Numerical Study of the Dispersion Properties of an X-band Backward Wave Oscillator with Rectangularly Rippled Wall Resonator

Ruhul Amin*a, Ghulam Saber, Rakibul Hasan Sagora

Abstract
Backward Wave Oscillators (BWOs) are devices that transform the electron beam energy into electromagnetic radiation at microwave frequencies. O-type BWOs consist of an axial electron beam propagating through a resonant cavity comprising of a slow wave structure (SWS). The electron beam is guided by a strong magnetic field while propagating through the cavity. The SWSs are used in BWOs in order to slow down the phase velocity of the electromagnetic wave so that the electron beam can resonantly interact with the wave which is the prerequisite of microwave generation. Dielectric loaded periodic structures were used as SWSs during the early stage of development of BWOs. However, this method of slowing down the wave suffers from dielectric breakdown as it cannot support high electric field. In order to prevent the dielectric breakdown, periodically corrugated metallic hollow waveguides are now being used as SWSs in BWOs. The sinusoidally corrugated SWS (SCSWS) has received the greatest attention among different corrugation profile. However, the fabrication process of this profile is complex and requires sophisticated tools. In an attempt to mitigate the fabrication problem, SWSs with simpler geometry have been proposed by the researchers. In this work, an earlier work on rectangularly corrugated SWS (RCSWS), which was investigated for non-relativistic electron beam has been extended for relativistic BWO in the X-band frequency range. The dispersion properties and the temporal growth rate of the axisymmetric transverse magnetic (TM) modes of have been analyzed numerically. In order to avoid the complicated boundary condition at the discontinuous boundaries of the rectangular radial profile, Fourier series has been used to approximate the axial profile of the SWS and the linear Rayleigh-Fourier theory has been utilized to determine the dispersion properties. The study shows that the growth rate of microwave for the RCSWS is somewhat lower than that for the case of SCSWS; however, a design tradeoff can be made to obtain a comparable performance.

Keywords: Backward Wave Oscillator, Rectangular Corrugation, Slow Wave Structure

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1. Introduction

The demand of Cherenkov type amplifiers and oscillators is growing due to their possible applications in varieties of civil and military fields such as high energy particle accelerators, industrial plasma heating, and high power radar (Benford, Swegle, & Schamiloglu, 2007; Blyakhman et al., 2007; Prater, 2004; Tantawi et al., 2005). With the evolution of the high power microwave (HPM) devices, BWOs and travelling wave tubes (TWTs) have drawn the attention of the research community as effective devices for the conversion of electron beam energy into microwave radiation. SWSs perform the task of reducing the velocity of the structure electromagnetic modes below the velocity of light so that annular electron beam can resonantly interact to produce microwave radiation. Periodically corrugated metallic walls which are currently used as SWS in BWOs and TWTs provide stability of operation, compact structure and better efficiency and output power (Bratman et al., 2010; Chen et al., 2002; Song et al., 2011; Xiao et al., 2010). SWSs can support waves both in forward and backward directions and are used in cross-field HPM devices as well (Dong, Dai-Bing, Fen, & Zhi-Kai, 2009; Lemke, Calico, & Clark, 1997). Recently, many researchers have performed extensive theoretical and experimental study on the BWOs with SCSWS (M. R. Amin et al., 1995; Minami et al., 1990). Efforts are being given by the BWO research community to enhance the output power and conversion efficiency of BWOs with SCSWS. V. Bratman et al. reported a peak output power of 4.3-5.3 GW with conversion efficiency 31-22% at 9.4 GHz and ~20ns pulse width (Bratman et al., 2011). Pulse compression technique has been proposed as a method of increasing the output power level. In some literatures wave propagation in cylindrical and rectangular SWS have been studied theoretically and experimentally (Amari, Vahldieck, Bornemann, & Leuchtmann, 1998; Jerby & Bekefi, 1993; Kesari, Jain, & Basu, 2005a, 2005b; Shi, Yuzheng, & Higo, 2001). Some of the researchers reported the dispersion characteristics of cold structures only that is the resonant coupling of electron beam and structure EM modes have not been considered (Hu, Tang, Chen, Lin, & Tong, 2001; Wang, Yang, Zhao, & Liang, 2005). Alternative radial profiles such as trapezoidal (M. R. Amin & Ogura, 2013; M. R. Amin, Ogura, Kojima, & Sagor, 2014), semi-circular (Saber, Sagor, & Amin, 2015), doubly rippled (M. R. Amin, 2013), oversized co-axial SWSs (Ogura, Yambe, Yamamoto, & Kobari, 2013) etc. are being investigated by the researchers nowadays. In (M. R. Amin & Ogura, 2007), rectangularly periodic cylindrical SWS was used with electron beam energy <100 keV and current around 200 A. Field matching technique with necessary boundary conditions had been utilized by the authors to obtain the dispersion characteristics. The analysis was done for low energy operation in the S-band frequency range with non-relativistic electron beam.

In this work, the earlier work (M. R. Amin & Ogura, 2007) on rectangularly corrugated SWS (RCSWS), which was investigated for non-relativistic electron beam has been extended by considering an infinitesimally thin annular relativistic electron beam typically used as the energy source for BWO operation. The parameters of the beam and SWS are so chosen that the device will operate in the X-band frequency range for the fundamental mode. The dispersion properties and the temporal growth rate of the axisymmetric transverse magnetic (TM) modes of have been analyzed numerically. In order to avoid the complicated boundary condition at the discontinuous boundaries of the rectangular radial profile, Fourier series has been used to approximate the axial profile of the SWS and the linear Rayleigh-Fourier theory has been utilized to determine the dispersion properties. The negative energy wave which is also known as slow space charge wave couples with the EM mode of SWS to produce instability which results in microwave radiation. Temporal growth rate (TGR) provides qualitative information about the strength of the generated microwave signal. In this study, the beam parameters have been varied and their effects on TGR and space charge wave modes (SCWM) have been studied. The zero-beam dispersion characteristics for different corrugation amplitude have been determined as well. Results obtained from the analysis reveal that the proposed RCSWS provide similar performance in terms of TGR and is far easier to fabricate than SCSWS which makes it an ideal candidate for practical implementations in BWO experiments. To the best of our knowledge, an X-band BWO with RCSWS has not been analysed so rigorously before.

2. Dispersion Relation of the RCSWS

The radial function, $R(z)$ defining the rectangular corrugation is given by,
\[
R(z) = R_0 + \left[ \frac{4h}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \left( (2n+1)(k_0 z + \frac{\pi}{2}) \right) \right]
\]  

where, \( R_0 \) is the average radius, \( h \) is the corrugation amplitude, \( k_0 = 2\pi/z_0 \) and \( z_0 \) is the period of corrugation. The axial variation of the radial function creates the periodic profile which as a consequence generates periodic dispersion curves. The waveguide wall has been assumed to be perfectly conducting and the infinitely strong magnetic field ensures the 1D motion of the electron beam in the axial direction. The intense relativistic electron beam (IREB) has been assumed to be infinitesimally thin.

The waveguide size parameters have been chosen such that the X-band regime operation of the BWO can be simulated. Based on the experimental results presented in (Butler, Wharton, & Furukawa, 1990; Carmel et al., 1992; Main et al., 1994) the selected beam parameters would not cause any beam transport related problems. The parameters are presented in table 1. A representative section of the RCSWS under investigation is depicted in fig. 1a. The axial profile of the SCSWS has also been given so that an analogy can be made. The structure parameters are given inside the figure. A 3D representation of the proposed RCSWS is presented in fig. 1b.

For beam free vacuum structure, the EM field components of the TM modes of the structure can be expressed by an infinite sum of the Floquet harmonics as:

\[
E_z(r,z,t) = \sum_{n=-\infty}^{\infty} E_{2n}(r)e^{i(k_{zn}z-\omega t)}
\]

where, \( k_{zn} = k_z + nK_0 \), \( n = 0, \pm 1, \pm 2, \cdots \) is the Floquet harmonic number and \( k_z \) is the axial wavenumber. Equation (2) signifies that an EM wave with angular frequency \( \omega \) in the SWS has infinite number of displaced spatial harmonic components with wavenumber \( k_{zn} = k_z + nK_0 \).

Using boundary conditions of EM fields and the continuity equation of the annular beam, the mathematical steps for deriving the dispersion relation of the BWO driven by an infinitesimally thin annular electron beam have been described in (Amin & Ogura, 2013). With IREB (16) of (Amin & Ogura, 2013) is,

\[
\sum_{m,n=-\infty}^{\infty} A_n [1 + (m-n)Q_n] K_n P_m^j + L_n P_m^N = \sum_{m,n=-\infty}^{\infty} D_{mn} \cdot A_n = 0
\]

where,

\[
Q_n = \frac{K_0 k_{zn}}{\beta_n^2}, \quad D_{mn} = [1 + (m-n)Q_n] (K_n P_m^j + L_n P_m^N),
\]

\[
P_m^j = \int_0^\pi \cos[(n-m)u] J_0(\beta_n R(z)) du \quad \text{and} \quad u = K_0 z
\]
\[ P_{mn}^N = \int_0^{\pi} \cos \left( (n-m)u \right) N_0 \left( \beta_n R(z) \right) \, du. \]

In (3) \( A_n \) is a column matrix represents the amplitude of Floquet harmonics. The dispersion relation results from the fact that in the solution of (5), at least some \( A_n \neq 0 \). Thus the generalized dispersion relation for the corrugated SWSs becomes,

\[ \det \left[ \sum_{m,n} D_{mn} \right] = 0 \]

Equation (4) is the required generalized expression for the analysis of dispersion characteristics.

### Table 1. The dimension parameters of the SCSWS and RCSWS

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( R_0 ) (cm)</th>
<th>( z_0 ) (cm)</th>
<th>( h ) (cm)</th>
<th>( K_0 ) (cm(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCSWS</td>
<td>1.50</td>
<td>1.67</td>
<td>0.41</td>
<td>3.76</td>
</tr>
<tr>
<td>RCSWS</td>
<td>1.50</td>
<td>1.67</td>
<td>0.28</td>
<td>3.76</td>
</tr>
</tbody>
</table>

3. Numerical Results

We have performed numerical calculations for both zero and non-zero beam dispersion characteristics. The zero beam dispersion characteristics can be determined from the real value of frequency and wavenumber while the non-zero beam dispersion characteristics require measurements of imaginary frequency. From the imaginary frequency TGR can be calculated which qualitatively signifies the microwave radiation strength.

3.1 Zero-beam dispersion characteristics

The operating frequency of BWOs can be roughly estimated from the intersection points of the dispersion curves and the beam lines. The dispersion curves for TM\(_{01}\), TM\(_{02}\) and TM\(_{03}\) modes have been determined numerically and presented in fig. 2a. Information regarding the dispersion properties of SWS is required in order to approximate the operating frequency before real experiment is performed. Therefore, the cold structure dispersion characteristics have been determined first. Fig. 2b presents the fundamental mode (TM\(_{01}\)) dispersion curve for different corrugation amplitude. The light line and beam lines are also drawn inside the figure in order to give an estimate of the probable oscillation frequencies of the BWO. It can be observed from the figure that the oscillation frequency decreases as the value of \( \beta \) is decreased. Where

\[ \beta = \sqrt{1 - \frac{1}{\left( 1 + \frac{\text{Beam Voltage}\,(\text{kV})}{\text{Light Velocity in Vacuum}} \right)^2}}. \]

Before comparing the TGR of two different SWSs, the zero-beam dispersion curves of the operating mode of both the SWSs should be matched. We have found that the TM\(_{01}\) mode dispersion curve for RCSWS with \( h=0.28\)cm matches well with the dispersion curve for SCSWS with \( h=0.41\)cm. Thus we have used \( h=0.28\)cm in rest of the analysis. Fig. 2b depicts the agreement of the dispersion curves for RCSWS and SCSWS. Information regarding the speed of axial energy transport can be obtained from the group velocity which is calculated from the derivative of the dispersion relation. The group velocities for TM\(_{01}\), TM\(_{02}\) and TM\(_{03}\) modes are presented in fig. 2c. If we determine the group velocity at the wavenumber where the dispersion curve intersects the beam lines, we would find that the value is negative. BWO has been named so because it operates in the negative group velocity regime. The energy transfer occurs in the opposite direction of the flow of electron beam.
3.2. Coupling of Beam Space Charge Wave with TM\textsubscript{01} Mode

The average radius, R\textsubscript{0} and the period of axial variation, z\textsubscript{0} play the dominant role in determining the oscillation frequency of the BWO. When the IREB is included in the analysis, the beam line split up into slow and fast SCWMs. The fast SCWM propagates at a velocity higher than the beam velocity whereas the slow SCWM propagates at lower velocity. The degree to which the slow and fast SCWM will go downward and upward from the beam line depends on the beam current and energy. The slow and fast SCWMs for $\beta = 0.70$ and $\beta = 0.60$ keeping beam current fixed at 0.5kA and beam radius at 0.8cm have been determined and depicted in fig. 3a and 3b. As the value of $\beta$ is decreased from 0.70 to 0.60, the slow and fast SCWMs moves further away from their corresponding beam line. The region of instability shrinks and the peak TGR reduces as well. As the beam line shifts in the right direction with the decrease of $\beta$, the oscillation frequency and the TGR reduces.
3.2b Temporal Growth Rate Study of the RCSWS

Microwave generation occurs due to the wave-particle interaction inside the BWO. From zero-beam analysis, it is possible to qualitatively estimate the oscillation frequency; however, information regarding the strength of the microwave radiation cannot be obtained. Inclusion of the IREB initiates the instability required for microwave generation. The magnitude of the imaginary frequency is called the TGR which provides information about the microwave signal strength. It has already been mentioned that the zero-beam dispersion curves of the operating mode should match in order to compare the TGR of RCSWS and SCSWS. The TGR for different values of $E$ has been presented in fig. 4a. As the value of $\beta$ is decreased the TGR shifts rightwards in the $k$-plane since the intersecting point of dispersion curve and beam line shifts rightwards. Figures 4b and 4c present the peak TGR (PTGR) for both RCSWS and SCSWS as functions of $E$ and beam current ($I_b$). The PTGR keeps on increasing as $\beta$ is increased and saturates at higher values of $\beta$. Similar phenomena can be observed in fig. 4c also. The PTGR of the proposed RCSWS can be increased by increasing the beam current; however, there is a certain limit above which the beam current cannot be increased due to transportation problem by IREB. From both the fig. 4b and 4c, it can be observed that the PTGR for the proposed RCSWS is very close to the PTGR for SCSWS. Therefore, RCSWS can be an ideal candidate as the SWS in BWO for real experiments.
The oscillation frequency estimated from the cold structure($I_b=0$) dispersion curve (fig. 2b) decreases due to the space charge effect of the IREB in BWO operation. The percentage of reduction of oscillation frequency from the estimated value is presented in table 2. The frequency corresponding to the PTGR for a specific value of $b$ is taken as the oscillation frequency for non-zero beam current.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>Percentage of reduction in oscillation frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ For $I_b=0$ kA</td>
<td>For $I_b=0.5$ kA</td>
</tr>
<tr>
<td>0.8</td>
<td>8.63</td>
</tr>
<tr>
<td>0.6</td>
<td>8.02</td>
</tr>
</tbody>
</table>

4. Discussion and Conclusion

A rigorous analysis on zero and non-zero beam dispersion characteristics of RCSWS has been presented. The cold structure dispersion characteristics provide information regarding the expected oscillation frequency while the non-zero beam dispersion characteristics give idea about the strength of the microwave signal as well as SCMWs. The SCWMs and PTGR have been calculated by varying the beam current and voltages and the zero-beam dispersion characteristics have been calculated for different corrugation amplitude. Results obtained from our analysis exhibit that the RCSWS provides almost similar performance as the SCSWS. In addition, while the fabrication of SCSWS requires sophisticated tools, the RCSWS can be fabricated easily using the sections of metallic cylinders with different inner radii. Thus it can concluded that the proposed RCSWS will be useful to the BWO research community.

References


