the cells of the skew shape λ^+/λ in "top-down reading order," i.e., reading the rows from left to right, starting from the top.

For example, if T is a tableau of shape $\lambda = \{5, 2, 1, 1\}$, the canonical packing of T is illustrated in Fig. 9.1. Here the circled elements represent the new cells added, and the letters denote labels in the original tableau.

Given T^+, we can construct a reduced decomposition $\Gamma(T^+)$ using the algorithm described in Section 5. Let $B^+ \in \mathscr{B}(\lambda^+)$ denote the balanced tableau obtained from $\Gamma(T^+)$ using the correspondence defined in Theorem 4.2. Finally, let B denote the tableau obtained from B^+ by successively deleting the labels L, $L-1$,..., $n+1$, using the "column-exchange" insertion-deletion procedure, described in Lemma 2.5. We illustrate the process with an example. If T and T^+ are as defined below, in Fig. 9.1, it is not difficult to check that the first six letters in $\Gamma(T^+)$ are 1 2 3 2 4 3, corresponding to the letter transpositions (12), (13), (14), (34), (15), (35). Hence B^+ can be represented by the first tableau in Fig. 9.2. Here the primed letters represent some rearrangement of the original letters a, b,..., i. Provided all of this works as desired, we will get a sequence of mappings

$$T \to T^+ \to \Gamma(T^+) \to B^+ \to B \tag{9.1}$$

whose composition is a map from $\mathscr{S}(\lambda)$ *to* $\mathscr{B}(\lambda)$. We claim that it does work:

$T^+ =$

a	b	c	d	e
f	g	10	11	
h	12	13		
i	14			
15				

FIG. 9.1. Canonical packing of a standard tableau.

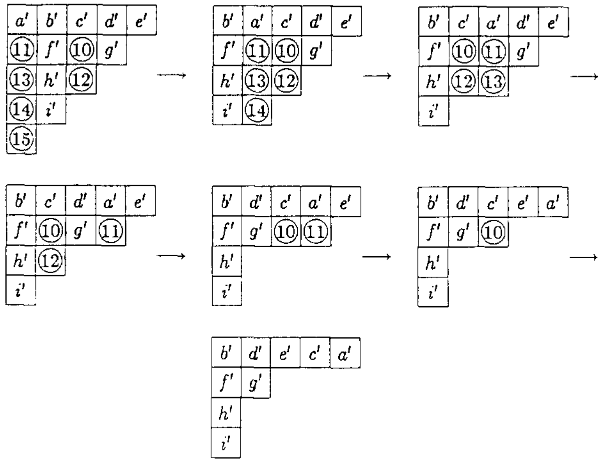


FIG. 9.2. Canonical unpacking of a balanced tableau.

LEMMA 9.2. *If the maps* (9.1) *are defined as above, then*

(i) *the construction of B from B^+ can be carried out, i.e., column-exchange deletion is valid at every step,*

(ii) *the shape of B is equal to λ (hence $B \in \mathscr{B}(\lambda)$), and*

(iii) *the map $T \to B$ is a bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$.*

Proof. Let S denote the standard skew subtableau of T^+ consisting of the circled elements. Then S has rows of length $\alpha_i = N - i - \lambda_i$, $1 \leqslant i \leqslant N-1$, where $N-1$ is the number of columns of T^+. We claim first that during the evacuation of T^+, the elements of S always remain in the same column. This can easily be shown by induction on $L-n$, the number of circled elements. If $\omega = \Gamma(S)$ denotes the word obtained by evacuating S completely, it is easy to describe ω explicitly and characterize the action of $\pi = \Pi(\omega)$ on the sequence $1\,2 \cdots n$. We can express ω as a product of sub-words of the form

$$k\,(k-1)\,(k-2) \cdots (k - \alpha_{N-k+1}) \tag{9.2}$$

for $k = 1, 2, \ldots, N-1$. In other words, π moves $k+1$ to the left α_{N-k} positions, sequentially, as $k = 1, 2, \ldots, N-1$. For example, if T is as illustrated in Fig. 9.1, then

$$\omega = [1]\,[2]\,[32]\,[43].$$

Here we have bracketed the subwords of the type represented in (9.2).

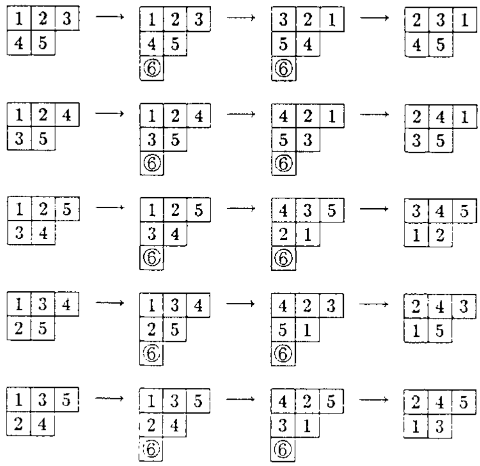


FIG. 9.3. Bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$ with $\lambda = \{3, 2\}$.

It follows from these observations that as the circled elements of T^+ are evacuated, all of the elements in row i introduce inversions $x > y$ with the same y, namely, $y = \bar{i} = N + 1 - i$. As a consequence, these elements also lie in row i of B^+. We conclude that the circled elements are increasing in each column of B^+. Furthermore, this last fact remains true (by induction) after each step of the column-exchange deletion process. We conclude that column-exchange deletion is always valid: it can only fail if at some stage the largest label is not the lowest entry in its column. Since the circled elements are increasing in each column, this can never happen. Property (ii) follows immediately from the fact that the uncircled elements of each row and column remain unchanged throughout the deletion process (although the rows and columns are permuted).

To prove (iii) it is only necessary to show that the preimage of a fixed arrangement of circled elements in B^+ is uniquely determined, i.e., if S' is any skew tableau with $\Gamma(S') = \Gamma(S) = \omega$, then $S' = S$. This is not true in general, but is true if S arises from a "canonical packing." It is not hard to show this directly (by induction, using the ideas developed in the first part of this proof). However, it is even easier to derive it from Corollary 8.5 of the last section.

Let μ be any subshape of $\lambda^{[N]}$. As in the statement of Corollary 8.5, let $\eta_\mu(\omega)$ denote the number of skew tableaux S' of shape $\lambda^{[N]}/\mu$ such that $\Gamma(S') = \omega$. Let $\sigma = \hat{\mathbf{1}}\pi^{-1}$, where $\pi = \Pi(\omega)$. By Corollary 8.5, $\eta_\mu(\omega)$ is equal to the coefficient $m_\mu(\sigma)$ in the expansion of $|\mathscr{R}(\sigma)|$. But is is easy to verify (using the fact that the inversions of σ correspond exactly to the uncircled elements of B^+) that $\bar{r}(\sigma) = \lambda$ and $\bar{c}(\sigma) = \lambda^*$. Hence $\bar{r}(\sigma)^* = \bar{c}(\sigma)$, and the only nonzero coefficient is $m_\lambda(\sigma)=1$. Hence $n_\lambda(\omega)=1$ and $\eta_\mu(\omega)=0$ if $\mu \neq \lambda$. This completes the proof. ∎

We can now state the main theorem of this section:

THEOREM 9.3. *Let Ω denote the map $T \to B$ defined by* (9.1). *Then Ω is a bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$.*

Figure 9.3 illustrates the steps in the bijection from $\mathscr{S}(\lambda)$ to $\mathscr{B}(\lambda)$ when $\lambda = \{3, 2\}$. The reader can check that the correspondence agrees with the one obtained in Lemma 3.3. We conjecture that this is always the case: Ω agrees with all of the special cases treated in Section 3. We have verified this for many examples, but so far do not have a general proof.

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