Abstract

We use the results of our paper *p-Fractals and power series—I* [P. Monsky, P. Teixeira, *p-Fractals and power series—I*. Some 2 variable results, J. Algebra 280 (2004) 505–536] to prove the ratio-
nality of the Hilbert–Kunz series of a large family of power series, including those of the form
\[ \sum_i f_i(x_i, y_i), \]
where the \( f_i(x_i, y_i) \) are power series with coefficients in a finite field. The methods are
effective, as we illustrate with examples. In the final section, which can be read independently
of the others, we obtain more precise results for the Hilbert–Kunz function of the 3 variable power
series \( z^D - h(x, y) \).

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1. Introduction

Throughout this paper \( p \) denotes a prime number and \( k \) a field of characteristic \( p \), and
the letter \( q \) is reserved for powers of \( p \). For ease of notation we often denote the list of
indeterminates \( x_1, \ldots, x_s \) by \( x \), and their \( q \)th powers \( x_1^q, \ldots, x_s^q \) by \( x^q \).

The *Hilbert–Kunz function* of \( f \in k[x] \) is the function
\[ n \mapsto e_n(f) = \deg(x^{p^n}, f) = \dim_k k[x]/(x^{p^n}, f). \]
This is the Hilbert–Kunz function of the local ring \( k[x]/(f) \) with respect to its maximal ideal. The Hilbert–Kunz series of \( f \) is the associated power series \( HKS(f) = \sum_{n=0}^{\infty} e_n(f) z^n \), and the Hilbert–Kunz multiplicity of \( f \) is the limit of \( e_n(f)/p^{(s-1)n} \) as \( n \to \infty \).

In many cases (see, for example, [2,3,6,9]) the Hilbert–Kunz series is a quotient of polynomials with integer coefficients. (Computer calculations strongly suggest that this is false in general—certain simple 5 variable \( f \) are likely counterexamples.) In this article we use the ideas and results of [8] to extend these rationality results to a large family of power series, including those of the form \( \sum_{i=1}^{n} f_i(x_i, y_i) \), where the \( f_i(x_i, y_i) \) are power series with coefficients in a finite field. (When \( n = 1 \) this is proved rather simply in [5] without the finiteness restriction on \( k \).)

Let \( \mathcal{F} \) be the set of rational numbers in \([0, 1]\) whose denominators are powers of \( p \). We are concerned with the following family of functions \( \mathcal{F} \to \mathbb{Q} \):

**Definition 1.1.** Suppose \( f \in k[x_1, \ldots, x_s] \). Then \( \varphi_f : \mathcal{F} \to \mathbb{Q} \) is the function \( a/q \mapsto q^{-s} \deg(x^d, f^n) \).

In the notation introduced in [8], \( \varphi_f \) is the function \( \varphi_1 \) with \( I = (x_1, \ldots, x_s) \), \( r = 1 \) and \( h_1 = f \). In that paper we introduced the concept of \( p \)-fractals: a function \( \varphi : \mathcal{F} \to \mathbb{Q} \) is a \( p \)-fractal if the \( \mathbb{Q} \)-subspace of \( \mathbb{Q}^\mathcal{F} \) spanned by the functions \( T_{q b} \varphi : t \mapsto \varphi((t + b)/q) \) (\( q \) a power of \( p \); \( b \in \mathbb{Z} \) with \( 0 \leq b < q \)) is finite dimensional. Equivalently, \( \varphi \) is a \( p \)-fractal if it is contained in a finite dimensional \( \mathbb{Q} \)-subspace of \( \mathbb{Q}^\mathcal{F} \) which is stable under the operators \( T_{p b} \) \((0 \leq b < p) \). We call such a subspace \( p \)-stable. Theorem 1 of [8] shows that all the functions \( \varphi_I \) (so in particular the \( \varphi_f \)) are \( p \)-fractals when \( s = 2 \) and \( k \) is finite.

The following simple proposition establishes the connection between \( p \)-fractals and the rationality of the Hilbert–Kunz series:

**Proposition 1.2.** Let \( f \in k[x_1, \ldots, x_s] \), and suppose the function \( \varphi_f \) is a \( p \)-fractal. Then the Hilbert–Kunz series of \( f \) is rational.

**Proof.** Let \( \varphi = \varphi_f \) and \( S = T_{p 0} \). Since \( \varphi \) is a \( p \)-fractal, the \( \mathbb{Q} \)-subspace \( V \) of \( \mathbb{Q}^\mathcal{F} \) spanned by \( S^n(\varphi) \), \( n = 0, 1, 2, \ldots \), is finite dimensional, and \( S \) maps \( V \to V \). So the restriction of \( S \) to \( V \) satisfies a polynomial identity of the form \( S^I = c_1 S^{l-1} + c_2 S^{l-2} + \cdots + c_I I \), where \( I \) is the identity map and \( c_i \in \mathbb{Q} \). This equation can be applied to any \( S^n(\varphi) \), and shows that the sequence \( (S^n(\varphi)) \) is linearly recurrent. Evaluating at \( 1 \), we get a linear recursion for \( S^n(\varphi)(1) = \varphi(1/p^n) = p^{-sn} e_n(f) \), and the proposition follows easily. \( \square \)

In the light of Proposition 1.2, we define:

**Definition 1.3.** A power series \( f \in k[x] \) is strongly rational if \( \varphi_f \) is a \( p \)-fractal.

We shall prove that strong rationality is preserved under certain operations: if \( f \in k[x_1, \ldots, x_s] \) and \( g \in k[y_1, \ldots, y_r] \) are strongly rational, then so are the product \( fg \), the sum \( f + g \) and powers of these power series. Using [8, Theorem 1] as a starting point, these results show, for example, the rationality of the Hilbert–Kunz series of power series
of the form $\sum_i f_i(x_i, y_i)$ and $\sum_i \prod_j f_{ij}(x_{ij}, y_{ij})$, where $f_i(x_i, y_i)$ and $f_{ij}(x_{ij}, y_{ij})$ are power series with coefficients in a finite field.

While the proofs that strong rationality is preserved by powers and products, in the sense described above, are relatively straightforward, the proof of the analogous result for sums will require some work—in particular, it will be necessary to use the representation ring introduced by Han and the first author in [3]. Section 2 summarizes a few definitions and results of [3], and introduces an endomorphism $\theta$ of the representation ring. In Section 3 we attach to each function $\varphi \colon \mathcal{F} \to \mathbb{Q}$ a certain sequence of elements of the representation ring, and find conditions on that sequence which are necessary and sufficient for $\varphi$ to be a $p$-fractal. In Section 4 we use these conditions to show that if $f(x)$ and $g(y)$ are strongly rational, then the same is true for $f(x) + g(y)$. We prove analogous results for products and powers; as we have noted, this together with [8, Theorem 1] gives our rationality results.

In Section 5 we show how our techniques can be used to effectively calculate the Hilbert–Kunz series of various power series. We conclude with an analysis of the Hilbert–Kunz function of $z^D - h(x, y)$, in Section 6.

2. The representation ring and the endomorphism $\theta$

We summarize here some of the definitions and results of [3], for the reader’s convenience. A $\mathbb{k}$-object is a finitely generated $\mathbb{k}[T]$-module on which $T$ acts nilpotently. $\Gamma$ is the Grothendieck group of the semigroup of isomorphism classes of $\mathbb{k}$-objects under the usual direct sum. We introduce a product on $\Gamma$ as follows: if two elements of $\Gamma$ are represented by the $\mathbb{k}$-objects $M$ and $N$, then their product is the image in $\Gamma$ of the $\mathbb{k}$-object $M \otimes \mathbb{k} N$, where $T$ acts distributively; namely $T(m \otimes n) = (Tm) \otimes n + m \otimes (Tn)$. Now $\Gamma$ endowed with this product is a commutative ring, called the representation ring. The zero and unity of $\Gamma$ are respectively the images of the zero module and $\mathbb{k}[T]/(T)$ in $\Gamma$.

For any nonnegative integer $n$, $\delta_n$ is the image of $M_n = \mathbb{k}[T]/(T^n)$ in $\Gamma$ (so in particular $\delta_0 = 0$ and $\delta_1 = 1$). The theory of modules over principal ideal domains shows that $(\Gamma, +)$ is a free abelian group with basis $\{\delta_1, \delta_2, \ldots\}$. In what follows we shall mostly use a second basis $\{\lambda_0, \lambda_1, \ldots\}$, where $\lambda_n = (-1)^n (\delta_n - 1) - \delta_n$.

The $\lambda_i$-coordinate of the image in $\Gamma$ of a $\mathbb{k}$-object $M$ is $(-1)^i \dim \mathbb{k} T^i M / T^{i+1} M$, or $(-1)^i (\dim \mathbb{k} M / T^{i+1} M - \dim \mathbb{k} M / T^i M)$. This is immediately seen for $M_n = \mathbb{k}[T]/(T^n)$, whose image in $\Gamma$ is $\delta_n = \sum_{i < n} (-1)^i \lambda_i$, and evidently the result extends additively to arbitrary $\mathbb{k}$-objects.

**Definition 2.1.** Let $f \in \mathbb{k}[x]$. Then $\langle f \rangle_n$ is the image in $\Gamma$ of the $\mathbb{k}$-object $\mathbb{k}[x]/(x^n)$, where $T$ operates by multiplication by $f$.

In terms of the $\lambda_i$,

$$\langle f \rangle_n = \sum_{i=0}^{q-1} (\deg(x^q, f^{i+1}) - \deg(x^q, f^i))(-1)^i \lambda_i,$$

where $q = p^n$. 

If \( g \) is another power series in a different set of variables, say \( g \in k[J_y] \), then the product \( \langle f \rangle_n \langle g \rangle_n \) is represented by the \( k \)-object \( k[x, y]/(x^q, y^q) \), where \( T \) operates by multiplication by \( f + g \). So \( \langle f \rangle_n \langle g \rangle_n = \langle f + g \rangle_n \).

**Definition 2.2.** \( \alpha : \Gamma \rightarrow \mathbb{Z} \) is the \( \mathbb{Z} \)-linear map \( \sum_{i=0}^{n} c_i \lambda_i \mapsto c_0 \).

Note that \( \alpha(\langle f \rangle_n) = \deg(x^q, f) = e_n(f) \), for all \( f \in k[x] \). Theorem 1.10 of [3] shows that \( \alpha(\lambda_i \lambda_j) = \delta_{i,j} \) (= 1 if \( i = j \); 0 otherwise).

We shall use the following multiplication formulas, from Lemma 3.3 and Theorems 2.5, 3.4 and 3.10 of [3]:

\[
\delta_i \delta_{jq} = i \cdot \delta_{jq} \quad (0 \leq i \leq q, \quad j \geq 1),
\]
\[
\lambda_i \lambda_{jq} = \lambda_{jq+i} \quad (0 \leq i < q),
\]
\[
\lambda_i \lambda_{jq-1} = \lambda_{jq-1-i} \quad (0 \leq i < q),
\]
\[
\lambda_q \lambda_{jq} = \lambda_{q(q-1)} + \lambda_{q(q+1)-1} + \lambda_{q(q+1)}, \quad \text{if } p \nmid j \text{ and } p \nmid j + 1,
\]
\[
\lambda_i \lambda_j = \sum_{k=j-i}^{\min(i+j, 2p-2i-j)} \lambda_k, \quad \text{if } i \leq j < p.
\]

In the remainder of this section we introduce a linear map \( \theta : \Gamma \rightarrow \Gamma \), and show that \( \theta \) is actually a ring homomorphism—a result essential to the next section. The existence of this endomorphism is an important result in its own right, which helps in understanding the structure of the representation ring.

**Definition 2.3.** \( \theta \) is the linear operator on \( \Gamma \) whose values on the basis \( \{\lambda_i\} \) are defined as follows:

\[
\theta(\lambda_i) = \begin{cases} 
\lambda_{pi}, & \text{if } i \text{ is even}, \\
\lambda_{pi+p-1}, & \text{if } i \text{ is odd}.
\end{cases}
\]

In what follows, let \( n \) be a fixed positive integer and \( q = p^n \).

**Lemma 2.4.** \( \theta^n(\Gamma) \) is a subring of \( \Gamma \).

**Proof.** \( \theta^n(\Gamma) \) is the \( E_q \) in Definition 3.12 of [3]. So the lemma follows immediately from Corollary 3.16 of that same paper. \( \square \)

**Definition 2.5.** \( \Gamma_n \) is the additive subgroup of \( \Gamma \) generated by \( \lambda_i \) with \( i < q = p^n \).

**Remark 2.6.** Theorem 3.2 of [3] shows that \( \Gamma_n \) is a subring of \( \Gamma \).

**Lemma 2.7.** \( \theta^n(\Gamma) \Gamma_n = \Gamma \).
Proof. It suffices to show that \( \lambda_i \in \theta^n(\Gamma)\Gamma_n \), for all \( i \). Write \( i = aq + b \), with \( 0 \leq b < q \). If \( a \) is even, Eq. (3) shows that \( \lambda_i = \lambda_{aq} \lambda_b = \theta^n(\lambda_a)\lambda_b \in \theta^n(\Gamma)\Gamma_n \), while if \( a \) is odd, Eq. (4) gives \( \lambda_i = \lambda_{q(a+1)-1} \lambda_{q-1-b} = \theta^n(\lambda_a)\lambda_{q-1-b} \in \theta^n(\Gamma)\Gamma_n \). \( \square \)

**Lemma 2.8.** If \( u \in \theta^n(\Gamma) \) and \( v \in \Gamma_n \), then \( \theta(uv) = \theta(u)\theta(v) \).

Proof. It is enough to verify that \( \theta(\lambda_{2qi}\lambda_j) = \theta(\lambda_{2qi})\theta(\lambda_j) \) and \( \theta(\lambda_{2qi-1}\lambda_j) = \theta(\lambda_{2qi-1})\theta(\lambda_j) \), for any \( i \) and \( j \) with \( j < q \). Those follow from simple calculations using Eqs. (3) and (4). \( \square \)

**Lemma 2.9.** Suppose \( 1 \leq j \leq p - 2 \). Then

- \( \lambda_{2q-1}\lambda_{q(j+1)-1} = \lambda_{q(j-1)} + \lambda_{q(j+1)-1} + \lambda_{q(j+1)} \);
- \( \lambda_{2q-1}\lambda_{qj} = \lambda_{qj-1} + \lambda_{qj} + \lambda_{q(j+2)-1} \).

Proof. Equations (3) and (4) show that \( \lambda_{2q-1}\lambda_{q(j+1)-1} = (\lambda_{q-1}\lambda_q)(\lambda_{q-1}\lambda_{qj}) = \lambda_q\lambda_{qj} \), so the first identity follows from Eq. (5). Also \( \lambda_{2q-1}\lambda_{qj} = \lambda_{q-1}\lambda_{q\lambda_{q(j-1)} + \lambda_{q-1}\lambda_{qj} + \lambda_{q(j+1)}} \), and Eqs. (3) and (4) give the second identity. \( \square \)

In what follows, let \( \mu_i = \theta^n(\lambda_i) \).

**Lemma 2.10.**

- If \( 1 \leq j \leq p - 2 \), \( \mu_1\mu_j = \mu_{j-1} + \mu_j + \mu_{j+1} \);
- \( \mu_1\mu_{p-1} = \mu_{p-2} \).

Proof. The first identity is simply a reformulation of Lemma 2.9. When \( p = 2 \), the second identity becomes \( (\lambda_{2q-1})^2 = \lambda_0 \), which is a special case of Eq. (4). On the other hand, when \( p \neq 2 \), \( \lambda_{2q-1}\lambda_{q(p-1)} = \lambda_q\lambda_{q-1}\lambda_{q(p-1)} = \lambda_q\lambda_{qp-1} = \lambda_q(p-1) = \lambda_{q(p-1)-1} \); this is precisely the second identity. \( \square \)

**Lemma 2.11.** If \( i \leq j < p \), then

\[
\mu_i\mu_j = \sum_{k=j-i}^{\min(i+j,2p-2i-j)} \mu_k.
\]

Proof. It suffices to prove the identity when \( i + j < p \) and to show that \( \mu_i\mu_{p-1} = \mu_{p-1-i} \), for \( i < p \). Both of these results are proved by induction on \( i \)—Lemma 2.10 gives the result for \( i = 1 \), and the inductive step relies on the identity \( \mu_i = \mu_1\mu_{i-1} - \mu_{i-1} - \mu_{i-2} \). The calculations are identical to those of Lemma 2.4 and Theorem 2.5 of [3]. \( \square \)

Comparing the above lemma and Eq. (6) we conclude the following:
Lemma 2.12. Suppose $i, j < p$. Then $\theta^n(\lambda_i \lambda_j) = \theta^n(\lambda_i) \theta^n(\lambda_j)$.

Theorem 2.13. $\theta$ is a ring homomorphism.

Proof. We will show by induction on $n$ that $\theta(uv) = \theta(u) \theta(v)$ for all $u$ and $v$ in $\Gamma_n$. Lemma 2.12 gives the result for $n = 1$, so now suppose the assertion holds for $n \geq 1$, and let $u, v \in \Gamma_{n+1}$. We may assume that $u = \lambda_a$ and $v = \lambda_b$, with $a, b < p^{n+1}$. Then as in the proof of Lemma 2.7 we may write $\lambda_a = \theta^n(\lambda_i) \lambda_j$ with $i < p$ and $j < p^n$. Similarly, we write $\lambda_b = \theta^n(\lambda_k) \lambda_l$ with $k < p$ and $l < p^n$. Then $\theta(uv) = \theta(\theta^n(\lambda_i) \theta^n(\lambda_k) \lambda_j \lambda_l)$. Since both $\theta^n(\Gamma)$ and $\Gamma_n$ are closed under multiplication, Lemma 2.8 shows that this is equal to $\theta(\theta^n(\lambda_i) \theta^n(\lambda_k)) \cdot \theta(\lambda_j \lambda_l)$. From Lemma 2.12 and the induction assumption it follows that $\theta(uv) = \theta(\theta^n(\lambda_i)) \cdot \theta(\lambda_j) \cdot \theta(\theta^n(\lambda_k)) \cdot \theta(\lambda_l)$, and one more application of Lemma 2.8 concludes the proof. \qed

3. Coherent sequences

In this section we associate to each function $\varphi : \mathcal{J} \to \mathbb{Q}$ a sequence of elements of the representation ring, and find necessary and sufficient conditions on that sequence in order for $\varphi$ to be a $p$-fractal. For our purposes we will need to allow rational coefficients, working with $\Gamma_\mathbb{Q} = \Gamma \otimes \mathbb{Z} \mathbb{Q}$ rather than $\Gamma$. The linear map $\alpha \otimes \mathbb{Z} 1 : \Gamma_\mathbb{Q} \to \mathbb{Q}$ will also be denoted by $\alpha$, by abuse of notation.

Definition 3.1. $\Lambda$ is the ring $(\Gamma_\mathbb{Q})^\mathbb{N}$ of sequences $u = (u_0, u_1, u_2, \ldots)$ with entries in $\Gamma_\mathbb{Q}$.

Definition 3.2. Given any function $\varphi : \mathcal{J} \to \mathbb{Q}$, $\mathcal{L}(\varphi)$ is the element of $\Lambda$ whose $n$th entry is

$$\mathcal{L}_n(\varphi) = \sum_{i=0}^{q-1} \left( \varphi \left( \frac{i+1}{q} \right) - \varphi \left( \frac{i}{q} \right) \right) (-1)^i \lambda_i,$$

where $q = p^n$. $\mathcal{L} : \mathcal{J}^\mathbb{Q} \to \Lambda$ is the map $\varphi \mapsto \mathcal{L}(\varphi)$.

In particular, if $f \in \mathbb{K}[x_1, \ldots, x_s]$ and $\varphi = \varphi_f$ (see Definition 1.1) then comparing the above definition to Eq. (1) we see that $\mathcal{L}_n(\varphi_f) = p^{-sn}(f)_n$. Consequently, $\alpha(\mathcal{L}_n(\varphi_f)) = p^{-sn} e_n(f)$. As noted after Eq. (1), if $g$ is a power series in the indeterminates $y_1, \ldots, y_s'$, then $(f + g)_n = (f)_n(g)_n$, and it follows at once that $\mathcal{L}(\varphi_{f+g}) = \mathcal{L}(\varphi_f) \mathcal{L}(\varphi_g)$.

Our first step towards finding the alternative characterization of $p$-fractals will be to describe the image of the map $\mathcal{L}$. Theorem 4.1 of [3] introduces an endomorphism $\psi_p$ of $\Gamma$ with the following property:

$$\psi_p(\delta_{pr+k}) = (p-k)\delta_r + k\delta_{r+1},$$

for all $r$ and $k$ with $0 \leq k \leq p$. It follows that
for all \( r \) and \( k \) with \( 0 \leq k < p \). On the \( \mathbb{k}[T] \)-module level, \( \psi_p \) of a \( \mathbb{k} \)-object has the same underlying module, but the new action of \( T \) is the \( p \)th power of the old action.

**Definition 3.3.** \( \psi \) is the endomorphism \( \psi_p \otimes \mathbb{Z} 1 \) of \( \Gamma_\mathbb{Q} \).

One can easily see that

\[
\alpha(\delta_i u) = \alpha(\delta_i \psi(u)),
\]

for all \( i \in \mathbb{N} \) and \( u \in \Gamma_\mathbb{Q} \). It suffices to verify the formula for \( u = \lambda_{pr+k} \), with \( 0 \leq k < p \).

Then, since \( \delta_i = \sum_{j<i} (-1)^j \lambda_j \) and \( \alpha(\lambda_i \lambda_j) = \delta_{i,j} \), both sides of (8) are zero if \( i \leq r \), and \((-1)^{pr+k} \) otherwise.

**Definition 3.4.** A sequence \( u = (u_0, u_1, u_2, \ldots) \) of \( \Lambda \) is coherent if the following properties hold:

1. Each \( u_n \) is a linear combination of \( \lambda_i \) with \( i < p^n \) (i.e., \( u_n \in \Gamma_n \otimes \mathbb{Z} \mathbb{Q} \));
2. \( \psi(u_{n+1}) = u_n \), for all \( n \).

\( \Lambda_0 \) is the \( \mathbb{Q} \)-subspace of \( \Lambda \) consisting of the coherent sequences.

Remark 2.6 and the fact that \( \psi \) is a ring homomorphism show that \( \Lambda_0 \) is actually a \( \mathbb{Q} \)-subalgebra of \( \Lambda \). A calculation using Eq. (7) shows that \( \mathcal{L}(\varphi) \) is coherent, for any \( \varphi \in \mathbb{Q}^\delta \). (For \( \varphi = \varphi_f \) this is more conceptually seen using the description of \( \psi_p \) on the \( \mathbb{k}[T] \)-module level.) In fact, \( \Lambda_0 \) consists precisely of the \( \mathcal{L}(\varphi) \), as the following lemma shows.

**Lemma 3.5.** \( \mathcal{L} : \mathbb{Q}^\delta \rightarrow \Lambda \) maps \( \mathbb{Q}^\delta \) onto \( \Lambda_0 \), and \( \ker \mathcal{L} \) is the 1-dimensional subspace of constant functions.

**Proof.** Suppose \( u = (u_0, u_1, u_2, \ldots) \) is a coherent sequence. Let \( \varphi : \mathcal{F} \rightarrow \mathbb{Q} \) be the map \( i/p^n \mapsto \alpha(\delta_i u_n) \). Note that Eq. (8) and the fact that \( u \) is coherent imply that \( \varphi \) is well defined. If we write \( u_n = \sum_j (-1)^j a_j \lambda_j \), then \( \alpha(\delta_i u_n) = a_0 + \cdots + a_{i-1} \). So

\[
\varphi\left(\frac{i+1}{p^n}\right) - \varphi\left(\frac{i}{p^n}\right) = \alpha(\delta_{i+1} u_n) - \alpha(\delta_i u_n) = a_i,
\]

and it follows that \( \mathcal{L}(\varphi) = u \), showing that \( \mathcal{L} \) maps \( \mathbb{Q}^\delta \) onto \( \Lambda_0 \).

To conclude the proof, note that \( \mathcal{L} \) maps a function \( \varphi \) to 0 if and only if \( \varphi((i+1)/p^n) - \varphi(i/p^n) = 0 \) for all \( n \) and \( i < p^n \), so the kernel of \( \mathcal{L} \) consists of the constant functions.  

We give \( \Lambda \) a structure of \( \Gamma \)-module, introducing a product \( \Gamma \times \Lambda \rightarrow \Lambda \) as follows:
Definition 3.6. Suppose $w \in \Gamma$ and $u = (u_0, u_1, u_2, \ldots) \in \Lambda$. Then

$$w \cdot u = (wu_0, \theta(w)u_1, \theta^2(w)u_2, \ldots).$$

Definition 3.7. $R : \Lambda \to \Lambda$ is the $\Gamma$-linear map taking $(u_0, u_1, u_2, \ldots)$ to $(v_0, v_1, v_2, \ldots)$, where $v_n = (-1)^p n^{\lambda p - 1} u_n$.

One sees directly from the above definition and Eq. (4) that if $\phi$ is a function $\mathcal{H} \to \mathbb{Q}$ and $\bar{\phi}$ is its “reflection” $t \mapsto \phi(1 - t)$, then

$$\mathcal{L}(\bar{\phi}) = R(\mathcal{L}(\phi)). \quad (9)$$

In particular, Lemma 3.5 then shows that $R$ stabilizes $A_0$.

Definition 3.8. $S : \Lambda \to \Lambda$ is the map $(u_0, u_1, u_2, \ldots) \mapsto (u_1, u_2, u_3, \ldots)$.

We shall refer to $S$ as the shift operator. Quick calculations using Eq. (3) show the following identities relating $R$ and $S$:

$$S(R(u)) = \lambda_1 S(u) \quad \text{if } p = 2, \quad S(R(u)) = \lambda_{p-1} R(S(u)) \quad \text{if } p > 2. \quad (10)$$

We now describe the action of the shift operator on $\mathcal{L}(\phi)$:

Lemma 3.9. Suppose $\phi$ is a function $\mathcal{H} \to \mathbb{Q}$. Then the shift operator acts on $\mathcal{L}(\phi)$ as follows:

$$S(\mathcal{L}(\phi)) = \sum_{k \text{ even}} \lambda_k \mathcal{L}(T_{p|k}\phi) + \sum_{k \text{ odd}} \lambda_k \mathcal{L}(\overline{T_{p|k}\phi}),$$

where $\overline{T_{p|k}\phi}$ denotes the “reflection” of $T_{p|k}\phi$, namely the map $t \mapsto T_{p|k}\phi(1 - t)$.

Proof. Fix a nonnegative integer $n$ and let $q = p^n$. Then

$$\mathcal{L}_{n+1}(\phi) = \sum_{i=0}^{pq-1} \left( \phi\left( \frac{i+1}{pq} \right) - \phi\left( \frac{i}{pq} \right) \right) (-1)^i \lambda_i,$$

and this sum can be split up into $p$ sums

$$s_k = \sum_{i=kq}^{kq+q-1} \left( \phi\left( \frac{i+1}{pq} \right) - \phi\left( \frac{i}{pq} \right) \right) (-1)^i \lambda_i \quad (0 \leq k < p).$$

Suppose first that $k$ is even. Shifting indices and using Eq. (3) we obtain
\[ s_k = \sum_{i=0}^{q-1} \left( \varphi \left( \frac{i+kq+1}{pq} \right) - \varphi \left( \frac{i+kq}{pq} \right) \right) (-1)^i \lambda_i \]

\[ = \lambda_{kq} \sum_{i=0}^{q-1} \left( \varphi \left( \frac{i+kq+1}{pq} \right) - \varphi \left( \frac{i+kq}{pq} \right) \right) (-1)^i \lambda_i, \]

which can be rewritten as

\[ s_k = \lambda_{kq} \sum_{i=0}^{q-1} \left( T_p | k \varphi \left( \frac{i+1}{q} \right) \right) - T_p | k \varphi \left( \frac{i}{q} \right) \right) (-1)^i \lambda_i = \lambda_{kq} \mathcal{L}_n \left( T_p | k \varphi \right). \]

Now suppose \( k \) is odd. Then manipulating indices and using Eq. (4) we get

\[ s_k = \sum_{i=0}^{q-1} \left( \varphi \left( \frac{kq+i-q}{pq} \right) - \varphi \left( \frac{kq+i}{pq} \right) \right) (-1)^i \lambda_{kq+i-q-1} \]

\[ = \lambda_{kq+i-q-1} \sum_{i=0}^{q-1} \left( T_p | k \varphi \left( 1 - \frac{i+1}{q} \right) \right) - T_p | k \varphi \left( 1 - \frac{i}{q} \right) \right) (-1)^i \lambda_i \]

\[ = \lambda_{kq+i-q-1} \mathcal{L}_n \left( \overline{T_p | k \varphi} \right). \]

In conclusion,

\[ \mathcal{L}_{n+1}(\varphi) = \sum_{k \text{ even}}^{p-1} \theta^n(\lambda_k) \mathcal{L}_n \left( T_p | k \varphi \right) + \sum_{k \text{ odd}}^{p-1} \theta^n(\lambda_k) \mathcal{L}_n \left( \overline{T_p | k \varphi} \right), \]

and the lemma follows.  \( \square \)

**Remark 3.10.** The expression for \( S(\mathcal{L}(\varphi)) \) given in Lemma 3.9 is unique, in the following sense: if \( S(\mathcal{L}(\varphi)) = \sum_{k=0}^{p-1} \lambda_k \mathcal{L}^{(k)} \) with \( \mathcal{L}^{(k)} \in \Lambda_0 \), then \( \mathcal{L}^{(k)} \) is equal to \( \mathcal{L}(T_p | k \varphi) \) or \( \mathcal{L}(\overline{T_p | k \varphi}) \), according as \( k \) is even or odd. In fact, the \( n \)th entry \( u_n \) of an element \( u \in \Lambda_0 \) is a linear combination of the \( \lambda_i \) with \( 0 \leq i < p^n \), and Eqs. (3) and (4) show that \( \theta^n(\lambda_k) u_n \) is a linear combination of the \( \lambda_i \) with \( kp^n \leq i < (k+1)p^n \). It follows that \( \sum_{k=0}^{p-1} \lambda_k \Lambda_0 \) is a direct sum.

If \( \varphi \) is a \( p \)-fractal, then one can easily see that there exists a finite dimensional subspace of \( \mathbb{Q}^g \) containing \( \varphi \) and stable under the operators \( T_q | b \) and reflections \( \psi \mapsto \overline{\psi} \). For \( \overline{\varphi} \) is also a \( p \)-fractal, by [8, Lemma 4.2], and therefore the subspace \( V \) of \( \mathbb{Q}^g \) spanned by \( \varphi, \overline{\varphi}, \) and all their transforms under the operators \( T_q | b \) is finite dimensional, and evidently stable under the \( T_q | b \). A simple calculation done in the proof of [8, Lemma 4.2] shows that \( T_q | b \overline{\psi} = T_q | q-1-b \overline{\psi} \), for any \( \psi \), so \( V \) is also stable under reflection. If \( M = \mathcal{L}(V) \), then \( M \) is a finite dimensional subspace of \( \Lambda_0 \) containing \( \mathcal{L}(\varphi) \), and Lemma 3.9 shows that
$S(M) \subseteq \sum_{k=0}^{p-1} \lambda_k M$. This is essentially the characterization of $p$-fractals that we were after.

**Definition 3.11.** A coherent sequence $u$ is regular if there exists a finite dimensional subspace $M$ of $A_0$ containing $u$ with $S(M) \subseteq \sum_{k=0}^{p-1} \lambda_k M$.

**Theorem 3.12.** $\varphi : \mathbb{I} \rightarrow \mathbb{Q}$ is a $p$-fractal if and only if $\mathcal{L}(\varphi)$ is regular.

**Proof.** The “only if” direction was shown before Definition 3.11. Now suppose $\mathcal{L}(\varphi)$ is regular, and let $M \ni \mathcal{L}(\varphi)$ be as in 3.11. By Eq. (10), $S(M + R(M))$ is contained in $\sum_{k=0}^{p-1} \lambda_k (M + R(M))$. So we may replace $M$ by $M + R(M)$; this allows us to assume that $V = \overline{V}$, where $V = \mathcal{L}^{-1}(M)$. Now $V$ contains $\varphi$, and Lemma 3.5 tells us that $V$ is finite dimensional. It remains to show that $V$ is $p$-stable. Suppose $\eta \in V$. Then $\mathcal{L}(\eta) \in M$, and $S(\mathcal{L}(\eta)) = \sum_{k=0}^{p-1} \lambda_k v^{(k)}$, with $v^{(k)} \in M$. Comparing with Lemma 3.9 and using Remark 3.10, we find that $T_{p|k}\eta (k \text{ even})$ and $\overline{T}_{p|k}\eta (k \text{ odd})$ are in $\mathcal{L}^{-1}(M) = V$. Since $V = \overline{V}$, all the $T_{p|k}\eta$ are in $V$, and $V$ is $p$-stable. $\square$

**Remark 3.13.** Note that if $S(M) \subseteq \sum_{k=0}^{p-1} \lambda_k M$ and $S(N) \subseteq \sum_{k=0}^{p-1} \lambda_k N$, then $S(M + N) \subseteq \sum_{k=0}^{p-1} \lambda_k (M + N)$ and $S(MN) \subseteq \sum_{k=0}^{p-1} \lambda_k MN$. Consequently, if $u$ and $v$ are regular, then so are $u + v$ and $uv$.

## 4. Operations preserving strong rationality

In this section we show that strong rationality is preserved under a number of operations. More precisely, if $f(x) \in k[x]$ and $g(y) \in k[y]$ are strongly rational, then so are powers of these series, the product $f(x)g(y)$ and the sum $f(x) + g(y)$.

**Proposition 4.1.** If $f$ is strongly rational, then so is any power $f^m$ of $f$.

**Proof.** Let $V$ be a finite dimensional $p$-stable subspace of $\mathbb{Q}^d$ containing $\varphi_f$ and the constant function 1. Let $\ell(t) = mt$, and define $V^*$ as in [8, Lemma 3.8]. Namely, $V^*$ consists of all $\varphi : \mathbb{I} \rightarrow \mathbb{Q}$ whose restriction to each interval $[d/m, (d+1)/m]$, $d = 0, \ldots, m-1$, is $t \mapsto \psi(\ell(t) - d)$ for some $\psi \in V$. Then $V^*$ is a finite dimensional $p$-stable subspace of $\mathbb{Q}^d$, by [8, Lemma 3.8], and it contains $\varphi_f^m$, since $\varphi_f^m(t) = \varphi_f(\ell(t))$ on $[0, 1/m]$ and $\varphi_f(t) = 1$ on each $[d/m, (d+1)/m]$ with $d > 0$. $\square$

**Proposition 4.2.** Suppose $f \in k[x_1, \ldots, x_s]$ and $g \in k[y_1, \ldots, y_s]$. If $f$ and $g$ are strongly rational, then so is $fg$.

**Proof.** Let $\varphi$ be an arbitrary power of $p$, and $0 \leq a \leq q$. Let $I$ and $J$ be the colon ideals $(x^d : f^a)$ and $(y^d : g^a)$. Then $((x^d, y^d) : f^a) = (I, y^d)$, and $((x^d, y^d) : (fg)^a) = (I, J)$. Then
5.1. Example 1

\[ \text{deg}(x^q, y^q, (fg)^a) = q^{s+s'} - \text{deg}(I, J) \]
\[ = q^{s+s'} - \text{deg}(I) \text{deg}(J) \]
\[ = q^{s+s'} - (q^s - \text{deg}(x^q, f^a))(q^{s'} - \text{deg}(y^q, g^a)) \]
\[ = q^s \text{deg}(x^q, f^a) + q^{s'} \text{deg}(y^q, g^a) - \text{deg}(x^q, f^a) \text{deg}(y^q, g^a), \]

and dividing by \( q^{s+s'} \) we conclude that \( \varphi_{fg} = \varphi_f + \varphi_g - \varphi_f \varphi_g \). The result then follows from the fact that the \( p \)-fractals form a subalgebra of \( \mathbb{Q}_p^2 \). \( \square \)

**Proposition 4.3.** Suppose \( f \in \mathbb{k}[x_1, \ldots, x_s] \) and \( g \in \mathbb{k}[y_1, \ldots, y_{s'}] \). If \( f \) and \( g \) are strongly rational, then so is \( f + g \).

**Proof.** In view of Theorem 3.12, it suffices to show that \( \mathcal{L}(\varphi_{f+g}) \) is regular whenever \( \mathcal{L}(\varphi_f) \) and \( \mathcal{L}(\varphi_g) \) are regular. But \( \mathcal{L}(\varphi_{f+g}) = \mathcal{L}(\varphi_f)\mathcal{L}(\varphi_g) \), as noted after Definition 3.2, and the result follows from Remark 3.13. \( \square \)

We proved in [8] that \( \varphi_f \) is a \( p \)-fractal for any power series \( f \) in two variables with coefficients in a finite field. This, combined with the above results, shows the rationality of the Hilbert–Kunz series of a large family of power series. In particular, we have the following:

**Theorem 4.4.** Suppose \( f_i \) and \( f_{ij} \) are 2 variable power series over a finite field. Then 
\[ \sum_i f_i(x_i, y_i) \] and \[ \sum_i \prod_j f_{ij}(x_{ij}, y_{ij}) \] are strongly rational. In particular, the Hilbert–Kunz series of these power series are rational.

5. Examples

In preparation for the examples that will follow, we introduce some notation. \( \Delta \) is the element \( \mathcal{L}(\varphi_{\Delta}) = \mathcal{L}(t) \) of \( A_0 \). So \( \Delta = (\delta_1, p^{-1}\delta_p, p^{-2}\delta_{p^2}, \ldots) \). Since \( \mathcal{L}(1-t) = -\mathcal{L}(t) \), \( R(\Delta) = -\Delta \). If \( v = (v_0, v_1, v_2, \ldots) \in A_0 \), then \( v \cdot \Delta = v_0(\delta_0)\Delta \), and in particular \( \Delta \cdot \Delta = \Delta \). This can be seen by writing \( v \) as \( \mathcal{L}(\varphi) \), for some \( \varphi \), and observing that \((-1)^i\lambda_i\delta_q = \delta_q \) for any \( i < q \)—an easy consequence of Eq. (2).

5.1. Example 1

Let \( \mathbb{k} = \mathbb{Z}/(3) \), \( f = y^3 - x^4 + x^2y^2 \) and \( g = xy(x + y) \). We shall calculate the Hilbert–Kunz series of \( f(x_1, y_1) + f(x_2, y_2) \) and of \( f(x_1, y_1) + g(x_2, y_2) \). In what follows we will indicate that two functions \( \varphi \) and \( \psi \) differ by a linear function by writing \( \varphi \approx \psi \). Set \( \varphi = \varphi_f \). In the first example of [8, Section 6] we found that \( 9T_{3|0}\varphi = \bar{\varphi}, 9T_{3|1}\varphi \approx \varphi \) and \( 9T_{3|2}\varphi \) is a constant function. Set \( a = \mathcal{L}(\varphi) \) and \( b = \mathcal{L}(\bar{\varphi}) = R(a) \). Lemma 3.9 tells us that \( 9S(a) = 9(\lambda_0\mathcal{L}(T(3|0)\varphi) + \lambda_1\mathcal{L}(T(3|1)\bar{\varphi})) = (b + (\text{constant}) \cdot \Delta) + \lambda_1(b + (\text{constant}) \cdot \Delta) \). Now \( 9a \) is the sequence \((9\lambda_0, 8\lambda_0 - \lambda_1, \ldots) \) while \( b = R(a) = (-\lambda_0, \ldots) \). So \( 9S(a) - b - \lambda_1b = (9\lambda_0, \ldots) \). It follows that:
\[ 9S(a) = b + \lambda_1 b + 9\Delta. \] (11)

Applying \( R \) and using Eq. (10) of Section 3 we find:

\[ 9S(b) = \lambda_2 a + \lambda_1 a - 9\lambda_2 \Delta. \] (12)

Set \( \psi = \varphi_g \). The results of [8, Example 2] show that \( 9T_{3|0} \psi \approx \psi, 9T_{3|1} \psi \approx \bar{\psi} \) and \( 9T_{3|2} \psi \) is a constant function. So if we set \( c = L(\psi) \), and note that \( 9c = (9\lambda_0, 7\lambda_0 - 2\lambda_1, \ldots) \), an argument like that of the last paragraph gives:

\[ 9S(c) = c + \lambda_1 c + (6\lambda_0 - 3\lambda_1) \Delta. \] (13)

To proceed further we define a \( \mathbb{Q}[z] \)-valued bilinear function \( r \). For sequences \( u = (u_0, u_1, \ldots) \) and \( v = (v_0, v_1, \ldots) \) in \( \Lambda \), set

\[ r(u, v) = (1 - 27z) \cdot \sum_{n=0}^{\infty} \alpha(u_n v_n)(81z)^n. \]

The remarks made after Definition 3.2 tell us that the Hilbert–Kunz series of \( f(x_1, y_1) + f(x_2, y_2) \) is \( (1 - 27z)^{-1} \cdot r(a, a) \). Similarly, the Hilbert–Kunz series of \( f(x_1, y_1) + g(x_2, y_2) \) is \( (1 - 27z)^{-1} \cdot r(a, c) \). The bilinear function \( r \) has the following basic properties:

(A) \( r(u, v) = (1 - 27z) \cdot \alpha(u_0 v_0) + z \cdot r(9S(u), 9S(v)) \);

(B) \( r(R(u), R(v)) = r(u, v) \);

(C) If \( u \) and \( v \) satisfy condition (1) of Definition 3.4, then \( r(\lambda_j u, \lambda_j v) = r(u, v) \) if \( j = i \), and \( r(\lambda_j u, \lambda_j v) = 0 \) otherwise;

(D) If \( u \in \Lambda_0 \), \( r(u, \Delta) \) is the constant \( \alpha(u_0) \). (To see this, note that since \( u \cdot \Delta = \alpha(u_0) \Delta \), \( \alpha(u_n \Delta_n) = \alpha(u_0 \Delta_0) = \alpha(u_0) \alpha(\Delta_n) = 3^{-n} \alpha(u_0) \).

Using (A)–(D), Eq. (11) and the fact that \( \alpha(a_0) = 1 \) we find:

\[
\begin{align*}
  r(a, a) &= 1 - 27z + z \cdot r(9S(a), 9S(a)) \\
  &= 1 - 27z + z(27z + 9, 9 + 9\Delta, \Delta + 9) \\
  &= 1 - 27z + z(2r(b, b) - 9 - 9 + 81) \\
  &= 1 + 36z + 2z \cdot r(a, a).
\end{align*}
\]

So \( r(a, a) = (1 + 36z)/(1 - 2z) \), and the Hilbert–Kunz series of \( h = f(x_1, y_1) + f(x_2, y_2) \) is \( (1 + 36z)/(1 - 2z)(1 - 27z) \). From this rational description of \( \text{HKS}(h) \) we can obtain the Hilbert–Kunz multiplicity \( \mu \) of \( h \) as follows. Since \( e_n(h) = \mu \cdot 27^n + O(9^n), \sum_{n=0}^{\infty} (e_n(h) - \mu \cdot 27^n)z^n \) converges on a neighborhood of \( 1/27 \). So \( \text{HKS}(h) - \mu/(1 - 27z) \) is holomorphic on this neighborhood, and \( \mu = \lim_{z \to 1/27} (1 - 27z) \text{HKS}(h) = (1 + 36/27)/(1 - 2/27) = 63/25. \)
Turning to \( f(x_1, y_1) + g(x_2, y_2) \), we compute \( r(a, c) \) and \( r(b, c) \) in similar fashion. Noting that \( \alpha(b_0) = -1 \) and \( \alpha(c_0) = 1 \) we find:

\[
\begin{align*}
 r(a, c) &= 1 - 27z + z \cdot r(9S(a), 9S(c)) \\
 &= 1 - 27z + z \cdot r(b + \lambda_1 b + 9\Delta, c + \lambda_1 c + 6\Delta - 3\lambda_1 \Delta) \\
 &= 1 - 27z + z(2r(b, c) - 6 + 3 + 9 + 54) \\
 &= 1 + 33z + 2z \cdot r(b, c),
\end{align*}
\]

\[
\begin{align*}
 r(b, c) &= -1 + 27z + z \cdot r(9S(b), 9S(c)) \\
 &= -1 + 27z + z \cdot r(\lambda_2 a + \lambda_1 a - 9\lambda_2 \Delta, c + \lambda_1 c + 6\Delta - 3\lambda_1 \Delta) \\
 &= -1 + 27z + z(r(a, c) - 3) \\
 &= -1 + 24z + z \cdot r(a, c).
\end{align*}
\]

The above equations yield \( r(a, c) = (1 + 31z + 48z^2)/(1 - 2z^2) \), and the Hilbert–Kunz series of \( f(x_1, y_1) + g(x_2, y_2) \) is \( (1 + 31z + 48z^2)/(1 - 2z^2)(1 - 27z) \). In particular, the Hilbert–Kunz multiplicity is \( \psi(x, y)/(h) \cdot 2 \cdot 48 \). Explicitly, \( E_\lambda \) is the class of \((x, y)\), while if \( \beta \neq \lambda \), \( E_\beta \) is the class of \((x + y)^2, (x + \lambda y)(\lambda x + \beta y))\).

In [8] we defined an involution \( R : X_2 \rightarrow X_2 \), and noted that \( R \) fixes all the \( E_\beta \), \( \beta \in \mathbb{k} \cup \{\infty\} \), except for \( E_\lambda \). Now let \( Y \) consist of all the \( E_\beta \), together with \( O \) and \( R(E_\lambda) \), so that \( Y \) is stable under \( R \). In [8] we also introduced “magnification operators” \( \tau_i = \tau_{p|l(i,i,i)} : X_2 \rightarrow X_2, 0 \leq i \leq p \). It is not hard to see that \( Y \) is stable under these operators—see [6].

Now to each \( C \) in \( Y \) we attach a \( \varphi_C : \mathcal{I} \rightarrow \mathbb{Q} \) (and an \( e_C = \mathcal{L}(\varphi_C) \) in \( \Lambda_0 \)) as follows. In [8] we attached a function \( \varphi_I : \mathcal{I} \rightarrow \mathbb{Q} \) to each ideal \( I \) in \( \mathbb{k}[x, y] \) whose image in \( B \) has finite colength; namely \( \varphi_I(a/q) = q^{-\deg(I[q], h^a)} \). When \( C = E_\lambda \) (respectively \( R(E_\lambda) \)) we set \( \varphi_C = \varphi(x, y) \) (respectively \( \varphi(x, y) \)). For any other \( C \) we choose an ideal \( I \supset (h) \) of \( \mathbb{k}[x, y] \) whose image in \( B \) lies in \( C \), and we set \( \varphi_C(t) = \varphi_I(t) - (\deg I)t \). This \( \varphi_C \) is independent of the choice of \( I \), and \( \varphi_C(0) = \varphi_C(1) = 0 \). Using the fact that \( R(C) = C \) we find that \( \varphi_{\overline{C}} = \varphi_C \), so that \( R(e_C) = e_C \).

Suppose now that \( C \) is some \( E_\beta \). Let \( \varphi_I \) be the function attached as above to \( \tau_I(C) \in Y \). Set \( e_i = L(\varphi_i) \) for even \( i \), and \( e_i = L(\overline{\varphi_i}) \) for odd \( i \). (Note that when \( \beta = \lambda \), \( \varphi_C = \varphi(x, y) \).
is constant on $[1/2, 1]$, so that $\varphi_i = 0$ for $i \geq p/2$.) Lemma 3.6 of [8] shows that $T_{p;i}(p^2 \varphi_C) \approx \varphi_i$. Invoking Lemma 3.9 we find that

$$p^2 S(e_C) = \sum \lambda_i e_i + \sum (-1)^i a_i \lambda_i \Delta, \quad (14)$$

where the $a_i \in \mathbb{Q}$ and the sums run over all $i < p$ when $\beta \neq \lambda$, and all $i < p/2$ when $\beta = \lambda$.

To completely determine the shift rules, we need to calculate the $a_i$. This is straightforward when $\beta = 0, 1$ or $\infty$. (In these cases, $\varphi_C(t) = 4(t - t^2)$ and $\tau_i(C) = C$, for all $i$.) For other $\beta$, it can be shown that the $a_i$ can be expressed in terms of certain “syzygy gaps.” In [7], the first author shows that each of these syzygy gaps is 0 or 2, a result that yields the following theorem, whose proof we omit:

**Theorem 5.1.** If $p = 2m + 1$, then $a_i = (8m - i)$.

### 5.2. Example 2

Suppose $\mathbb{k} = \mathbb{Z}/7$, $f = x^3 + y^4$ and $g = x^4 + xy^3$. We shall calculate the Hilbert–Kunz series of $f(x_1, y_1) + g(x_2, y_2)$. Let $\xi, \eta, \varphi$ and $\psi$ be $\varphi_{x_3}, \varphi_{y_3}, \varphi_f$ and $\varphi_g$, respectively. Set $u = \mathcal{L}(\varphi)$, so that $u = \mathcal{L}(\xi) \mathcal{L}(\eta)$. We begin by working out the shift rule for $u$.

Evidently $\xi(t) = \min(3t, 1)$. So $7T_{7|2}\xi \approx \xi$, while $T_{7|i}\xi$ is linear for all other $i$. Lemma 3.9 then tells us that $7S(\mathcal{L}(\xi)) = \lambda_2 \mathcal{L}(\xi)$ + a linear combination of the $\lambda_i \Delta$. Now $7\mathcal{L}(\xi)$ is the sequence $(7\lambda_0, 3\lambda_0, 3\lambda_1, 3\lambda_2, \ldots )$. We conclude:

$$7S(\mathcal{L}(\xi)) = \lambda_2 \mathcal{L}(\xi) + (3\lambda_0 - 3\lambda_1) \Delta.$$

Similarly $\eta(t) = \min(4t, 1)$, and so $7T_{7|1}\eta \approx \eta$, while $T_{7|i}\eta$ is linear for all other $i$. Then $7S(\mathcal{L}(\eta)) = \lambda_1 \mathcal{L}(\eta)$ + a linear combination of the $\lambda_i \Delta$. Since $7\mathcal{L}(\eta) = (7\lambda_0, 4\lambda_0 - 3\lambda_1, \ldots )$, we find:

$$7S(\mathcal{L}(\eta)) = \lambda_1 \mathcal{L}(\eta) + (4\lambda_0 - 4\lambda_1) \Delta.$$

Multiplying the last two displayed formulas, using Eq. (6) and noting that $\Delta \cdot \Delta = \mathcal{L}(\xi) \cdot \mathcal{L}(\eta) \cdot \Delta = \mathcal{L}(\xi) \cdot \Delta = \Delta$, we find:

$$49S(u) = (\lambda_1 + \lambda_2 + \lambda_3)u + (21\lambda_0 - 16\lambda_1 + 9\lambda_2 - 2\lambda_3) \Delta. \quad (15)$$

Now set $a = \mathcal{L}(\psi)$ and $b = R(a)$. We shall work out the shift rules for $a$ and $b$. A linear change of variables takes $g$ into $h = xy(x + y)(x + 3y)$, and we are in the situation described in the paragraphs preceding this example. Let $M \in X_2(h)$ be the ideal class $E_3$ of the ideal $(x, y)$. Let $M^* = R(E_3)$ and $D = t_2(E_3)$. Then $a = e_M$ and $b = e_{M^*}$; let $d = e_D$.

One can show that the $\tau_i(M)$, $0 \leq i \leq 6$, are the classes $M, M^*, D, M, O, O$ and $O$, respectively. Equation (14) and Theorem 5.1 then give us:

$$49S(a) = (\lambda_0 + \lambda_1)a + \lambda_3 b + \lambda_2 d + (3\lambda_0 - 2\lambda_1 + \lambda_2)8 \Delta. \quad (16)$$

Applying $R$ we find:
\[ 49S(b) = \lambda_3 a + (\lambda_5 + \lambda_6)b + \lambda_4 d + (-\lambda_4 + 2\lambda_5 - 3\lambda_6)8\Delta. \] (17)

Furthermore it may be shown that the \( \tau_i(D) \), \( 0 \leq i \leq 6 \), are the classes \( D, M, M^*, D, M, M^* \) and \( D \), respectively. Equation (14) and Theorem 5.1 then give the shift rule:

\[ 49S(d) = (\lambda_4 + \lambda_5)a + (\lambda_1 + \lambda_2)b + (\lambda_0 + \lambda_3 + \lambda_6)d + (3\lambda_0 - 2\lambda_1 + \lambda_2 - \lambda_4 + 2\lambda_5 - 3\lambda_6)8\Delta. \] (18)

If \( v \) and \( v' \) are in \( \Lambda \), set

\[ r(v, v') = (1 - 343z) \cdot \sum_{n=0}^{\infty} \alpha(v_n v'_n) (2401z)^n. \]

Results analogous to (A)–(D) of Example 1 evidently hold, and the Hilbert–Kunz series of \( f(x_1, y_1) + g(x_2, y_2) \) is \( (1 - 343z)^{-1} \cdot r(u, a) \), since \( u = L(\varphi_f) \) and \( a = L(\varphi_g) \). Note that \( \alpha(u_0), \alpha(a_0), \alpha(b_0) \) and \( \alpha(d_0) \) are 1, 1, -1 and 0, respectively. (\( R(d) = d \) gives the last of these facts.) As in Example 1 we find:

\[ r(u, a) = 1 - 343z + z \cdot r(49S(u), 49S(a)), \]
\[ r(u, b) = -1 + 343z + z \cdot r(49S(u), 49S(b)), \]
\[ r(u, d) = z \cdot r(49S(u), 49S(d)). \]

Using the shift rules (15)–(18) and arguing as in Example 1 we get:

\[ r(u, a) = 1 + 490z + z(r(u, a) + r(u, b) + r(u, d)), \]
\[ r(u, b) = -1 + 339z + z \cdot r(u, a), \]
\[ r(u, d) = 831z + z(2r(u, b) + r(u, d)). \]

The above system gives \( r(u, a) = (1 + 488z + 679z^2 + 339z^3)/(1 - 2z - z^3) \). We conclude that the Hilbert–Kunz series of \( f(x_1, y_1) + g(x_2, y_2) \) is

\[
\frac{1 + 488z + 679z^2 + 339z^3}{(1 - 343z)(1 - 2z - z^3)}.
\]

6. The Hilbert–Kunz function of \( z^D - h(x, y) \)

We next investigate in more detail the Hilbert–Kunz function of \( g = z^D - h(x, y) \), where \( h \neq 0 \) is in the maximal ideal of \( k[[x, y]] \). We allow \( k \) to be infinite. Write \( D = p^c E \), with \( \gcd(E, p) = 1 \). A key result of this section, Theorem 6.7, is an estimate in a (one sided) neighborhood of \( 1/E \) for a certain function \( \psi : \mathcal{F} \to \mathbb{Q} \) constructed from \( h \). Then we will see how to express the \( e_n(g) \) in terms of values of \( \psi \) at two sequences of points, each of
which approaches $1/E$ as $n \to \infty$. We shall make use of a contraction operator having
$1/E$ as its fixed point, together with ideal class ideas from [8].

We may write $h = \prod_{i=1}^{r} h_i^{d_i}$, with the $h_i$ pairwise prime irreducibles. Let $G = \prod_{i=1}^{r} h_i$, and $X_2(G)$ be as in [8]. Choose a constant $K$ such that every ideal class $C$ in $X_2(G)$ contains an ideal which is the image of some $I \subseteq \mathbb{k}[x, y]$ with $\deg I \leq K$. Remark 3.3 of [8] shows this is possible. Given $C$ we fix such an $I = I_C$ and let $\Phi_C$ be the function $\mathcal{I} \to \mathbb{Q}$ attached to $I$, i.e., the function $(a_1/q, \ldots, a_r/q) \mapsto q^{-2} \deg(I[q], \prod_{i=1}^{r} h_i^{d_i})$. Note that $|\Phi_C| \leq \deg I \leq K$. If $C$ is the trivial class, $\Phi_C$ is linear, while if $C$ is the class of $(x, y)$, then $\Phi_C \approx \Phi_G$, where $\Phi_G(a_1/q, \ldots, a_r/q) = q^{-2} \deg(x^{q}, y^{q}, \prod_{i=1}^{r} h_i^{d_i})$.

**Definition 6.1.** Suppose $a = (a_1, \ldots, a_r)$ with $0 \leq a_i < d_i$. Let $X(a)$ consist of all $t = (t_1, \ldots, t_r)$ in $\mathcal{I}$ with each $d_i t_i$ in $[a_i, a_i + 1]$; the $X(a)$ cover $\mathcal{I}$. $\Phi : \mathcal{I} \to \mathbb{Q}$ is of $h$-type if for each $X(a)$ there is a $C$ in $X_2(G)$ such that the restriction of $\Phi$ to $X(a)$ is $t \mapsto \Phi_C(dt - a) + (linear)$, where $dt$ denotes the component-wise product of the vectors $d = (d_1, \ldots, d_r)$ and $t$.

Note that $\Phi_h : \mathcal{I} \to \mathbb{Q}$ with $\Phi_h(c_1/q, \ldots, c_r/q) = q^{-2} \deg(x^{q}, y^{q}, \prod_{i=1}^{r} h_i^{d_i})$ is of $h$-type. For if $a = (0, \ldots, 0)$, then on $X(a)$, $\Phi_h(t) = \Phi_G(dt)$, while $\Phi_h$ is the constant function 1 on all other $X(a)$.

**Lemma 6.2.** If $\Phi$ is of $h$-type, then for all $b = (b_1, \ldots, b_r)$ with $0 \leq b_i < p$, $\Psi = p^{2}T_{p|b} \Phi$ is of $h$-type.

**Proof.** What follows is essentially contained in Lemma 3.8 of [8]. Fix $a$ as in Definition 6.1 and write $a + db$ as $pa^* + b^*$, with $0 \leq b_i^* < p$. Note that if $t$ is in $X(a)$, then $(t + b)/p$ is in $X(a^*)$. Choose $C$ so that $\Phi(u) \approx \Phi_C(du - a^*)$ on $X(a^*)$. So $\Phi((t + b)/p) \approx \Phi_C(d(t + b)/p - a^*)$ on $X(a)$. Multiplying by $p^2$ we find that on $X(a)$, $\Psi(t) \approx p^2 \Phi_C((dt - a + b^*)/p) = p^2 T_{p|b} \Phi_C(dt - a)$. But [8] tells us that $p^2 T_{p|b} \Phi_C \approx \Phi_{C'}$, where $C' = \tau_{p|b}C$. Since this result holds for all $a$, $\Psi$ is of $h$-type.

Returning to the first paragraph of this section, we write $D = p^c E$ with $\gcd(E, p) = 1$. We shall assume $E \neq 1$. Let $\alpha$ be the order of $p$ in $(\mathbb{Z}/(E))^*$. Set $P = p^\alpha$ and write $P - 1$ as $E\gamma$.

**Definition 6.3.** $\Psi = p^{2c} T_{p^c|0, \ldots, 0} \Phi_h$ and $\Psi_n = J^n(\Psi)$, where $J$ is the operator $p^{2}T_{p|\gamma}$.

Let $\psi$ and $\psi_n : \mathcal{I} \to \mathbb{Q}$ be the compositions of $\Psi$ and $\Psi_n$ with the diagonal map $t \mapsto (t, \ldots, t)$. If we take $\varphi_h : \mathcal{I} \to \mathbb{Q}$ to be the function $a/q \mapsto q^{-2} \deg(x^{q}, y^{q}, h^a)$, then $\varphi_h$ is the composition of $\Phi_h$ and the diagonal map. It follows that $\psi = p^{2c} T_{p^c|0} \varphi_h$, and that $\psi_n = J^n(\psi)$, where $J$ is the operator $p^{2}T_{p|\gamma}$.

Note that the map $[0, 1] \mapsto [0, 1], w \mapsto (w + \gamma)/P$, stabilizes $\mathcal{I}$, has slope $1/P$ and fixes $E^{-1}$. So this map takes $E^{-1} + w$ to $E^{-1} + w/P$. Our plan is to use this contraction mapping to study the function $\psi$ in a right (respectively left) neighborhood of $E^{-1}$ in $\mathcal{I}$. 


Lemma 6.2 shows that each $\Psi_n$ is of $h$-type. So the restriction of $\Psi_n$ to any $X(a)$ has distance $\leq K$ from a linear function in the uniform metric (since each $|\Phi_C| \leq K$). Choose an open right neighborhood $(0, \epsilon)$ of 0 such that all points $(E^{-1} + w, \ldots, E^{-1} + w)$, with $w$ in this interval and $E^{-1} + w$ in $\mathcal{J}$, lie in a single $X(a)$. Then there exist $\lambda_n$ and $u_n$ such that for all such $w$, $|\Psi_n(E^{-1} + w, \ldots, E^{-1} + w) - \lambda_n w - u_n| \leq K$. In other words:

**Lemma 6.4.** $|\psi_n(E^{-1} + w) - \lambda_n w - u_n| \leq K$ whenever $w$ is small and positive and $E^{-1} + w$ is in $\mathcal{J}$.

**Remark 6.5.** One gets a similar result for small negative $w$, but it may not be possible to choose the same constants $\lambda_n$ and $u_n$. When no $d_i$ is a multiple of $E$, however, the point $(E^{-1}, \ldots, E^{-1})$ is an interior point of an $X(a)$, and so Lemma 6.4 holds for all $w$ in a small 2-sided neighborhood of 0.

**Lemma 6.6.** Suppose we are in the situation of Lemma 6.4. Then there are real numbers $\lambda$ and $u$ such that $\lambda_n = \lambda P^n + O(1)$ and $u_n = u P^{2n} + O(1)$.

**Proof.** Replacing $w$ by $w/P$ in Lemma 6.4 and multiplying by $P^2$ (noting that $\psi_{n+1}(E^{-1} + w) = P^2 \psi_n(E^{-1} + w/P)$, since $\psi_{n+1} = J(\psi_n)$), we find:

$$|\psi_{n+1}(E^{-1} + w) - P \lambda_n w - P^2 u_n| \leq P^2 K.$$ 

Combining this inequality with the inequality obtained by replacing $n$ by $n + 1$ in Lemma 6.4 we find that $|(\lambda_{n+1} - P \lambda_n) w + (u_{n+1} - P^2 u_n)| \leq P^2 K + K$. Since this holds for all $w$ in a fixed interval, $\lambda_{n+1} - P \lambda_n$ is $O(1)$, and consequently $u_{n+1} - P^2 u_n$ is $O(1)$. Then $\lambda_{n+1}/P^{n+1} - \lambda_n/P^n$ is $O(P^{-n})$; we conclude that $\lambda_n/P^n$ converges to some $\lambda$, and that $\lambda_n/P^n = \lambda + O(P^{-n})$. This gives the result for $\lambda_n$; that for $u_n$ is proved similarly. □

We now derive the desired estimate for $\psi$ in a neighborhood of $E^{-1}$.

**Theorem 6.7.** Let $u$ and $\lambda$ be as in Lemma 6.6. Then for $w > 0$, with $E^{-1} + w$ in $\mathcal{J}$, we have $\psi(E^{-1} + w) = u + \lambda w + O(w^2)$.

**Proof.** Fix $\epsilon > 0$ so that the result of Lemma 6.4 holds for $w$ in $(0, \epsilon)$. If $w$ is in $(0, \epsilon)$, choose $n$ so that $P^n w$ is in $[\epsilon/P, \epsilon)$. Then $\psi(E^{-1} + w) = P^{-2n} \psi_n(E^{-1} + P^n w)$. Now Lemma 6.4, with $w$ replaced by $P^n w$, together with Lemma 6.6, tells us that $\psi_n(E^{-1} + P^n w) - P^{2n} \lambda w - P^{2n} u$ is bounded independently of $w$. Thus $|\psi(E^{-1} + w) - \lambda w - u| \leq (\text{constant}) P^{-2n}$. Since $P^{-2n} = \omega^2(P^n w)^2$, and $|P^n w| \geq \epsilon/P$, we get the result. □

**Remark 6.8.** There is a similar result when $w < 0$, with a possibly different $\lambda$. When $E$ does not divide any $d_i$, Remark 6.5 shows that Theorem 6.7 holds for $w$ in a 2-sided neighborhood of 0.

We now estimate $e_n(g)$, where $g = z^D - h$. Recall that $D = p^c E$, with $\gcd(E, p) = 1$. If $n \geq c$ write $p^{n-c} = s_r E + r_n$, with $0 \leq r_n < E$. Then $q = p^n = s_n D + p^c r_n$. Note that $p^c s_n/q$ is close to $E^{-1}$ for large $n$. Observe in fact:
\[
\frac{pc^c s_n}{q} = \frac{pc^c}{q} \left( q - \frac{pc^c r_n}{D} \right) = \frac{1}{E} - \frac{p^{2c} r_n}{Dq},
\]

\[
\frac{pc^c (s_n + 1)}{q} = \frac{1}{E} - \frac{p^{2c} r_n}{Dq} + \frac{pc^c D}{Dq} = \frac{1}{E} + \frac{p^{2c} (E - r_n)}{Dq}.
\]

**Lemma 6.9.** Suppose that \( E > 1 \). Then:

1. \( \deg(x^q, y^q, h^s_n) = (up^{-2c})q^2 - (\lambda^+ r_n/D)q + O(1) \), for some \( u \) and \( \lambda^- \).
2. \( \deg(x^q, y^q, h^{s_n+1}) = (up^{-2c})q^2 + (\lambda^+ r_n/D)q + O(1) \), for some \( \lambda^+ \).

If \( E \) does not divide any \( d_i \), then \( \lambda^+ = \lambda^- \).

**Proof.**

\[ \deg(x^q, y^q, h^s_n) = q^2 \psi_h(s_n/q) = p^{-2c}q^2 \psi(pcs_n/q). \]

The observation above, combined with Remark 6.8, shows that \( \psi(pcs_n/q) = u - \lambda^- p^{2c} r_n/(Dq) + O(q^{-2}) \), for some \( u \) and \( \lambda^- \). Multiplying by \( p^{-2c}q^2 \) we get (1). A similar argument, using Theorem 6.7 itself, shows that \( \deg(x^q, y^q, h^{s_n+1}) = (u^* p^{-2c})q^2 + (\lambda^+ r_n/D)q + O(1) \), for some \( u^* \) and \( \lambda^+ \). It is easy to see that \( \deg(x^q, y^q, h^{s_n+1}) - \deg(x^q, y^q, h^s_n) = O(q) \), and it follows that \( u^* = u \). When \( E \) does not divide any \( d_i \) we use Remark 6.8 to see that \( \lambda^+ = \lambda^- \). \( \square \)

We now estimate \( e_n = e_n(g) \) for \( n \geq c \). An easy calculation shows that

\[ e_n = p^c ((E - r_n) \deg(x^q, y^q, h^s_n) + r_n \deg(x^q, y^q, h^{s_n+1})) \).

When \( E = 1 \), the \( r_n \) are all 0. So \( e_n = D \deg(x^q, y^q, h^q/D) \), and \( e_{n+1} = p^2 e_n \) for \( n \geq c \). Suppose \( E > 1 \). Lemma 6.9 then gives:

\[ e_n = (p^{-2c} Du)^q q^2 - \frac{(\lambda^- - \lambda^+) r_n (E - r_n)}{E} \cdot q + O(1). \]

We have proved:

**Theorem 6.10.** Let \( r_n \) be the remainder when \( p^{n-c} \) is divided by \( E \). Then there exist \( \mu \) and \( \mu_1 \) such that \( e_n(g) = \mu p^{2n} - \mu_1 r_n (E - r_n) p^n + O(1) \). If \( E = 1 \) or \( E \) does not divide any \( d_i \) then \( \mu_1 = 0 \).

When \( k \) is finite one can say more:

**Theorem 6.11.** Suppose \( k \) is finite. Then \( \mu \) and \( \mu_1 \) are rational and the \( O(1) \) term in Theorem 6.10 is eventually periodic.

**Proof.** This is immediate from Theorem 6.10 and the rationality of the Hilbert–Kunz series of \( g \), proved earlier. One can also give the argument sketched next, which avoids the heavy machinery used in proving rationality. Since \( k \) is finite, \( X_2(G) \) is a finite set, according to [8]. The functions \( \Psi_n \) of Definition 6.3 are all of \( h \)-type. Fix \( a = (a_1, \ldots, a_r) \). Then there must exist \( M \) and \( L \), \( L > 0 \), with \( \Psi_M \approx \Psi_{M+L} \) on \( X(a) \). Composing with the
diagonal map one finds that $\psi_M \approx \psi_{M+L}$ on a right neighborhood of $1/E$ and on a left neighborhood of $1/E$. The information obtained in this way about $\psi$, together with the formulas for $\text{deg}(x^q,y^q,h^{sn})$ and $\text{deg}(x^q,y^q,h^{sn+1})$ in terms of values of $\psi$, leads quickly to the desired results. □

The $p^n$ term in Theorem 6.10 can occur. Here are two examples. (The first, well known, goes back to Kunz.)

(1) Suppose $p \neq 5$ and $g = z^5 - x^5y^4$, so that $E = 5$ and the $d_i$ are 5 and 4. Then $\mu = 5$ and $\mu_1 = 1/5$.

(2) Suppose $p = 7$ and $g = z^{14} - x^6y^6(x^2 - y^2)$, so that $E = 2$ and the $d_i$ are 6, 6, 1, and 1. It can be shown that for $n \geq 2$, $e_n = (74/7)49^n - 6 \cdot 7^n - 42$. So $\mu = 74/7$ and $\mu_1 = 6$.

We conclude with a few remarks. Brenner [1] has results similar in character to Theorems 6.10 and 6.11. He requires homogeneity and normality, and so he never gets a $p^n$ term. On the other hand, he can treat the homogeneous coordinate ring of an arbitrary non-singular curve. (Also, the ideal used to define his Hilbert–Kunz functions need not be the homogeneous maximal ideal.) In order to get results like Theorem 6.11, Brenner, like us, has to assume that $k$ is finite. Whether this condition is really needed is unknown.

Our results, Brenner’s results, and the results of Huneke, McDermott and Monsky [4], suggest that analogues of Theorems 6.10 and 6.11 may hold for general Hilbert–Kunz functions in dimension 2. [4] gives a result like Theorem 6.10 in a general setting, but assuming normality, so that no $p^n$ term occurs. Very little is known when the normality assumption is dropped. The situation is of course completely different in higher dimensions.

References