On the existence of positive solutions
for a class of singular boundary value problems

Anmin Mao\textsuperscript{a,b,*}, Shixia Luan\textsuperscript{a}, Yanheng Ding\textsuperscript{a}

\textsuperscript{a} Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, PR China
\textsuperscript{b} Department of Mathematics, Qufu Normal University, Shandong 273165, PR China

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Abstract

In this paper, by the use of a fixed point theorem, many new necessary and sufficient conditions for
the existence of positive solutions in $C[0,1] \cap C^1[0,1] \cap C^2(0,1)$ or $C[0,1] \cap C^2(0,1)$ are presented
for singular superlinear and sublinear second-order boundary value problems. Singularities at $t = 0,\ t = 1$ will be discussed.
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1. Introduction

In the study of nonlinear phenomena many mathematical models give rise to the singular boundary value problem

$$
\begin{align*}
    u'' + p(t)u^\lambda &= 0, & 0 < t < 1, \\
    \alpha u(0) - \beta u'(0) &= 0, \\
    \gamma u(1) + \delta u'(1) &= 0,
\end{align*}
$$
\tag{1.0}

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\textsuperscript{*} Corresponding author.

E-mail address: maoanmin@mail.amss.ac.cn (A. Mao).

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where $\lambda \in \mathbb{R}$, $\alpha, \beta, \gamma, \delta \geq 0$; $\rho = \gamma \beta + \alpha \gamma + \alpha \delta > 0$; $p \in C(0, 1)$; and $p > 0$ on $(0, 1)$. Such a problem has been studied extensively, see, for example, [1–7] and [10–14]. In [1], where $\lambda < 0$, $\beta = 0$ and $\delta = 0$, Taliaferro showed that (1.0) has a solution in $C[0, 1] \cap C^2(0, 1)$ if and only if

$$\int_0^1 t(1-t)p(t)dt < \infty.$$  

Moreover, this solution is in $C^1[0, 1] \cap C^2(0, 1)$ if and only if

$$\int_0^{1/2} t^2 p(t)dt < \infty \quad \text{and} \quad \int_{1/2}^1 (1-t)^2 p(t)dt < \infty.$$  

In [3], where $0 < \lambda < 1$, $\beta = 0$ and $\delta = 0$, Zhang showed (1.0) has a solution in $C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ if and only if

$$\int_0^1 t^\lambda (1-t)^\lambda p(t)dt < \infty.$$  

But necessary and sufficient conditions for the existence of positive solution of superlinear BVPs (1.0) still remain unknown. Our goal in this paper is to study BVPs (1.0) in the case of $\lambda > 1$ and to examine the more general BVPs

$$u'' + p(t)f(u) = 0, \quad 0 < t < 1,$$

$$\begin{align*}
\alpha u(0) - \beta u'(0) &= 0, \\
\gamma u(1) + \delta u'(1) &= 0,
\end{align*}$$

where $f(u)$ is superlinear or sublinear, $\alpha, \beta, \gamma, \delta \geq 0$, and $\rho = \gamma \beta + \alpha \gamma + \alpha \delta > 0$. Our approach differs from those in [1,3] in that in this paper we use fixed points theorem. We are interested in establishing necessary and sufficient condition for the existence of a nonnegative solution to singular superlinear BVPs (1.0) and BVPs (1.1)–(1.2).

In our discussion, by a solution of BVPs (1.1)–(1.2) we mean a function $u \in C[0, 1] \cap C^2(0, 1)$ which satisfies boundary condition (1.2) as well as Eq. (1.1) on $(0, 1)$. A solution of BVPs (1.1)–(1.2) is also called a $C[0, 1]$ solution. If in addition there is a solution $u(t) \in C[0, 1]$, i.e., $u'(0+)$ and $u'(1-)$ both exist, we call it a $C^1[0, 1]$ solution. We call a solution $u(t)$ positive if $u(t) > 0$ holds for $t \in (0, 1)$.

We shall need the following well-known results (see, for example, Theorems 2.1 and 2.2 in [8]). Let $X$ be a Banach space with the norm $\| \cdot \|$, $\Omega$ be bounded open set in $X$ such that $\theta \in \Omega$, and $A : \overline{\Omega} \cap P \to P$ is completely continuous map, $P \subset X$ a cone in $X$.

**Lemma 1.1.** If $Ax \neq \lambda x$ for $x \in \partial \Omega \cap P$, $\lambda \geq 1$, then $i(A, \Omega \cap P, P) = 1$.

**Lemma 1.2.** Assume that $A : \overline{\Omega} \cap P \to P$ is completely continuous, and there exists $B : \partial \Omega \cap P \to P$ which is completely continuous, such that (1) $\inf_{x \in \partial \Omega \cap P} \|Bx\| > 0$; (2) $x - Ax \neq \lambda Bx$ for $x \in \partial \Omega \cap P$, and $\lambda \geq 0$; then $i(A, \Omega \cap P, P) = 0$. 
Remark (See [9]). A map is compact, if the image of every bounded set in the domain is a relatively compact set in the range. By a map being completely continuous, we mean that it is continuous and compact.

2. Main results

The boundary value problem

\[
\begin{align*}
  u'' &= 0, \\
  \alpha u(0) - \beta u'(0) &= 0, \\
  \gamma u(1) + \delta u'(1) &= 0,
\end{align*}
\]

has a Green’s function \( G : [0, 1] \times [0, 1] \to [0, \infty) \), namely

\[
G(t, s) = \begin{cases} 
\frac{1}{\rho} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1, \\
\frac{1}{\rho} (\gamma + \delta - \gamma s)(\beta + \alpha t), & 0 \leq t \leq s \leq 1.
\end{cases}
\]

(2.1)

It is clear that

\[
G(t, s) \leq G(s, s), \quad G(t, s) \leq G(t, t), \quad 0 \leq t, s \leq 1.
\]

(2.2)

**Theorem 2.1.** Assume the following conditions hold.

\((H_1)\) \( f : [0, \infty) \to [0, \infty) \) is continuous and nondecreasing in \( u \), \( f(u) > 0 \) on \( (0, \infty) \), and there exists \( \lambda_0 > 1 \) such that

\[
 f(tu) \leq t^{\lambda_0} f(u), \quad \forall t \geq 1, \; u \in (0, \infty).
\]

(2.3)

\((H_2)\) \( p : (0, 1) \to [0, \infty) \) is continuous, \( \int_0^1 G(s, s)p(s)ds < \infty \), and there exists \( \theta_1 \in (0, 1/2) \) such that

\[
0 < \int_{\theta_1}^{1-\theta_1} G(s, s)p(s)ds.
\]

\((H_3)\) \( 0 \leq \lim_{u \to 0^+} \frac{f(u)}{u} < M_1, \; m_1 < \lim_{u \to \infty} \frac{f(u)}{u} \leq \infty \), where

\[
M_1 = \left( \max_{t \in [0, 1]} \int_0^1 G(t, s)p(s)ds \right)^{-1},
\]

\[
m_1 = \left( \min_{t \in [\theta_1, 1-\theta_1]} \int_{\theta_1}^{1-\theta_1} G(t, s)p(s)ds \right)^{-1}.
\]
Then a necessary and sufficient condition for BVPs (1.1)–(1.2) to have a positive solution in $C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ is that

$$0 < \int_0^1 p(s) f(G(s, s)) \, ds < \infty.$$  \hfill (2.4)

**Remark 2.1.** \((H_2')\) is equivalent to

\((H_{-2}')\) $p \in C((0, 1), [0, +\infty))$, and there exists $t_0 \in (0, 1)$ with $p(t_0) > 0$.

**Proof.** First by \((H_1)\), we have

$$f(tu) \geq t^{\lambda_0} f(u), \quad t \in (0, 1), \ u \in [0, \infty).$$ \hfill (2.5)

In fact, for $t \in (0, 1)$, $t^{-1} > 1$, so $f(u/t) \leq (1/t)^{\lambda_0} f(u)$, let $v = u/t$, i.e., $u = tv$ and $f(v) \leq (1/t)^{\lambda_0} f(tv)$, i.e., $f(tv) \geq t^{\lambda_0} f(v)$.

Necessity. Let

$$u \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$$

be a nontrivial positive solution of (1.1)–(1.2).

(I) $\beta > 0$, $\delta > 0$. From (1.2), we know $u(t)$ must satisfy one of the following three cases:

(1) $u(0) > 0$, $u(1) > 0$, $u'(0) = 0$, $u'(1) < 0$;
(2) $u(0) > 0$, $u(1) > 0$, $u'(0) > 0$, $u'(1) = 0$;
(3) $u(0) > 0$, $u(1) > 0$, $u'(0) > 0$, $u'(1) < 0$.

For case (3), there is $t_0 \in (0, 1)$ such that $u'(t_0) = 0$. From $u'' < 0$, we see that $u'(t) \leq 0$ for $t \in [t_0, 1]$ and $u'(t) \geq 0$ for $t \in [0, t_0]$. This implies

$$0 \leq \int_0^1 p(s) \, ds = \int_0^{t_0} p(s) \, ds + \int_{t_0}^1 p(s) \, ds$$

$$\leq f^{-1}(u(0)) \int_0^{t_0} p(s) f(u(s)) \, ds + f^{-1}(u(1)) \int_{t_0}^1 p(s) f(u(s)) \, ds$$

$$= f^{-1}(u(0)) \int_0^{t_0} (-u''(s)) \, ds + f^{-1}(u(1)) \int_{t_0}^1 (-u''(s)) \, ds$$

$$= f^{-1}(u(0))u'(0) + f^{-1}(u(1))(-u'(1)) < \infty.$$  \hfill (2.6)

Therefore, we have

$$0 < \int_0^1 p(s) f(G(s, s)) \, ds < \infty.$$


The proof for cases (1) and (2) are analogous to that of the case (3).

(II) $\beta = 0, \delta = 0$. In this case, $G(s, s) = s(1-s), s \in (0, 1), u(0) = u(1) = 0,$ and $u(t) > 0$ on $(0, 1)$. Since $u'' < 0$ it is easy to get $u(t) \geq u(1/2)t(1-t), t \in [0, 1],$ i.e., $t(1-t) \leq u(t)(u(1/2))^{-1}.$ Let $r \geq \max\{1, (u(1/2))^{-1}\},$ then

$G(t, t) \leq ru(t), \quad t \in [0, 1].$

So by (H1),

$$\int_0^1 p(s) f(G(s, s)) ds \leq \int_0^1 p(s) f(ru(s)) ds \leq r^{\lambda_0} \int_0^1 p(s) f(u(s)) ds$$

$$= r^{\lambda_0} \int_0^1 (-u'') ds = r^{\lambda_0}[u'(0) - u'(1)] < \infty.$$

On the other hand, note $0 < G(s, s) < 1$ on $[\theta_1, 1 - \theta_1]$ by (H2) and (2.5). Then

$$\int_0^{1-\theta_1} p(s) f(G(s, s)) ds \geq \int_{\theta_1}^{1-\theta_1} p(s) f(G(s, s)) ds \geq \int_{\theta_1}^{1-\theta_1} p(s)(G(s, s))^{\lambda_0} f(1) ds$$

$$= \int_{\theta_1}^{1-\theta_1} p(s)G(s, s)(G(s, s))^{\lambda_0-1} f(1) ds > 0.$$

So (2.4) holds.

(III) $\beta > 0, \delta = 0$. In this case, $G(s, s) = (1-s)(\beta + \alpha s)/(\beta + \alpha) \leq (1-s), s \in [0, 1], \alpha = u(0) > 0, u(1) = 0, u'(0) = \alpha(\beta)^{-1}u(0) > 0.$ Since $u'' < 0$ it is easy to get $u(s) \geq u(0)(1-s) \geq u(0)G(s, s), s \in [0, 1],$ i.e., $G(s, s) \leq (u(0))^{-1}u(s)$ on $[0, 1].$ Let $r > \max\{1, (u(0))^{-1}\},$ then $G(s, s) \leq ru(s).$ So

$$\int_0^1 p(s) f(G(s, s)) ds \leq \int_0^1 p(s) f(ru(s)) ds \leq r^{\lambda_0} \int_0^1 p(s) f(u(s)) ds$$

$$= r^{\lambda_0} \int_0^1 (-u'') ds = r^{\lambda_0} \int_0^1 [u'(0) - u'(1)] < \infty.$$

Similarly,

$$\int_0^1 p(s) f(G(s, s)) ds > 0.$$

Then (2.4) follows.

(IV) $\beta = 0, \delta > 0$. In this case $G(s, s) \leq s$ on $[0, 1],$ and $u(0) = 0, u(1) > 0, u'(1) = -\gamma \delta^{-1}u(1) \leq 0.$ Since $u'' < 0$ holds. We have $u(s) \geq u(1)s$ on $[0, 1].$ Similarly to (III), (2.4) holds.
Sufficiency. Suppose that \((2.4)\) holds, we define a cone
\[ C^+[0,1] = \{ u \in C[0,1] \mid u(t) \geq 0, \ t \in [0,1] \}. \]
We define \(T : C^+[0,1] \to C[0,1]\) by
\[ Tu(t) = \int_0^1 G(t,s)p(s)f(u(s))ds, \quad \forall u \in C^+[0,1]. \]
It is easy to verify that the hypotheses on the function \(p(s)\) and \(f(u)\) imply operator \(T\) is well defined, and for every \(u \in C^+[0,1]\), \(Tu(t)\) is nonnegative and continuous on \(C[0,1]\). Hence
\[ T : C^+[0,1] \to C^+[0,1]. \]
It is well known that the fixed point of the equation
\[ Tu = u, \quad u \in C^+[0,1], \quad (2.6) \]
is the solution of BVPs \((1.1)-(1.2)\). Next we will look for the fixed point.
The following discussion will be divided into three parts.
1. We first show that \(T : C^+[0,1] \to C^+[0,1]\) is completely continuous.
Let \(D \subset C^+[0,1]\) be bounded, i.e., \(\|u\| \leq M\) for all \(u \in D\) and some \(M > 0\). It is clear that if \(u \in C^+[0,1]\) satisfies \(u \in D\) by \((H_1)\), we have
\[ |Tu(t)| \leq \int_0^1 G(s,s)p(s)f(u(s))ds \leq \int_0^1 G(s,s)p(s)f(M)ds, \quad (2.7) \]
so \(T(D)\) is uniformly bounded.
Next we prove that \(|(Tu)'(t)| \in L'(0,1)\) for every \(u \in D\). In fact, for \(u \in D\),
\[ |(Tu)'(t)| = \left| -\frac{\gamma}{\rho} \int_0^t (\beta + \alpha s)p(s)f(u(s))ds + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma s)p(s)f(u(s))ds \right| \leq \frac{\gamma}{\rho} \int_0^t (\beta + \alpha s)p(s)f(u(s))ds + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma s)p(s)f(u(s))ds \leq f(M)g(t), \quad (2.8) \]
where
\[ g(t) := \frac{\gamma}{\rho} \int_0^t (\beta + \alpha s)p(s)ds + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma s)p(s)ds, \]
and
\[\int_0^1 |g(t)| \, dt = \int_0^1 \frac{\gamma}{\rho} \, dt + \int_0^1 (\beta + \alpha s) p(s) \, ds + \int_0^1 \frac{\alpha}{\rho} \, dt + \int_0^1 (\gamma + \delta - \gamma s) p(s) \, ds\]
\[= \int_0^1 \frac{\gamma}{\rho} (\beta + \alpha s) p(s) \, ds + \int_0^1 \frac{\alpha}{\rho} (\gamma + \delta - \gamma s) p(s) \, ds\]
\[\leq 2 \int_0^1 (\gamma + \delta - \gamma s)(\beta + \alpha s) p(s) \, ds = 2 \int_0^1 G(s, s) p(s) \, ds < \infty.\] (2.9)

From (2.8) and (2.9), we get
\[0 \leq \int_0^1 |(Tu)'(t)| \, dt < \infty,\] (2.10)
therefore, for any \(0 \leq t_1 < t_2 \leq 1, u \in D\), we have
\[|Tu(t_1) - Tu(t_2)| = \left| \int_{t_1}^{t_2} (Tu)'(t) \, dt \right| < \int_{t_1}^{t_2} |(Tu)'(t)| \, dt.\] (2.11)

By the absolute continuity of the integral and (2.8) and (2.9), it is easy to find that \(T(D)\) is equicontinuous.

From the Ascoli–Arzela theorem, \(T(D)\) is relatively compact. This completes the proof that \(T\) is compact.

Next, we prove \(T : C^+[0, 1] \to C^+[0, 1]\) is continuous.
Assume that \(u_n, u^* \in C^+[0, 1]\) and \(u_n \to u^*\). Then there exists \(M_0 > 0\) such that \(\|u^*\| < M_0, \|u_n\| < M_0\) for every \(n > 0\). Let us now show \(Tu_n \to Tu^*(n \to \infty)\). If \(t \in [0, 1]\), we have
\[|Tu_n(t) - Tu^*(t)| \leq \int_0^1 G(t, s) p(s) \left| f(u_n(s)) - f(u^*(s)) \right| \, ds\]
\[\leq \int_0^1 G(s, s) p(s) \left| f(u_n(s)) - f(u^*(s)) \right| \, ds.\]

Let \(g_n(s) = G(s, s) p(s) f(u_n(s)) - f(u^*(s))), n = 1, 2, \ldots;\) by (H1), we have
\[|g_n(s)| \leq 2G(s, s) p(s) f(M_0) \text{ a.e. on } [0, 1].\] (2.12)
Set \( F(s) = 2G(s, s)p(s)f(M_0) \). From (H2), we get
\[
0 \leq \int_0^1 F(s) \, ds < \infty
\]
and
\[
|g_n(s)| \leq F(s) \quad \text{a.e. on}[0, 1]
\]
for every \( n \geq 1 \). We claim that \( g_n(s) \to 0 \) a.e. on \([0, 1]\). In fact, for any \( s \in (0, 1) \), since \( p(s)f(u) \) is continuous on \([0, \infty)\), for any \( \varepsilon > 0 \) there is \( \delta_1 > 0 \) such that \( |u_1 - u_2| < \delta_1 \) implies that \( |p(s)f(u_1) - p(s)f(u_2)| < \varepsilon / G(s, s) \). By \( u_n(s) \to a^*(s) \), there exists \( N > 0 \) such that \( n \geq N \) implies that \( |u_n(s) - a^*(s)| < \delta_1 \) and \( |p(s)f(u_n(s)) - p(s)f(a^*(s))| < \varepsilon / G(s, s) \). Thus we have
\[
|g_n(s) - 0| = G(s, s)p(s)|f(u_n(s)) - f(a^*(s))| < \varepsilon, \quad \forall n \geq N,
\]
therefore \( g_n(s) \to 0 \), a.e. on \([0, 1] \). It follows from Lebesgue’s theorem that \( |Tu_n(t) - Tu^*(t)| \to 0 \) as \( n \to \infty \) uniformly in \( t \in [0, 1] \), which shows \( T \) is continuous.

So we conclude that operator \( T \) is completely continuous.

2. We define a cone \( K \subset C^+[0, 1] \) by
\[
K = \left\{ u \in C^+[0, 1] \mid \exists \rho > 0: u(t) \leq \rho G(t, t), \quad \min_{t \in [\theta_1, 1 - \theta_1]} u(t) \geq \sigma \| u \| \right\},
\]
where \( \sigma = \min \left\{ \frac{\delta_1 + \theta_1 \gamma}{\delta + \theta_1 \gamma}, \frac{\delta_1 + \theta_1 \delta}{\delta + \theta_1 \alpha} \right\} \). By [9, Theorem 1.5.1], it follows that \( K \) is a cone.

To see \( T(K) \subset K \), for \( u \in K \), \( \exists \rho > 1 \) such that \( u(t) \leq \rho G(t, t) \) and for \( t \in [0, 1] \), we get
\[
Tu(t) = \int_0^1 G(t, s)p(s)f(u(s)) \, ds \leq \int_0^1 r_n^0 G(t, s)p(s)f(G(s, s)) \, ds.
\]
Let \( r_{Tu} = r_n^{1/0} \int_0^1 p(s)f(G(s, s)) \, ds \). By (2.5), we know \( r_{Tu} > 0 \), so
\[
Tu(t) \leq r_{Tu} G(t, t), \quad \text{for} \ t \in [0, 1].
\]
In addition, for \( t \in [0, 1] \),
\[
Tu(t) = \int_0^1 G(t, s)p(s)f(u(s)) \, ds \leq \int_0^1 G(s, s)p(s)f(u(s)) \, ds, \tag{2.13}
\]
hence
\[
\|Tu\| \leq \int_0^1 G(s, s)p(s)f(u(s)) \, ds.
\]
Since \( \theta_1 \leq t \leq 1 - \theta_1 \) implies
\[
G(t,s) = \begin{cases} 
\frac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma s}, & s \leq t \\
\frac{\beta + \alpha t}{\beta + \alpha s}, & t \leq s
\end{cases}
\]

\[
\delta + \frac{\gamma}{\gamma + \delta},
\]

i.e.,
\[
G(t,s) \geq \sigma, \quad \text{for } \theta_1 \leq t \leq 1 - \theta_1, \quad s \in [0, 1].
\]

Hence, if \( u \in K \), we have
\[
\min_{\theta_1 \leq t \leq 1 - \theta_1} Tu(t) = \min_{\theta_1 \leq t \leq 1 - \theta_1} \int_0^1 G(t,s)p(s)f(u(s))ds
\geq \sigma \int_0^1 G(s,s)p(s)f(u(s))ds \geq \sigma \|Tu\|
\]

i.e., \( Tu \in K \). Then \( T(K) \subset K \) and \( T : K \to K \) is completely continuous.

3. By the first part of (H3), there are \( R_1 > 0 \), \( \varepsilon_0 > 0 \) such that \( 0 < u \leq R_1 \) implies \( f(u)/u \leq (M_1 - \varepsilon_0) \). Therefore, we have
\[
f(u) \leq (M_1 - \varepsilon_0)u \leq (M_1 - \varepsilon_0)R_1, \quad 0 < u \leq R_1.
\]

Set \( \Omega_1 = \{ u \in C[0, 1] | \|u\| < R_1 \} \), for any \( u \in \partial \Omega_1 \cap K \), we have
\[
\|Tu\| = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)p(s)f(u(s))ds \leq (M_1 - \varepsilon_0)R_1 \max_{r \in [0,1]} \int_0^1 G(t,s)p(s)ds
\leq R_1 - \varepsilon_0 R_1 \max_{r \in [0,1]} \int_0^1 G(t,s)p(s)ds < R_1,
\]

then for \( u \in \partial \Omega_1 \cap K \) and \( \lambda \geq 1 \), we have
\[
Tu \neq \lambda u. \quad (2.15)
\]

In fact, if not, there exist \( u_0 \in \partial \Omega_1 \cap K \) and \( \lambda_0^* \geq 1 \) such that \( Tu_0 = \lambda_0^*u_0 \), then \( \|Tu_0\| \geq \|u_0\| \), which is a contradiction. According to Lemma 1.1, we have
\[
i(T, \Omega_1 \cap K, K) = 1. \quad (2.16)
\]

By the second part of (H3), \( m_1 < \lim_{u \to \infty} f(u)/u \leq \infty \), there exist \( \xi > \theta_1 R_1, \varepsilon_1 > 0 \) such that
\[
f(u) \geq (m_1 + \varepsilon_1)u, \quad u \geq \xi.
\]

Let \( R_2 = \sigma^{-1} \xi \), and \( \Omega_2 = \{ u \in C[0, 1] | \|u\| < R_2 \} \), then
\[
\min_{0 \leq t \leq 1} \left\{ u(t) : t \in [\theta_1, 1 - \theta_1] \right\} \geq \sigma \|u\| = \xi, \quad \forall u \in \partial \Omega_2 \cap K. \quad (2.17)
\]
We define $B : C[0, 1] \to C[0, 1]$ by

$$Bu = u, \quad \forall u \in C[0, 1].$$

It is easy to verify $B : \partial \Omega_2 \cap K \to K$ is completely continuous and $\inf_{u \in \partial \Omega_2 \cap K} \|Bu\| > 0$. We now prove that

$$u - Tu \neq \lambda Bu, \quad \text{for } \lambda \geq 0 \text{ and } u \in \partial \Omega_2 \cap K. \quad (2.18)$$

In fact, if not, there are $\lambda^* \geq 0$ and $u_0 \in \partial \Omega_2 \cap K$ such that $u_0 - Tu_0 = \lambda^* Bu_0$. So $\lambda^* > 0$, otherwise there is a fixed point in $\partial \Omega_2 \cap K$ and this would complete the proof. Let $\eta = \min\{u_0(t) | t \in [\theta_1, 1 - \theta_1]\}$. Then if $t \in [\theta_1, 1 - \theta_1]$, we have

$$u_0(t) = \int_0^1 G(t, s)p(s)f(u_0(s)) \, ds + \lambda_0^* Bu_0(t)$$

$$\geq \int_{\theta_1}^{1-\theta_1} G(t, s)p(s)f(u_0(s)) \, ds + \lambda_0^* Bu_0(t)$$

$$\geq (m_1 + \varepsilon_1) \int_{\theta_1}^{1-\theta_1} G(t, s)p(s)u_0(s) \, ds + \lambda_0^* u_0(t)$$

$$\geq \eta (m_1 + \varepsilon_1) \int_{\theta_1}^{1-\theta_1} G(t, s)p(s) \, ds + \lambda_0^* u_0(t)$$

$$\geq \eta + \eta \varepsilon_1 \int_{\theta_1}^{1-\theta_1} G(t, s)p(s) \, ds + \lambda_0^* u_0(t).$$

Therefore

$$u_0(t) > \eta, \quad t \in [\theta_1, 1 - \theta_1],$$

which is a contradiction. According to Lemma 1.2, we get

$$i(T, \Omega_2 \cap K, K) = 0. \quad (2.19)$$

(2.16) and (2.19) together implies

$$i(T, (\Omega_2 \setminus \Omega_1) \cap K, K) = i(T, \Omega_2 \cap K, K) - i(T, \Omega_1 \cap K, K) = 0 - 1 = -1.$$  

Consequently, according to [9, Theorem 2.3.2], $T$ has a fixed point $u^*$ in $(\Omega_2 \setminus \Omega_1) \cap K$, satisfying $0 < R_1 \leq \|u^*\| \leq R_2$. Since $u^* \in K$, there exists $r_\omega > 1$ such that $u^*(t) \leq r_\omega G(t, t)$, then

$$\int_0^1 |(u^*)''(t)| \, dt = \int_0^1 p(t)f(u^*(t)) \, dt \leq (r_\omega)^2 \int_0^1 p(t)f(G(t, t)) \, dt < \infty. \quad (2.20)$$
It follows from (2.20) that $u^* \in C^1[0, 1]$, so $u^*$ is a $C^1[0, 1]$ positive solution of (1.1), (1.2).

This completes the proof of sufficiency. \qed

**Corollary 1.** Let $p$ be as above, $0 < \int_0^1 G(s,s)p(s) \, ds < \infty$, and $\lambda > 1$. Then a necessary and sufficient condition for (1.0) to have a positive solution in $C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ is that

$$0 < \int_0^1 (G(s,s))^\lambda p(s) \, ds < \infty.$$ 

Moreover, by the proof of Theorem 2.1 we get the following result.

(H-1’) $f : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing in $u$, $f(u) > 0$ on $(0, \infty)$.
(H-2’) $p : (0, 1) \to [0, \infty)$ is continuous and there exists $t_0 \in (0, 1)$ with $p(t_0) > 0$.
(H-3’) $\lim_{u \to 0^+} \frac{f(u)}{u} = 0$, $\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty$.

**Theorem 2.2.** Assume (H1), (H2), and (H-3’) hold. Then a necessary and sufficient condition for BVPs (1.1)–(1.2) to have a positive solution in $C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ is that

$$\int_0^1 p(s)f(G(s,s)) \, ds < \infty.$$ 

**Proof.** Clearly (H1)–(H3) hold, and result follows from Theorem 2.1. We omit the detail. \qed

**Remark 2.3.** In [10] $\lim_{u \to 0^+} \frac{f(u)}{u} = 0$, $\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty$, so our functions are more general than that in [10].

**Theorem 2.3.** Assume (H-1’)–(H-3’) hold. Then a sufficient condition for BVPs (1.1)–(1.2) to have a positive solution in $C[0, 1] \cap C^2(0, 1)$ is that

$$\int_0^1 p(s)G(s,s) \, ds < \infty.$$ 

**Proof.** By (H-2’), there exists $\theta_1 \in (0, 1/2)$ such that

$$0 < \int_{\theta_1}^{1-\theta_1} G(s,s)p(s) \, ds < \infty.$$
So (H2), (H3) hold, and in the proof of the sufficiency of Theorem 2.1 only make the change of replacing cone $K$ with

$$K' = \left\{ u \in C^+ [0, 1] \left| \min_{t \in [\theta_1, 1-\theta_1]} u(t) \geq \sigma \| u \| \right. \right\}.$$

Then result follows. We omit the detail. $\square$

**Remark 2.4.** In [10] only results related to $C^1[0, 1]$ positive solutions are given.

**Remark 2.5.** Results of BVPs (1.1)–(1.2) of superlinear type are more difficult to get compared to the sublinear type in [1–3].

Next, we shall study BVPs (1.1)–(1.2) in the sublinear case.

**Theorem 2.4.** Assume

(H′ 1 ) $f : [0, \infty) \to [0, \infty)$ is continuous and $f(u) > 0$ on $(0, \infty)$, $f$ is nondecreasing in $u$, there exists $\lambda_1$, $0 < \lambda_1 < 1$, such that

$$f(tu) \geq t^{\lambda_1} f(u), \quad t \in (0, 1), \quad u \in [0, \infty).$$

Suppose (H2) holds, and

(H′ 3 ) $0 \leq \lim_{u \to \infty} \frac{f(u)}{u} < M_1$, $m_1 < \lim_{u \to 0+} \frac{f(u)}{u} \leq \infty,$

where

$$M_1 = \left( \max_{t \in [0, 1]} \frac{1}{0} \int_0^1 G(t, s) p(s) ds \right)^{-1}, \quad m_1 = \left( \min_{t \in [\theta_1, 1-\theta_1]} \frac{1-\theta_1}{\theta_1} \int_{\theta_1}^{1-\theta_1} G(t, s) p(s) ds \right)^{-1}.$$

Then a necessary and sufficient condition for BVPs (1.1)–(1.2) to have a positive solution in $C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ is that

$$0 < 1 \int_0^1 p(s) f(G(s, s)) ds < \infty. \quad (2.21)$$

**Proof.** By (H′ 1 ), we have

$$f(tu) \leq t^{\lambda_1} f(u), \quad t \geq 1, \quad u \in [0, \infty).$$

In fact, when $t = 1$, the above inequality holds, when $t > 1$, $1/t \in (0, 1)$, then $f(u/t) \geq (1/t)^{\lambda_1} f(u)$, let $v = u/t$, i.e., $u = tv$, so $f(v) \geq (1/t)^{\lambda_1} f(tv)$, i.e., $f(tv) \leq t^{\lambda_1} f(v)$.

The proof of necessity is almost the same as that in Theorem 2.1. We will show the proof of the sufficiency,

**Sufficiency.** We base the proof on the argument in Theorem 2.1 and need only show completely continuous operator $T : K \to K$ has a fixed point.

By the first part of (H′ 3 ), there exist $r_1 > 0$ and $\varepsilon_1 > 0$ such that $u \geq r_1$ implies $f(u) \leq (M_1 - \varepsilon_1) u$. Let $c = \max\{ f(x) \mid 0 \leq x \leq r_1 \}$, then

$$f(u) \leq c + (M_1 - \varepsilon_1) u, \quad u \in [0, \infty).$$
Chose $R_3 > c \varepsilon_1^{-1}$, let $\Omega_3 = \{u \in C[0, 1] \mid \|u\| < R_3\}$. Then for $u \in \partial \Omega_3 \cap K$, we have

$$\max_{t \in [0, 1]} Tu(t) = \max_{t \in [0, 1]} \int_0^1 G(t, s) p(s) f(u(s)) \, ds$$

$$\leq (c + (M_1 - \varepsilon_1)\|u\|) \max_{t \in [0, 1]} \int_0^1 G(t, s) p(s) \, ds$$

$$\leq M_1\|u\| \max_{t \in [0, 1]} \int_0^1 G(t, s) p(s) \, ds + (c - \varepsilon_1 R_3) \int_0^1 G(s, s) p(s) \, ds$$

$$= R_3 + (c - \varepsilon_1 R_3) \int_0^1 G(s, s) p(s) \, ds < R_3,$$

i.e., $\|Tu\| < \|u\|$. It is easy to verify $Tu \neq \lambda u$ for $u \in \partial \Omega_3 \cap K$ and $\lambda \geq 1$. According to Lemma 1.1, we have

$$i(T, \Omega_3 \cap K, K) = 1.$$

From the second part of $(H'_3)$, there exist $R_4, 0 < R_4 < R_3$ and $\varepsilon_3 > 0$ such that $0 < u \leq R_4$ implies $(m_1 + \varepsilon_3) \leq f(u)/u$. Let $\Omega_4 = \{u \in C[0, 1] \mid \|u\| < R_4\}$. We define $B : C[0, 1] \rightarrow C[0, 1]$ by

$$Bu = u, \quad \forall u \in C[0, 1].$$

It is easy to verify $B : \partial \Omega_4 \cap K \rightarrow K$ is completely continuous and $\inf_{u \in \partial \Omega_4 \cap K} \|Bu\| > 0$. We now show that

$$u - Tu \neq \lambda Bu, \quad \text{for } \lambda \geq 0 \text{ and } u \in \partial \Omega_4 \cap K. \quad (2.33)$$

In fact, if not, there exist $u_0 \in \partial \Omega_4 \cap K$ and $\lambda_0^n \geq 0$ such that $u_0 - Tu_0 = \lambda_0^n Bu_0$. Since $\lambda_1 = 0$ implies a fixed point in $\partial \Omega_4 \cap K$ and this would complete the proof. So assume $\lambda_0^n > 0$. Let $\tau_2 = \min\{u_0(t) \mid \theta_1 \leq t \leq 1 - \theta_1\}$, then if $t \in [\theta_1, 1 - \theta_1]$, we have

$$u_0(t) = \int_0^1 G(t, s) p(s) f(u_0(s)) \, ds + \lambda_0^n Bu_0(t)$$

$$\geq \int_{\theta_1}^{1 - \theta_1} G(t, s) p(s) f(u_0(s)) \, ds + \lambda_0^n u_0(t)$$

$$\geq (m_1 + \varepsilon_3) \tau_2 \int_{\theta_1}^{1 - \theta_1} G(t, s) p(s) \, ds + \lambda_0^n u_0(t).$$
\[ \begin{align*}
\geq t_2 + \varepsilon_3 t_2 \int_{\theta_1}^{1-\theta_1} G(t,s) p(s) ds + \lambda_0^* u_0(t)
\end{align*} \]

therefore,

\[ u_0(t) > t_2, \quad t \in [\theta_1, 1 - \theta_1], \]

which is a contradiction. So (2.33) holds. According to Lemma 1.2, we get

\[ i(T, \Omega_4 \cap K, K) = 0. \tag{2.34} \]

From \( \Omega_4 \subset \Omega_3 \) and (2.32), (2.34), we have

\[ i(T, (\Omega_3 \setminus \Omega_4) \cap K, K) = i(T, \Omega_3 \cap K, K) - i(T, \Omega_4 \cap K, K) = 1 - 0 = 1. \]

Consequently, by [9, Theorem 2.3.2], \( T \) has a fixed point \( u^* \) in \( (\Omega_3 \setminus \Omega_4) \cap K \), satisfying \( R_4 \leq \|u^*\| \leq R_3 \), and \( u^* \) is also a \( C^1[0, 1] \) positive solution.

This completes the proof. \( \square \)

**Corollary 2.** Let \( p \) be as above, \( 0 < \int_0^1 G(s,s) p(s) ds < \infty \) and \( 0 < \lambda < 1 \). Then a necessary and sufficient condition for (1.0) to have a positive solution in \( C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1) \) is that

\[ 0 < \int_0^1 (G(s,s))^\lambda p(s) ds < \infty. \]

We list the following assuming

\((H-4')\) \( \lim_{u \to -\infty} \frac{f(u)}{u} = 0, \lim_{u \to 0^+} \frac{f(u)}{u} = +\infty. \)
\((H-5)\) \( f : [0, \infty) \to [0, \infty) \) is continuous and \( f(u) > 0 \) on \( (0, \infty) \), \( f \) is nondecreasing in \( u. \)
\((H-6)\) \( p : (0, 1) \to [0, \infty) \) is continuous and there exists \( t_0 \in (0, 1) \): \( p(t_0) > 0. \)

We get the following results from Theorem 2.4.

**Theorem 2.5.** Assume \((H-1'), (H_2), \) and \((H-4')\) hold. Then a necessary and sufficient condition for (1.0) to have a positive solution in \( C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1) \) is that

\[ 0 < \int_0^1 f \left( G(s,s) \right) p(s) ds < \infty. \]

**Theorem 2.6.** Assume \((H-5), (H-6), (H-4')\) hold. Then a sufficient condition for BVPs (1.1), (1.2) to have a positive solution in \( C[0, 1] \cap C^2(0, 1) \) is that

\[ \int_0^1 G(s,s) p(s) ds < \infty. \]
Remark 2.6. Theorem 2.5 contains the results in [3] and Theorem 2(4) in [10].

Remark 2.7. In this paper function \( f(u) \) and the boundary conditions are more general than that in [1–3,10] where \( f(u) \) only satisfies \( \lim_{u \to 0^+} \frac{f(u)}{u} = 0 \) (or \( \infty \)), \( \lim_{u \to +\infty} \frac{f(u)}{u} = \infty \) (or 0), and only the cases \( \beta = 0, \delta = 0 \) are considered. In addition, our method is different from those in [1–3,10].

Example 2.1. The singular boundary value problem
\[
\begin{cases}
u'' + t^{-1/2}(1-t)^{-1/3}u^\lambda = 0, & 0 < t < 1, \quad \lambda > 1, \\ au(0) - \beta u'(0) = 0, & \gamma u(1) + \delta u'(1) = 0, \quad \beta, \delta > 0
\end{cases}
\]
has a solution \( u^* \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1) \) with \( u^*(t) > 0 \) on \((0, 1)\). To see this, we will apply Theorem 2.1 with \( p(t) = t^{-1/2}(1-t)^{-1/3}, \ f(u) = u^\lambda \) (\( \lambda > 1 \)). Clearly (H1) holds. Note that
\[
\int_0^1 p(s)G(s,s)ds = \int_0^{1/2} \frac{1}{\rho} \frac{(\gamma + \delta - \gamma s)(\beta + \alpha s)}{(1-s)^{1/3}s^{1/2}} ds \\
= \int_0^{1/2} \frac{1}{\rho} \frac{(\gamma(1-s) + \delta)(\beta + \alpha s)}{s^{1/2}} ds \\
+ \int_{1/2}^1 \frac{1}{\rho} \frac{(\gamma(1-s) + \delta)(\beta + \alpha s)}{s^{1/2}} ds < +\infty.
\]
Consequently (H2) holds (with \( \theta_1 = 1/4 \)).

Also note that (H3) holds since \( \lim_{u \to 0^+} \frac{f(u)}{u} = 0 \) and \( \lim_{u \to +\infty} \frac{f(u)}{u} = +\infty \). Finally note that \( \int_0^1 p(s)f(G(s,s))ds = \int_0^1 p(s)(G(s,s))^\lambda ds < +\infty \). The result now follows from Theorem 2.1.

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